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VISCOELASTIC AND BOUNDARY FEEDBACK DAMPING: PRECISE ENERGY DECAY RATES WHEN CREEP MODES ARE DOMINANT

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Dedicated to John Nohel on the occasion of his sixty-fifth birthday

ABSTRACT. For a linear Volterra equation of scalar type in a Banach space, sufficient conditions are given for the operator norms of three associated resolvent kernels to be integrable with respect to a weight on the positive half-line. The results and methods extend those introduced by Prüss for integrability with respect to ordinary Lebesgue measure. The estimates are applied to the estimation of precise decay rates for energy in a viscoelastic solid when the memory kernel decays algebraically and creep modes dominate the oscillating modes. It is shown that boundary feedback is ineffective in promoting decay in such cases.

1. Introduction. We give sufficient conditions for three resolvent kernels associated with the problem

$$\begin{gathered}\ddot{\mathbf{u}}(t) = E\mathbf{L}\mathbf{u}(t) + \frac{d}{dt}\int_0^t a(t-\tau)\mathbf{L}\mathbf{u}(\tau)\,d\tau \quad \left(\dot{} = \frac{d}{dt}\right), \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{u}_1\end{gathered} \tag{1.1}$$

to be integrable with respect to certain weight functions on $\mathbb{R}^+ \equiv [0,\infty)$. Here $E > 0$, $\mathbf{L}$ is the generator of a strongly continuous cosine family in a Banach Space $\mathbf{X}$, and a satisfies

$$\begin{gathered}a \in C(0,\infty) \cap L^1(0,1) \text{ is positive, nonincreasing and} \\ \text{log-convex on } (0,\infty) \text{ with } 0 = a(\infty) < a(0^+) \leq \infty.\end{gathered} \tag{1.2}$$

The resolvents in question are defined formally by

$$\mathbf{u}(t) = \mathbf{U}(t)\mathbf{u}_0 + \mathbf{W}(t)\mathbf{u}_1, \quad \mathbf{V} = \dot{\mathbf{U}}. \tag{1.3}$$

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(In fact, $\dot{\mathbf{W}} = \mathbf{U}$ as well.) We shall assume

$$\int_1^\infty |\dot{a}(t)|\rho(t)\,dt < \infty \tag{1.4}$$

for some subexponential weight function $\rho(t)$ (such as $\rho(t) = (1+t)^r$, $0 \le r < \infty$) as well as a stability condition (e.g., $\mathbf{L} < \mathbf{0}$ in the Hilbert space case) and show, for example, that

$$\int_0^\infty \|\mathbf{U}(t)\|(1+t)\rho(t)\,dt < \infty \tag{1.5}$$

($\|\cdot\|$ = the operator norm in $\mathbf{X}$). Using resolvent formulas we will be able to deduce corresponding integrability results for solutions of (1.1).

Our results extend those for the special case $\rho(t) \equiv 1$ given by Prüss [**18, 19**]. Indeed, our proofs are based on the methods developed in [**18, 19**], together with local analyticity (a method for establishing integrability in a weighted space for scalar functions [**13**]) and the Paley-Wiener Lemma in Banach space [**9**].

Our results have natural applications to problems in linear viscoelasticity. For example, the viscoelastic wave equation (in a solid)

$$u_{tt}(x,t) = Eu_{xx} + \frac{d}{dt}\int_0^t a(t-\tau)u_{xx}(x,\tau)\,d\tau \tag{1.6}$$

fits our framework, and we get results on rates of energy decay. We develop this connection and compare the decay rates obtainable for (1.6) with homogeneous boundary conditions to those for the same equations with stabilizing boundary feedback (as examined, e.g., in [**15**], [**14**, Chapter 6]). We show, in particular, that such feedback is nearly irrelevant to the decay rate unless the kernel $a(t)$ decays exponentially; in the latter case, feedback can even slow the rate of decay.

As the analysis of [**11**] shows, decay rates correspond essentially to the largest real part of the singularities of the Laplace transform of the solution. When the dominant singularity has a nonzero imaginary part, we get oscillations that can be damped via boundary feedback. Viscoelastic materials exhibit "creep modes" that appear as real singularities that are insensitive to such feedback; the phenomena studied in this paper relate mainly to cases where these singularities are dominant.

Our main results on resolvents are stated in Section 2 and proved in Section 3. Section 4 concerns integrability and energy decay of solutions of (1.1), while Section 5 is devoted to the example (1.6) with and without boundary feedback.

2. Statement of main result on integrability of resolvents. In this section we state our main result, Theorem 2.1, on integrability properties of the operator resolvent $\mathbf{U}$, its derivative $\mathbf{V}$, and its definite integral $\mathbf{W}$. This result extends to weighted L^1 spaces recent results of Prüss [**18, 19**] on the integrability of resolvents. Theorem 2.1 is actually a corollary of the more technical results stated and proved in Section 3.

Throughout Sections 2 and 3, $\mathbf{X}$ denotes a Banach space, and $\mathbf{L}$ is a closed linear operator in $\mathbf{X}$ with dense domain $\mathcal{D}(\mathbf{L})$ which generates a strongly continuous cosine family $\mathbf{C}(t)$. We consider the first order $\mathbf{X}$-valued integrodifferential equation

$$\dot{\mathbf{u}}(t) = \int_0^t A(t-\tau)\mathbf{L}\mathbf{u}(\tau)\,d\tau + \mathbf{f}(t),\ t \geq 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{2.1}$$

where $\mathbf{f} \in C(\mathbb{R}^+, \mathbf{X})$, and the scalar kernel A has the form $A(t) = E + a(t)$ with $E > 0$ and a satisfying (1.2). Under these conditions [**18**, Theorem 6], problem (2.1) admits a *resolvent* $\mathbf{U}(t)$, that is, a strongly continuous family $\{\mathbf{U}(t)\}_{t\geq 0}$ in $\mathcal{L}(\mathbf{X})$, the bounded linear operators in $\mathbf{X}$, such that $\mathbf{U}(0) = \mathbf{I}$ (the identity operator), $\mathbf{U}(t)$ commutes with $\mathbf{L}$, and the resolvent equation

$$\dot{\mathbf{U}}(t)\mathbf{x} = \int_0^t A(t-\tau)\mathbf{L}\mathbf{U}(\tau)\mathbf{x}\,d\tau,\ t \geq 0, \quad \mathbf{x} \in \mathcal{D}(\mathbf{L}), \tag{2.2}$$

is satisfied. In addition, Prüss [**18, 19**] showed that, under these conditions, $\mathbf{U}(t)$ and its derivative $\mathbf{V}(t) = \dot{\mathbf{U}}(t)$ (strong derivative) are integrable (in a sense to be made precise below) on $\mathbb{R}^+$. $\mathbf{U}$ and $\mathbf{V}$ are important in the study of the abstract equation (2.1) since the solutions are given by various "variation of constants formulae." For example, all solutions of (2.1) are given by

$$\mathbf{u}(t) = \mathbf{U}(t)\mathbf{u}_0 + \int_0^t \mathbf{U}(t-\tau)\mathbf{f}(\tau)\,d\tau,$$

while $\mathbf{V}(t)$ occurs in a similar formula for solutions of the integrated version of (2.1) [**2**]. (See also [**20**] where similar formulae arise in the study of bounded solutions to infinite delay equations on the whole line $\mathbb{R}$.)

As stated in the Introduction, we are interested in precise decay rates when $a(t)$ decays subexponentially. For this purpose we consider weights on $\mathbb{R}^+$ of the following type (see [**9**]):

(W) The function $\rho(t)$ is a (regular) *weight* on $\mathbb{R}^+$ if ρ is positive, continuous and nondecreasing on $\mathbb{R}^+, \rho(0) = 1$, $\rho(t+s) \le \rho(t)\rho(s)$ and $\rho_* := -\lim_{t\to\infty} t^{-1} \log \rho(t) = 0$.

Interesting examples of weights satisfying (W) are provided by

$$\begin{aligned} &\rho_1(t) = (1+t)^r, \quad r \ge 0, \\ &\rho_2(t) = (1+\log(1+t))^\gamma \rho_1(t), \quad \gamma \ge 0, \\ &\rho_3(t) = \exp(t^\alpha)\rho_2(t), \quad 0 \le \alpha < 1. \end{aligned} \tag{2.3}$$

We let $L^1(\mathbb{R}^+;\rho)$ denote the space of complex measurable functions φ that are integrable with respect to ρ, that is, for which

$$\int_0^\infty |\varphi(t)|\rho(t)\,dt < \infty.$$

When $\rho(t) \equiv 1$ we denote $L^1(\mathbb{R}^+;\rho)$ by $L^1(\mathbb{R}^+)$. Laplace transforms of functions in $L^1(\mathbb{R}^+;\rho)$ are denoted by $L^1(\mathbb{R}^+;\rho)^\wedge$.

Since $\mathbf{X}$ is infinite dimensional, there are several different notions of integrability for resolvents (cf. [**19**] when $\rho(t) \equiv 1$):

Definition 2.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces, let $\{\mathbf{S}(t)\}_{t\ge 0} \subseteq \mathcal{L}(\mathbf{X},\mathbf{Y})$ be a strongly measurable family of bounded linear operators, and let $\rho(t)$ satisfy (W). Then $\mathbf{S}(t)$ is

(i) *ρ-strongly integrable* if

$$\int_0^\infty \|\mathbf{S}(t)\mathbf{x}\|\rho(t)\,dt < \infty, \quad \text{for each } \mathbf{x} \in \mathbf{X}.$$

(ii) ρ-*integrable* if there is a $\varphi \in L^1(\mathbb{R}^+;\rho)$ such that $\|\mathbf{S}(t)\|_{\mathcal{L}(\mathbf{X},\mathbf{Y})} \leq \varphi(t)$ a.e. on $\mathbb{R}^+$.

(iii) ρ-*uniformly integrable* if $\mathbf{S} \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X},\mathbf{Y});\rho)$.

The integrability results for $\mathbf{V}(t) = \dot{\mathbf{U}}(t)$, as well as the integral $\mathbf{W}(t)$ defined (strongly) by

$$\mathbf{W}(t) = \int_0^t \mathbf{U}(\tau)\,d\tau,$$

are stated in terms of the space (see [**19**])

$$\mathbf{X}_1 = \{\mathbf{x} \in \mathbf{X} : \mathbf{C}(t)\mathbf{x} \in C^1(\mathbb{R}^+, \mathbf{X})\}$$

with norm

$$\|\mathbf{x}\|_1 = \|\mathbf{x}\| + \sup_{0\leq t\leq 1} \|\dot{\mathbf{C}}(t)\mathbf{x}\|. \tag{2.4}$$

It is well known that $\mathbf{X}_1$ is a Banach space. Moreover, for any $\omega_1 > \omega_0(\mathbf{L})$, where $\omega_0(\mathbf{L}) \geq 0$ is the growth type of $\mathbf{C}(t)$ defined by

$$\omega_0(\mathbf{L}) = \inf\{\omega \geq 0 \mid \text{ there exists } M \in \mathbb{R}^+ \text{ such that } \|\mathbf{C}(t)\| \leq Me^{\omega|t|} \text{ for } t \in R\},$$

the estimate

$$\|\dot{\mathbf{C}}(t)\mathbf{x}\| \leq Ke^{\omega_1 t}\|\mathbf{x}\|_1, \quad t \geq 0,\ \mathbf{x} \in \mathbf{X}_1, \tag{2.5}$$

holds for some $K = K(\omega_1) \geq 1$. (These facts concerning cosine families may be found in [**21**] and [**8**, Chapters 2 and 3].) In the following, $\hat{A}(s) = E/s + \hat{a}(s)$ denotes the Laplace transform of $A(t)$.

