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AN APPLICATION OF FINITE HILBERT TRANSFORMS IN THE DERIVATION OF A STATE SPACE MODEL FOR AN AEROELASTIC SYSTEM

TERRY L. HERDMAN[1] AND JANOS TURI[2]

ABSTRACT. Dynamic modeling of various aeroelastic control systems requires, at some point in the derivation of the model, an application of Söhngen's inversion formula for finite Hilbert transforms to obtain a desired representation for the solution of the airfoil equation. Conditions on initial data to guarantee well-posedness of the resulting model equations must be matched with those needed to justify the validity of the inversion formula. We show that this compatibility can be achieved by assuming that the circulation history belongs to a weighted L_2 space. The resulting state space formulation provides a suitable setting for control design for the aeroelastic system.

1. Introduction. In recent years the feasibility and advantages of active control surfaces to reduce maneuver, gust and fatigue loads and dampen vibration that contributes to flutter have been extensively studied [**1, 2, 21, 25**]. A systematic procedure for control design requires the development of a "realistic" mathematical model that predicts the dynamic behavior of the physical system. The development of state space models for aeroelastic systems, including unsteady aerodynamics, is potentially important for the design and development of highly maneuverable aircraft.

In [**8**] a complete dynamic model was formulated in terms of a functional differential equation of neutral-type for the elastic motions of a three-degree-of-freedom "typical" airfoil section, with flap, in a two-dimensional, incompressible flow (Theodorsen's problem). In subsequent papers [**10, 11, 12, 13, 24**] the well-posedness of the

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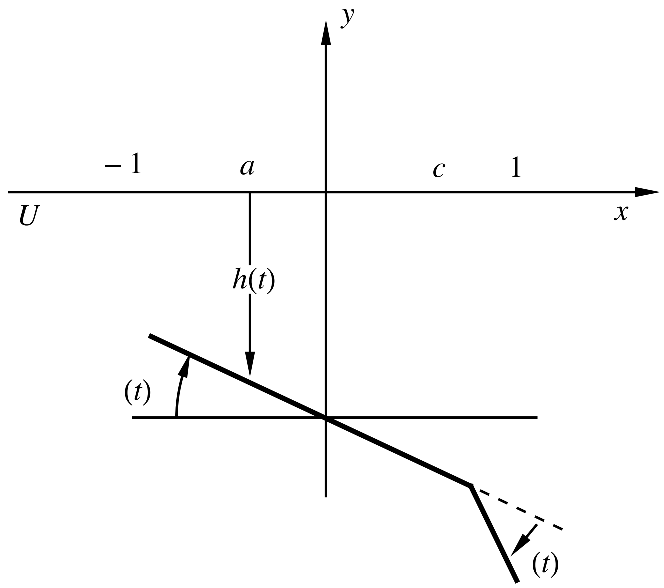
modeling neutral equation was studied in a product space framework. The analysis showed that the dynamic model extends to a well-posed state-space model on the product spaces $\mathbf{R}^8 \times L_p, p \geq 1$, if and only if $p < 2$. It is well known that the product space framework can be very useful in investigating control and identification problems for hereditary systems [**5, 6, 19, 20**]. Since the ultimate goal of the modeling process is to generate a framework for the design of active control schemes for flutter suppression, the above results would suggest considering histories belonging to $L_p, p < 2$. On the other hand, in the derivation of the evolution equation for the circulation on the airfoil, one has to assume that the circulation history belongs to L_p for $p > 2$ in order to guarantee the applicability of Söhngen's inversion formula for obtaining a representation for the solution of the airfoil equation [**7, 9, 23**]. In view of the above observations it is natural to ask whether it is possible to find a state-space such that Söhngen's inversion formula is applicable and, at the same time, the resulting neutral equation is well-posed.

In this note we study the derivation of the model equations assuming that the circulation history belongs to a weighted L_2-space. The motivation for this is that, recently [**14, 18**], the well-posedness of the finite delay version of the model equations was established on the product space $\mathbf{R}^7 \times L_{2,g}$ (g denotes the weight-function). Here we show that $L_{2,g}$ is appropriate (see Theorem 3.9 below) for the derivation of the model (i.e., representation for the solution of the airfoil equation can be obtained by using Söhngen's formula). One consequence of this result is that it provides a candidate for the state-space for the infinite delay neutral equation [**8**] which can then be used to study the flutter-suppression problem. Well-posedness of the infinite delay neutral equation on a weighted product space will be studied elsewhere.

2. Problem formulation. For a detailed discussion of the mathematical model we refer the interested reader to [**8**] and the references therein. However, for the sake of completeness, we recall the essential features of the derivation of the model. The "typical airfoil" is pictured below.



