JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 2, Number 3, Summer 1990

SOME RESULTS ON NONLINEAR HEAT EQUATIONS FOR MATERIALS OF FADING MEMORY TYPE

PH. CLÉMENT AND G. DA PRATO

1. Introduction. In this paper we consider a model for the heat conduction for a material covering an n-dimensional bounded set Ω with boundary $\partial\Omega, n = 1, 2, 3$.

(1.1)
$$\begin{cases} \frac{d}{dt}\Big(b_0 u(t,x) + \int_0^t \beta(t-s)u(s,x)\,ds\Big) = c_0 \Delta u(t,x), & t > 0, x \in \Omega, \\ u(0,x) = x, \quad x \in \Omega, \end{cases}$$

where $u(t,x)$ is the temperature of the point x at time t (we assume that the temperature is 0 for $x \in \partial\Omega$), b_0 is the *specific heat* and c_0 the *thermal conductivity*. We assume that the specific heat has a term of fading memory type $\int_0^t \beta(t-s)u(s,x)\,ds$, whereas the thermal conductivity is constant. Concerning the kernel β we assume only that it is locally integrable in $[0,\infty[$; this will allow us to consider kernels as $\beta(t) = e^{-\omega t}t^{\alpha-1}, \omega \geq 0, \alpha \in]0,1[$.

Model (1.1) (including also a memory term for the thermal conductivity) has been introduced in [**7**] and studied in [**1**] and [**5**].

We write problem (1.1) in abstract form in the Banach space $X = C(\overline{\Omega})$,

$$\begin{cases} \frac{d}{dt}(u(t) + (\beta * u)(t)) = Au(t), & t > 0, \\ u(0) = x, \end{cases} \tag{1.2}$$

where $u(t) = u(t,\cdot)$ and A is the realization in $C(\overline{\Omega})$ of the Laplace operator Δ with Dirichlet boundary conditions.

In order to study (1.2), we assume that A generates an analytic semigroup and that β is Laplace transformable with Laplace transform $\hat{\beta}(\lambda)$ analytic in a sector $S_{\omega,\theta} = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg(\lambda - \omega)| < \theta\}$ with $\omega \in \mathbf{R}$ and $\theta \in]\pi/2, \pi[$. Then the Laplace transform $\hat{u}(\lambda)$ of u is given formally by

$$\hat{u}(\lambda) := F(\lambda)x = R(\lambda + \lambda\hat{\beta}(\lambda), A)x. \tag{1.3}$$

Work of the second author was partially supported by the Italian National Project M.P.I. 40% "Equazioni di Evoluzione e Applicazioni Fisico-Matematiche."

In Section 2, by proceeding as in [**3**] and [**6**], we solve problem (1.2) by means of a resolvent operator $R(t)$ obtained by inverting its formal Laplace transform $F(\lambda)$. We remark that if $\beta \in W^{1,1}_{\text{loc}}(0,\infty)$, then problem (1.2) can be easily studied as a perturbation of heat equation. The main difference of our results with respect to [**3**] and [**6**] is that when β is not regular there is also a lack of regularity for $R(t)x$. Indeed it can happen that, even if $x \neq 0$ is very regular (say $x \in D(A^\infty)$), $R(\cdot)x$ is not differentiable in 0. For this reason we introduce in Section 3 a new notion of strict solution in order to study the inhomogeneous problem

$$\begin{cases} \frac{d}{dt}(u(t) + (\beta * u)(t)) = Au(t) + f(t), \quad t > 0, \\ u(0) = x, \end{cases} \tag{1.4}$$

where $f : [0,T] \to X$ is continuous.

