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A KAC-FEYNMAN INTEGRAL EQUATION FOR CONDITIONAL WIENER INTEGRALS

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ABSTRACT. Let $F(x) = \exp\{\int_0^t \theta(s, \int_0^s h(u)\,dx(u))\,ds\}$, for x an element of Wiener space $C[0,T]$ and potential function $\theta(\cdot,\cdot) : [0,T] \times \mathbf{R} \to \mathbf{C}$. In this paper we show that the conditional Wiener integral, $E(F|X)$, with conditioning function $X(x) = \int_0^t h(u)\,dx(u)$, satisfies the Kac-Feynman integral equation. We also consider vector-valued conditioning functions $X(x)$, as well as potentials $\theta(s,\cdot)$ that are Fourier-Stieltjes transforms of Borel measures on $\mathbf{R}$.

1. Introduction. Let $(C[0,T], \mathcal{F}, m_w)$ denote Wiener space where $C[0,T]$ is the space of all continuous functions x on $[0,T]$ with $x(0) = 0$. Let $F(x)$ be a Wiener integrable function on $C[0,T]$, and let $X(x)$ be a Wiener measurable function on $C(0,T]$. In [**10**], Yeh introduced the concept of the conditional Wiener integral of F given X, $E(F|X)$, and for the case $X(x) = x(T)$ obtained some very useful results including a Kac-Feynman integral equation. Further work involving conditional Wiener integrals include [**3, 4, 8,** and **11**]. In [**9**], Park and Skoug extended the theory to include very general conditioning functions, including conditioning functions of the form $X(x) = (\int_0^T \alpha_1(s)\,dx(s), \dots, \int_0^T \alpha_n(s)\,dx(s))$.

A very important class of functions in quantum mechanics are functions on $C[0,T]$ of the form

$$G(x) = \exp\left\{\int_0^T \theta(s, x(s))\,ds\right\}$$

where $\theta : [0,T] \times \mathbf{R} \to \mathbf{C}$. These functions are clearly contained in the class of functions of the form

(1.1)
$$F(x) = \exp\left\{\int_0^T \theta\left(s, \int_0^s h(u)\,dx(u)\right) ds\right\}, \quad h \in L_2[0,T],\ h \neq 0 \text{ a.e.}$$

In Section 3 below, with $a(t) \equiv \int_0^t h^2(u)\,du$, and with appropriate conditions on the potential function θ, we show that the function

$$H(t,\xi) \equiv (2\pi a(t))^{-1/2} \exp\left\{-\frac{\xi^2}{2a(t)}\right\} \cdot E\left(\exp\left\{\int_0^t \theta\left(s, \int_0^s h(u)\,dx(u)\right) ds\right\} \Bigg| \int_0^t h(u)\,dx(u) = \xi\right) \tag{1.2}$$

satisfies the Kac-Feynman integral equation

$$\begin{aligned} H(t,\xi) = {} & (2\pi a(t))^{-1/2} \exp\left\{-\frac{\xi^2}{2a(t)}\right\} \\ & + \int_0^t [2\pi(a(t)-a(s))]^{-1/2} \int_{\mathbf{R}} \theta(s,\eta) H(s,\eta) \\ & \cdot \exp\left\{-\frac{(n-\xi)^2}{2(a(t)-a(s))}\right\} d\eta ds \end{aligned} \tag{1.3}$$

on $(0,T) \times \mathbf{R}$.

Using an inversion formula, Yeh [**10**] derived the Kac-Feynman integral equation for time-independent continuous potential functions $\theta(\xi)$ with conditioning function $X(x) = x(t)$. We state his result at the beginning of Section 3 below. In [**4**], Chung and Kang, using the same inversion formula, obtained similar results for bounded potentials $\theta(s,\xi)$ and with $X(x) = x(t)$. In Section 2 of this paper we derive a simple formula for expressing conditional Wiener integrals in terms of ordinary (i.e., nonconditional) Wiener integrals; see equations (2.6) and (2.7) below. We then use this formula, instead of the inversion formula, to derive the Kac-Feynman integral equation. Even for very general potential $\theta(s,\xi)$, our approach is much simpler than using the inversion formula method. In addition, as we will see in Section 4, this simplification allows us to evaluate conditional expectations with vector-valued conditioning functions.

In Section 5 we consider the case where the potentials $\theta(s,\cdot)$'s are Fourier-Stieltjes transforms of Borel measures on $\mathbf{R}$. Finally, we state the following well-known integration formula,

$$(1.4)\quad \left[\frac{b}{2\pi}\right]^{1/2}\int_{\mathbf{R}}\exp\left\{-\frac{bu^2}{2}+iuv\right\}du=\exp\left\{-\frac{v^2}{2b}\right\},\quad \operatorname{Re} b>0$$

which we use in Section 5.