With these preliminaries we have

THEOREM 2.1. *Let* $A(t) = E + a(t)$ *where* $E > 0$ *and* (1.2) *holds. Set* $\mu = \sqrt{A(0+)}$, *and* $\kappa = -\dot{A}(0+)/2\mu$ *when* $\mu < \infty$. *Assume that* $\mathbf{L}$ *is invertible and that* $s/\hat{A}(s) \in \mathcal{R} :=$ *resolvent set of* $\mathbf{L}$ *for* $s \in \Pi := \{s : \Re s \geq 0\}$, $s \neq 0$, *in case* $\omega_0(\mathbf{L}) > 0$. *Let* $\rho(t)$ *be a weight satisfying* (W), *and assume that* $a \in AC^2_{\mathrm{loc}}(0,\infty)$ *satisfies*

$$\int_1^\infty (|\dot{a}(t)| + t\ddot{a}(t) + |t^2\dddot{a}(t)|)\rho(t)\,dt < \infty. \tag{2.6}$$

(i) *If* $\mu + \kappa = \infty$, *then* $(1+t)\mathbf{U}(t) \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); \rho)$ *and both* $\mathbf{LW}(t)$ *and* $(1+t)^2\mathbf{V}(t)$ *belong to* $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); \rho)$.

(ii) *If* $\mu + \kappa < \infty$ *and* $\omega_0(\mathbf{L}) < \kappa/\mu^2$, *then* $(1+t)\mathbf{U}(t)$ *is* ρ*-integrable in* $\mathcal{L}(\mathbf{X})$ *and both* $\mathbf{LW}(t)$ *and* $(1+t)^2\mathbf{V}(t)$ *are* ρ*-integrable in* $\mathcal{L}(\mathbf{X}_1, \mathbf{X})$.

We remark that $s/\hat{A}(s) \in \mathcal{R}$, $s \in \Pi$, $s \neq 0$, automatically holds when $\omega_0(\mathbf{L}) = 0$ since (see [**21**, Proposition 2.6]), $z^2 \in \mathcal{R}$ whenever a complex number z satisfies $\Re z > \omega_0(\mathbf{L})$, and $\beta(s) = (s/\hat{A}(s))^{1/2}$ satisfies $\Re\beta(s) > 0$, $s \in \Pi$, $s \neq 0$; this can be deduced from $\Re\hat{A}(s) > 0$, $(\Im s)(\Im\hat{A}(s)) < 0$, $\Re s \geq 0$, $\Im s \neq 0$ [**10**].

We finish this section with a discussion of Theorem 2.1 and, especially, hypothesis (2.6).

First note that if $\rho(t)$ is a differentiable weight satisfying (W) and

$$t\dot{\rho}(t) \leq M\rho(t), \quad t \geq 1, \tag{2.7}$$

for some $M < \infty$, then

$$\int_1^\infty |\dot{a}(t)|\rho(t)\,dt < \infty, \tag{2.8}$$

together with monotonicity conditions on $a(t)$, can be used to show that the other integrals in (2.6) are also finite. For example, since a is nonincreasing and convex, an integration by parts yields

$$\int_1^\infty t\rho(t)\ddot{a}(t)\,dt \leq -\rho(1)\dot{a}(1) + \int_1^\infty |\dot{a}(t)|(t\dot{\rho}(t) + \rho(t))\,dt < \infty$$

when (2.7) and (2.8) hold. If, in addition, $-\dot{a}(t)$ is convex, then $t^2\dddot{a}(t) \in L^1(\mathbb{R}^+; \rho)$ also follows from (2.7) and (2.8). (Use the fact that $(1+t)\rho(t)$ satisfies an inequality of the form (2.7) when $\rho(t)$ does.) Note that the weights ρ_1 and ρ_2 in (2.3) satisfy (2.7).

On the other hand, for general weights satisfying (W), (2.6) does not follow from (2.8) and monotonicity conditions on a. An example of this is provided by taking $\rho(t) = \exp(t^\alpha)$, $0 < \alpha < 1$, and $a(t) = \int_t^\infty (1+\tau)^{-p}/\rho(\tau)\,d\tau$ with $1 < p \leq 1 + \alpha$, for which it is easily checked that $\dot{a}(t) \in L^1(\mathbb{R}^+; \rho)$ but $t\ddot{a}(t) \notin L^1(\mathbb{R}^+; \rho)$.

Finally, we note that in the case where $\rho(t) \equiv 1$, Theorem 2.1 sharpens the results of Prüss in [**18**] and [**19**] since there the conclusion is that $\mathbf{U}$ and $\mathbf{V}$ are integrable, whereas Theorem 2.1 deals with the integrability of $(1+t)\mathbf{U}(t)$ and $(1+t)^2\mathbf{V}(t)$. One power of $(1+t)$ in our results depends crucially on the fact that $E > 0$, and, in fact, the conclusions of Theorem 2.1 do not in general hold when $E = 0$. The reason for this will become clear in the proofs that appear in the next section. However, analogues of Theorem 2.1 for $\mathbf{U}(t)$ and $\mathbf{V}(t)$ in the case when $E = 0$ can also be derived. The basic change in the conclusion is that $(1+t)\mathbf{U}(t)$ must be replaced by $\mathbf{U}(t)$ and $(1+t)^2\mathbf{V}(t)$ by $(1+t)\mathbf{V}(t)$. We leave it to the interested reader to state precisely and prove these results.

3. Proof of Theorem 2.1 and some extensions. This section is organized as follows. First, in Section 3.1 we prove Theorem 2.1(ii), that is, the regular case where $\mu + \kappa < \infty$. In Section 3.2 we turn to the singular case, i.e., $\mu + \kappa = \infty$. Here we prove the more general Theorem 3.2 and show that Theorem 2.1(i) follows from Theorem 3.2 and the decomposition Proposition 3.1.

3.1 *Proof of Theorem* 2.1 *in the regular case.* In this section we assume that $\mu + \kappa < \infty$ and prove Theorem 2.1(ii).

(a) Proof that $(1+t)\mathbf{U}(t)$ is ρ-integrable in $\mathcal{L}(\mathbf{X})$. Following the proof of the regular case of Theorem 11 in [**18**], write $\mathbf{U}(t) = \mathbf{U}_0(t) + \mathbf{U}_1(t)$, where

$$\mathbf{U}_1(t) = \mathbf{C}(\mu t)\exp(-\kappa t/\mu).$$

(We remark that, due to the weight $\rho(t)$, we cannot in general assume w.l.o.g. that $\mu = 1$ as is done in [**18**].) By definition of $\omega_0(\mathbf{L})$, if $\epsilon \in (0, \kappa\mu^{-2} - \omega_0(\mathbf{L}))$, there exists $M > 0$ so that $\|\mathbf{C}(t)\| \leq M\exp\{(\kappa\mu^{-2} - \epsilon)t\}$, $t \geq 0$. Hence,

$$\|\mathbf{U}_1(t)\| \leq Me^{-\epsilon\mu t}, \quad t \geq 0, \tag{3.1}$$

and it suffices to prove that

$$(1+t)\mathbf{U}_0(t) \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); \rho). \tag{3.2}$$

To do this, use the fact that $\hat{\mathbf{C}}(s) = s(s^2 - \mathbf{L})^{-1}$ and the operational calculus to get

$$\hat{\mathbf{U}}_1(s) = \frac{1}{\mu}\{\hat{g}^{-1}(s) - \hat{g}(s)\mathbf{L}\}^{-1}, \quad s \in \Pi, \tag{3.3}$$

where $\hat{g}(s) = \mu^2(\mu s + \kappa)^{-1}$ is the transform of $g(t) = \mu \exp(-\kappa t/\mu)$. We now obtain a convolution equation for $\mathbf{U}_0(t)$, and use the Paley-Wiener Lemma for abstract equations with weights due to Gripenberg [**9**, Theorem 2] to deduce (3.2). Following the argument in [**18**], note that

$$\begin{aligned} \hat{\mathbf{U}}_0 &= \hat{\mathbf{U}} - \hat{\mathbf{U}}_1 = (\hat{\mathbf{U}}_1^{-1} - \hat{\mathbf{U}}^{-1})\hat{\mathbf{U}}_1\hat{\mathbf{U}} \\ &\equiv \hat{\mathbf{R}}(s)\hat{\mathbf{U}}(s). \end{aligned} \tag{3.4}$$

Now, we can divide both sides of (3.4) by $(1+\kappa\hat{A}(s))$ and rearrange as in [**18**] to see that $\mathbf{U}_0$ satisfies the convolution equation whose transform is

$$\hat{\mathbf{U}}_0(s) = \hat{\mathbf{R}}_1(s) + \hat{\mathbf{R}}_2(s)\hat{\mathbf{U}}_0(s), \tag{3.5}$$

where

$$\begin{aligned} \hat{\mathbf{R}}_1(s) &= (1 + \kappa\hat{A}(s))^{-1}\hat{\mathbf{R}}(s)\hat{\mathbf{U}}_1(s), \\ \hat{\mathbf{R}}_2(s) &= (1 + \kappa\hat{A}(s))^{-1}(\hat{\mathbf{R}}(s) + \kappa\hat{A}(s)). \end{aligned}$$

Notice also that

$$\hat{\mathbf{U}}_1^{-1}(s) - \hat{\mathbf{U}}^{-1}(s) = \frac{\kappa}{\mu} + \{(\hat{g}^{-1}(s)\hat{A}(s) - \mu)\hat{g}(s)\mathbf{L}\}.$$

Making the substitutions

$$s\hat{A}(s) = \mu^2 + \hat{a}(s), \quad \hat{\hat{a}}(s) = s\hat{a}(s) + 2\kappa\mu, \tag{3.6}$$

we can now rearrange terms as in [**18**] to verify that

$$\begin{aligned} \hat{\mathbf{R}}(s) = [&\mu^{-4}\{\mu^2\hat{\hat{a}}(s) + 2\kappa\mu\hat{a}(s) + \kappa^2\hat{A}(s)\} \\ &- \mu^3\{\mu\hat{a}(s) + \kappa\hat{A}(s)\}\hat{\mathbf{U}}_1^{-1}(s)]\hat{\mathbf{U}}_1(s) \\ = &\,\mu^{-4}\{\mu^2\hat{\hat{a}}(s) + 2\kappa\mu\hat{a}(s) + \kappa^2\hat{A}(s)\}\hat{\mathbf{U}}_1(s) \\ &- \mu^{-3}\{\mu\hat{a}(s) + \kappa\hat{A}(s)\}. \end{aligned}$$

Next, we use Theorem 6.1 of [**13**] to deduce the following lemma for the scalar functions appearing in the expression (3.5). $V(\mathbb{R}^+;\rho)$ denotes the Banach algebra formed by adjoining the unit δ (point mass at 0) to $L^1(\mathbb{R}^+;\rho)$, and $V(\mathbb{R}^+;\rho)^\wedge$ denotes the algebra of Laplace transforms of these measures.