In Section 4, assuming, in addition, that β is nonnegative and nonincreasing and that $||e^{tA}|| \leq e^{\omega t}$, for some $\omega \leq 0$, we prove the estimate

$$||R(t)|| \leq s_{\omega+\beta}(t), \tag{1.5}$$

where $s_{\omega+\beta}$ is the solution of the integral equation

$$s_{\omega+\beta}(t) + \int_0^t (\omega + \beta)(t-\sigma) s_{\omega+\beta}(\sigma) d\sigma = 1. \tag{1.6}$$

This result enables us to solve (see Section 5) the semilinear problem,

$$\begin{cases} \frac{d}{dt}(u(t) + (\beta * u)(t)) = Au(t) + F(u(t)), \quad t > 0, \\ u(0) = x, \end{cases} \tag{1.7}$$

where $F : X \to X$ is locally Lipschitz and such that

$$||x|| \leq ||x - \delta F(x)||, \quad \forall\, \delta > 0, \quad \forall\, x \in X. \tag{1.8}$$

We recall that nonlinear integrodifferential equations of this type have been discussed, when β is regular, by several authors (see [**2, 1**] and the references quoted therein). But in the above papers it is assumed that the nonlinear term is monotone; moreover, only the existence of weak solutions is stated.

We have also studied the positivity of the solutions. More precisely, under the hypotheses of Section 4 we can show that, if Q is a closed convex cone in X such that $e^{tA}(Q) \subset Q$ and if $x \in Q$, then the solution of (1.4) remains on Q. A similar result holds for problem (1.7).

Finally, in Section 6, we have discussed the physical example (1.1) also when a nonlinear perturbation term occurs. In a subsequent paper we shall consider the more general case in which also a memory term related to conductivity appears.

2. Construction of the resolvent $\mathbf{R(t)}$. Let X be a complex Banach space (norm $||\cdot||$), $A : D(A) \subset X \to X$ a closed linear operator and $\beta : [0, \infty[\to \mathbf{R}$ a Laplace transformable function. We shall denote by $\rho(A)$ the resolvent set of A, by $\sigma(A)$ the spectrum of A, by $R(\lambda, A)$ the resolvent of A and by $\hat{\beta}(\lambda)$ the Laplace transform of β. For any $\theta \in]0, \pi[$ we shall denote by $S_{\omega,\theta}$ the sector

$$S_{\omega,\theta} = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg(\lambda - \omega)| < \theta\}.$$

We are here concerned with the Volterra integrodifferential equation

$$\begin{cases} \frac{d}{dt}(u(t) + (\beta * u)(t)) = Au(t), & t > 0, \\ u(0) = x, \end{cases} \tag{2.1}$$

where $x \in X$ and $(\beta * u)(t) = \int_0^t \beta(t-s)u(s)\,ds$. We assume

(2.2)

$\exists M > 0, \omega \in \mathbf{R}, \theta \in]\pi/2, \pi[$ and $\alpha \in]0, 1[$ such that

(i) $\rho(A) \supset S_{\omega,\theta}$ and $||R(\lambda, A)|| \le M/|\lambda - \omega|, \quad \forall\, \lambda \in S_{\omega,\theta}$

(ii) There exists an analytic extension of $\hat{\beta}(\lambda)$ in $S_{\omega,\theta}$ (still denoted by $\hat{\beta}(\lambda)$) such that $||\hat{\beta}(\lambda)|| \le M/|\lambda - \omega|^{\alpha}, \quad \forall\, \lambda \in S_{\omega,\theta}$.

We fix once and for all a maximal analytic extension of $\hat{\beta}(\lambda)$ (still denoted by $\hat{\beta}(\lambda)$) and we denote by Ω its domain of definition. Set

$$\rho_F = \{\lambda \in \Omega; \lambda + \lambda\hat{\beta}(\lambda) \in \rho(A)\} \tag{2.3}$$

and

$$F(\lambda) = R(\lambda + \lambda\hat{\beta}(\lambda), A), \quad \forall\, \lambda \in \rho_F. \tag{2.4}$$

Let us remark that we do not assume that $D(A)$ is dense in X and that β is right differentiable at 0. Examples of kernels fulfilling hypotheses (2.2) are $\beta(t) = e^{-\omega t}t^{\alpha-1}, \omega \geq 0, \alpha \in]0,1[$.