The downwash function

$$w_a(t,x)=\begin{cases} -\dot{h}(t)-(x-a)\dot{\theta}(t)-U\theta(t), & -1<x<c \\ -\dot{h}(t)-(x-a)\dot{\theta}(t)-U\theta(t) & \\ \qquad -(x-c)\dot{\beta}(t)-U\beta(t), & c<x<1 \end{cases}$$

represents the vertical velocity component that the fluid must have in order not to penetrate the airfoil. The functions $h(t), \theta(t), \beta(t)$ denote the plunge, pitch angle and flap angle respectively. The constant U denotes the undisturbed stream velocity and c is the x coordinate of the joint point between the airfoil and the flap.

The disturbance velocity is given by the gradient of the potential function $\varphi(t,x,y)$ which satisfies Laplace's equation

$$\frac{\partial^2}{\partial x^2}\varphi(t,x,y)+\frac{\partial^2}{\partial y^2}\varphi(t,x,y)=0, \quad t>0,$$

together with the following boundary conditions:

(*Flow Tangency Condition*)

$$\frac{\partial}{\partial y}\varphi(t,x,0^+) = w_a(t,x), \quad -1 < x < 1, \quad t \geq 0,$$

(*Zero Pressure Discontinuity*)

$$\frac{\partial}{\partial t}\varphi(t,x,0^+) + U\frac{\partial}{\partial x}\varphi(t,x,0^+) = 0, \quad |x| \geq 1,\ t \geq 0,$$

(*Kutta Condition*)

$$\frac{\partial}{\partial t}\varphi(t,1^-,0^+) + U\frac{\partial}{\partial x}\varphi(t,1^-,0^+) = 0, \quad t > 0.$$

The goal is to derive the relationship between the motion of the airfoil given by $w_a(t,x)$ (input) and the resulting forces on the airfoil (output). These forces are given by known integrals of the pressure which can be computed from φ. We wish to find a solution of Laplace's equation which satisfies the above boundary conditions and has the form

$$\varphi(t,x,y) = -\frac{1}{2\pi}\int_{-1}^{\infty} \gamma(t,\alpha)\tan^{-1}\big(\frac{y}{x-\alpha}\big)\,d\alpha,$$

where the integral is taken in the Cauchy sense [**7**]. The function $\gamma(t,x), t > 0, -1 < x < \infty$, represents the circulation per unit distance (angular velocity of the fluid) and is decomposed into the circulation per unit distance on the airfoil,$\gamma_a(t,x)$ for $-1 < x < 1$, and in the wake, $\gamma_w(t,x)$ for $1 \leq x < \infty$. Thus, the solution φ has the form

$$\begin{aligned}\varphi(t,x,y) = -\frac{1}{2\pi}\Bigg[&\int_{-1}^{1} \gamma_a(t,\alpha)\tan^{-1}\big(\frac{y}{x-\alpha}\big)\,d\alpha \\ &+ \int_{1}^{\infty} \gamma_w(t,\alpha)\tan^{-1}\big(\frac{y}{x-\alpha}\big)\,d\alpha\Bigg].\end{aligned} \tag{2.1}$$

The Flow Tangency Condition and (2.1) yield

$$\frac{\partial}{\partial y}\varphi(t,x,0^+) = -\frac{1}{2\pi}\Bigg[\int_{-1}^{1}\frac{\gamma_a(t,\alpha)}{x-\alpha}d\alpha + \int_{1}^{\infty}\frac{\gamma_w(t,\alpha)}{x-\alpha}\,d\alpha\Bigg] = w_a(t,x) \tag{2.2}$$

for $-1 < x < 1$. Taking the partial derivative of φ in (2.1) with respect to x, first for $-1 < x < 1$ and then for $1 < x < \infty$, leads to the identity

$$\frac{\partial}{\partial x}\varphi(t,x,0^+) = \begin{cases} -\frac{1}{2}\gamma_a(t,x), & -1 < x < 1 \\ -\frac{1}{2}\gamma_w(t,x), & 1 < x < \infty. \end{cases} \tag{2.3}$$

For each $\psi \in L_1(-\infty, 0)$ and $\eta \in \mathbf{R}$ define the extended (takes into account $t < 0$) total airfoil circulation function $\Gamma(-\infty, \infty) \to \mathbf{R}$ by

$$\Gamma(t) = \begin{cases} \int_{-\infty}^{t} \psi(s)\, ds, & t < 0 \\ \eta + \int_0^t \frac{\partial}{ds} \int_{-1}^{1} \gamma_a(s, \alpha)\, d\alpha\, ds, & t \geq 0. \end{cases} \tag{2.4}$$

Here it is assumed that the function $t \to \int_{-1}^{1} \gamma_a(t, \alpha)\, d\alpha$ is absolutely continuous. This assumption and (2.4) imply that Γ is absolutely continuous on $[0, +\infty), \Gamma(0^+) = \eta, \dot{\Gamma}(s) = \psi(s)$ for $s < 0, \dot{\Gamma}(\cdot) \in L_1(-\infty, T]$ for all $T \geq 0$ and $\lim_{t \to -\infty} |\Gamma(t)| = 0$.