LEMMA 3.1. *Under the assumptions of Theorem* (2.1), $\hat{A}(s)(1+\kappa\hat{A}(s))^{-1}$, $\hat{a}(s)(1+\kappa\hat{A}(s))^{-1}$ *and* $\hat{\dot{a}}(s)(1+\kappa\hat{A}(s))^{-1}$ *all belong to* $L^1(\mathbb{R}^+;(1+t)\rho(t))^\wedge$.

PROOF. First, note that $\kappa\hat{A}(s)(1+\kappa\hat{A}(s))^{-1}$ is the Laplace transform of the integral resolvent

$$r_1(t) = \kappa A(t) - r_1 * \kappa A(t) = \kappa A(t) - \kappa A * r_1(t).$$

As an application of Theorem 6.1 of [**13**], it is shown on p. 770 of that paper that our hypotheses imply that $r_1 \in L^1(\mathbb{R}^+;(1+t)\rho(t))$, and the proof for $\hat{A}(s)(1+\kappa\hat{A}(s))^{-1}$ is complete.

A closely analogous argument to that on p. 770 of [**13**] yields the claim for $\varphi_2(s) = \hat{a}(s)(1+\kappa\hat{A}(s))^{-1}$. Namely, near $s_0 = 0$, write

$$\varphi_2(s) = s\hat{a}(s)(s+\kappa\mu^2+\kappa\hat{\dot{a}}(s))^{-1},$$

and note that $\varphi_2(s)$ is locally analytic w.r.t. $V(\mathbb{R}^+;\rho)$ at $s_0 = 0$, and that the order of dependence of φ_2 on s with respect to $\hat{\dot{a}}(s)$ at $(0,\hat{\dot{a}}(0))$ is at least $m = 1$ [**13**, Definition 4.1]. Near nonzero points of Π as well as near ∞, write

$$\varphi_2(s) = \hat{a}(s)(1+\kappa\mu^2 s^{-1} + \kappa s^{-1}\hat{\dot{a}}(s))^{-1};$$

thus, $\varphi_2(s)$ is locally analytic w.r.t. $V(\mathbb{R}^+;\rho)$ on $\bar{\Pi} = \Pi \cup \{\infty\}$ and $\varphi_2(0) = 0$. Finally, note that, by (2.6) and $\mu + \kappa < \infty$, $|\dot{a}(t)| + t\ddot{a}(t) \in L^1(\mathbb{R}^+;\rho)$, so $\dot{a}$ satisfies condition (6.8) of [**13**] with $m = 1$ and $z_0 = s_0 = 0$. Thus, using [**13**, Theorem 6.1] with $m = 1$ and $z_0 = s_0 = 0$, we conclude that $\varphi_2 \in L^1(\mathbb{R}^+;(1+t)\rho(t))^\wedge$.

The same reasoning can be used to show that $\varphi_3(s) = \hat{\dot{a}}(s)(1+\kappa\hat{A}(s))^{-1} \in L^1(\mathbb{R}^+;(1+t)\rho(t))^\wedge$. Alternatively, we can write $\varphi_3(s) =$

$\hat{\hat{a}}(s)(1-\hat{r}_1(s))$ and use the fact that $\hat{\hat{a}}(s)$ and $\hat{r}_1(s)$ both belong to $L^1(\mathbb{R}^1;(1+t)\rho(t))^\wedge$ to conclude that φ_3 does too. $\square$

Returning to the proof of (3.2), use (3.1) and Lemma 3.1 to conclude that $\hat{\mathbf{R}}_1(s)$ and $\hat{\mathbf{R}}_2(s)$ both belong to $L^1(\mathbb{R}^+,\mathcal{L}(\mathbf{X});(1+t)\rho(t))^\wedge$. (Here and below we use the fact that the convolution of a locally integrable function with a strongly continuous operator ($\mathbf{U}_1$ here) is measurable.)

As in [**18**] we see that

$$\mathbf{I}-\dot{\mathbf{R}}_2(s)=(1+\kappa\hat{A}(s))^{-1}(s-\hat{A}(s)\mathbf{L})\hat{\mathbf{U}}_1(s),\quad s\in\Pi,$$

so $\mathbf{I}-\hat{\mathbf{R}}_2(s)$ is invertible for each $s\in\Pi$. Now, by [**9**, Theorem 2], there exists $\mathbf{Q}_1\in L^1(\mathbb{R}^+,\mathcal{L}(\mathbf{X});(1+t)\rho(t))$ such that

$$\mathbf{Q}_1(t)=\mathbf{R}_2(t)+\mathbf{Q}_1*\mathbf{R}_2(t)=\mathbf{R}_2(t)+\mathbf{R}_2*\mathbf{Q}_1(t),\qquad t\geq 0.$$

Thus, solving (3.6) for $\hat{\mathbf{U}}_0$, we see that $\hat{\mathbf{U}}_0(s)=\hat{\mathbf{R}}_1(s)+\hat{\mathbf{Q}}_1(s)\hat{\mathbf{R}}_1(s)$ and (3.2) is proved. $\square$

(b) Proof that $(1+t)^2\mathbf{V}(t)$ and $\mathbf{LW}(t)$ are ρ-integrable in $\mathcal{L}(\mathbf{X}_1,\mathbf{X})$. First consider $\mathbf{V}(t)$. Pick $\omega_1\in(\omega_0(\mathbf{L}),\kappa\mu^{-2})$ and note that, by (2.5), $\|\dot{\mathbf{C}}(t)\|_{\mathcal{L}(\mathbf{X}_1,\mathbf{X})}\leq Me^{\omega_1 t}$. Define

$$\mathbf{V}_1(t)=\mu\dot{\mathbf{C}}(\mu t)\exp(-\kappa t/\mu)$$

and write $\mathbf{V}(t)=\mathbf{V}_0(t)+\mathbf{V}_1(t)$. Since

$$\|\mathbf{V}_1(t)\|_{\mathcal{L}(\mathbf{X}_1,\mathbf{X})}\leq\mu M\exp(\mu[\omega_1-\kappa\mu^{-2}]t)$$

is exponentially decaying, it suffices to show that

$$(1+t)^2\mathbf{V}_0(t)\in L^1(\mathbb{R}^+,\mathcal{L}(\mathbf{X}_1,\mathbf{X});\rho). \tag{3.7}$$

To prove (3.7), note that $\hat{\dot{\mathbf{C}}}(s)=\mathbf{L}(s^2-\mathbf{L})^{-1}$, so

$$\hat{\mathbf{V}}_1(s)=\hat{\dot{\mathbf{C}}}(\hat{g}^{-1}(s))=\mathbf{L}(\hat{g}^{-2}(s)-\mathbf{L})^{-1}, \tag{3.8}$$

where, as before, $\hat{g}(s) = \mu^2(\mu s+\kappa)^{-1}$. Combining (3.3) and (3.8) yields

$$\mathbf{L}^{-1}\hat{\mathbf{V}}_1(s) = \mu\hat{g}(s)\hat{\mathbf{U}}_1(s). \tag{3.9}$$

Since $\hat{\mathbf{V}}(s) = \hat{A}(s)\mathbf{L}(s - \hat{A}(s)\mathbf{L})^{-1}$,

$$\begin{aligned}\hat{\mathbf{V}}_0(s) &= (\hat{\mathbf{V}}_1^{-1}(s) - \hat{\mathbf{V}}^{-1}(s))\hat{\mathbf{V}}_1(s)\hat{\mathbf{V}}(s)\\ &= \{\hat{g}^{-2}(s) - s/\hat{A}(s)\}\mathbf{L}^{-1}\hat{\mathbf{V}}_1(s)\hat{\mathbf{V}}(s),\end{aligned}$$

and after some manipulation we see that

$$\hat{\mathbf{V}}_0(s) = \hat{\mathbf{R}}_4(s) + \hat{\mathbf{R}}_3(s)\hat{\mathbf{V}}_0(s), \tag{3.10}$$

where

$$\begin{aligned}\hat{\mathbf{R}}_3(s) &= \kappa^2\mu^{-3}\hat{g}(s)\hat{\mathbf{U}}_1(s) + \mu\hat{r}_0(s)\hat{\mathbf{U}}_1(s),\\ \hat{\mathbf{R}}_4(s) &= \hat{\mathbf{R}}_3(s)\hat{\mathbf{V}}_1(s),\end{aligned}$$

and $r_0(t)$ is the scalar function whose transform is

$$\hat{r}_0(s) = \varphi_0(s) = \hat{g}(s)\{\hat{g}^{-2}(s) - s/\hat{A}(s) - \kappa^2/\mu^4\}.$$

Theorem 6.1 of [**13**] can be used to show

$$r_0(t) \in L^1(\mathbb{R}^+; (1+t)^2\rho(t)). \tag{3.11}$$

Assuming (3.11) for the moment, we now use Gripenberg's version of the Paley-Wiener Lemma to obtain (3.7). To do this, first observe that, by (3.1), (3.11), and the definition of g, $\mathbf{R}_3 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)^2\rho(t))$. To see that $\mathbf{I} - \hat{\mathbf{R}}_3(s)$ is invertible for $s \in \Pi$, note that $\hat{\mathbf{V}}(s) - \hat{\mathbf{V}}_1(s) = \hat{\mathbf{V}}_0(s) = \hat{\mathbf{R}}_3(s)\hat{\mathbf{V}}(s)$; so $\mathbf{I} - \hat{\mathbf{R}}_3(s) = \hat{\mathbf{V}}_1(s)\hat{\mathbf{V}}^{-1}(s)$ and clearly this is invertible for $s \in \Pi$. By [**9**, Theorem 2], there exists $\mathbf{Q}_2 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)^2\rho(t))$ such that $\mathbf{Q}_2 = \mathbf{R}_3 + \mathbf{Q}_2 * \mathbf{R}_3$. Since $\|\mathbf{R}_3(t)\|_{\mathcal{L}(\mathbf{X}_1,\mathbf{X})} \le \|\mathbf{R}_3(t)\|_{\mathcal{L}(\mathbf{X})}$, $\mathbf{R}_4 = \mathbf{R}_3 * \mathbf{V}_1 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); (1+t)^2\rho(t))$; hence, solving (3.10) for $\mathbf{V}_0$ gives $\mathbf{V}_0 = \mathbf{R}_4 + \mathbf{Q}_2 * \mathbf{R}_4$, and (3.7) is proved once we verify (3.11).

Turning to $\mathbf{LW}(t)$, note that from (2.2) and the definitions of $\mathbf{V}$ and $\mathbf{W}$, $\mathbf{L}\hat{\mathbf{W}}(s) = \hat{\mathbf{V}}(s)/(s\hat{A}(s))$. We claim that

$$\frac{1}{s\hat{A}(s)} = \frac{1}{\mu^2} + \hat{k}(s) \quad \text{where } k \in L^1(\mathbb{R}^+; \rho). \tag{3.12}$$

To verify (3.12), simply note that $1/(s\hat{A}(s)) - 1/\mu^2 = 1/(\mu^2 + \hat{a}(s)) - 1/\mu^2$ is locally analytic w.r.t. $V(\mathbb{R}^+;\rho)$ on all of $\bar{\Pi}$ and equal to zero at ∞, so (3.12) is a consequence of Proposition 2.3 in [**13**]. Thus, $\mathbf{LW}(t) = \mu^{-2}\mathbf{V}(t) + k * \mathbf{V}(t)$ is ρ-integrable in $\mathcal{L}(\mathbf{X}_1, \mathbf{X})$.