LEMMA 2.1. *Assume* (2.2). *Then there exists an* $r > 0$ *such that, setting* $\omega_\theta = \omega + r\sec\theta$, *one has* $\rho_F \supset S_{\omega_\theta,\theta}$ *and*

$$||F(\lambda)|| \leq \frac{2M}{|\lambda - \omega|}, \quad \forall\, \lambda \in S_{\omega_\theta,\theta} \tag{2.5}$$

$$F(\lambda) = R(\lambda, A)[1 + \lambda\hat{\beta}(\lambda)R(\lambda, A)]^{-1}, \quad \forall\, \lambda \in S_{\omega_\theta,\theta}. \tag{2.6}$$

Finally, there exists $M_1 > 0$ *such that*

$$||AF(\lambda)|| \leq M_1, \quad \forall\, \lambda \in S_{\omega_\theta,\theta}. \tag{2.7}$$

PROOF. Given $y \in X$ and $\lambda \in S_{\omega,\theta}$, consider the equation

$$\lambda x + \lambda\hat{\beta}(\lambda)x - Ax = y. \tag{2.8}$$

Setting $\lambda x - Ax = z$ (2.8) reduces to

$$z + \lambda\hat{\beta}(\lambda)R(\lambda, A)z = y. \tag{2.9}$$

By (2.2) there exists an $r > 0$ such that

$$||\lambda\hat{\beta}(\lambda)R(\lambda, A)|| \leq \frac{1}{2}, \quad \forall\, \lambda \in S_{\omega_\theta,\theta}. \tag{2.10}$$

Now (2.5) and (2.6) follow by a standard fixed point argument.

It remains to prove (2.7). Recalling (2.6),

$$\begin{aligned} AF(\lambda) &= (\lambda + \lambda\hat{\beta}(\lambda))F(\lambda) - 1 \\ &= \lambda F(\lambda) + \lambda\hat{\beta}(\lambda)R(\lambda, A)[1 + \lambda\hat{\beta}(\lambda)R(\lambda, A)]^{-1} - 1 \end{aligned} \tag{2.11}$$

so that (2.7) follows from (2.5) and (2.10). □

We now set

$$R(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}F(\lambda)\,d\lambda, \quad t > 0, \tag{2.12}$$

where $\gamma = \gamma^- \cup \gamma^+, \gamma^\pm = \{\lambda \in \mathbf{C} : \lambda = \omega_\theta + \rho e^{\pm i\theta}, \rho \geq 0\}$ is oriented counterclockwise.

The following result is proved as in [**3, 6**].

Proposition 2.2. *Assume* (2.2) *and let* $R(t)$ *be defined by* (2.12). *Then the following statements hold*

(i) *There exists* $K > 0$ *such that*

$$||R(t)|| \leq Ke^{\omega_\theta t}, \quad t \geq 0, \tag{2.13}$$

$$||R'(t)|| \leq \frac{K}{t}e^{\omega_\theta t}, \quad t \geq 0. \tag{2.14}$$

(ii) *We have*

$$\lim_{t\to 0} R(t)x = x, \quad \forall\, x \in \overline{D(A)}. \tag{2.15}$$

Thus $R(\cdot)x, \beta * R(\cdot)x \in C([0,\infty[;X)$, *for all* $x \in \overline{D(A)}$.

(iii) R *is analytic in the sector* $S_{0,\theta-\pi/2}$.

(iv) *For all* $t > 0$ *and* $x \in X, R(t)x \in D(A)$ *and* $AR(\cdot)$ *is analytic in the sector* $S_{0,\theta-\pi/2}$.

(v) *For all* $t > 0$,

$$R'(t) + \int_0^t \beta(s)R'(t-s)\,ds = AR(t). \tag{2.16}$$

Proposition 2.3. *If* $x \in D(A)$ *and* $Ax \in \overline{D(A)}$ *we have*

$$\lim_{t\to 0}\frac{d}{dt}(R(t)x + (\beta * R(\cdot)x)(t)) = Ax. \tag{2.17}$$

Thus $R(\cdot)x + (\beta * R)(\cdot)x \in C^1([0,\infty[;X)$ *and* $AR(\cdot)x \in C([0,\infty[;X)$.

