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REMARKS ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF VOLTERRA FUNCTIONAL EQUATIONS IN L^p SPACES

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1. Existence and uniqueness problems for the Volterra integral equations were discussed by many authors. Usually the solutions were sought in the space of continuous functions.

In the literature on the subject there is not much more than the classical result for integral equations in L^p or L^2 spaces which one can find in [**5, 10**]. Similar results for multidimensional integral equations in L^2 spaces appeared in [**2**]. Recently, the author of the present paper has shown [**8, 9**] that Bielecki's technique of weighted norms [**3**] (which was successfully employed by many authors dealing with integral equations in the space of continuous functions—see [**6**] and a review paper [**4**]) can be applied fairly well to integral equations in L^p spaces.

The aim of the present note is to show how the comparison method works in the case of L^p_{loc} spaces (for an abstract formulation of the method consult the paper [**7**]) and to discuss different approaches to the problem in the case of a linear comparison function Ω (see Assumption A).

2. Let $a, b \in \mathbf{R}^n$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$; we write $a \le b$ if $a_i \le b_i$, $i = 1, 2, \dots, n$. Put

$$I = [a, b) = \{t : t \in \mathbf{R}^n,\ a \le t < b\}.$$

The case when $b = (+\infty, \dots, +\infty)$ is accepted. We call this set an interval. Let B be a Banach space with a norm $|\cdot|$. The symbol $L^p_{\text{loc}}(I, B)$ will denote the space of all locally Bochner integrable functions x (Bochner integrable on every compact subset I_c of I) for

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which the integrals

$$\int_{I_c} |x(s)|^p \, ds, \qquad I_c \subset I \ (I_c\text{-compact}), \quad p \geq 1,$$

are finite. From now on we will write

$$\int_{a_1}^{t_1} \int_{a_2}^{t_2} \cdots \int_{a_n}^{t_n} x(s_1, s_2, \dots, s_n) \, ds_n \, ds_{n-1} \cdots ds_1 \quad \text{as} \quad \int_a^t x(s) \, ds.$$

Let the operator $F : L^p_{\mathrm{loc}}(I, B) \to L^p_{\mathrm{loc}}(I, B)$ be given. We consider the equation

$$x(t) = (Fx)(t), \quad t \in I. \tag{1}$$

We assume

Assumption A. Suppose that the function $\Omega : I \times \mathbf{R}_+ \to \mathbf{R}_+$, $\mathbf{R}_+ = [0, +\infty)$ has the following properties:

(i) $\Omega(t, \cdot)$ is nondecreasing and continuous for almost all t in I.

(ii) $\Omega(\cdot, u)$ is measurable for all $u \in \mathbf{R}_+$, and, for every constant M, there is a locally integrable function $\gamma_M : I \to \mathbf{R}_+$ such that

$$\Omega(t, u) \leq \gamma_M(t), \quad t \in I, \ 0 \leq u \leq M.$$

(iii) For any nondecreasing and continuous function $q : I \to \mathbf{R}_+$, there is a nonnegative and continuous solution u_0 of the inequality

$$u(t) \geq 2^{p-1} \cdot \int_a^t \Omega(s, u(s)) \, ds + q(t), \quad t \in I. \tag{2}$$

(iv) The function $u(t) \equiv 0$, $t \in I$, is the only solution of the equation

$$u(t) = \int_a^t \Omega(s, u(s)) \, ds, \quad t \in I. \tag{3}$$

(v) For every $x, y \in L^p_{\mathrm{loc}}(I, B)$, the inequality

$$|(Fx)(t) - (Fy)(t)|^p \le \Omega\left(t, \int_a^t |x(s) - y(s)|^p \, ds\right), \quad t \in I, \tag{4}$$

holds.