It remains to give the

Proof of (3.11). By the first part of (3.6) and the definition of $r_0(t)$, it is easy to verify that

$$\varphi_0(s) = \frac{s}{\mu(\mu s + \kappa)}(\mu s + 2\kappa) - \frac{\mu^2 s^2}{(\mu s + \kappa)(\mu^2 + \hat{a}(s))}$$

is locally analytic w.r.t. $V(\mathbb{R}^+;\rho)$ on Π, and that the order of dependence on the transforms used in this expression at $s_0 = 0$ is at least 2. At ∞ use the second part of (3.6) to write

$$\begin{aligned}\varphi_0(s) &= \frac{1}{\mu(\mu + \kappa/s)}\left\{\frac{(\mu s + 2\kappa)(\mu^2 + \hat{a}(s)) - \mu^3 s}{\mu^2 + \hat{a}(s)}\right\} \\ &= \frac{1}{\mu(\mu + \kappa/s)}\left\{\frac{\mu\hat{\dot{a}}(s) + 2\kappa\hat{a}(s)}{\mu^2 + \hat{a}(s)}\right\}.\end{aligned}$$

Write $a(t) = a_1(t) + a_2(t)$ where a_1 has compact support and $a_2(t)$ vanishes on some interval $(0, T)$. Then, using (2.6), it is easy to check that all the functions whose transforms appear in the representation for $\varphi_0(s)$ satisfy (6.8) of [**13**] with $m = 2$, $z_0 = s_0 = 0$, and (3.11) follows from Theorem 6.1 of [**13**]. $\square$

Remarks. An examination of the proof of Lemma 3.1 shows that the assumption that $t^2\dddot{a}(t) \in L^1((1,\infty);\rho)$ was never used. Hence, the conclusion that $(1+t)\mathbf{U}(t)$ is ρ- integrable remains true with (2.6) replaced by the weaker assumption that $\dot{a}(t)$ and $t\ddot{a}(t) \in L^1((1,\infty);\rho)$.

It is also interesting to note that the assumption that $a(t)$ be log-convex may be dropped in Theorem 2.1(ii) if we are willing to relax the requirement of strong continuity and settle for resolvents that are only ρ-integrable and that are defined by their respective Laplace transforms. An examination of the proof above shows that we have proved

THEOREM 2.1 (ii)$'$. *Let the hypotheses and notation of Theorem* 2.1(ii) *hold except that* (1.2) *is replaced by*

$$a \in AC^1[0,\infty) \text{ with } a(0) > 0, \dot{a}(0) < 0 \text{ and } a(t) \to 0 \text{ as } t \to \infty,$$

(2.6) *is replaced by*

$$\int_0^\infty (|\dot{a}(t)| + |t\ddot{a}(t)|)\rho(t)\,dt < \infty,$$

and

$$1 + \kappa \hat{A}(s) \neq 0, \quad s \in \Pi,\ s \neq 0,$$

is assumed to hold. We must now assume that $s/\hat{A}(s) \in \mathcal{R}$ *for* $s \in \Pi$, $s \neq 0$, *even when* $\omega_0(\mathbf{L}) = 0$. *Then there is a strongly measurable family* $\mathbf{U}(t)$ *in* $\mathcal{L}(\mathbf{X})$ *defined by* $\hat{\mathbf{U}} = (s - \hat{A}(s)\mathbf{L})^{-1}$, $s \in \Pi$, $s \neq 0$, *such that* $(1+t)\mathbf{U}(t)$ *is* ρ*-integrable in* $\mathcal{L}(\mathbf{X})$.

If, in addition, $a \in AC^2_{\mathrm{loc}}(0,\infty)$ *and*

$$\int_0^\infty (|\dot{a}(t)| + |t\ddot{a}(t)| + |t^2\dddot{a}(t)|)\rho(t)\,dt < \infty,$$

then there are strongly measurable families $\mathbf{V}(t)$ *and* $\mathbf{LW}(t)$ *in* $\mathcal{L}(\mathbf{X}_1, \mathbf{X})$ *defined by* $\hat{\mathbf{V}}(s) = \hat{A}(s)\mathbf{L}(s - \hat{A}(s)\mathbf{L})^{-1}$, $\mathbf{L}\hat{\mathbf{W}}(s) = s^{-1}\mathbf{L}(s - \hat{A}(s)\mathbf{L})^{-1}$, $s \in \Pi$, $s \neq 0$, *respectively, such that* $(1+t)^2\mathbf{V}(t)$ *and* $\mathbf{LW}(t)$ *are* ρ*-integrable in* $\mathcal{L}(\mathbf{X}_1, \mathbf{X})$.

We note that if $-\mathbf{L}$ is a positive definite self-adjoint operator in a Hilbert space $\mathbf{X}$, and if a satisfies the conditions of Theorem 2.1(ii)$'$ and is positive, nonincreasing and convex, then $\mathbf{U}(t)$ and $\mathbf{V}(t)(-\mathbf{L})^{-1/2}$ are bounded and strongly continuous in $\mathbf{X}$ by [**1, 2**].

3.2 *The singular case.* We now consider the singular case where $\mu + \kappa = \infty$. In Theorem 3.1 we show that if $A_1(t) = E + a_1(t)$ is as in Theorem 2.1(i), but with a_1 decaying exponentially fast as $t \to \infty$, then the conclusion of Theorem 2.1(i) holds for any weight $\rho_1(t)$ satisfying (W). Theorem 3.2 shows that the conclusion of Theorem 2.1(i) holds for "perturbed" kernels of the form $A(t) = A_1(t) + a_2(t)$, when A_1 is as in Theorem 3.1, and $a_2(t)$ satisfies no monotonicity conditions, but vanishes near $t = 0$ and has derivatives belonging to appropriate weighted L^1-spaces. Theorem 2.1(i) follows from the decomposition

Proposition 3.1 which shows that if A is as in Theorem 2.1(i), then it may be decomposed in the form $A(t) = A_1(t) + a_2(t)$ so that Theorem 3.2 applies.

(a) Exponentially decaying kernels. We prove

THEOREM 3.1. *Let $A_1(t) = E + a_1(t)$ where $E > 0$ and a_1 satisfies* (1.2) *with $\dot{a}_1(0+) = -\infty$, and $a_1 \in AC^1_{\mathrm{loc}}(0,\infty)$ if $\mu_1 = \sqrt{A_1(0+)} < \infty$. Assume that*

$$e^{\epsilon t}\dot{a}_1(t) \in L^1(\mathbb{R}^+) \quad \textit{for some } \epsilon > 0. \tag{3.13}$$

Assume that $\mathbf{L}$ generates a C_0 cosine family, $\mathbf{L}$ is invertible, and

$$s/\hat{A}_1(s) \in \mathcal{R} \quad \textit{for } s \in \Pi,\ s \neq 0,\ \textit{in case } \omega_0(\mathbf{L}) > 0. \tag{3.14}$$

Let $\mathbf{U}_1, \mathbf{V}_1, \mathbf{W}_1$ be the analogues of $\mathbf{U}, \mathbf{V}, \mathbf{W}$, respectively, with the kernel $A(t)$ replaced by $A_1(t)$. Then, for any weight $\rho_1(t)$ satisfying (W), *$\mathbf{U}_1(t) \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); \rho_1)$ and both $\mathbf{L}\mathbf{W}_1(t)$ and $\mathbf{V}_1(t)$ belong to $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); \rho_1)$.*

PROOF. The proof of Theorem 3.1 combines the technique used by Prüss to prove the singular kernel case of Theorem 11 of [**18**] with the methods used in Section 3.1. We begin by recalling some notation from and consequences of results in Section 2 of [**18**].

As an easy consequence of [**18**, Proposition 2] we get

LEMMA 3.2. *Let A_1 be as in Theorem* 3.1 (*without the requirement that $\dot{a}_1(0+) = -\infty$*), *and set*

$$\alpha_1(s) = (s\hat{A}_1(s))^{-1/2} \quad (\textit{principal branch}).$$

Then we can write

$$\frac{1}{\alpha_1(s)} = \frac{1}{\mu_1} + \hat{\dot{k}}_1(s) \tag{3.15}$$

where $\dot{k}_1$ is nonnegative. Moreover,

$$\int_0^\infty e^{rt}\dot{k}_1(t)\,dt = \frac{1}{\alpha_1(r)} - \frac{1}{\mu_1} < \infty \tag{3.16}$$

for $-\infty < r \leq \eta$, *where* η *is any number such that* $0 < \eta \leq \epsilon$ *and* $E - \eta\hat{a}_1(-\eta) > 0$.

PROOF. The representation (3.15) for $\Re s > 0$ is simply Proposition 2(iv) of [**18**] applied to the kernel $A_1(t)$. Since $\dot{k}_1 \geq 0$, the real point of the axis of convergence of $\hat{\dot{k}}_1(s)$ is a singularity of $\hat{\dot{k}}_1(s)$ by [**22**; p. 58, Theorem 5b], so (3.16) holds. □

Next set

$$\beta_1(s) = s/\alpha_1(s),$$

and define $h_0(s,x)$ and $h(s,x)$ for the kernel $A_1(t)$ as on p. 327 of [**18**] by

$$h_0(s,x) = \exp(-x\beta_1(s)), \quad h(s,x) = \frac{1}{\alpha_1(s)} h_0(x,s). \tag{3.17}$$

By Theorems 3 and 4 of [**18**], we can write

$$\hat{w}_{0t}(s,x) = h_0(s,x), \quad \hat{w}_t(s,x) = h(s,x), \tag{3.18}$$

where $w_0(t,x)$ and $w(t,x)$ are the functions in Theorem 3 of [**18**] corresponding to $A_1(t)$. Notice, in particular, that, for each $x > 0$, $w_0(t,x)$ and $w(t,x)$ are nondecreasing and continuous functions of $t \geq 0$ that are absolutely continuous for $t \neq x/\mu_1$.