The Zero Pressure Discontinuity Condition, (2.3) and (2.4) yield the equality

$$0 = \dot{\Gamma}(t) + \frac{\partial}{\partial t}\left[\int_1^x \gamma_w(t, \alpha)\, d\alpha\right] = U \gamma_w(t, x) \tag{2.5}$$

for $1 < x < \infty, t > 0$, and it follows from (2.5) that

$$\gamma_w(t, x) = -\frac{1}{U}\dot{\Gamma}\left[t + \frac{(1-x)}{U}\right]. \tag{2.6}$$

Substituting (2.6) into (2.2) we obtain the evolution equation

$$-\frac{1}{\pi}\int_{-1}^{1} \frac{\gamma_a(t, \alpha)}{x - \alpha}\, d\alpha = 2w_a(t, x) - \frac{1}{\pi}\int_0^{\infty} \frac{\dot{\Gamma}(t - \sigma)}{x - 1 - \sigma U}\, d\sigma. \tag{2.7}$$

The Kutta Condition and (2.3) lead to the finiteness condition

$$\gamma_a(t, 1^-) = \frac{-\dot{\Gamma}(t)}{U}. \tag{2.8}$$

In order to obtain the desired input-output relation between w_a and the resulting forces on the airfoil, a key step is to find an inversion formula for equation (2.7), that is, solve for γ_a. The inversion for equation (2.7), often called the airfoil equation, will be discussed in the next section. Here we remark that, under certain conditions, we get the following equation for the circulation on the airfoil:

$$\int_{-\infty}^{0} K(s)\dot{\Gamma}(t+s)\, ds = 2\int_{-1}^{1} \left(\frac{1+s}{1-s}\right)^{1/2} w_a(t, s)\, ds + \int_{-\infty}^{0} \psi(s)\, ds - \eta, \tag{2.9}$$

where $K(s) \equiv ((Us-2)/Us))^{1/2}$.

In [**8**] the evolution equation (2.9) was coupled to the rigid-body dynamics of the airfoil in order to obtain a complete set of functional differential equations that describes the composite system. The resulting model for the aeroelastic system has the form

$$\frac{d}{dt}\left[Ax(t)+\int_{-\infty}^{0} A(s)x(t+s)\,ds\right] = Bx(t)+\int_{-\infty}^{0} B(s)x(t+s)\,ds \tag{2.10}$$

for $t > 0$, where $x(t) = \operatorname{col}(h(t), \theta(t), \beta(t), \dot{h}(t), \dot{\theta}(t), \dot{\beta}(t), \Gamma(t), \dot{\Gamma}(t))$. The 8×8 matrix A is singular (each entry of the last row is zero), while the 8×8 matrix function $A(s)$ is weakly singular ($A_{88}(s) = ((Us-2)/Us)^{1/2}$) at $s=0$. The finite delay version of (2.10),

$$\frac{d}{dt}\left[Ax(t)+\int_{-r}^{0} A(s)x(t+s)\,ds\right] = Bx(t)+\int_{-r}^{0} B(s)x(t+s)\,ds, \tag{2.11}$$

has been studied in [**10–13**], in the state spaces $\mathbf{R}^8 \times L_p(-r,0)$. It is to be noted that an appropriate initial condition for (2.11) must contain the "past history" of ψ. If one defines the operators D and L by

$$D\varphi = A\varphi(0) + \int_{-r}^{0} A(s)\varphi(s)\,ds$$

and

$$L\varphi = B\varphi(0) + \int_{-r}^{0} B(s)\varphi(s)\,ds,$$

then the initial value problem associated with (2.10) becomes

$$\frac{d}{dt}Dx_t = Lx_t, \quad t > 0, \tag{2.12}$$

with initial data

$$Dx_0(\cdot) = \eta,\ x_0(s) = \varphi(s), \quad \text{for } -r \le s \le 0. \tag{2.13}$$

System (2.12)–(2.13) belongs to a general class of neutral functional differential equations (NFDE's) which have "nonatomic" D operators.

Burns, Herdman and Turi [**10, 11**] have obtained sufficient conditions to assure the well-posedness for this general class of NFDE's. In particular, they have shown that the NFDE (2.12)–(2.13), which represents the finite delay approximation for the aeroelastic system, is well-posed on $\mathbf{R}^8 \times L_p(-r, 0)$ if and only if $p < 2$.