Observe that $L^p_{\mathrm{loc}}(I, B)$ is a complete locally convex space with the family of seminorms

$$||x||_t = \left(\int_a^t |x(s)|^p \, ds\right)^{1/p}, \quad t \in I.$$

3. Take any $x_0 \in L^p_{\mathrm{loc}}(I, B)$, and define a sequence $\{F^k x_0\}$, $k = 0, 1, \dots,$ of iterations of x_0 by the operator $F : F^0 x_0 = x_0$, $F^{k+1} x_0 = F(F^k x_0)$.

Let u_0 be a solution of inequality (2) with q defined by the equation

$$q(t) = 2^{p-1} \cdot \int_a^t |x_0(s) - (Fx_0)(s)|^p \, ds, \quad t \in I. \tag{5}$$

Define the sequence $\{u_k\}$, $k = 0, 1, \dots,$ by the formula

$$u_{k+1}(t) = \int_a^t \Omega(s, u_k(s)) \, ds, \quad t \in I, \ k = 0, 1, \dots . \tag{6}$$

Observe that the sequence $\{u_k\}$ is nonincreasing and, by condition (iv) of Assumption A and the Dini theorem, it converges to zero uniformly on every compact subset of I.

Now we are in a position to formulate the main result of the paper:

Theorem 1. *If Assumption* A *holds, then there is a unique solution of equation* (1)*, say x^*, and x^* is the limit in $L^p_{\mathrm{loc}}(I, B)$ of the sequence $\{F^k x_0\}$, $k = 0, 1, \dots$. Moreover,*

$$\int_a^t |x^*(s) - (F^k x_0)(s)|^p \, ds \le u_k(t), \quad t \in I, \ k = 0, 1, \dots , \tag{7}$$

with u_k defined by equation (6).

Proof. We show first that, for $x_k = F^k x_0$,

$$\int_a^t |x_k(s) - x_0(s)|^p \, ds \le u_0(t), \quad t \in I, \ k = 0, 1, \dots . \tag{8}$$

We do this by induction. It is clear that (8) holds for $k = 0$. Assuming that (8) holds for a given k, we get

$$|x_{k+1}(s) - x_0(s)|^p \le 2^{p-1} \cdot (|(Fx_k)(s) - (Fx_0)(s)|^p + |(Fx_0) - x_0(s)|^p)$$

because

$$(\alpha + \beta)^p \le 2^{p-1} \cdot (\alpha^p + \beta^p) \quad \text{for every} \quad \alpha, \beta \ge 0 \text{ and } p \ge 1.$$

Using (4) and (8) yields

$$\begin{aligned} |x_{k+1}(s) - x_0(s)|^p &\le 2^{p-1} \cdot \Omega\biggl(s, \int_a^s |x_k(\xi) - x_0(\xi)|^p \, d\xi\biggr) \\ &\quad + 2^{p-1} \cdot |(Fx_0)(s) - x_0(s)|^p \\ &\le 2^{p-1} \cdot \Omega(s, u_0(s)) + 2^{p-1} \cdot |(Fx_0)(s) - x_0(s)|^p. \end{aligned}$$

The integration of this inequality over the interval $[a, t]$ gives

$$\int_a^t |x_{k+1}(s) - x_0(s)|^p \, ds \le 2^{p-1} \cdot \int_a^t \Omega(s, u_0(s)) \, ds + q(t) \le u_0(t),$$

which, together with the induction assumption, means that (8) holds for all $k = 0, 1, \dots$.

Now we prove that

$$\int_a^t |x_{k+m}(s) - x_k(s)|^p \, ds \le u_k(t), \quad t \in I, \ k, m = 0, 1, \dots , \tag{9}$$

doing this by induction again (with respect to k). By (8) we see that (9) holds for $k = 0$. Assuming that (9) holds for a specified k and any $m = 0, 1, \dots$, we get

$$\begin{aligned} |x_{k+1+m}(s) - x_{k+1}(s)|^p &= |(Fx_{k+m})(s) - (Fx_k)(s)|^p \\ &\le \Omega\biggl(s, \int_a^s |x_{k+m}(\xi) - x_k(\xi)|^p \, d\xi\biggr) \\ &\le \Omega(s, u_k(s)). \end{aligned}$$