PROOF. From Proposition 2.2,

$$\frac{d}{dt}(R(t)x + (\beta * R(\cdot)x)(t)) = AR(t)x = R(t)Ax, \quad t > 0.$$

Since $Ax \in \overline{D(A)}$, (2.17) follows from (2.15). □

PROPOSITION 2.4. *If $x \in D(A)$, then $R(\cdot)x + (\beta * R)(\cdot)x$ is Lipschitz continuous. Moreover, there is a $K' > 0$ such that*

$$|R'(t)x| \leq K' t^{\alpha-1}|x|. \tag{2.18}$$

PROOF. Let $x \in D(A)$; if $t > 0$, by (2.16), we have

$$\frac{d}{dt}(R(t)x + (K * R(\cdot)x)(t)) = AR(t)x = R(t)Ax.$$

Thus, by (2.16), $R(\cdot)x + (\beta * R)(\cdot)x$ is Lipschitz continuous. Moreover,

$$\begin{aligned} R'(t)x &= \frac{1}{2i\pi}\int_\gamma \lambda e^{\lambda t} F(\lambda)x \, d\lambda = \frac{1}{2i\pi}\int_\gamma e^{\lambda t}(\lambda F(\lambda) - I)\, x d\lambda \\ &= \frac{1}{2i\pi}\int_\gamma e^{\lambda t}(AF(\lambda)x - \lambda \hat{K}(\lambda)F(\lambda)x)\, d\lambda \\ &= R(t)Ax - \frac{1}{2i\pi}\int_\gamma e^{\lambda t}\lambda \hat{K}(\lambda)F(\lambda)x \, d\lambda. \end{aligned}$$

The first term is bounded near 0 by (2.13). Concerning the second one,

$$\left|\left|\frac{1}{2i\pi}\int_\gamma e^{\lambda t}\lambda \hat{K}(\lambda)F(\lambda)x\, d\lambda\right|\right| \leq M\frac{e^{\omega_0 t}}{\pi}\int_0^\infty e^{\rho t\cos\eta}\rho^{-\alpha}d\rho||x||,$$

and the conclusion follows. □

PROPOSITION 2.5. *Assume* (2.2)*, let* $z \in X$ *and set* $v(t) = \int_0^t R(s)z\,ds$. *Then*

(i) *For all* $T > 0, v \in L^\infty(0,T : D(A)) \cap W^{1,\infty}(0,T : X)$.

(ii) *If* $z \in \overline{D(A)}$, *then* $v \in C(0,T : D(A)) \cap C^1(0,T : X)$.

PROOF. Let $\rho > \omega$, then, by taking the Laplace transforms, one can check the identity

$$v(t) = R(\rho, A)\{\rho v(t) - R(t)z - (\beta * R(\cdot)z)(t)\},$$

and the conclusion follows. □

We now want to characterize those elements x of X such that $R(\cdot)x$ is Hölder continuous. This problem is connected with the asymptotic behavior of $||\lambda F(\lambda)x - x||$, as the following lemma shows.

PROPOSITION 2.6. *Assume* (2.2) *and let* $R(t)$ *be defined by* (2.12). *Let* $x \in \overline{D(A)}$, *and* $\gamma \in]0,1[$, *then the following assertions are equivalent*:

(i) $\forall\, \eta \in]0,\theta[$, *there exists a constant* $K_1(\eta) > 0$ *such that*

$$||R(re^{\pm i\eta})x - x|| \leq K_1(\eta)e^{\omega_\theta r \cos\eta} r^\gamma, \quad \forall\, r > 0. \tag{2.19}$$

(ii) $\forall\, \eta \in]0,\theta[$, *there exists a constant* $K_2(\eta) > 0$ *such that*

$$||R'(re^{\pm i\eta})x|| \leq K_2(\eta)e^{\omega_\theta r \cos\eta} r^{\gamma-1}, \quad \forall\, r > 0. \tag{2.20}$$

(iii) $\forall\, \eta \in]0,\theta[$, *there exists a constant* $K_3(\eta) > 0$ *such that*

(2.21)

$$||\lambda F(\lambda)x - x|| \leq K_3(\eta)|\lambda - \omega|^{-\gamma}, \quad \text{for } \lambda = \omega_\theta + \rho e^{\pm i(\pi/2+\eta)}, \quad \forall\, \rho > 0$$

where the constants $K_i(\eta), i = 1,2,3$, *are increasing in* η.