Now let ω_1 be a positive number satisfying $\omega_1 > \omega_0(\mathbf{L})$ (the growth type of $\mathbf{C}(t)$), and fix a positive number $\omega > \omega_1 - \beta_1(-\eta)$ where $\eta > 0$ is chosen as in Lemma 3.2. Define $\mathbf{U}_{1,\omega}(t)$ and $\mathbf{R}_{1,\omega}(t)$ as in (7.7), (7.9) of [**18**], respectively, by

$$\mathbf{U}_{1,\omega}(t) = \int_0^\infty e^{-\omega\tau}\mathbf{C}(\tau) w_t(t,\tau)\, d\tau,$$
$$\mathbf{R}_{1,\omega}(t) = \int_0^\infty e^{-\omega\tau}\mathbf{C}(\tau) w_{0t}(t,\tau)\, d\tau.$$

Similarly, define $\mathbf{V}_{1,\omega}(t)$ by

$$\mathbf{V}_{1,\omega}(t) = \int_0^\infty e^{-\omega\tau}\dot{\mathbf{C}}(\tau) w_{0t}(t,\tau)\, d\tau.$$

Note that, by the definition of $\omega_0(\mathbf{L})$ and (2.5), there is a $K > 1$ such that

$$\|\mathbf{C}(t)\| \le Ke^{\omega_1 t}, \quad \|\dot{\mathbf{C}}(t)\|_{\mathcal{L}(\mathbf{X}_1,\mathbf{X})} \le Ke^{\omega_1 t}, \quad t \ge 0.$$

(Here $\|\mathbf{C}(t)\|$ is the norm in $\mathcal{L}(\mathbf{X})$.) Combining this with the fact that w_t and w_{0t} are nonnegative, we can deduce that

$$\mathbf{U}_{1,\omega}(t) \text{ and } \mathbf{R}_{1,\omega}(t) \text{ belong to } L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); e^{\eta t}), \tag{3.19}$$

$$\mathbf{V}_{1,\omega}(t) \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); e^{\eta t}). \tag{3.20}$$

To verify the first claim in (3.19), simply note that

$$\begin{aligned}\int_0^\infty e^{\eta t}\|\mathbf{U}_{1,\omega}(t)\|\,dt &\le K\int_0^\infty\int_0^\infty e^{(\omega_1-\omega)\tau}e^{\eta t}w_t(t,\tau)\,d\tau\,dt\\ &= K\alpha_1^{-1}(-\eta)\int_0^\infty e^{[\omega_1-\beta_1(-\eta)-\omega]\tau}\,d\tau < \infty,\end{aligned}$$

where we have used (3.17) and (3.18). The other claims in (3.19) and (3.20) are proved in an analogous fashion.

Since $\hat{\mathbf{C}}(s) = s(s^2 - \mathbf{L})^{-1}$, $s > \omega_0(\mathbf{L})$, some calculation, as in [**18**], using (3.17) and (3.18) (see [**18**, formula (8.3)]), yields

$$\mathbf{U}_1(t) = \mathbf{S}_1(t) + \mathbf{S}_2 * \mathbf{U}_1(t), \tag{3.21}$$

where

$$\begin{aligned}\hat{\mathbf{S}}_1(s) &= h(s)\hat{\mathbf{U}}_{1,\omega}(s),\\ \hat{\mathbf{S}}_2(s) &= \omega(1+h(s))\hat{\mathbf{R}}_{1,\omega}(s).\end{aligned}$$

Here, as on p. 341 of [**18**], $h(s)$ denotes

$$h(s) = \beta_1(s)[\omega + \beta_1(s)]^{-1}.$$

Similarly, as in the derivation of (3.10), we can write $\hat{\mathbf{V}}_1 - \hat{\mathbf{V}}_{1,\omega} = (\hat{\mathbf{V}}_{1,\omega}^{-1} - \hat{\mathbf{V}}_1^{-1})\hat{\mathbf{V}}_{1,\omega}\hat{\mathbf{V}}_1$. Using (3.17), (3.18) and $\hat{\dot{\mathbf{C}}}(s) = \mathbf{L}(s^2 - \mathbf{L})^{-1}$ for $\Re s > \omega_0(\mathbf{L})$, we get that $\hat{\mathbf{V}}_1 - \hat{\mathbf{V}}_{1,\omega} = (\omega^2 + 2\omega\beta_1)\mathbf{L}^{-1}\hat{\mathbf{V}}_{1,\omega}\hat{\mathbf{V}}_1$. Since $\mathbf{L}^{-1}\hat{\mathbf{V}}_{1,\omega} = (\omega + \beta_1)^{-1}\hat{\mathbf{R}}_{1,\omega}$, it is easy to verify that

$$\mathbf{V}_1(t) = \mathbf{V}_{1,\omega}(t) + \mathbf{S}_2 * \mathbf{V}_1(t). \tag{3.22}$$

Next note that, with $\eta > 0$ as above,

$$\begin{aligned}\int_0^\infty e^{\eta t}\left|\int_0^\infty e^{-\omega x} w_{0t}(t,x)\,dx\right| dt \\ &= \int_0^\infty \int_0^\infty e^{\eta t} w_{0t}(t,x)\,dt\; e^{-\omega x}\,dx \\ &= \int_0^\infty e^{-x(\omega+\beta_1(-\eta))}\,dx < \infty,\end{aligned}$$

where we have used $w_{0t} \geq 0$, (3.17), (3.18) and $\omega + \beta_1(-\eta) > \omega_1 > 0$. But using (3.17), (3.18) again, we see that

$$\begin{aligned}[][\omega + \beta_1(s)]^{-1} &= \int_0^\infty e^{-(\omega+\beta_1(s))x}\,dx \\ &= \int_0^\infty e^{-st}\int_0^\infty e^{-\omega x} w_{0t}(t,x)\,dx\,dt, \quad s > 0.\end{aligned}$$

Thus, the scalar function $h(s) = 1 - \omega(\omega + \beta_1(s))^{-1}$ belongs to $V(\mathbb{R}^+; e^{\eta t})^\wedge$, so, by (3.19),

$$\mathbf{S}_1(t) \text{ and } \mathbf{S}_2(t) \text{ belong to } L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); e^{\eta t}). \tag{3.23}$$

Now, as shown on p. 341 of [**18**],

$$\mathbf{I} - \hat{\mathbf{S}}_2(s) = \frac{h(s)}{\beta_1(s)}\left(\frac{s}{\hat{A}_1(s)} - \mathbf{L}\right)\hat{\mathbf{R}}_{1,\omega}(s), \quad s \in \Pi,$$

so, by hypothesis (3.14), $\mathbf{I} - \hat{\mathbf{S}}_2(s)$ is invertible for $s \in \Pi$. Since $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); e^{\eta t}) \subseteq L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); \rho_1(t))$ whenever ρ_1 is a weight satisfying (W), [**9**, Theorem 2] gives a $\mathbf{P}_1 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); \rho_1(t))$ such that $\mathbf{P}_1 = \mathbf{S}_2 + \mathbf{P}_1 * \mathbf{S}_2 = \mathbf{S}_2 + \mathbf{S}_2 * \mathbf{P}_1$. Thus, solving (3.21) and (3.22) for $\mathbf{U}_1$ and $\mathbf{V}_1$, respectively, and using (3.23) and (3.20), we obtain $\mathbf{U}_1 = \mathbf{S}_1 + \mathbf{P}_1 * \mathbf{S}_1 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); \rho_1(t))$ and $\mathbf{V}_1 = \mathbf{V}_{1,\omega} + \mathbf{P}_1 * \mathbf{V}_{1,\omega} \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); \rho_1(t))$.

Finally, $\mathbf{L}\hat{\mathbf{W}}_1 = \hat{\mathbf{V}}_1(s)/\alpha_1^2(s) = (1/\mu_1 + \hat{\hat{k}}_1(s))^2\hat{\mathbf{V}}_1(s)$ by (3.15), and, using (3.16), we also get

$$\mathbf{L}\mathbf{W}_1 \in \mathcal{L}^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); \rho_1(t)). \;\square$$

(b) Perturbed kernels. The conclusion of Theorem 3.1, together with local analyticity and Gripenberg's Paley-Wiener Lemma, can be used to obtain

THEOREM 3.2. *Let* $A_1(t) = E + a_1(t)$ *and* $\mathbf{L}$ *satisfy all the hypotheses of Theorem* 3.1. *Assume that* $\rho(t)$ *is a weight satisfying* (W), *and let* $A(t) = A_1(t) + a_2(t)$ *where* $a_2 \in AC^2_{\text{loc}}(0,\infty)$ *is such that* $a_2(t) = 0$ *on some interval* $(0,T)$, $a_2(t) \to 0$, $t \to \infty$, *and*

$$\int_T^\infty (|\dot{a}_2(t)| + t|\ddot{a}_2(t)| + t^2|\dddot{a}_2(t)|)\rho(t)\,dt < \infty. \tag{3.24}$$

In addition, assume that $s/\hat{A}(s) \in \mathcal{R}$ *for* $s \in \Pi$, $s \neq 0$. *Then* $\mathbf{U}, \mathbf{V}$ *and* $\mathbf{W}$ *satisfy the conclusion of Theorem* 2.1(i).

PROOF. We first remark that the existence of the strongly continuous resolvent $\mathbf{U}(t)$ corresponding to $A(t)$ is guaranteed by Corollary 4 of [**18**].

Since $\hat{\mathbf{U}} - \hat{\mathbf{U}}_1 = (\hat{\mathbf{U}}_1^{-1} - \hat{\mathbf{U}}^{-1})\hat{\mathbf{U}}_1\hat{\mathbf{U}} = \hat{a}_2\mathbf{L}\hat{\mathbf{U}}_1\hat{\mathbf{U}}$, an easy calculation shows that

$$\hat{\mathbf{U}}(s) = \hat{\mathbf{S}}_4(s) + \hat{\mathbf{S}}_3(s)\hat{\mathbf{U}}(s), \tag{3.25}$$

where

$$\hat{\mathbf{S}}_3(s) = \frac{s^2\hat{a}_2(s)}{\hat{A}(s)}\hat{\mathbf{W}}_1(s),$$
$$\hat{\mathbf{S}}_4(s) = \hat{\mathbf{U}}_1(s) - \frac{s\hat{a}_2(s)}{\hat{A}(s)}\hat{\mathbf{W}}_1(s).$$

(Of course, throughout this proof $\mathbf{U}_1$, $\mathbf{V}_1$ and $\mathbf{W}_1$ correspond to the kernel A_1.) Also, using $\hat{\mathbf{V}}(s) = \hat{A}(s)\mathbf{L}\hat{\mathbf{U}}(s) = \hat{A}(s)\mathbf{L}(s - \hat{A}(s)\mathbf{L})^{-1}$ and the corresponding expression for $\hat{\mathbf{V}}_1(s)$, we easily see that

$$\hat{\mathbf{V}}(s) = \hat{\mathbf{V}}_1(s) + \hat{\mathbf{S}}_3(s)\hat{\mathbf{V}}(s). \tag{3.26}$$

Theorem 6.1 of [**13**] can be used to show that the scalar functions occurring in the expressions for $\hat{\mathbf{S}}_3$ and $\hat{\mathbf{S}}_4$ satisfy

$$\frac{s\hat{a}_2(s)}{\hat{A}(s)} \in L^1(\mathbb{R}^+;(1+t)\rho(t))^\wedge, \tag{3.27}$$

$$\frac{s^2\hat{a}_2(s)}{\hat{A}(s)} \in L^1(\mathbb{R}^+;(1+t)^2\rho(t))^\wedge. \tag{3.28}$$

To verify (3.28), set $\varphi(s) = s^2\hat{a}_2(s)\hat{A}^{-1}(s)$, and note that, since $a_2(t) = 0$ on $(0,T)$, φ can be rewritten as

$$\varphi(s) = s^2\hat{a}_2(s)[E + s\hat{a}_1(s) + \hat{\dot{a}}_2(s)]^{-1}.$$

Clearly, this expression is locally analytic w.r.t. $V(\mathbb{R}^+;\rho)$ on Π, and the order of dependence on the transforms used in the expression at $s_0 = 0$ is at least 2. Near ∞, recall that $\alpha_1^2 = s\hat{A}_1$, and use the operational calculus and Lemma 3.2 to write

$$\begin{aligned}\varphi(s) &= s^3\hat{a}_2(s)[E + s\hat{a}_1(s) + s\hat{a}_2(s)]^{-1}\\ &= \hat{\ddot{a}}_2(s)[\alpha_1^2(s) + \hat{\dot{a}}_2(s)]^{-1}\\ &= \hat{\ddot{a}}_2(s)\left(\frac{1}{\mu_1} + \hat{\dot{k}}_1(s)\right)^2\left\{1 + \hat{\dot{a}}_2(s)\left(\frac{1}{\mu_1} + \hat{\dot{k}}_1(s)\right)^2\right\}^{-1}.\end{aligned}$$

By (3.13) and (3.16), $a_1(t)$ and $\dot{k}_1(t)$ belong to $L^1(\mathbb{R}^+;(1+t)^l\rho(t))$ for any $l \geq 0$. Since (3.24) also holds, all the functions whose transforms appear in the representations for $\varphi(s)$ satisfy condition (6.8) of [**13**] with $m = 2$, $z_0 = s_0 = 0$, and (3.28) follows from [**13**, Theorem 6.1]. The proof of (3.27) is similar (here the order of dependence at $s_0 = 0$ is 1) and details are left to the reader.