3. Inversion of the evolution equation (2.7). An evolution equation of the general form of (2.7) occurs often in the model derivation of a lifting surface in an incompressible flow and is referred to as the airfoil equation. The airfoil equation has been studied extensively by several authors (see [**2–4, 7–9, 15–17, 22–23** and references therein]). A brief history of the airfoil equation, including references to earlier work, can be found in Cheng and Rott [**6**]. H. Söhngen used the theory of the finite Hilbert transformation to establish the validity of an inversion formula for the airfoil equation in the L_p-space where $p > 1$. Generalizations for this inversion formula, which were motivated by actual requirements for the solution of problems that arise in the study of linearized conical supersonic flow, can be found in [**16**]. Our goal here is to find sufficient conditions for the function $\dot{\Gamma}_t$, which appears in (2.7), that will allow us to use Söhngen's inversion formula. In particular, we wish to establish the validity of the inversion formula when $\dot{\Gamma}_t$ belongs to a weighted L_2-space. The recent well-posedness results for the system (2.12)–(2.13) on weighted product spaces (see [**14, 18**]) is the motivation for studying the inversion formula in this setting. We wish to solve

Problem A. Find the solution $\gamma_a(t, x)$ of (2.7) for $-1 < x < 1, t > 0$ which satisfies (2.8) for $t > 0$.

The right-hand side of (2.7), for each $t > 0$, is the finite Hilbert transformation of $\gamma_a(t, \cdot)$ which we denote by $\mathcal{F}_x[\gamma_a(t, \alpha)]$, that is

$$\mathcal{F}_x[\gamma_a(t,\alpha)] \equiv \frac{1}{\pi}\int_{-1}^{1} \frac{\gamma_a(t,\alpha)}{\alpha - x}\, d\alpha. \tag{3.1}$$

The problem of finding the solution γ_a now becomes the problem of finding an inversion formula (in an appropriate L_p space) for the finite

Hilbert transformation

$$\mathcal{F}_x[\gamma_a(t,\alpha)] = 2w_a(t,x) - \frac{1}{\pi}\int_0^\infty \frac{\dot{\Gamma}(t-\sigma)}{x-1-\sigma U}\, d\sigma \equiv F(t,x). \tag{3.2}$$

In order to find an L_p solution to (3.2), $p > 1$, it must be assumed that the function $F(t,\cdot)$ itself belongs to the class L_p [**23**]. Clearly, $w_a(t,\cdot) \in L_p(-1,1)$ for all $p > 1$. Consequently, the values of $p > 1$ for which $F(t,\cdot) \in L_p$ are determined by the integral term (viewed as a function of x) that appears in (3.2). The change of variables $s = 1+\sigma U$, together with (2.4), yield

$$\begin{aligned} -\frac{1}{\pi}\int_0^\infty \frac{\dot{\Gamma}(t-\sigma)}{x-1-\sigma U} d\sigma &= -\frac{1}{\pi}\int_1^\infty \frac{\dot{\Gamma}(t+(1-s)/U)}{x-s}\, ds \\ &= -\frac{1}{\pi}\int_1^\infty \frac{\dot{\Gamma}_t \circ \beta(s)}{s-x}\, ds, \end{aligned} \tag{3.3}$$

where $\dot{\Gamma}_t : (-\infty,0] \to \mathbf{R}, \beta : [-1,\infty) \to (-\infty,0]$ are defined by $\dot{\Gamma}_t(u) = \dot{\Gamma}(t+u)$ and $\beta(s) = (1-s)/U$, respectively. Extend $\dot{\Gamma}_t$ to $(-\infty,\infty)$ by defining $\dot{\Gamma}_t(u) = 0$ *outside the interval* $(-\infty,0]$. This extension allows the integral term on the right-hand side of (3.3) to be viewed as the (infinite) Hilbert transformation

$$\mathcal{H}_x[\dot{\Gamma}_t \circ \beta(s)] \equiv \frac{1}{\pi}\int_{-\infty}^\infty \frac{\dot{\Gamma}_t \circ \beta(s)}{s-x}\, ds. \tag{3.4}$$

It follows from known L_p-properties of Hilbert transforms (see [**23**]), together with (3.2)–(3.3), that $F(t,\cdot) \in L_p(-1,1)$ if and only if $\dot{\Gamma}_t \circ \beta(\cdot) \in L_p(-\infty,\infty)$. The latter condition holds if and only if $\dot{\Gamma}_t \in L_p(-\infty,0)$. This observation yields the following result.

Lemma 3.1. *If* $\dot{\Gamma}_t \in L_p(-\infty,0)$ *for* $p > 1$ *then* $F(t,\cdot)$, *defined by* (3.2)*, belongs to* $L_p(-1,1)$.