PROOF. (i)⇒(iii). It is sufficient to prove (iii) for $\lambda = \omega_\theta + \rho e^{\pm i(\pi/2+\eta-\varepsilon)}$, $\forall\, \rho > 0$, with $\varepsilon \in]0,\eta[$ and $\eta \in]0,\theta[$. Set

$$I_{\pm i\eta} := \{z \in \mathbf{C} : z = re^{\pm i\eta}, r > 0\}.$$

We consider the case $\lambda = \omega + \rho e^{i(\pi/2+\eta-\varepsilon)}$, the other case being similar. First we define

$$Q(\lambda)x = \int_{I_{\pm\eta}} e^{-\lambda z} R(z)x\, dz, \quad x \in X. \tag{2.22}$$

$Q(\lambda)$ is well defined and analytic on the sector $S_{0,\eta+\pi/2}$; thus, $Q(\lambda)x = F(\lambda)x$. It follows that

$$\lambda F(\lambda)x - x = \frac{1}{2i\pi} \int_{I_{\pm\eta}} \lambda e^{-\lambda z}(R(z)x - x)\, dz$$

which yields (iii) by a simple computation.

(iii)⇒(ii). We consider only the case $z = re^{i\eta}$, the other case being similar. Let $\eta \in]0, \theta[, r > 0$, and x satisfying (2.21). From Proposition 2.2, we have, for $r > 0$,

$$R'(re^{i\eta})x = \frac{1}{2i\pi} \int_\gamma \lambda e^{\lambda z} F(\lambda)x\, d\lambda = \frac{1}{2i\pi} \int_\gamma e^{\lambda z}(\lambda F(\lambda)x - x)\, d\lambda,$$

and (ii) follows.

(ii)⇒(i). We only consider the case $z = re^{i\eta}$. We have

$$\begin{aligned} |R(re^{i\eta})x - x| &= \lim_{\varepsilon \to 0} \left| \int_\varepsilon^r R'(re^{i\eta})x\, dr \right| \\ &\leq \lim_{\varepsilon \to 0} (r - \varepsilon) K_2(\eta) e^{\omega_\theta r \cos\eta} r^{\gamma - 1}, \end{aligned}$$

and the proof is complete. □

The next proposition states a relation among the assumptions of Proposition 2.5 and real interpolation spaces $D_A(\gamma, \infty)$ introduced in [**4**]. Let us recall the definition of $D_A(\gamma, \infty), \gamma \in]0, 1[$; we set

$$||x||_{\gamma,\eta} = \operatorname*{Sup}_{\rho > 0} \{||\lambda^\gamma R(\lambda, A)x||;\ \lambda = \omega_\theta + \rho e^{\pm i\eta}\}, \quad \eta \in]0, \theta[. \tag{2.23}$$

It is well known that the norms $\{||x||_{\gamma,\eta}; \eta \in]0, \theta[\}$ are equivalent.

PROPOSITION 2.7. *Assume* (2.2), *and let* $R(t)$ *be defined by* (2.12). *Let* $x \in \overline{D(A)}$, *and* $\gamma \in \,]0,\alpha]$; *then the following assertions are equivalent*:

(i) $x \in D_A(\gamma,\infty)$.

(ii) $\forall\, \eta \in \,]0,\theta[$, *there exists a constant* $K_3(\eta) > 0$ *such that* (2.21) *holds.*

PROOF. (i)⇒(ii). Let $x \in D_A(\gamma,\infty), \lambda = \omega_\theta + \rho e^{\pm i\eta}$. Then

$$\begin{aligned}\lambda F(\lambda)x - x &= AF(\lambda)x - \lambda\hat{\beta}(\lambda)F(\lambda)x \\ &= [1+\lambda\hat{\beta}(\lambda)R(\lambda,A)]^{-1}AR(\lambda,A)x - \lambda\hat{\beta}(\lambda)F(\lambda)x.\end{aligned} \tag{2.24}$$

Thus there exists a constant $C > 0$ such that

$$||\lambda F(\lambda)x - x|| \le C\left\{|\lambda|^\gamma ||x||_{\gamma,\eta} + \frac{1}{|\lambda-\omega|^\alpha}||x||\right\}.$$

Since $\gamma \le \alpha$, this completes the proof of the first implication.