For each $l \geq 0$, $(1+t)^l\rho(t)$ is a weight satisfying (W) so, by Theorem 3.1,

$$\mathbf{U}_1 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)^l\rho(t)), \qquad l \geq 0.$$

Since

$$\mathbf{W}_1(t) = \int_0^t \mathbf{U}_1(\tau)\,d\tau = -\int_t^\infty \mathbf{U}_1(\tau)\,d\tau,$$

it is easy to check that $\mathbf{W}_1(t)$ also belongs to $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)^l\rho(t))$, $l \geq 0$, and then, using (3.27) and (3.28), we see that $\mathbf{S}_4(t)$ belongs to $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)\rho(t))$, and $\mathbf{S}_3(t)$ belongs to $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)^2\rho(t))$.

Since

$$\mathbf{I} - \hat{\mathbf{S}}_3(t) = \left[(s - \hat{A}_1(s)\mathbf{L}) - \frac{s\hat{a}_2(s)}{\hat{A}(s)}\right](s - \hat{A}_1(s)\mathbf{L})^{-1},$$

we can use the fact that

$$\frac{s(1 - \hat{a}_2(s)/\hat{A}(s))}{\hat{A}_1(s)} = \frac{s}{\hat{A}(s)}$$

belongs to the resolvent set $\mathcal{R}$ for $s \in \Pi$, $s \neq 0$, and the invertibility of $\mathbf{L}$ to see that $\mathbf{I} - \hat{\mathbf{S}}_3(s)$ is invertible for $s \in \Pi$. By [**9**, Theorem 2], there exists $\mathbf{P}_2 \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)^2\rho(t))$ such that $\mathbf{P}_2 = \mathbf{S}_3 + \mathbf{P}_2 * \mathbf{S}_3$, and it follows that $\mathbf{U} = \mathbf{S}_4 + \mathbf{P}_2 * \mathbf{S}_4$ belongs to $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}); (1+t)\rho(t))$, and $\mathbf{V} = \mathbf{V}_1 + \mathbf{P}_2 * \mathbf{V}_1$ belongs to $L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); (1+t)^2\rho(t))$.

Finally, since $\mathbf{L}\hat{\mathbf{W}}(s) = \hat{\mathbf{V}}(s)/(s\hat{A}(s))$, we get $\mathbf{LW}(t) \in L^1(\mathbb{R}^+, \mathcal{L}(\mathbf{X}_1, \mathbf{X}); \rho(t))$ provided that

$$\frac{1}{s\hat{A}(s)} \in V(\mathbb{R}^+; \rho)^\wedge. \tag{3.29}$$

To verify (3.29), note that

$$\frac{1}{s\hat{A}(s)} = [E + s\hat{a}_1(s) + \hat{a}_2(s)]^{-1}$$

is locally analytic w.r.t. $V(\mathbb{R}^+; \rho)$ on Π. At ∞ use Lemma 3.2 to write

$$\begin{aligned}\frac{1}{s\hat{A}(s)} &= [\alpha_1^2(s) + \hat{a}_2(s)]^{-1}\\ &= \left(\frac{1}{\mu_1} + \hat{\hat{k}}_1(s)\right)^2 \left\{1 + \hat{a}_2(s)\left(\frac{1}{\mu_1} + \hat{\hat{k}}_1(s)\right)^2\right\}^{-1},\end{aligned}$$

and note that this expression is locally analytic w.r.t. $V(\mathbb{R}^+; \rho)$ at ∞. (3.29) now follows from Proposition 2.3 of [**13**], and the proof of Theorem 3.2 is complete. □

(c) Proof of Theorem 2.1(i). We conclude this section by stating and proving the following decomposition proposition which shows that Theorem 2.1(i) is a special case of Theorem 3.2.

PROPOSITION 3.1. *Let $A(t) = E + a(t)$ where $E > 0$, and a satisfies (1.2) with $\dot{a}(0+) = -\infty$ and $a \in AC^J_{\mathrm{loc}}(0,\infty)$, where $J = 0, 1$ or 2. Let $\mathbf{L}$ generate a C_0 cosine family, let $\mathbf{L}$ be invertible, and assume that $s/\hat{A}(s) \in \mathcal{R}$ for $s \in \Pi$, $s \neq 0$, in case $\omega_0(\mathbf{L}) > 0$. In addition, assume that*

$$\text{(3.30)} \qquad \begin{aligned} &\text{(i) } \omega_0(\mathbf{L}) = 0 \textit{ or } \text{(ii) } J = 1 \textit{ or } 2 \textit{ and } \ddot{a} \textit{ is (or extends to) a} \\ &\textit{function of bounded variation on } [c,\infty),\ c > 0. \end{aligned}$$

(We normalize $\ddot{a}$ to be left continuous in (3.30(ii)), $J = 1$.) Then A can be decomposed as

$$A(t) = E + a_1(t) + a_2(t) = A_1(t) + a_2(t),$$

where

$$\text{(3.31)} \qquad a_1(t) \in AC^J_{\mathrm{loc}}(0,\infty) \textit{ satisfies } (1.2),$$

$$\text{(3.32)} \qquad \begin{aligned} &\textit{there is a } T > 0 \textit{ such that } a_2(t) \equiv 0 \textit{ on } [0,T], a_1 \in C^\infty[T,\infty) \\ &\textit{with } \dddot{a}_1 \leq 0 \textit{ on } [T,\infty), \textit{ and there is an } \epsilon > 0 \textit{ such that} \\ &a_1(t) = o(e^{-\epsilon t}),\ t \to \infty, j = 0, 1, \ldots, \textit{ and} \end{aligned}$$

(3.14) *holds.*

PROOF. If (3.30(ii)) holds, to see that (3.14) can be satisfied, assume (3.31), (3.32) for the moment and recall that, by (3.17) and (3.18) with $x = 1$ (for the kernel $A(t)$ in place of $A_1(t)$), $e^{-\beta(s)} = \hat{w}_{0t}(s,1)$, where $w_{0t}(t,1)$ belongs to $L^1(\mathbb{R}^+)$ by [**18**, Theorem 3(v)]. Thus, by the Riemann-Lebesgue Lemma and Lindelöf's Principle [**3**, p. 2], $\exp(-\beta(s)) \to 0$, i.e., $\Re\beta(s) \to \infty$ as $s \to \infty$, uniformly in Π. Since $\beta^2(s) \in \mathcal{R}$ when $\Re\beta(s) > \omega_0(\mathbf{L})$, this fact and a compactness argument using the assumption that $\beta^2(s) \in \mathcal{R}$, $s \in \Pi$, $s \neq 0$, show that

$$D \equiv \operatorname{dist}(\beta^2(\Pi), \sigma(\mathbf{L})) > 0.$$

Since $\Re\hat{a}(s) \geq 0$ and $(\Im s)(\Im\hat{a}(s)) \leq 0$, with the same inequalities holding for a_1, (3.32) yields

$$
\begin{aligned}
|\beta^2(s) - \beta_1^2(s)| &= \left|\frac{s\hat{a}_2(s)}{\hat{A}(s)\hat{A}_1(s)}\right| \\
&\leq \left|\frac{s^3\hat{a}_2(s)}{E^2}\right| = \left|\frac{\widehat{d\ddot{a}_2}(s)}{E^2}\right| \\
&\leq E^{-2}\int_T^\infty |d\ddot{a}_2(t)|.
\end{aligned} \tag{3.33}
$$

(Again, the convention is $\ddot{a}_2(T) = \ddot{a}_2(T-)$.) Thus, in addition to (3.31) and (3.32), we will get (3.14) if

$$\int_T^\infty |d\ddot{a}_2(t)| < E^2 D/2. \tag{3.34}$$

To obtain (3.31) and (3.32) write

$$a_1(t) = \begin{cases} a(t), & 0 < t \leq T \\ a(T)\exp(\int_T^t q(x)\,dx), & T < t < \infty, \end{cases} \tag{3.35}$$

where T and q are to be chosen. We have

$$\dot{a}_1 = qa_1, \quad \ddot{a}_1 = (\dot{q} + q^2)a_1, \quad \dddot{a}_1 = (\ddot{q} + 3q\dot{q} + q^3)a_1, \quad \text{etc.} \tag{3.36}$$

If (3.30(i)) holds, T is an arbitrary positive number. If (3.30(ii)) holds, choose T so large that

$$\int_T^\infty |d\ddot{a}(t)| < E^2 D/4. \tag{3.37}$$

Now pick $q \in C^\infty[T,\infty)$ with bounded derivatives of all orders, with $(-1)^j q^{(j-1)}(t) > 0$, $j = 1, 2, 3, t > T$, subject to the identities obtained by equating $a^{(j)}(T+)$ to $a_1^{(j)}(T+)$ in (3.36), $1 \leq j \leq \max\{J, 1\}$, and $\ddot{a}(T+) = \ddot{a}_1(T+)$ under hypothesis (3.30(ii)) with $J = 1$. We require as well that

$$\frac{\dot{a}(T+)}{a(T)} = q(T) < q(\infty) < -\epsilon < 0. \tag{3.38}$$

Note that $q(T) = \dot{a}(T+)/a(T) < 0$, and that

$$\dot{q}(T) = \frac{\ddot{a}(T+)a(T) - \dot{a}^2(T)}{a^2(T)} \geq 0$$

when $J = 2$ or (3.30(ii)) holds, by log-convexity, so our sign conditions on $q^{(j-1)}$ are consistent with the matching conditions at T.