In order to obtain an inversion formula for (3.2) we employ the following result [**23**].

Theorem 3.2. *Equation* (3.2) *with* $F(t,\cdot) \in L_p(-1,1)$ *for some* $p \geq 4/3$ *has the solution*

$$\gamma_a(t,x) = -\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-s^2}{1-x^2}\right)^{1/2}\frac{F(t,s)}{s-x}\,ds + C(t)(1-x^2)^{-1/2}, \tag{3.5}$$

for $-1 < x < 1$, *where* $C(t)$ *is dependent only on* t. *Moreover, the term* $C(t)(1-x^2)^{-1/2}$ *represents the general solution of the homogeneous equation corresponding to* (3.5).

Although equation (3.5) provides an inversion formula for $\gamma_a(t,x)$, this representation has shown little promise to produce the desired evolution equation (2.9) for the circulation on the airfoil. The following result, which follows from Theorem 3.2, is often employed to obtain this evolution equation [**3, 4, 7, 8, 15**].

Corollary 3.3. *If* $\dot{\Gamma}_t \in L_p(-\infty,0)$, *for some* $p > 2$, *then equation* (3.2) *has the solution*

(3.6)
$$\gamma_a(t,x) = -\frac{1}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\int_{-1}^{1}\left(\frac{1+s}{1-s}\right)^{1/2}\frac{F(t,s)}{s-x}\,ds + K(t)(1-x^2)^{-1/2},$$

for $-1 < x < 1$, *where* $K(t)$ *is dependent only on* t.

Proof. We combine (3.5) with the identity

$$\left(\frac{1-s^2}{1-x^2}\right)^{1/2} = \left(\frac{1-x}{1+x}\right)^{1/2}\left(\frac{1+s}{1-s}\right)^{1/2}\left(1-\frac{s-x}{1-x}\right)$$

to obtain

$$\begin{aligned}\gamma_a(t,x) = -\frac{1}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\bigg[&\int_{-1}^{1}\left(\frac{1+s}{1-s}\right)^{1/2}\frac{F(t,s)}{s-x}\,ds \\ &+ \int_{-1}^{1}\left(\frac{1+s}{1-s}\right)^{1/2}\frac{F(t,x)}{(1-x)}\,ds\bigg] + C(t)(1-x^2)^{-1/2}.\end{aligned} \tag{3.7}$$

The assumption that $\dot{\Gamma}_t \in L_p(-\infty,0), p > 2$, and Lemma 3.1 imply that $F(t,\cdot) \in L_p(-1,1)$ for $p > 2$. Therefore, $F(t,s)/(1-s)^{1/2}$ belongs to $L_1(-1,1)$, and we have that

$$\begin{aligned} -\frac{1}{\pi}\Big(\frac{1-x}{1+x}\Big)^{1/2} &\int_{-1}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}\frac{F(t,s)}{(1-x)}\,ds \\ &= \frac{-1}{\pi(1-x^2)^{1/2}}\int_{-1}^{1}(1+s)^{1/2}\frac{F(t,s)}{(1-s)^{1/2}}\,ds = \frac{-\hat{C}(t)}{\pi(1-x^2)^{1/2}}, \end{aligned}$$

where $\hat{C}(t) = \int_{-1}^{1}(1+s)^{1/2}(F(t,s)/(1-s)^{1/2})\,ds$. Equation (3.6) follows by letting $K(t) = C(t) - \hat{C}(t)/\pi$. $\square$

Theorem 3.4. *If* $\hat{\Gamma}_t \in L_p(-\infty,0)$, *for some* $p > 2$, *then* Problem A *has the solution* (3.6), *where* $K(\cdot) = 0$.

Proof. Corollary 3.3 implies that $\gamma_a(t,x)$ has representation (3.6). Now we show that $\gamma_a(t,x)$, given by (3.6), satisfies (2.8) (the "finiteness" condition) if and only if $K(\cdot)$ in (3.6) is zero (a.e.). Recalling that [**4**]

$$\int_{-1}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}\frac{1}{s-x}\,dx = -\pi,$$

we have

$$\gamma_a(t,x) = \Big(\frac{1-x}{1+x}\Big)^{1/2}\Bigg[\frac{1}{\pi}\int_{-1}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}\frac{F(t,s)-F(t,x)}{s-x}\,ds + F(t,x)\Bigg]. \tag{3.8}$$