The integration of this inequality over the interval $[a,t] = I_t$ results in

$$\int_a^t |x_{k+1+m}(s) - x_{k+1}(s)|^p \, ds \le \int_a^t \Omega(s, u_k(s)) \, ds = u_{k+1}(t).$$

This, together with the induction assumption, shows that (9) is proved. However, (9) means that the sequence $\{x_k\}$ is a Cauchy sequence in $L^p_{\mathrm{loc}}(I,B)$, so it converges to some element $x^* \in L^p_{\mathrm{loc}}(I,B)$. The inequality (7) is implies by (9) when $m \to +\infty$. Now,

$$\begin{aligned}
&\int_a^t |x^*(s) - (Fx^*)(s)|^p \, ds \\
&\qquad \le \int_a^t [|x^*(s) - x_{k+1}(s)| + |x_{k+1}(s) - (Fx^*)(s)|]^p \, ds \\
&\qquad \le 2^{p-1} \cdot \int_a^t |x^*(s) - x_{k+1}(s)|^p \, ds \\
&\qquad \quad + 2^{p-1} \cdot \int_a^t |(Fx_k)(s) - (Fx^*)(s)|^p \, ds \\
&\qquad \le 2^{p-1} \cdot u_{k+1}(t) + 2^{p-1} \cdot \int_a^t \Omega(s, u_k(s)) \, ds = 2^p \cdot u_{k+1}(t), \\
&\qquad \qquad t \in I.
\end{aligned}$$

Taking the limit as $k \to \infty$ gives that x^* is a solution of equation (1).

To prove the uniqueness of x^*, let us assume that there is another solution x^{**} of (1). Let u_0^* be a continuous solution of the inequality

$$u(t) \ge \int_a^t \Omega(s, u(s)) \, ds + \int_a^t |x^{**}(s) - x^*(s)|^p \, ds, \quad t \in I. \tag{10}$$

We have

$$x^*(t) = (Fx^*)(t), \qquad x^{**}(t) = (Fx^{**})(t),$$

and, by (4),

$$|x^{**}(s) - x^*(s)|^p \le \Omega\Big(s, \int_a^s |x^{**}(\xi) - x^*(\xi)|^p \, d\xi\Big), \tag{11}$$

$$\begin{aligned} \int_a^t |x^{**}(s)-x^*(s)|^p\,ds &\le \int_a^t \Omega\bigg(s,\int_a^s |x^{**}(\xi)-x^*(\xi)|^p\,d\xi\bigg)\,ds \\ &\le \int_a^t \Omega(s,u_0^*(s))\,ds, \quad t\in I, \end{aligned} \tag{12}$$

because, by the definition of u_0^*,

$$\int_a^t |x^{**}(s)-x^*(s)|^p\,ds \le u_0^*(t), \quad t\in I.$$

Put

$$u_{k+1}^*(t)=\int_a^t \Omega(s,u_k^*(s))\,ds, \quad t\in I,\ k=0,1,\dots\,.$$

One can see easily that the sequence $\{u_k^*\}$ behaves as the sequence $\{u_k\}$—it is nonincreasing and converges to zero. Moreover, by the definition of u_0^*, (11) and (12), it follows that

$$\int_a^t |x^{**}(s)-x^*(s)|^p\,ds \le u_k^*(t), \quad t\in I,\ k=0,1,\dots\,;$$

this implies that $x^{**}=x^*$. Thus, the proof of the theorem is complete. □

4. Let us now consider the very important special case when the function Ω is linear with respect to the second variable, i.e., the case when

$$\Omega(t,u)=M(t)u, \quad t\in I, \tag{13}$$

with some locally integrable function $M: I\to \mathbf{R}_+$. The conditions (i)–(iv) of Assumption A are evidently satisfied. For u_0 we can take the function

$$u_0(t)=q(t)\exp\bigg(2^{p-1}\cdot\int_a^t M(s)\,ds\bigg), \quad t\in I. \tag{14}$$

This is implied by the following (see [**1**]):