2. A simple formula for generalized conditional Wiener integrals. Let $h \in L_2[0,T]$ with $h \neq 0$ a.e. on $[0,T]$. Then, the stochastic integral

$$(2.1)\qquad z(t)\equiv\int_0^t h(s)\,dx(s),\quad 0\le t\le T,$$

is a Gaussian process with mean zero and covariance

$$E(z(s)z(t))=\int_0^{s\wedge t}h^2(u)\,du$$

where $s\wedge t$ is the minimum of s and t. For each partition $\tau=\tau_n=\{t_1,\dots,t_n\}$ of $[0,T]$ with $0=t_0<t_1<\cdots<t_n=T$, let $Z_\tau : C[0,T]\to\mathbf{R}^n$ be defined by

$$Z_\tau(x)=\left(\int_0^{t_1}h(s)\,dx(s),\dots,\int_0^{t_n}h(s)\,dx(s)\right)=(z(t_1),\dots,z(t_n)).$$

Let $a(t)\equiv\int_0^t h^2(s)\,ds$, and define the function $[z]$ on $[0,T]$ by

$$(2.2)\qquad \begin{gathered}[z](t)=z(t_{j-1})+\frac{a(t)-a(t_{j-1})}{a(t_j)-a(t_{j-1})}(z(t_j)-z(t_{j-1})),\\ t_{j-1}\le t\le t_j,\ j=1,\dots,n.\end{gathered}$$

Similarly, for $\vec{\xi}=(\xi_1,\dots,\xi_n)\in\mathbf{R}^n$, define the function $[\vec{\xi}]$ of $\vec{\xi}$ on $[0,T]$ by

$$(2.3)\qquad \begin{gathered}[\vec{\xi}](t)=\xi_{j-1}+\frac{a(t)-a(t_{j-1})}{a(t_j)-a(t_{j-1})}(\xi_j-\xi_{j-1}),\\ t_{j-1}\le t\le t_j,\ j=1,\dots,n,\ \text{and } \xi_0=0.\end{gathered}$$

Theorem 1. *Let $\{x(t), 0 \le t \le T\}$ be the standard Wiener process, and let $z(t)$ and $x(t)$ be related as above. Then the process $\{z(t) - [z](t), 0 \le t \le T\}$ and $Z_\tau(x) = (z(t_1), \ldots, z(t_n))$ are (stochastically) independent.*

Proof. Let $t_{j-1} \le t \le t_j$. Then $E[(z(t) - [z](t))z(t_i)] = 0$ for $i = 1, \ldots, n$. Thus, $\{z(t) - [z](t), t_{j-1} \le t \le t_j\}$ is independent of $Z_\tau(x)$. Since j was arbitrary, the proof is complete. □

Corollary 2. *The process $\{z(t) - [z](t), 0 \le t \le T\}$ vanishes at each partition point t_j, $j = 1, \ldots, n$, and the processes $\{z(t) - [z](t), t_{j-1} \le t \le t_j\}$, $j = 1, \ldots, n$ are independent Gaussian processes.*

We note that $\{z(t) - [z](t), t_{j-1} \le t \le t_j\}$ behaves very much like Brownian bridge. Indeed, it becomes the Brownian bridge process if $h(t) \equiv 1$ on $[0, T]$. Also note that the Gaussian process $\{z(t), 0 \le t \le T\}$ doesn't have stationary increments unless h is constant.

The next theorem is crucial to the development of our simple formula for conditional Wiener integrals.

Theorem 3. *Let $z(t)$ and $x(t)$ be related as in Theorem 1. If $F(z(\cdot)) = F(\int_0^{\cdot} h(u)\, dx(u))$ is Wiener integrable with respect to x on $C[0, T]$, then, for any Borel measurable set B in $\mathbf{R}^n$,*

$$\mu_\tau(B) \equiv \int_{Z_\tau^{-1}(B)} F(z(\cdot)) m_W\,(dx) = \int_B E[F(z(\cdot) - [z](\cdot) + [\vec{\xi}](\cdot))] P_{Z_\tau}\,(d\vec{\xi}) \tag{2.4}$$

where $P_{Z_\tau}(B) = m_W(Z_\tau^{-1}(B))$.