Using (3.2), the expression for the downwash function $w_a(t,x)$, and assuming that $x > c$, we can write the integral term on the right-hand side of (3.8) as

$$\begin{aligned} &-\frac{1}{\pi^2}\int_{-1}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}\int_{0}^{\varepsilon}\frac{\dot{\Gamma}(t-\sigma)}{(1-s+\sigma U)(1-x+\sigma U)}\,d\sigma\,ds \\ &-\frac{1}{\pi^2}\int_{-1}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}\int_{\varepsilon}^{\infty}\frac{\dot{\Gamma}(t-\sigma)}{(1-s+\sigma U)(1-x+\sigma U)}\,d\sigma\,ds \\ &-\frac{2}{\pi}\int_{-1}^{c}\Big(\frac{1+s}{1-s}\Big)^{1/2}(-\dot{\theta}(t) + \frac{(x-c)\dot{\beta}(t)+U\beta(t)}{s-x})\,ds \\ &-\frac{2}{\pi}\int_{c}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}(-\dot{\theta}(t)-\dot{\beta}(t))\,ds \equiv I + II + III + IV, \end{aligned} \tag{3.9}$$

where $\varepsilon > 0$. Changing the order of integration in I yields

$$I = -\frac{1}{\pi^2}\int_0^{\varepsilon} \frac{\dot{\Gamma}(t-\sigma)}{1-x+\sigma U}\int_{-1}^{1}\left(\frac{1+s}{1-s}\right)^{1/2}\frac{1}{1+\sigma U - s}\,ds\,d\sigma, \tag{3.10}$$

where

$$\begin{aligned}\int_{-1}^{1}\Big(\frac{1+s}{1-s}\Big)^{1/2}\frac{1}{1+\sigma U - s}\,ds \\ &= \frac{1}{(\sigma U)^{1/2}}\Bigg[2(2+\sigma U)^{1/2}\Big\{\tan^{-1}\Big(\frac{\sigma U}{2+\sigma U}\Big)^{1/2} \\ &\qquad - \tan^{-1}\Big(-\frac{2+\sigma U}{\sigma U}\Big)^{1/2}\Big\} - \pi(\sigma U)^{1/2}\Bigg] \\ &\equiv \frac{1}{(\sigma U)^{1/2}}R(\sigma).\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) and noting that

$$\lim_{\sigma\to 0^+} R(\sigma) = 2\sqrt{2}\cdot\frac{\pi}{2} = \sqrt{2}\pi$$

and, assuming the continuity of $\dot{\Gamma}$ at t,

$$\lim_{\sigma\to 0^+}\dot{\Gamma}(t-\sigma)R(\sigma) = \sqrt{2}\pi\dot{\Gamma}(t),$$

we have that

$$\begin{aligned} I &= -\frac{1}{\pi^2}\Bigg[\int_0^{\varepsilon}\frac{\sqrt{2}\pi\dot{\Gamma}(t)}{(\sigma U)^{1/2}}\cdot\frac{1}{(1-x)+\sigma U}\,d\sigma \\ &\qquad + \int_0^{\varepsilon}\frac{\dot{\Gamma}(t-\sigma)R(\sigma)-\sqrt{2}\pi\dot{\Gamma}(t)}{(\sigma U)^{1/2}(1-x+\sigma U)}\,d\sigma\Bigg] \\ &= -\frac{1}{\pi^2}\frac{2\sqrt{2}\pi\dot{\Gamma}(t)}{U(1-x)^{1/2}}\tan\frac{(\varepsilon u)^{1/2}}{(1-x)^{1/2}} \\ &\qquad + o(1)\frac{2}{(1-x)^{1/2}}\tan\frac{(\varepsilon U)^{1/2}}{(1-x)^{1/2}}.\end{aligned} \tag{3.12}$$

Concerning terms II, III, and IV in (3.9), we have the estimates

$$|II| \leq \frac{1}{\pi^2}\pi \cdot \frac{1}{\varepsilon^2 U^2} \cdot ||\dot{\Gamma}||_{L_1(-\infty,t)}, \tag{3.13}$$

$$|III| \leq \frac{2}{\pi}\pi\Big[|\dot{\theta}(t)| + \frac{(1-c)|\dot{\beta}(t)| + U|\beta(t)|}{|c-x|}\Big] \tag{3.14}$$

and

$$|IV| \leq \frac{2}{\pi}\pi(|\dot{\theta}(t)| + |\dot{\beta}(t)|), \tag{3.15}$$

respectively. Finally, observing that

$$\lim_{x\to 1^-}\Big(\frac{1-x}{1+x}\Big)^{1/2} F(t,x) = 0, \tag{3.16}$$

expressions (3.8)–(3.16) imply that

$$-\frac{\dot{\Gamma}(t)}{U} = \lim_{x\to 1^-}\Big(\frac{1-x}{1+x}\Big)^{1/2}\Big(\frac{-1}{\pi^2}\Big)\frac{2\sqrt{2}\pi\dot{\Gamma}(t)}{U(1-x)^{1/2}}\tan\frac{(\varepsilon U)^{1/2}}{(1-x)^{1/2}}, \tag{3.17}$$

and the statement of the theorem follows. □

The circulation on the airfoil, $\Gamma(t)$, is obtained by integrating the circulation per unit distance, $\gamma_a(t,x)$, over the interval $-1 < x < 1$, that is,

$$\Gamma(t) = \int_{-1}^{1} \gamma_a(t,x)\,dx, \quad t > 0. \tag{3.18}$$

If $\dot{\Gamma}_t \in L_p(-\infty,0), p > 2$, then Theorem 3.4, (3.2) and (3.6) yield the evolution equation (2.9) for the circulation on the airfoil.