(ii)⇒(i). By (2.24), we have

$$R(\lambda,A)x = [1+\lambda\hat{\beta}(\lambda)R(\lambda,A)]\{\lambda F(\lambda)x - x + \lambda\hat{\beta}(\lambda)F(\lambda)x\}, \tag{2.25}$$

and now the conclusion follows easily. □

We end this section with an approximation result which will be used later. Let A_n be the Yosida approximation of A, i.e., $A_n = nJ_n - n$, where $J_n = nR(n,A)$. Set

$$\rho_{F_n} = \{\lambda \in \Omega; \lambda + \lambda\hat{\beta}(\lambda) \in \rho(A_n)\} \tag{2.26}$$

$$F_n(\lambda) = R(\lambda + \lambda\hat{\beta}(\lambda), A_n), \quad \forall\, \lambda \in \rho_{F_n} \tag{2.27}$$

$$R_n(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}F_n(\lambda)\,d\lambda, \quad t > 0. \tag{2.28}$$

PROPOSITION 2.8. *Assume* (2.2), *and let* $R(t)$ *be defined by* (2.12) *and* $R_n(t)$ *by* (2.28). *Then*

$$||R_n(t)|| \le K e^{\omega_\theta t}, \quad t \ge 0, \tag{2.29}$$

and

$$\lim_{n\to\infty} R_n(t) = R(t), \quad \forall\, t > 0 \text{ in } \mathcal{L}(X) \tag{2.30}$$

uniformly on bounded sets of $]0, \infty[$.

3. The nonhomogeneous problem. We are here concerned with the problem

$$\begin{cases} \frac{d}{dt}(u(t) + (\beta * u)(t)) = Au(t) + f(t), & t > 0, \\ u(0) = x, \end{cases} \tag{3.1}$$

where $x \in X, f \in C([0,T]; X)$ and A and β verify (2.2).

We denote by $R(t)$ the resolvent defined by (2.12). We say that $u \in C([0,T]; X)$ is a *mild solution* of problem (3.1) if it satisfies the integral equation

$$u(t) = R(t)x + \int_0^t R(t-s) f(s)\, ds, \quad t \ge 0. \tag{3.2}$$

We want now to define a *strict solution* of (3.1). Remark that if $A = 0$ and $f = 0$, it is not in general true that $u(t) = R(t)x$ is of class C^1. Thus the following definition seems to be natural.

DEFINITION. u is called a *strict solution* of (3.1) if $u \in C([0,T]; D(A))$, $u + \beta * u \in C^1([0,T]; X)$ and fulfills (3.1).

PROPOSITION 3.1. *Assume* (2.2), *and let* $f \in C^\delta([0,T]; X)$, *for some* $\delta \in \,]0,1[, x \in D(A), Ax + f(0) \in \overline{D(A)}$. *Then the mild solution* u *to* (3.1) *is a strict solution.*

PROOF. Set

$$u(t) = u_1(t) + u_2(t) + u_3(t) + u_4(t), \tag{3.3}$$

where

$$u_1(t) = R(t)x \tag{3.4}$$

$$u_2(t) = \int_0^t R(t-s)[f(s) - f(t)]\,ds \tag{3.5}$$

$$u_3(t) = \int_0^t R(s)[f(t) - f(0)]\,ds \tag{3.6}$$

$$u_4(t) = \int_0^t R(s)f(0)\,ds. \tag{3.7}$$

Since

$$u_4(t) = A^{-1}[R(t)f(0) + (\beta * R)(\cdot)f(0)(t) - f(0)], \tag{3.8}$$

we have

$$\begin{aligned} &A(u_1(t) + u_4(t)) \\ &\quad = R(t)(Ax + f(0)) + (\beta * R)(\cdot)f(0)(t) - f(0) \in C([0,T];X). \end{aligned} \tag{3.9}$$

By Proposition 2.3,

$$R(\cdot)x + (\beta * R)(\cdot)x \in C^1([0,\infty[;X), \text{ and } AR(\cdot)x \in C([0,\infty[;X).$$

Thus we have only to check that v is a strict solution of (3.1) with $x = 0$. Set

$$v_n(t) = \int_0^t R_n(t-s)f(s)\,ds, \tag{3.10}$$

where $R_n(t)$ is defined in (2.28). We have

$$\begin{aligned} \frac{d}{dt}(v_n(t)) &= (1 - R_n(t))f(t) + \int_0^t \frac{d}{dt}R_n(t-s)[f(s) - f(t)]\,ds \\ &=: z_n(t) + w_n(t). \end{aligned} \tag{3.11}$$