Proof. First assume that F is the characteristic function of a Wiener measurable set A, i.e., $F(z) = \chi_A(z)$. Then

$$\begin{aligned}\int_{Z_\tau^{-1}(B)} \chi_A(z) m_W\,(dx) &= m_W(A \cap Z_\tau^{-1}(B)) \\ &= \int_B m_W(z \in A \mid Z_\tau(x) = \vec{\xi}) P_{Z_\tau}\,(d\vec{\xi}) \\ &= \int_B m_W(z - [z] + [\vec{\xi}] \in A \mid Z_\tau(x) = \vec{\xi}) P_{Z_\tau}\,(d\vec{\xi}).\end{aligned}$$

But $z - [z]$ and $Z_\tau(x)$ are independent by Theorem 1. Hence,

$$\begin{aligned}\int_{Z_\tau^{-1}(B)} \chi_A(z) m_W\,(dx) &= \int_B m_W(z - [z] + [\vec{\xi}] \in A) P_{Z_\tau}\,(d\vec{\xi}) \\ &= \int_B E[\chi_A(z - [z] + [\vec{\xi}])] P_{Z_\tau}\,(d\vec{\xi}).\end{aligned}$$

Therefore, the result holds for measurable characteristic functions. The general case follows by the usual argument in integration theory. □

Let $F(\int_0^\bullet h(u)\,dx(u)) \in L_1(C[0,T], m_W)$. Then the conditional Wiener integral of F given Z_τ, denoted by

$$E\left(F\left(\int_0^\bullet h(u)\,dx(u)\right) \middle| Z_\tau(x) = \xi\right),$$

is a Lebesgue measurable function of $\vec{\xi}$, unique up to null sets in $\mathbf{R}^n$, satisfying the equation

$$\begin{aligned}\int_{Z_\tau^{-1}(B)} F\left(\int_0^\bullet h(u)\,dx(u)\right) m_W\,(dx) \\ = \int_B E\left(F\left(\int_0^\bullet h(u)\,dx(u)\right) \middle| Z_\tau(x) = \vec{\xi}\right) P_{Z_\tau}\,(d\vec{\xi})\end{aligned} \tag{2.5}$$

for all Borel sets B in $\mathbf{R}^n$.

From (2.4) and (2.5) we obtain a very simple formula for conditional Wiener integrals, namely, that

$$E(F(z) | Z_\tau(x) = \vec{\xi}) = E[F(z - [z] + [\vec{\xi}])] \tag{2.6}$$

for a.e. $\vec{\xi}$ in $\mathbf{R}^n$. In particular, if τ consists of a single point T, then (2.6) becomes

$$E\bigg(F\bigg(\int_0^{\cdot} h(u)\,dx(u)\bigg)\bigg|\int_0^T h(u)\,dx(u)=\xi\bigg) = E\left[F\bigg(\int_0^{\cdot} h(u)\,dx(u)-\frac{a(\cdot)}{a(T)}\int_0^T h(u)\,dx(u)+\frac{a(\cdot)}{a(T)}\xi\bigg)\right]. \tag{2.7}$$

Formulas (2.6) and (2.7) are very useful in application because they allow us to evaluate conditional Wiener integrals by first expressing them in terms of ordinary Wiener integrals and then evaluating these Wiener integrals. We give two examples illustrating this technique.

Example 1. Let $F(x)=\int_0^T\int_0^t h(u)\,dx(u)\,dt$, where h is as before. Then

$$E\bigg(\int_0^T\int_0^t h(u)\,dx(u)\,dt|Z_\tau(x)=\vec{\xi}\bigg) = E\bigg[\int_0^T (z(t)-[z](t)+[\vec{\xi}](t))\,dt\bigg] \\ = \int_0^T [\vec{\xi}](t)\,dt = \sum_{j=1}^n \int_{t_{j-1}}^{t_j}\bigg(\xi_{j-1}+\frac{a(t)-a(t_{j-1})}{a(t_j)-a(t_{j-1})}(\xi_j-\xi_{j-1})\bigg)\,dt.$$

In particular, if $h(t)\equiv 1$, then $a(t)=t$ and, hence, the above equals $(1/2)\sum_{j=1}^n(\xi_j+\xi_{j-1})(t_j-t_{j-1})$, which agrees with the computation in [**8**].