Lemma B. *If a function $D : \mathbf{R}_+ \to \mathbf{R}$ has nondecreasing derivative $\mathcal{D}'$ and the function $M : I \to \mathbf{R}_+$ is locally integrable, then*

$$\int_a^t M(s)\mathcal{D}'\bigg(\int_a^s M(\xi)\,d\xi\bigg)\,ds \le \mathcal{D}\bigg(\int_a^t M(s)\,ds\bigg) - \mathcal{D}(0). \tag{15}$$

Indeed, applying this inequality for $\mathcal{D}(z) = \exp(z)$, we get

$$\begin{aligned}
2^{p-1}&\int_a^t M(s)q(s)\exp\left(2^{p-1}\cdot\int_a^s M(\xi)\,d\xi\right)ds + q(t)\\
&\le q(t)\bigg[\int_a^t 2^{p-1}M(s)\exp\left(2^{p-1}\cdot\int_a^s M(\xi)\,d\xi\right)ds + 1\bigg]\\
&\le q(t)\exp\left(2^{p-1}\cdot\int_a^t M(s)\,ds\right).
\end{aligned}$$

Now the sequence $\{u_k\}$ can be evaluated by formula (6), and clearly the assertion of the theorem holds if the condition (v) of Assumption A is satisfied.

Observe that the function Ω defined by the formula (13) appears, for instance, when the operator F has the form

$$(Fx)(t) = g(t) + \int_a^t f(t,s,x(s))\,ds, \quad t \in I, \tag{16}$$

and the function $f : I \times I_t \times B \to B$ satisfies the Lipschitz condition of the form

$$|f(t,s,x) - f(t,s,y)| \le L(t,s)|x-y|, \quad t \in I,\ s \in I_t,\ x,y \in B, \tag{17}$$

with L being a measurable and nonnegative function such that

$$\int_a^t \left(\int_a^s L^q(s,\xi)\,d\xi\right)^{p/q} ds < +\infty, \quad t \in I,\ p > 1;\ \frac{1}{p} + \frac{1}{q} = 1.$$

Now, using the Hölder inequality, we have

$$\begin{aligned} |(Fx)(t)-(Fy)(t)|^p &\le \bigg(\int_a^t |f(t,s,x(s))-f(t,s,y(s))|\,ds\bigg)^p \\ &\le \bigg(\int_a^t L(t,s)|x(s)-y(s)|\,ds\bigg)^p \\ &\le \bigg(\int_a^t L^q(t,s)\,ds\bigg)^{p/q}\cdot\int_a^t |x(s)-y(s)|^p\,ds \\ &= M(t)\cdot\int_a^t |x(s)-y(s)|^p\,ds, \end{aligned}$$

for

$$M(t)=\begin{cases} \Big(\int_a^t L^q(t,s)\,ds\Big)^{p/q}, & t\in I,\ p>1,\\ \operatorname{ess-sup}_{s\in I_t} L(t,s), & t\in I,\ p=1. \end{cases} \tag{18}$$

It is assumed that the function M is locally integrable over I. The inequality

$$|(Fx)(t)-(Fy)(t)|^p \le M(t)\int_a^t |x(s)-y(s)|^p\,ds,\quad t\in I, \tag{19}$$

Lemma B, applied for functions $\mathcal{D}_i(z)=z^i/i!$, and the induction assumption imply

(20)

$$|(F^kx)(t)-(F^ky(t)|^p \le \frac{M(t)}{(k-1)!}\bigg(\int_a^t M(s)\,ds\bigg)^{k-1}\cdot\int_a^t |x(s)-y(s)|^p\,ds,$$

$t\in I$, for any $x,y\in L^p_{\mathrm{loc}}(I,B)$ and $k=1,2,\dots$.