With these choices, the sign and matching conditions on q, together with (3.38), yield (3.31) and (3.32), so the proof is complete under (3.30(i)). When (3.30(ii)) holds, the matching conditions and (3.32), (3.37) imply

$$\begin{aligned}
\int_T^\infty |d\ddot{a}_2(t)| &= \int_{T+}^\infty |d\ddot{a}_2(t)| \\
&= \int_{T+}^\infty |d\ddot{a}(t) - \dddot{a}_1(t)\,dt| \\
&\leq \int_{T+}^\infty |d\ddot{a}(t)| - \int_T^\infty \dddot{a}_1(t)\,dt \\
&= \int_{T+}^\infty |d\ddot{a}(t)| + \ddot{a}(T+) \\
&\leq 2\int_T^\infty |d\ddot{a}(t)| < E^2 D/2
\end{aligned}$$

so (3.34) holds and we are done. $\square$

4. Asymptotic behavior in viscoelastic materials. In this section and the next we regard (1.1) as a model for vibrations in a viscoelastic solid. Here $\mathbf{X}$ is a Hilbert space and $\mathbf{L}$ a negative definite self-adjoint linear operator with spectrum in $(-\infty, -\lambda_0]$, where $\lambda_0 > 0$. When $\ddot{a}$ is bounded, this is a special case of the model developed and studied by Dafermos [**4**]. More recently, J.E. Lagnese [**14**] and G. Leugering [**15**] have used energy methods to study decay rates for models similar to those of Dafermos but with boundary feedback conditions that improve the decay rates of oscillations. Our purpose is to show that, when $a(t)$ does not decay exponentially, the results of Sections 2 and 3 above yield decay rates without boundary feedback that are essentially as rapid as those obtained with feedback. We also show that there are cases (with exponentially decaying $a(t)$) where such "stabilizing" feedbacks lead to slower decay. These phenomena reflect the fact that energy in a viscoelastic medium dissipates in part through a "creep" mechanism that interacts with boundary forces in a way that is entirely different from that exhibited in purely elastic materials. Attempts to separate the creep behavior from oscillatory motion analytically (with applications to controllability, for example) are underway [**6, 16, 17**].

We assume throughout this section that $A(t)$ and ρ satisfy the conditions of Theorem 2.1. The cosine family $\mathbf{C}(t) = \cos \mathbf{M}t$, with $\mathbf{M} = (-\mathbf{L})^{1/2}$, can be defined via the spectral theorem, and one sees that $\omega_0 = 0$ and $\mathbf{X}_1$ is the domain, $\mathcal{D}(\mathbf{M})$, of $\mathbf{M}$ with norm equivalent to $\|\mathbf{M}\mathbf{x}\|$. We take $\mathbf{u}_0 \in \mathbf{X}_1$, $\mathbf{u}_1 \in \mathbf{X}$. (Alternatively, in the regular case, one can use Theorem 2.1(ii)$'$ and the remarks following that theorem to deduce the estimates needed below and weaken the log-convexity requirement on $a(t)$ to ordinary convexity.)

The results of Section 2 lead to the following representation theorem and estimate.

THEOREM 4.1. *Under the above assumptions, the function*

$$\mathbf{u}(t) \equiv \mathbf{U}(t)\mathbf{u}_0 + \mathbf{W}(t)\mathbf{u}_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbf{X}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbf{X}_1) \tag{4.1}$$

is the unique mild solution of the integrated form

$$\mathbf{u}(t) = \mathbf{u}_0 + \mathbf{u}_1 t + \int_0^t \int_0^\tau A(\tau - r)\mathbf{L}\mathbf{u}(r)\, dr\, d\tau \tag{4.2}$$

of (1.1) *in* $\mathbf{X}$, *and*

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \mathbf{V}(t)\mathbf{u}_0 + \mathbf{U}(t)\mathbf{u}_1 \\ &= \mathbf{V}(t)\mathbf{M}^{-1}(\mathbf{M}\mathbf{u}_0) + \mathbf{U}(t)\mathbf{u}_1, \quad t > 0. \end{aligned} \tag{4.3}$$

Moreover, we have the estimate

$$\int_0^\infty (\|\mathbf{u}(t)\|_1 + \|\dot{\mathbf{u}}(t)\|)\rho(t)\, dt \le C(\|\mathbf{u}_0\|_1 + \|\mathbf{u}_1\|), \tag{4.4}$$

for some constant C.

PROOF. The assertions involved in (4.1), (4.2) and (4.3) follow directly from the properties of $\mathbf{U}$ and $\mathbf{V}$ developed, e.g., in [**1**] and [**2**]. For (4.4) the estimate is

$$
\begin{aligned}
\int_0^\infty &(\|\mathbf{u}(t)\|_1 + \|\dot{\mathbf{u}}(t)\|)\rho(t)\,dt \\
&\le \int_0^\infty (\|\mathbf{U}(t)\mathbf{M}\mathbf{u}_0\| + \|\mathbf{M}\mathbf{W}(t)\mathbf{u}_1\| \\
&\quad + \|\mathbf{V}(t)\mathbf{M}^{-1}\mathbf{M}\mathbf{u}_0\| + \|\mathbf{U}(t)\mathbf{u}_1\|)\rho(t)\,dt \\
&\le (\|\mathbf{u}_0\|_1 + \|\mathbf{u}_1\|)\int_0^\infty (\|\mathbf{U}(t)\| + \|\mathbf{M}\mathbf{W}(t)\| \\
&\quad + \|\mathbf{V}(t)\mathbf{M}^{-1}\|)\rho(t)\,dt \\
&\equiv C(\|\mathbf{u}_0\|_1 + \|\mathbf{u}_1\|),
\end{aligned}
\tag{4.5}
$$

where the operator norms are in $\mathcal{L}(\mathbf{X})$ and Theorem 2.1 has been used in the last step. (In Case (ii) of Theorem 2.1, we modify the estimate to reflect that the operators are ρ-integrable. Actually, the operator norms of $\mathbf{U}(t), \mathbf{M}\mathbf{W}(t)$ and $\mathbf{V}(t)\mathbf{M}^{-1}$ are measurable here, as may be seen from a spectral representation of these resolvents.) $\square$

Now define

$$
\begin{aligned}
\mathcal{E}(t) = &\frac{1}{2}A(t)\|\mathbf{M}\mathbf{u}(t)\|^2 + \frac{1}{2}\|\dot{\mathbf{u}}(t)\|^2 \\
&- \frac{1}{2}\int_0^t \dot{a}(t-\tau)\|\mathbf{M}(\mathbf{u}(t) - \mathbf{u}(\tau))\|^2\,d\tau.
\end{aligned}
\tag{4.6}
$$

Leugering [**15**] studied a case of (1.1) but with stabilizing boundary feedback (in a history space setting, with corresponding modification of (4.6)) with $a \in C^2[0,\infty)$, but not necessarily log-convex.

He showed that

$$\int_0^\infty \mathcal{E}(t)(1+t)^j \, dt < \infty$$

($j =$ positive integer) provided

$$\int_0^\infty a(t)(1+t)^{j-1} \, dt < \infty.$$

We shall derive a comparable estimate for the homogeneous problem (1.1), where there is no initial history.

LEMMA 4.2. *Under the hypothesis of Theorem* 4.1, *if* $a(0+) < \infty$, *then* $\mathcal{E}(t) \le \mathcal{E}(0)$, $t \ge 0$.

PROOF. If $\mathbf{Lu}_0$ and $\mathbf{u}_1$ belong to $\mathcal{D}(\mathbf{M})$, then $\mathbf{u}(t) \in C^1(\mathbb{R}^+, \mathcal{D}(\mathbf{M}))$ by [**5**]. Then $\|\mathbf{M}(\mathbf{u}(t) - \mathbf{u}(\tau))\| = O(t-\tau)$, $\tau \to t$, and the standard energy computation [**4**] yields

$$\dot{\mathcal{E}}(t) = \frac{1}{2}\dot{a}(t)\|\mathbf{Mu}(t)\|^2 - \frac{1}{2}\int_0^t \ddot{a}(t-\tau)\|\mathbf{M}(\mathbf{u}(t) - \mathbf{u}(\tau))\|^2 \, d\tau$$

for $t > 0$, so our result follows. An approximation argument, using the continuity of $\mathbf{Mu}(t)$ and $\dot{\mathbf{u}}(t)$ in $\mathbf{X}$ as functions of $\mathbf{Mu}_0$ and $\mathbf{u}_1$, together with Fatou's lemma, completes the proof in the general case. □

THEOREM 4.3. *Under the assumptions of Theorem* 4.1, *if* $a(0+) < \infty$, *then we have*

$$\int_0^\infty \mathcal{E}(t)\rho(t) \, dt < \infty.$$

PROOF. By Theorem 4.1 and Lemma 4.2,

$$\frac{1}{2}\int_0^\infty \{A(t)\|\mathbf{Mu}(t)\|^2 + \|\dot{\mathbf{u}}(t)\|^2\}\rho(t) \, dt < \infty.$$

For the integral (viscoelastic stored energy) term, the estimate is

$$\begin{aligned}
&\int_0^\infty \rho(t) \int_0^t |\dot{a}(\tau)| \left\| \mathbf{M}(\mathbf{u}(t) - \mathbf{u}(t-\tau)) \right\|^2 \, d\tau \, dt \\
&\qquad \le 2 \int_0^\infty |\dot{a}(\tau)| \int_\tau^\infty \rho(t)(\|\mathbf{Mu}(t)\|^2 + \|\mathbf{Mu}(t-\tau)\|^2) \, dt \, d\tau \\
&\qquad \le 2 \int_0^\infty |\dot{a}(\tau)| \, d\tau \int_0^\infty \|\mathbf{M}(\mathbf{u}(t))\|^2 \rho(t) \, dt \\
&\qquad\quad + 2 \int_0^\infty |\dot{a}(\tau)| \int_0^\infty \rho(t+\tau) \|\mathbf{Mu}(t)\|^2 \, dt \, d\tau \\
&\qquad \le 2a(0) \int_0^\infty \|\mathbf{Mu}(t)\|^2 \rho(t) \, dt \\
&\qquad\quad + 2 \int_0^\infty |\dot{a}(\tau)| \rho(\tau) \, d\tau \int_0^\infty \|\mathbf{Mu}(t)\|^2 \rho(t) \, dt < \infty. \qquad \square
\end{aligned}$$

5. An example with boundary feedback. To see the limitation of boundary feedback in the stabilization of creep deformations, we consider a special problem consisting of the viscoelastic wave equation

$$u_{tt}(x,t) = E u_{xx}(x,t) + \frac{d}{dt} \int_0^t a(t-\tau) u_{xx}(x,\tau) \, d\tau, \qquad 0 \le t < \infty, \ 0 \le x \le 1, \tag{5.1}$$

with boundary and initial conditions

$$u(0,t) = 0, \ t > 0; \quad u(x,0) = u_0(x), \ u_t(x,0) = 0, \ 0 \le x \le 1, \tag{5.2}$$

and the feedback condition

$$E u_x(1,t) + \frac{d}{dt} \int_0^t a(t-\tau) u_x(1,\tau) \, d\tau = -k u_t(1,t), \tag{5.3}$$

where a satisfies (1.2) and (to permit comparison to [**11, 12**] and for simplicity) is completely monotone on $(0,\infty)$. We assume that $E > 0$, $k \ge 0$ and, to make existence equations easy,

$$u_0 \in C_c^\infty(0,1). \tag{5.4}$$

Let

$$\mathcal{E}_1(t) = \int_0^1 [u_x^2(x,t) + u_t^2(x,t)]\, dt.$$

When $k = 0$, (5.1)–(5.3) is an example of (1.1) with $\mathbf{X} = L^2(0,1)$, $\mathbf{L} = d^2/dx^2$ on

$$\mathcal{D}(\mathbf{L}) \equiv \{u \in H^2(0,1) : u(0) = u'(1) = 0\}.$$

From Section 4 we get $\mathbf{u}(t) = \mathbf{U}(t)\mathbf{u}_0$, $\dot{\mathbf{u}}(t) = \mathbf{V}(t)\mathbf{u}_0$, so the estimate of Theorem 2.1 yields (as in (4.5))

$$\int_0^\infty \mathcal{E}_1^{1/2}(t)(1+t)\rho(t)\, dt \le C\mathcal{E}_1^{1/2}(0), \tag{5.5}$$

provided (2.6) holds.