Example 2. Let $F(x)=\exp\{\int_0^T\int_0^t h(u)\,dx(u)\,dt\}$. Then

$$E\bigg(\exp\bigg\{\int_0^T\int_0^t h(u)\,dx(u)\,dt\bigg\}\bigg|Z_\tau(x)=\vec{\xi}\bigg) \\ = E\bigg[\exp\bigg\{\int_0^T (z(t)-[z](t)+[\vec{\xi}](t))\,dt\bigg\}\bigg] \\ = \exp\bigg\{\int_0^T [\vec{\xi}](t)\,dt\bigg\}E\bigg[\exp\bigg\{\int_0^T (z(t)-[z](t))\,dt\bigg\}\bigg].$$

Hence, for each fixed $y \in C[0,T]$, we have, as expected,

$$\lim_{||\tau||\to 0} E\bigg(\exp\bigg\{ \int_0^T \int_0^t h(u)\, dx(u)\, dt \bigg\} \bigg| Z_\tau(x) = Z_\tau(y) \bigg) = \exp\bigg\{ \int_0^T \int_0^t h(u)\, dy(u)\, dt \bigg\}.$$

3. The Kac-Feynman integral equation. Under the assumption that V is a nonnegative continuous function on $\mathbf{R}$ satisfying the condition

$$\int_{\mathbf{R}} V(\xi) \exp\bigg\{ -\frac{\xi^2}{2t} \bigg\}\, d\xi < \infty \quad \text{for every } t > 0,$$

Yeh [**10**] has shown that the function U defined on $(0,\infty) \times \mathbf{R}$ by

(3.1)
$$U(t,\xi) = E\bigg(\exp\bigg\{ -\int_0^t V(x(s))\, ds \bigg\} \bigg| x(t) = \xi \bigg) (2\pi t)^{-1/2} \exp\bigg\{ -\frac{\xi^2}{2t} \bigg\}$$

satisfies the Kac-Feynman integral equation

(3.2)
$$U(t,\xi) = (2\pi t)^{-1/2} \exp\bigg\{ -\frac{\xi^2}{2t} \bigg\} - \int_0^t [2\pi(t-s)]^{-1/2} \int_{\mathbf{R}} V(\eta) U(s,\eta) \exp\bigg\{ -\frac{(\xi-\eta)^2}{2(t-s)} \bigg\}\, d\eta\, ds,$$

whose solution is given by

$$U(t,\xi) = \sum_{k=0}^{\infty} (-1)^k U_k(t,\xi), \quad (t,\xi) \in (0,\infty) \times \mathbf{R},$$

where the sequence $\{U_k\}$ is defined inductively by

$$U_0(t,\xi) = (2\pi t)^{-1/2}\exp\left\{-\frac{\xi^2}{2t}\right\},$$

$$U_{k+1}(t,\xi) = \int_0^t [2\pi(t-s)]^{-1/2}\int_{\mathbf{R}} V(\eta)U_k(s,\eta)\exp\left\{-\frac{(\xi-\eta)^2}{2(t-s)}\right\} d\eta\, ds$$

for $k = 0, 1, \ldots$.

The following theorem is a substantial improvement over the above result.

Theorem 4. *Let* $h \in L_2[0,T]$ *with* $h \neq 0$ a.e. *on* $[0,T]$. *Let* $a(t) = \int_0^t h^2(u)\,du$, *and let* $\theta(t,\xi)$ *be a complex-valued Lebesgue measurable function on* $[0,T]\times\mathbf{R}$ *such that*

$$(3.3)\quad F(x) = \exp\left\{\int_0^t \theta\left(s, \int_0^s h(u)\,dx(u)\right) ds\right\} \in L_1(C[0,T], m_W),$$

for each $t \in [0,T]$, *and*

$$(3.4)\qquad \int_{\mathbf{R}} |\theta(s,\xi)|\exp\left\{-\frac{\xi^2}{2a(t)}\right\} d\xi < \infty$$

for all $(s,t) \in (0,T]^2$. *Then the function* H *defined on* $(0,T]\times\mathbf{R}$ *by* (1.2) *satisfies the Kac-Feynman integral equation* (1.3) *whose solution is given by*

$$(3.5)\qquad H(t,\xi) = \sum_{k=0}^{\infty} H_k(t,\xi), \quad (t,\xi) \in (0,T]\times\mathbf{R},$$

where the sequence $\{H_k\}$ *is given inductively by*

$$H_0(t,\xi) = [2\pi a(t)]^{-1/2}\exp\left\{-\frac{\xi^2}{2a(t)}\right\},$$

$$(3.6)\quad H_{k+1}(t,\xi) = \int_0^t [2\pi(a(t)-a(s))]^{-1/2} \cdot \int_{\mathbf{R}} \exp\left\{-\frac{(\eta-\xi)^2}{2(a(t)-a(s))}\right\}\theta(s,\eta)H_k(s,\eta)\,d\eta\,ds.$$

Proof. For $(t,\xi) \in (0,T] \times \mathbf{R}$ let

$$I = I(t,\xi) = E\left(\exp\left\{ \int_0^t \theta\left(s, \int_0^s h(u)\,dx(u)\right) ds \right\} \Big| \int_0^t h(u)\,dx(u) = \xi \right). \tag{3.7}$$