5. The inequality (20) is very important in a classical discussion of the existence and uniqueness problems for equation (1). The following conclusions can be drawn from (20) immediately:

(a) The sequence $\{F^k x_0\}$, $k = 0, 1, \dots$, of iterations of $x_0 \in L^p_{\mathrm{loc}}(I, B)$ by F converges absolutely for all $t \in I$ for which $M(t)$ is finite. Indeed, put, in (20), $x = x_0$ and $y = Fx_0$ to get

$$|(F^{k+1}x_0)(t) - (F^k x_0)(t)|^p \leq \frac{M(t)}{(k-1)!}\left(\int_a^t M(s)\,ds\right)^{k-1} \cdot \int_a^t |(Fx_0)(s) - x_0(s)|^p\,ds.$$

From this inequality it follows that the series

$$x_0(t) + \sum_{i=0}^{\infty}[(F^{k+1}x_0)(t) - (F^k x_0)(t)]$$

converges absolutely if $M(t)$ is finite. Let

$$x^*(t) = \lim_{k\to\infty}(F^k x_0)(t), \quad t \in I. \tag{21}$$

From the Lebesgue theorem it follows that $\{F^k x_0\}$ converges to x^* in $L^p_{\mathrm{loc}}(I, B)$. This and the inequality

$$|(F^{k+1}x_0)(t) - (Fx^*)(t)|^p \leq M(t)\int_a^t |(F^k x_0)(s) - x^*(s)|^p\,ds$$

(which follows from (19)) imply that x^* is a solution of equation (1).

(b) x^* defined by (21) is the unique solution of equation (1). Indeed, put, in (20), $x = x^*$ and $y = x^{**}$, x^{**}–is supposed to be another solution of (1); then

$$|x^*(t) - x^{**}(t)|^p \leq \frac{M(t)}{(k-1)!}\left(\int_a^t M(s)\,ds\right)^{k-1} \cdot \int_a^t |x^*(s) - x^{**}(s)|^p\,ds,$$

for $k = 1, 2, \dots$. This inequality implies that $x^*(t) = x^{**}(t)$ for all t for which $M(t)$ is finite.

(c) The error for $F^k x_0$ approximation to x^* has the form

$$|(F^k x_0)(t) - x^*(t)|^p \leq \frac{M(t)}{(k-1)!}\left(\int_a^t M(s)\,ds\right)^{k-1} \cdot \int_a^t |x(s) - x^*(s)|^p\,ds, \tag{22}$$

$t \in I$, for $k = 1, 2, \dots$. Indeed, we get this by putting $x = x_0$ and $y = x^*$ in (20).

The integration of (22) over the interval $[a, t]$ leads us to the inequality

$$\int_a^t |(F^k x_0)(s) - x^*(s)|^p \, ds \le \frac{1}{k!}\left(\int_a^t M(s)\, ds\right)^k \cdot \int_a^t |x_0(s) - x^*(s)|^p \, ds. \tag{23}$$

6. However, there is another conclusion of (20) which permits for different treatment of equation (1). Integrating (20) gives

$$\int_a^t |(F^k x)(s) - (F^k y)(s)|^p \, ds \le \frac{1}{k!}\left(\int_a^t M(s)\, ds\right)^k \cdot \int_a^t |x(s) - y(s)|^p \, ds, \quad t \in I,\ k = 0, 1, \dots . \tag{24}$$

Take any $b' < b$ and assume that k' is sufficiently large to get

$$\alpha^p := \frac{1}{k'!}\left(\int_a^{b'} M(s) ds\right)^{k'} < 1.$$

For such b' and k' (k' depends on b') we can conclude that the operator $F^{k'}$ considered in $L^p(I_{b'}, B)$ is a contraction with the coefficient α so that it has a unique fixed point $x^* \in L^p(I_{b'}, B)$.

It is easy to observe that x^* is also a unique fixed point of the operator F considered in $L^p(I_{b'}, B)$ (i.e., a solution of equation (1) defined on $I_{b'}$).