Now $z_n(t) = f(t) - R_n(t)[f(t) - f(0)] + R_n(t)f(0)$; since $f(0) \in \overline{D(A)}R(\cdot), f(0)$ is continuous in $[0,T]$ by Proposition (2.2); moreover, it is easy to check that $R(\cdot)(f(\cdot) - f(0))$ is also continuous in $[0,T]$. So,

$$\begin{aligned} \lim_{n\to\infty} z_n(t) =& (1 - R(t))f(t) \quad \text{in } C([0,T];X), \\ & (1 - R(\cdot))f(\cdot) \in C([0,T];X). \end{aligned} \tag{3.12}$$

Moreover, by recalling (2.14) and using the hypothesis $f \in C^\delta([0,T];X)$, one sees that there exists a constant C such that

$$\left|\left|\frac{d}{dt}R_n(t-s)[f(s)-f(t)]\right|\right| \le C|t-s|^{\delta-1}.$$

It follows that

(3.13)

$$\lim_{n\to\infty} w_n(t) = \int_0^t \frac{d}{dt}R(t-s)[f(s)-f(t)]\,ds =: w(t) \quad \text{in } C([0,T];X),$$

and so $v \in C^1([0,T];X)$. Since $v(0)=0$, we also have $\beta * v \in C^1([0,T];X)$, and, consequently, $v \in C^1([0,T];D(A))$. This implies that u is a strict solution of (3.1). □

4. Some additional properties of $\mathbf{R(t)}$. In this section, we prove some additional estimates for the resolvent $||R(t)||$, which will be used in the next section. Also, we consider a closed convex cone Q in X and give sufficient conditions in order that $R(t)(Q) \subset Q$.

We assume, besides (2.2),

$$\text{(4.1)} \qquad \begin{cases} \text{(i) } \exists\, \omega \le 0 \text{ such that } ||e^{tA}|| \le e^{\omega t}, \text{ for all } t \ge 0, \\ \text{(ii) } \beta \text{ is nonnegative and nonincreasing.} \end{cases}$$

For any kernel K we denote by s_K the solution of the integral equation

$$\text{(4.2)} \qquad s_K + K * s_K = 1.$$

It is well known (see for instance [**1**]) that, if K is nonnegative and nonincreasing, then $s_K(t) \ge 0$ for all $t \ge 0$.

Proposition 4.1. *Assume* (2.2) *and* (4.1). *Let $R(t)$ be defined by* (2.12). *Then the following estimate holds*:

$$\text{(4.3)} \qquad ||R(t)|| \le s_{\beta+\omega}(t), \quad \forall\, t \ge 0,$$

where $s_{\beta+\omega}$ is defined in (4.2).

If, moreover, $e^{tA}(Q) \subset Q$*, then* $R(t)(Q) \subset Q$*,* $\forall\, t \geq 0$*.*

PROOF. In view of Proposition 2.7, it suffices to prove that

$$||R_n(t)|| \leq s_{[n\omega/(n+\omega)+\beta]}(t) \quad \forall\, t \geq 0, \tag{4.4}$$

where $R_n(t)$ is defined by (2.28).

Let $x \in X$, and let $u_n(t) = R_n(t)x$; then $R_n(t)x$ is the solution of the problem

$$\begin{cases} nu_n(t) + \frac{d}{dt}(u_n(t) + (\beta * u_n)(t)) = nJ_n u_n(t), \quad t > 0, \\ u_n(0) = x, \end{cases} \tag{4.5}$$

which is equivalent to

$$u_n + (\beta + n) * u_n = x + 1 * nJ_n u_n \tag{4.6}$$

and also to

$$u_n = s_{n+\beta} x + s_{n+\beta} * nJ_n u_n. \tag{4.7}$$

Since $s_{n+\beta} \geq 0$, it follows that

$$||u_n(t)|| \leq s_{n+\beta}(t)||x|| + \frac{n^2}{n+\omega}\int_0^t s_{n+\beta}(t-s)||u_n(s)||\, ds, \tag{4.8}$$