By differentiating the function $\exp\{\int_0^s \theta(u, \int_0^u h(v)\,dx(v))\,du\}$ with respect to s and then integrating the derivative on $[0,t]$, we obtain

$$\begin{aligned} &\exp\left\{ \int_0^t \theta\left(s, \int_0^s h\,dx\right) ds \right\} \\ &\qquad = 1 + \int_0^t \theta\left(s, \int_0^s h\,dx\right) \exp\left\{ \int_0^s \theta\left(u, \int_0^u h\,dx\right) du \right\} ds. \end{aligned} \tag{3.8}$$

Next, taking conditional expectations and then using (3.7), (2.7) and the Fubini theorem, we obtain

$$\begin{aligned} I(t,\xi) = 1 + \int_0^t E\Bigg[\theta\left(s, \int_0^s h\,dx - \frac{a(s)}{a(t)} \int_0^t h\,dx + \frac{a(s)}{a(t)}\xi\right) \\ \cdot \exp\left\{ \int_0^s \theta\left(u, \int_0^u h\,dx - \frac{a(u)}{a(t)} \int_0^t h\,dx + \frac{a(u)}{a(t)}\xi\right) du \right\}\Bigg] ds. \end{aligned} \tag{3.9}$$

Also we note that

$$\begin{aligned} &\int_0^s \theta\left(u, \int_0^u h\,dx - \frac{a(u)}{a(t)} \int_0^t h\,dx + \frac{a(u)}{a(t)}\xi\right) du \\ &\quad = \int_0^s \theta\Bigg(u, \int_0^u h\,dx - \frac{a(u)}{a(s)} \int_0^s h\,dx + \frac{a(u)}{a(s)} \\ &\qquad\qquad \cdot \left[\int_0^s h\,dx - \frac{a(s)}{a(t)} \int_0^t h\,dx + \frac{a(s)}{a(t)}\,\xi \right]\Bigg) du. \end{aligned} \tag{3.10}$$

But, for $0 < u < s < t$, the Gaussian random variables

$$\int_0^s h\,dx - \frac{a(s)}{a(t)} \int_0^t h\,dx \sim N\left(0, \sigma^2 \equiv a(s)\left[1 - \frac{a(s)}{a(t)}\right]\right)$$
$$\int_0^u h\,dx - \frac{a(u)}{a(s)} \int_0^s h\,dx \sim N\left(0, a(u)\left[1 - \frac{a(u)}{a(s)}\right]\right)$$

are independent, and so we may use (3.10) in (3.9) to obtain
(3.11)

$$\begin{aligned} I(t,\xi) = 1 + \int_0^t (2\pi\sigma^2)^{-1/2} \int_{\mathbf{R}} \exp\left\{-\frac{(\eta - \xi a(s)/a(t))^2}{2\sigma^2}\right\}\theta(s,\eta) \\ \cdot E\bigg(\exp\bigg\{\int_0^s \theta\bigg(u, \int_0^u h\,dx - \frac{a(u)}{a(s)}\int_0^s h\,dx \\ + \frac{a(u)}{a(s)}\eta\bigg)\,du\bigg\}\bigg)\,d\eta\,ds \\ = 1 + \int_0^t (2\pi\sigma^2)^{-1/2} \int_{\mathbf{R}} \exp\left\{-\frac{(\eta - \xi a(s)/a(t))^2}{2\sigma^2}\right\}\theta(s,\eta) \\ \cdot E\bigg(\exp\bigg\{\int_0^s \theta\bigg(u, \int_0^u h\,dx\bigg)\,du\bigg\}\bigg|\int_0^s h\,dx = \eta\bigg)\,d\eta\,ds. \end{aligned}$$

Also, it is not hard to verify that
(3.12)

$$\begin{aligned} (2\pi a(t))^{-1/2}\exp\left\{-\frac{\xi^2}{2a(t)}\right\}(2\pi\sigma^2)^{-1/2}\exp\left\{-\frac{(\eta - \xi a(s)/a(t))^2}{2\sigma^2}\right\} \\ \cdot (2\pi a(s))^{1/2}\exp\left\{\frac{\eta^2}{2a(s)}\right\} \\ = [2\pi(a(t)-a(s))]^{-1/2}\exp\left\{-\frac{(\eta-\xi)^2}{2(a(t)-a(s))}\right\}. \end{aligned}$$

Thus, multiplying equation (3.11) by $(2\pi a(t))^{-1/2}\exp\{-\xi^2/2a(t)\}$, and then, using (3.12), (3.7) and (1.2), we obtain the integral equation (1.3).