At this point, we wish to note that the inversion formula (3.6) in Corollary 3.3 and the conclusion for Theorem 3.4 are valid if the condition $\dot{\Gamma}_t \in L_p(-\infty,0)$, for some $p > 2$, is replaced by the condition $F(t,\cdot) \in L_p(-1,1)$, for some $p > 2$. Also, the equation for the circulation on the airfoil (3.9) is valid when $F(t,\cdot) \in L_p(-1,1)$, for some $p > 2$.

Corollary 3.5. *If* $F(t,\cdot) \in L_p(-1,1)$, *for some* $p > 2$, *then equation* (3.2) *has the solution* $\gamma_a(t,x)$, $-1 < x < 1$, *given by* (3.6).

Theorem 3.6. *If* $F(t,\cdot) \in L_p(-1,1)$, *for some* $p > 2$, *then Problem* A *has the solution* $\gamma_a(t,x)$, $-1 < x < 1$, *given by* (3.6) *with* $K(\cdot) = 0$.

The importance of Theorem 3.6 compared to Theorem 3.4 can be seen in the following remark.

Remark 3.7. If one uses Theorem 3.4 to derive the evolution equation (2.9) and couple this to the rigid-body dynamics of the airfoil, then the resulting composite system (finite delay) would be described by (2.12)–(2.13). The results found in [**10**–**13**] establish that (2.12)–(2.13) is well-posed on $\mathbf{R}^8 \times L_p(-r,0)$, for $p > 1$, if and only if $x_0(\cdot) = \varphi(\cdot) \in L_p(-r,0)$, for $p < 2$. Since $x_0 = \varphi(\cdot) = (h(\cdot),\theta(\cdot),\beta(\cdot),\dot{h}(\cdot),\dot{\theta}(\cdot),\dot{\beta}(\cdot),\Gamma(\cdot),\dot{\Gamma}(\cdot))$, it follows that $\varphi \in L_p(-r,0)$ requires that $\dot{\Gamma}_0 = \psi$ must belong to $L_p(-r,0)$. To assure the validity of the inversion formula (3.6) and the desired solution for Problem A, we must have $\psi \in L_p(-\infty,0)$, for some $p > 2$. On the other hand, such an assumption would imply $\psi \in L_p(-r,0)$, for some $p > 2$, and the system (2.12)–(2.13) would not be well-posed on the corresponding product space $\mathbf{R}^8 \times L_p(-r,0)$.

To address the problem of finding a space on which the derivation of (3.6) is valid and a compatible product space on which the corresponding system (2.12)–(2.13) is well-posed, we consider the space $\mathbf{R}^7 \times L_{2,g}(-r,0)$. The motivation to consider this space was supplied by the recent results in [**14**] and [**18**] which yield the well-posedness of (2.12)–(2.13) on $\mathbf{R}^7 \times L_{2,g}(-r,0)$. Here the weight function is defined by $g(\sigma) = (-\sigma)^{-1/2}$, for $\sigma < 0$. It is to be noted that the following lemma holds for a general class of weight functions. However, for simplicity we choose g as given above.

Lemma 3.8. *If* $\dot\Gamma_t \in L_{2,g}(-r,0)$ *then* $F(t,x)$ *defined by* (3.2) *belongs to* $L_p(-1,1)$ *for some* $p > 2$.

Proof. Equation (3.2) together with $c(x) \equiv (1-x)/U$ yield

$$F(t,x) = \frac{1}{\pi U}\int_0^r \frac{\dot\Gamma(t-\sigma)}{c(x)+\sigma}\,d\sigma + \frac{1}{\pi U}\int_r^\infty \frac{\dot\Gamma(t-\sigma)}{c(x)+\sigma}\,d\sigma. \tag{3.19}$$