Putting $x = x_0$, $y = x^*$, $t \in I_{b'}$ in (24), we get

$$\int_a^t |(F^k x_0)(s) - x^*(s)|^p \, ds \le \frac{1}{k!}\left(\int_a^t M(s)\, ds\right)^k \int_a^t |x_0(s) - x^*(s)|^p \, ds, \quad k = 0, 1, \dots . \tag{25}$$

This means that the sequence $\{F^k x_0\}$ converges in $L^p(I_{b'}, B)$ to x^*.

To obtain the unique solution of equation (1) in the whole space $L^p_{\mathrm{loc}}(I, B)$ it is enough to employ the continuation process. It is clear that, for the global solution x^*, the inequality (25) holds for all $t \in I$ and $k = 0, 1, \dots$. Observe that the term

$$\int_a^t |x_0(s) - x^*(s)|^p \, ds$$

appearing on their right hand side of inequalities (22), (23) and (25) can be easily eliminated by employing the Gronwall inequality. Indeed,

$$\begin{aligned} |x^*(s) - x_0(s)|^p &\le 2^{p-1} \cdot |(Fx^*(s) - (Fx_0)(s)|^p \\ &\quad + 2^{p-1} \cdot |(Fx_0)(s) - x_0(s)|^p \\ &\le 2^{p-1} \cdot M(s) \int_a^s |x^*(\xi) - x_0(\xi)|^p \, d\xi \\ &\quad + 2^{p-1} \cdot |(Fx_0)(s) - x_0(s)|^p. \end{aligned}$$

Integrating this inequality over the interval $[a, t]$ obtains

$$\begin{aligned} \int_a^t |x^*(s) - x_0(s)|^p \, ds &\le \int_a^t \left(2^{p-1} M(s) \int_a^s |x^*(\xi) - x_0(\xi)|^p \, d\xi \right) ds \\ &\quad + 2^{p-1} \int_a^t |(Fx_0)(s) - x_0(s)|^p \, ds, \end{aligned}$$

which implies the inequality

$$\begin{aligned} &\int_a^t |x^*(s) - x_0(s)|^p \, ds \\ &\qquad \le 2^{p-1} \int_a^t |(Fx_0(s) - (x_0)(s)|^p \, ds \cdot \exp\left(2^{p-1} \int_a^t M(s) \, ds \right). \end{aligned}$$

Finally observe that, in $L^p(I_{b'}, B)$, one can introduce the metric

$$d_{p,k'} = \sum_{i=0}^{k'-1} \beta^i \cdot ||F^i x - F^i_y||_p, \qquad \beta = \frac{1}{\sqrt[k']{\alpha}}.$$

It is easy to check that F is a contraction in $L^p(I_{b'}, B)$ with respect to this metric. Indeed, we have

$$\begin{aligned} d_{p,k'}(Fx, Fy) &= \sum_{i=0}^{k'-1} \beta^i \cdot ||F^{i+1}x - F^{i+1}y||_p \\ &= \sum_{i=0}^{k'-2} \beta^i \cdot ||F^{i+1}x - F^{i+1}y||_p + \alpha\beta^{k'-1}||x-y||_p \\ &= \sum_{i=1}^{k'-1} \beta^{i-1} \cdot ||F^i x - F^i y||_p + \beta^{-1}||x-y||_p \\ &= \beta^{-1} \sum_{i=0}^{k-1} \beta^i \cdot ||F^i x - F^i y||_p = \beta^{-1} d_{p,k'}(x,y). \end{aligned}$$

7. Now we will present conditions under which the existence and uniqueness result for equation (1) can be established simply by an application of the Banach contraction principle when the appropriate norm is employed. This will be an extension of Bielecki's technique for the $L^p_{\mathrm{loc}}(I, B)$ spaces.