To prove (3.5), first assume that $\theta(s,\xi)$ is bounded, say

$$|\theta(s,\xi)| \le M \quad \text{on } [0,T]\times\mathbf{R}.$$

Then, by induction, one can easily verify that

$$|H_k(t,\xi)| \le \frac{(Mt)^k}{k!}H_0(t,\xi) \quad \text{for } k = 0,1,2,\dots .$$

Thus,

$$\sum_{k=0}^{\infty}|H_k(t,\xi)| \le H_0(t,\xi)\exp\{Mt\},$$

and so, in view of (3.6), $\sum_{k=0}^{\infty} H_k(t,\xi)$ satisfies the integral equation. Thus, the proof is complete when θ is bounded. If θ is unbounded, we can use the truncation method to get the same result. $\square$

We also have that

$$E\bigg(\exp\bigg\{\int_0^t \theta\bigg(s, \int_0^s h(u)\,dx(u)\bigg)\,ds\bigg\}\bigg) = \int_{\mathbf{R}} H(t,\xi)\,d\xi.$$

4. Vector-valued conditional expectations. In this section we consider the vector-valued conditional expectation

$$J \equiv E\bigg(\exp\bigg\{\int_0^T \theta\bigg(s, \int_0^s h(u)\,dx(u)\bigg)\,ds\bigg\}\bigg|\int_0^{t_j} h(u)\,dx(u) = \xi_j,\ j=1,\dots,n\bigg) \tag{4.1}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and θ is as in Theorem 4. We first note that, by (2.6), Corollary 2 and (2.2),

$$\begin{aligned} J &= E\bigg(\exp\bigg\{\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \theta(s, z(s) - [z](s) + [\vec{\xi}](s))\,ds\bigg\}\bigg) \\ &= \prod_{j=1}^n E\bigg(\exp\bigg\{\int_{t_{j-1}}^{t_j} \theta(s, z(s) - [z](s) + [\vec{\xi}](s))\,ds\bigg\}\bigg) \\ &= \prod_{j=1}^n E\bigg(\exp\bigg\{\int_{t_{j-1}}^{t_j} \theta\bigg(s, \int_{t_{j-1}}^s h(u)\,dx(u) + \xi_{j-1}\bigg)\,ds\bigg\}\bigg| \\ &\qquad\qquad \cdot \int_{t_{j-1}}^{t_j} h(u)\,dx(u) = \xi_j - \xi_{j-1}\bigg). \end{aligned} \tag{4.2}$$

Proceeding as in Section 3, it is easy to show that the function $H^*(t,\xi;t_{j-1},\xi_{j-1})$, as a function of (t,ξ) defined on $[t_{j-1},t_j]\times\mathbf{R}$ by

(4.3)
$$H^*(t,\xi;t_{j-1},\xi_{j-1}) = E\bigg(\exp\bigg\{ \int_{t_{j-1}}^{t} \theta\Big(s, \int_{t_{j-1}}^{s} h\,dx + \xi_{j-1}\Big)\,ds \bigg\} \bigg| \int_{t_{j-1}}^{t} h\,dx = \xi - \xi_{j-1} \bigg) \cdot [2\pi(a(t)-a(t_{j-1}))]^{-1/2} \exp\bigg\{ -\frac{(\xi-\xi_{j-1})^2}{2(a(t)-a(t_{j-1}))} \bigg\},$$

satisfies the Kac-Feynman integral equation

$$\begin{aligned} H^*(t,\xi;t_{j-1},\xi_{j-1}) &= [2\pi(a(t)-a(t_{j-1}))]^{-1/2} \exp\bigg\{ -\frac{(\xi-\xi_{j-1})^2}{2(a(t)-a(t_{j-1}))} \bigg\} \\ &\quad + \int_{t_{j-1}}^{t} [2\pi(a(t)-a(s))]^{-1/2} \int_{\mathbf{R}} \exp\bigg\{ -\frac{(\eta-\xi)^2}{2(a(t)-a(s))} \bigg\} \\ &\quad \cdot \theta(s,\eta) H^*(s,\eta;t_{j-1},\xi_{j-1})\,d\eta\,ds \end{aligned}$$

whose solution is readily given by

$$H^*(t,\xi;t_{j-1},\xi_{j-1}) = \sum_{k=0}^{\infty} H_k^*(t,\xi;t_{j-1},\xi_{j-1}),$$

where the sequence $\{H_k^*\}$ is given inductively by

$$\begin{aligned} H_0^*(t,\xi;t_{j-1},\xi_{j-1}) &= [2\pi(a(t)-a(t_{j-1}))]^{-1/2} \\ &\quad \cdot \exp\bigg\{ -\frac{(\xi-\xi_{j-1})^2}{2(a(t)-a(t_{j-1}))} \bigg\}, \\ H_{k+1}^*(t,\xi;t_{j-1},\xi_{j-1}) &= \int_{t_{j-1}}^{t} [2\pi(a(t)-a(s))]^{-1/2} \\ &\quad \cdot \int_{\mathbf{R}} \exp\bigg\{ -\frac{(\eta-\xi)^2}{2(a(t)-a(s))} \bigg\} \\ &\quad \cdot \theta(s,\eta) H_k^*(s,\eta;t_{j-1},\xi_{j-1})\,d\eta\,ds. \end{aligned}$$