For the second term on the right-hand side of (3.19) we have the estimate

$$\left|\frac{1}{\pi U}\int_r^\infty \frac{\dot\Gamma(t-\sigma)}{c(x)+\sigma}\,d\sigma\right| \le \frac{1}{\pi U r}\int_r^\infty |\dot\Gamma(t-\sigma)|\,d\sigma = \frac{1}{\pi U r}N(t), \tag{3.20}$$

where $N(t)$ is defined by

$$N(t) = ||\dot\Gamma||_{L_1(-\infty,t)}.$$

For the first term on the right-hand side of (3.19), we have the estimate

$$\begin{aligned}
&\left|\frac{1}{\pi U}\int_0^r \frac{\dot\Gamma(t-\sigma)}{c(x)+\sigma}\,d\sigma\right| \\
&\qquad \le \frac{1}{\pi U}\int_0^r \left|\frac{(g(-\sigma))^{-1/2}\dot\Gamma(g(-\sigma))^{1/2}(t-\sigma)}{c(x)+\sigma}\right| d\sigma \\
&\qquad \le \frac{1}{\pi U}\left[\int_0^r \left(\frac{g(-\sigma)^{-1/2}}{c(x)+\sigma}\right)^2 d\sigma\right]^{1/2}\left[\int_0^r g(-\sigma)(\dot\Gamma(t-\sigma))^2\,ds\right]^{1/2} \\
&\qquad \le \frac{1}{\pi U}B(x)M(t),
\end{aligned} \tag{3.21}$$

where $B(x)$ and $M(t)$ are defined by

$$B(x) = \left[\int_0^r \frac{1}{g(-\sigma)(c(x)+\sigma)^2}\,d\sigma\right]^{1/2} \tag{3.22}$$

and

$$M(t) = ||\dot\Gamma_t||_{L_{2,g}(-r,0)}.$$

The estimates (3.20), (3.21), together with equation (3.19), yield

$$|F(t,x)| \leq \frac{1}{\pi U}[B(x)M(t) + N(t)],$$

for $t > 0, -1 < x < 1$. Consequently, $F(t,\cdot) \in L_p(-1,1)$, for some $p > 2$, if $B(\cdot) \in L_p(-1,1)$ for some $p > 2$.

In order to show that B, given by (3.22) with $g(\sigma) = (-\sigma)^{-1/2}$, belongs to L_p for some $p > 2$, we first find an estimate for $B^2(x), -1 < x < 1$. A change of variables together with integration by parts yields

$$\begin{aligned} B^2(x) &= \int_0^r \frac{\sqrt{\sigma}}{(c(x)+\sigma)^2}\,d\sigma = 2\int_0^{\sqrt{r}} \frac{v^2}{(c(x)+v^2)^2}\,dv \\ &= \frac{-\sqrt{r}}{c(x)+r^2} + \frac{1}{\sqrt{c(x)}}\tan^{-1}\Big(\frac{\sqrt{r}}{\sqrt{c(x)}}\Big). \end{aligned}$$

The desired estimate is given by

$$B^2(x) \leq \frac{\pi}{2}(c(x))^{-1/2}.$$

For $0 < \varepsilon < 1$, it follows that

$$\begin{aligned} \int_{-1}^1 B^{2(1+\varepsilon)}(x)\,dx &\leq \int_{-1}^1 \left[\frac{\pi}{2}\Big(\frac{U}{1-x}\Big)^{1/2}\right]^{1+\varepsilon} dx \\ &= \Big(\frac{\pi}{2}U^{1/2}\Big)^{1+\varepsilon} 2\left[-\frac{(1-x)^{((1-\varepsilon)/2)}}{1-\varepsilon}\right]_{-1}^1 \end{aligned}$$

that is, $B(\cdot) \in L_p(-1,1)$, for $2 \leq p < 4$. The following result is an immediate consequence of Lemma 3.8 and Theorem 3.6. □

Theorem 3.9. *If* $\dot{\Gamma}_t \in L_{2,g}(-r,0)$ *then Problem* A *has the solution* $\gamma_a(t,x), -1 < x < 1$, *given by* (3.6) *with* $K(\cdot) = 0$.

The circulation on the airfoil defined by (3.18), together with Theorem 3.9, yield the evolution equation (2.9) for the circulation on the airfoil (see [**8**]).

4. Conclusion. The assumption that $\dot{\Gamma}_t \in L_{2,g}(-r,0)$ was shown to be sufficient to assure the validity of the (Söhngen) inversion formula (3.6) for the solution $\gamma_a(t,x)$ of the evolution equation (2.7). Therefore, this assumption is sufficient to obtain (2.12)–(2.13) as the dynamic model that describes the aeroelastic system. The results presented here, together with the results given in [**14**] and [**18**], provide a suitable setting for control design. We also note that these results give insight concerning the selection of an appropriate state space for the corresponding infinite delay problem.

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Interdisciplinary Center for Applied Mathematics, Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061

Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609

current address: Programs in Mathematical Sciences, University of Texas at Dallas, Richardson, TX 75083.