We shall take $\rho(t) = (1+t)^{j+\eta}$, where j is a nonnegative integer and $0 \le \eta < 1$, so that (2.6) reduces to

$$\int_1^\infty |\dot{a}(t)|(1+t)^{j+\eta}\, dt < \infty \tag{5.6}$$

(see (2.8)).

When $k > 0$, we can proceed as in [**12**] (i.e., directly via the complex inversion formula) to define $u(x,t)$ in terms of its Laplace transform $\hat{u}(x,s)$, to establish existence. In particular, one gets the explicit formula

$$\hat{u}(1,s) = \frac{\int_0^1 u_0(y)\sinh \beta y\, dy}{\alpha \cosh \beta + k \sinh \beta}, \quad \Re \ge 0, \tag{5.7}$$

with $\alpha = \alpha(s) = (s\hat{A}(s))^{1/2}$, $\alpha(0) > 0$, and $\beta = s/\alpha$. Using (5.4) we find that $u(1,\cdot) \in L^2 \cap C^\infty[0,\infty)$ and $u(1,t) = \int_0^1 u_x(x,t)\, dx$, so that

$$\mathcal{E}_1^{1/2}(t) \ge |u(1,t)|, \tag{5.8}$$

by the Schwarz inequality.

Now suppose the feedback condition (5.3) enables us to improve (5.5) to

$$\int_0^\infty \mathcal{E}_1^{1/2}(t)(1+t)^{j+\eta+1+\epsilon}\, dt < \infty \tag{5.5+ϵ}$$

with $\eta < \eta + \epsilon < 1$. Then, by (5.8), we have

$$\begin{aligned} &\lim_{s\to 0+} s^{-\eta-\epsilon}[\hat{u}^{j+1}(1,s) - \hat{u}^{j+1}(1,0)] \\ \text{(5.9)}\qquad &\quad = \lim_{s\to 0+} \int_0^\infty u(1,t)t^{(j+1+\eta+\epsilon)} \frac{e^{-st}-1}{(st)^{\eta+\epsilon}}\, dt = 0 \end{aligned}$$

by dominated convergence. For expressions like (5.9) we use right-hand derivatives at 0 and we say $u \in C^{j+1+\eta+\epsilon}$. By decomposing $\hat{u}(1,s)$, we shall show that if $j \geq 0$ (see note below), then

$$\text{(5.6+}\epsilon'\text{)}\qquad \int_1^\infty |\dot{a}(t)| t^{j+\eta+\epsilon'}\, dt < \infty$$

for every positive $\epsilon' < \epsilon$, so that (5.5+ϵ') holds for all such ϵ', even when $k = 0$.

Write $a(t) = b_1(t) + b_2(t)$, where both b_1 and b_2 are positive, decreasing and convex, but $b_1(t) = 0$ for $t > 1$ and $b_2(0+) < \infty$ (for example, b_2 can be linear on $(0,1)$). Then, with $A_0 = E + b_2(0+)$, we have

$$\frac{1}{\hat{A}(s)} = \frac{s}{A_0 + s\hat{b}_1(s) + \hat{\dot{b}}_2(s)}.$$

Since $\hat{b}_1$ is analytic at 0 and $b_2 = a$ for $t > 1$, so that

$$\text{(5.10)}\qquad \hat{b}_2(s) \in C^{j+\eta}$$

(as in (5.9)), it follows that

$$\text{(5.11)}\qquad w(s) \equiv \hat{A}(s)^{-1} \in C^{j+1+\eta}.$$

(Here and below we use the formula

$$(sg(s))^{(j+1)}|_{s=0} = (j+1)g^{(j)}(0) \qquad \text{when } g \in C^{\,j}[0,s_1) \cap C^{j+1}(0,s_1).)$$

In the present case,

$$g(s) = (A_0 + s\hat{b}_1(s) + \hat{\dot{b}}_2(s))^{-1}.$$

The hard terms in the proof of (5.11) are

$$E^{-2}s^{-\eta}\left[(j+1)\left(\hat{\dot{b}}_2^{(j)}(s)-\hat{\dot{b}}_2^{(j)}(0)\right)+s\hat{\dot{b}}_2^{(j+1)}(s)\right].$$

The first of these tends to 0 by (5.10), while

$$\left|s^{1-\eta}\hat{\dot{b}}_2^{(j+1)}(s)\right|\le\int_0^\infty (st)^{1-\eta}e^{-st}t^{j+\eta}|\dot{b}_2(t)|\,dt\to 0,\quad s\to 0+,$$

again by dominated convergence.

Now write

$$\begin{aligned}\hat{u}(1,s)&=\frac{1}{\hat{A}(s)}\frac{\int_0^1 u_0(y)\beta^{-1}\sinh\beta y\,dy}{\cosh\beta+\frac{k}{\hat{A}(s)}\beta^{-1}\sinh\beta}\\&\equiv w(s)\frac{h(sw(s))}{C(sw(s))+kw(s)S(sw(s))}\\&\equiv\Phi(s,w(s))\equiv h(0)w(s)+\Psi(s,w(s)).\end{aligned}\tag{5.12}$$

Then $\Phi(s,w)$ is analytic at $(0,0)$ with $\Phi(0,0)=0$, $\Phi_w(0,0)=h(0)=\int_0^1 yu_0(y)\,dy$, so Ψ has the form $\Psi(s,w)=w\tilde{\Psi}(s,w)$ with $\tilde{\Psi}$ analytic at $(0,0)$ and $\tilde{\Psi}(0,0)=0$. Using (5.11) we get $\Psi(s,w(s))\in C^{j+1+\lambda}$ for all $\lambda<1$, so (5.9) and (5.12) yield

$$w(s)\in C^{j+1+\eta+\epsilon}.\tag{5.13}$$

Now decompose $w(s)$ as

$$w(s)=w_1(s)-\hat{b}_1(s)w_1(s)w(s),\tag{5.14}$$

where $w_1(s)=s(A_0+\hat{\dot{b}}_2(s))^{-1}$. As above, $w_1(s)\in C^{j+1+\eta}$, and $w_1(s)=O(s)$ $(s\to 0+)$, so it is clear that

$$\hat{b}_1(s)w_1(s)w(s)\in C^{j+1+\lambda},\quad \lambda<1,$$

and, in particular, for $\lambda=\eta+\epsilon$. Thus, (5.13) and (5.14) lead to

$$w_1(s)\in C^{j+1+\eta+\epsilon}.\tag{5.15}$$

But

$$w_1(s) = -\frac{s\hat{\dot{b}}_2(s)}{E^2} + \frac{A_0 s}{E^2} + \frac{w_1(s)}{E^2}(\hat{\dot{b}}_2(s) - \hat{\dot{b}}_2(0))^2. \tag{5.16}$$

When $j \geq 1$ or $\epsilon < \eta$, the last term here is in $C^{j+1+\eta+\epsilon}$, so (5.15) gives us

$$\begin{aligned} 0 &= \lim_{s\to 0+} s^{-\eta-\epsilon}\left[(s\hat{\dot{b}}_2(s))^{(j+1)} - (s\hat{\dot{b}}_2(s))^{(j+1)}|_{s=0}\right] \\ &= \lim_{s\to 0+}\left[\frac{j+1}{s^{\eta+\epsilon}}\int_0^\infty (e^{-st}-1)(-t)^j \dot{b}_2(t)\,dt \right. \\ &\qquad \left. + s^{1-\eta-\epsilon}\int_0^\infty e^{-st}(-t)^{j+1}\dot{b}_2(t)\,dt\right]. \end{aligned}$$

Since both terms inside the bracket have the same sign, both tend to zero. In particular,

$$l(s) \equiv \frac{1-e^{-1/2}}{s^{\eta+\epsilon}}\int_{1/2s}^{1/s} t^j |\dot{b}_2(t)|\,dt$$

is bounded for $s \in (0,1)$. But

$$l(2^{-n}) \geq (1-e^{-1/2})(2^{\epsilon-\epsilon'})^n \int_{2^{n-1}}^{2^n} t^{j+\eta+\epsilon'}|\dot{b}_2(t)|\,dt,$$

when $\epsilon < \epsilon'$, since

$$s^{-\eta-\epsilon} \geq s^{\epsilon'-\epsilon} t^{\eta+\epsilon'} = (2^n)^{(\epsilon-\epsilon')} t^{\eta+\epsilon'}$$

in this situation. Finally, then,

$$\int_1^\infty t^{j+\eta+\epsilon'}|\dot{a}(t)|\,dt = \int_1^\infty t^{j+\eta+\epsilon'}|\dot{b}_2(t)|\,dt \leq M\sum_{n=1}^\infty 2^{-n(\epsilon-\epsilon')} < \infty,$$

as claimed.

NOTE 1. If $j = 0$, then $\left(\hat{\dot{b}}_2(s) - \hat{\dot{b}}_2(0)\right)^2$ adds only 2η degrees of smoothness to the last term in (5.16). If $\eta > 0$, this is enough, since we can assume that η is maximal in (5.6) and get a contradiction to (5.5+ϵ). The case $j = \eta = 0$ is not covered.

NOTE 2. When $A(t) = E+\gamma(1+t)^{-r}$, $j < r < j+1$, $j =$ nonnegative integer, we can derive an asymptotic expansion for $u(1,t)$ valid for t near ∞ which gives the precise dependence on $k \geq 0$ for fixed initial data $u_0(y)$. Namely, use (5.7) to develop $\hat{u}(1,s)$ in an asymptotic series valid near $s = 0$, and then use the theory relating the behavior of $\hat{u}(1,s)$ near $s = 0$ to that of $u(1,t)$ near $t = \infty$ (e.g., [**7**, Theorem 37.1]) to deduce an asymptotic expansion for $u(1,t)$ valid for t near ∞. An easy calculation shows that the dominant term in this expansion for $u(1,t)$ is actually independent of $k \geq 0$.

NOTE 3. For the case where

$$\int_0^\infty e^{\delta t} a(t)\,dt < \infty$$

for some positive δ, we can even arrange matters so that the exponential decay rate when $k > 0$ is *slower* than that for $k = 0$. For example, with $A(t) = E + \gamma e^{-\delta t}$ and an appropriate choice of the parameters E, γ, δ, one sees from the numerical estimates of [**11**, Section 4] that the singularity for $\hat{u}(1,s)$ with the largest real part is the "Class 4" eigenvalue on the negative real axis that moves to the right from $\sigma_* = -E\delta/(E+\gamma)$ as k increases from zero.

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