Using (4.1), (4.2), and (4.3), we finally have the formula

$$E\bigg(\exp\bigg\{\int_0^T \theta\bigg(s, \int_0^s h(u)\,dx(u)\bigg)\,ds\bigg\}\bigg|\int_0^{t_j} h(u)\,dx(u) = \xi_j,\; j = 1, \dots, n\bigg)$$
$$= \prod_{j=1}^n [2\pi(a(t_j) - a(t_{j-1}))]^{1/2} \exp\bigg\{\frac{(\xi_j - \xi_{j-1})^2}{2(a(t_j) - a(t_{j-1}))}\bigg\} \cdot H^*(t_j, \xi_j; t_{j-1}, \xi_{j-1}).$$

We note that the functions H and H^* are related by $H(t,\xi) = H^*(t,\xi;0,0)$.

5. Potentials which are Fourier-Stieltjes transforms. In this section we consider a class $\mathcal{G}$ of potentials $\theta(s,\xi)$ which has played an important role in Feynman integration theory [**7**]. Also see [**1, 2, 5,** and **6**].

Let $\mathcal{G}$ be the set of all $\mathbf{C}$-valued functions θ on $[0,T] \times \mathbf{R}$ of the form

$$\theta(s,\xi) = \int_{\mathbf{R}} e^{i\xi v}\,d\sigma_s(v), \tag{5.1}$$

where $\{\sigma_s : 0 \le s \le T\}$ is a family from $M(\mathbf{R})$, the space of $\mathbf{C}$-valued countably additive (and hence bounded) Borel measures on $\mathbf{R}$, satisfying

(i) For every $B \in \mathcal{B}(\mathbf{R})$, $\sigma_s(B)$ is Borel measurable in s,

(ii) $||\sigma_s|| \in L_1[0,T]$.

Theorem 5. *Let h be as in Theorem 4, and let $\theta \in \mathcal{G}$ be given by* (5.1). *Let $F : C[0,T] \to \mathbf{C}$ be given by*

$$F(x) = \exp\bigg\{\int_0^t \theta\bigg(s, \int_0^s h(u)\,dx(u)\bigg)\,ds\bigg\}.$$

Then $H(t,\xi)$ given by (1.2) *satisfies the integral equation* (1.3).

Proof. Since $|\theta(s,\xi)| \le ||\sigma_s|| \in L_1[0,T]$ for all $(s,\xi) \in [0,T] \times \mathbf{R}$ it follows that conditions (3.3) and (3.4) hold. Thus the hypotheses of Theorem 4 are satisfied, and so $H(t,\xi)$ satisfies the Kac-Feynman integral equation (1.3). □

Theorem 6. *Let* h *and* θ *be as in Theorem* 5. *Then the conditional Wiener integral*

$$I(t,\xi) = E\biggl(\exp\biggl\{\int_0^t \theta\biggl(s, \int_0^s h(u)\,dx(u)\biggr)\,ds\biggr\}\biggm| \int_0^t h(u)\,dx(u) = \xi\biggr)$$

has the series expansion

$$\begin{aligned} I(t,\xi) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \int_{\mathbf{R}^n} \exp\biggl\{ i\xi \sum_{j=1}^{n} \frac{a(s_j)}{a(t)} v_j \\ - \frac{1}{2} \sum_{k=1}^{n+1} \biggl(\Bigl[a(s_k) - a(s_{k-1})\Bigr] \biggl[\sum_{j=k}^{n} v_j - \sum_{j=1}^{n} \frac{a(s_j)}{a(t)} v_j \biggr]^2 \biggr) \biggr\} \\ \cdot\, d\sigma_{s_1}(v_1) \cdots d\sigma_{s_n}(v_n)\, d\vec{s} \end{aligned} \tag{5.2}$$

where $\Delta_n(t) = \{\vec{s} = (s_1, \dots, s_n) \in [0,t]^n : 0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t\}$.