We have

Theorem 2. *If the function $M : I \to \mathbf{R}_+$ is locally integrable and the conditions*

$$|(Fx)(t) - (Fy)(t)|^p \le M(t) \int_a^t |x(s) - y(s)|^p \, ds, \quad t \in I, \tag{26}$$

for every $x, y \in L^p_{\mathrm{loc}}(I, B)$, and

$$\int_a^t |(Fo)(s)|^p \, ds \le C \cdot \exp\left(\int_a^t M(s)\, ds\right), \quad t \in I, \tag{27}$$

where C is some constant, are fulfilled, then there is, in $L^p_{\mathrm{loc}}(I, B)$, a solution x^* of equation (1) *and a constant Q such that*

$$\int_a^t |x^*(s)|^p \, ds \le Q \cdot \exp\left(\int_a^t M(s)\, ds\right), \quad t \in I. \tag{28}$$

The solution x^* is unique in the class of functions $x \in L^p_{\mathrm{loc}}(I,B)$ satisfying the condition

$$\sup_{t\in I}\left\{\int_a^t |x(s)|^p\,ds\cdot\exp\left(-\int_a^t M(s)\,ds\right)\right\}<+\infty. \tag{29}$$

Proof. Let $L^p_M(I,B)$ denote a subspace of all $x \in L^p_{\mathrm{loc}}(I,B)$ for which (29) holds. Introduce in $L^p_M(I,B)$ a family of norms defined by the formula

$$||x||_\lambda=\left(\sup_{t\in I}\left\{\int_a^t |x(s)|^p\,ds\cdot\exp\left(-\lambda\int_a^t M(s)\,ds\right)\right\}\right)^{1/p},\quad \lambda>1. \tag{30}$$

It is easy to check that $F : L^p_M(I,B) \to L^p_M(I,B)$. Indeed, for $x \in L^p_M(I,B)$, we have

$$\begin{aligned}
&\int_a^t |(Fx)(s)|^p\,ds\\
&\le \int_a^t [|(Fx)(s)-(Fo)(s)|+|(Fo)(s)|]^p\,ds\\
&\le \int_a^t 2^{p-1}(|(Fx)(s)-(Fo)(s)|^p+|(Fo)(s)|^p)\,ds\\
&\le 2^{p-1}\int_a^t\left(M(s)\int_a^s |x(\xi)|^p\,d\xi\right)ds+2^{p-1}\int_a^t |(Fo)(s)|^p\,ds\\
&\le 2^{p-1}\int_a^t M(s)P\cdot\exp\left(\int_a^s M(\xi)\,d\xi\right)ds+2^{p-1}\cdot C\exp\left(\int_a^t M(s)\,ds\right)\\
&\le 2^{p-1}(P+C)\exp\left(\int_a^t M(s)\,ds\right),
\end{aligned}$$

where C and P are some constants (P depends on x). Now we show easily that F is a contraction in $L^p_M(I,B)$.

In fact, for every $x,y \in L^p_M(I,B)$,

$$|(Fx)(s)-(Fy)(s)|^p \le M(s)\int_a^s |x(\xi)-y(\xi)|^p\,d\xi.$$

After integration of this inequality over the interval $[a,t]$, we get

$$\begin{aligned}\int_a^t |(Fx)(s)-(Fy)(s)|^p\,ds &\le \int_a^t \biggl(M(s)\int_a^s |x(\xi)-y(\xi)|^p\,d\xi\biggr)ds \\ &\le \int_a^t \biggl(M(s)\exp\biggl(\lambda\int_a^s M(\xi)\,d\xi\biggr)\cdot\exp\biggl(-\lambda\int_a^s M(\xi)\,d\xi\biggr) \\ &\qquad\qquad \cdot\int_a^s |x(\xi)-y(\xi)|^p\,d\xi\biggr)ds \\ &\le ||x-y||_\lambda^p\cdot\frac{1}{\lambda}\exp\biggl(\lambda\int_a^t M(\xi)\,d\xi\biggr).\end{aligned}$$

Hence, we find that

$$||Fx-Fy||_\lambda^p \le \frac{1}{\lambda}||x-y||_\lambda^p$$

and

$$||Fx-Fy||_\lambda \le \sqrt[p]{\frac{1}{\lambda}}\cdot||x-y||_\lambda$$

which means that F is a contraction because $\lambda > 1$. Thus, the proof of the theorem is complete. $\square$

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