Proof. We first note that

$$\begin{aligned} I(t,\xi) &= E\biggl(\sum_{n=0}^{\infty} \frac{1}{n!} \biggl\{ \int_0^t \theta\biggl(s, \int_0^s h(u)\,dx(u)\biggr)\,ds \biggr\}^n \biggm| \\ &\qquad \cdot \int_0^t h(u)\,dx(u) = \xi \biggr) \\ &= E\biggl(\biggl[1 + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \biggl\{ \prod_{j=1}^{n} \theta\biggl(s_j, \int_0^{s_j} h(u)\,dx(u)\biggr) \biggr\}\, d\vec{s} \biggr] \biggm| \\ &\qquad \cdot \int_0^t h(u)\,dx(u) = \xi \biggr). \end{aligned}$$

Next, using (2.7) and then the Fubini theorem, we obtain

$$
\begin{aligned}
I(t,\xi) &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)} E\Big[\prod_{j=1}^{n}\theta\Big(s_j,\int_0^{s_j} h\,dx-\frac{a(s_j)}{a(t)} \\
&\qquad\qquad\qquad\cdot\int_0^t h\,dx+\frac{a(s_j)}{a(t)}\xi\Big)\Big]\,d\vec{s} \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)} E\Big[\prod_{j=1}^{n}\theta\Big(s_j,\sum_{k=1}^{j}\int_{s_{k-1}}^{s_k} h\,dx-\frac{a(s_j)}{a(t)} \\
&\qquad\qquad\qquad\cdot\sum_{k=1}^{n+1}\int_{s_{k-1}}^{s_k} h\,dx+\frac{a(s_j)}{a(t)}\xi\Big)\Big]\,d\vec{s}.
\end{aligned}
$$

We then use a well-known Wiener integration formula, equation (5.1) and the Fubini theorem to obtain

$$
\begin{aligned}
I(t,\xi) &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)}\int_{\mathbf{R}^{n+1}}(2\pi)^{-\frac{(n+1)}{2}}\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n+1}u_j^2\Big\} \\
&\cdot\Big\{\prod_{j=1}^{n}\theta\Big(s_j,\sum_{k=1}^{j}[a(s_k)-a(s_{k-1})]^{1/2}u_k-\frac{a(s_j)}{a(t)}\sum_{k=1}^{n+1}[a(s_k)-a(s_{k-1})]^{1/2}u_k \\
&\qquad\qquad+\frac{a(s_j)}{a(t)}\xi\Big)\Big\}\,du_1\cdots du_{n+1}\,d\vec{s} \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)}\int_{\mathbf{R}^{n+1}}\int_{\mathbf{R}^{n}}(2\pi)^{-\frac{(n+1)}{2}}\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n+1}u_j^2\Big\} \\
&\cdot\exp\Big\{i\sum_{j=1}^{n}v_j\Big(\sum_{k=1}^{j}[a(s_k)-a(s_{k-1})]^{1/2}u_k \\
&\qquad-\frac{a(s_j)}{a(t)}\sum_{k=1}^{n+1}[a(s_k)-a(s_{k-1})]^{1/2}u_k+\frac{a(s_j)}{a(t)}\xi\Big)\Big\} \\
&\qquad\cdot d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)du_1\cdots du_{n+1}\,d\vec{s} \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)}\int_{\mathbf{R}^{n}}\exp\Big\{i\xi\sum_{j=1}^{n}\frac{a(s_j)}{a(t)}v_j\Big\}(2\pi)^{-\frac{(n+1)}{2}}
\end{aligned}
$$

$$\cdot \int_{\mathbf{R}^{n+1}} \exp\bigg\{ -\frac{1}{2}\sum_{j=1}^{n+1} u_j^2 + i\sum_{j=1}^{n} v_j\bigg(\sum_{k=1}^{j}[a(s_k)-a(s_{k-1})]^{1/2}u_k - \frac{a(s_j)}{a(t)}\sum_{k=1}^{n+1}[a(s_k)-a(s_{k-1})]^{1/2}u_k\bigg)\bigg\} \cdot du_1\cdots du_{n+1}\, d\sigma_{s_1}(v_1)\cdots d\sigma_{s_n}(v_n)\, d\vec{s}.$$

Finally, we use equation (1.4) to carry out the integrations with respect to $u_1, u_2, \dots, u_{n+1}$, respectively, which yields equation (5.2) as desired. □

Remark . Note that, in view of (5.2) and the fact that

$$H(t,\xi) = I(t,\xi)(2\pi a(t))^{-1/2}\exp\left\{-\frac{\xi^2}{2a(t)}\right\},$$

we have, for potentials θ in $\mathcal{G}$, obtained a series expansion for the solution of the integral equation (1.3).

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