which implies, by a classical argument,

$$||u_n(t)|| \leq \phi_n(t)||x||, \tag{4.9}$$

where ϕ_n is the solution to the integral equation

$$\phi_n - \frac{n^2}{n+\omega} s_{n+\beta} * \phi_n = s_{n+\beta}. \tag{4.10}$$

Since the Laplace transform of ϕ_n and s_n are given, respectively, by

$$\hat{\phi}_n(\lambda) = \frac{\hat{s}_{n+\beta}(\lambda)}{1 - \frac{n^2}{n+\omega}\hat{s}_{n+\beta}(\lambda)} \tag{4.11}$$

and

$$\hat{s}_n(\lambda) = \frac{1}{\lambda + n + \lambda\hat{\beta}(\lambda)}, \tag{4.12}$$

we have

$$\hat{\phi}_n(\lambda) = \frac{1}{\lambda + \frac{n\omega}{n+\omega} + \lambda\hat{\beta}(\lambda)} = \hat{s}_{[n\omega/(n+\omega)+\beta]}(\lambda), \tag{4.13}$$

which implies (4.4). Finally, to prove the last statement it suffices to remark that, by (4.6), it follows that $u_n(t) \in Q$, for all $t \geq 0$, since $J_n(Q) \subset Q$. □

5. Semilinear equations. Let X be a complex Banach space and Q a closed convex cone in X. For any $r > 0$ we shall denote by B_r the ball $B_r = \{z \in X; ||z|| \leq r\}$. Let $A : D(A) \subset X \to X$ be a closed linear operator, $\beta : [0, \infty[\to \mathbf{R}$ a Laplace transformable function and $F : X \to X$ a nonlinear mapping.

We are concerned here with the semilinear problem

$$\begin{cases} \frac{d}{dt}(u(t) + (\beta * u)(t)) = Au(t) + F(u(t)), & t > 0, \\ u(0) = x. \end{cases} \tag{5.1}$$

We assume (2.2), (4.1) (with $\omega = 0$, for simplicity) and, concerning F,

$$\begin{cases} \text{(i) For all } r > 0, \text{ there exists } M_r > 0 \text{ such that} \\ \qquad ||F(x) - F(y)|| \leq M_r ||x - y||, \quad \forall\, x, y \in B_r. \\ \text{(ii) For all } \delta > 0 \text{ and all } x \in X, ||x|| \leq ||x - \delta F(x)||. \\ \text{(iii) } F(0) = 0. \end{cases} \tag{5.2}$$

We say that $u \in C([0, T]; X)$ is a *mild solution* of problem (5.1) if u fulfills the integral equation

$$u(t) = R(t)x + \int_0^t R(t-s)F(u(s))\, ds, \tag{5.3}$$

where the resolvent $R(t)$ is defined by (2.12).

In the following lemma, we gather, for later use, some properties of the nonlinear mapping F.

LEMMA 5.1. *Let F be a mapping in X such that hypotheses* (5.2) *are fulfilled. For any $r > 0$, set $\delta_r = M_{2r}/2$. Then, if $\delta \in]0, \delta_r[$, the mapping $1 - \delta F : B_{2r} \to X$ is one-to-one and $(1 - \delta F)(B_{2r}) \supset B_r$. Define a mapping $J_{\delta,r} : B_r \to X$, for all $r > 0$ and $\delta \in]0, \delta_r[$, by setting*

$$J_{\delta,r}(x) = (1 - \delta F)^{-1}(x), \quad x \in B_r. \tag{5.4}$$

Then

$$||J_{\delta,r}(x)|| \leq ||x||, \quad \forall\, x \in B_r, \tag{5.5}$$

$$\lim_{\delta \to 0} J_{\delta,r}(x) = x, \quad \forall\, x \in B_r. \tag{5.6}$$

PROOF. The first statement follows from (5.2)(i) and the Contraction Principle. Moreover, (5.5) follows from (5.2)(ii) and (5.3) is easily checked. □

We set, finally,

$$F_{\delta,r}(x) = F(J_{\delta,r}(x)) = \frac{1}{\delta}(J_{\delta,r}(x) - x), \quad x \in B_r,\ \delta \in]0, \delta_r[. \tag{5.7}$$

By (5.5), it follows that

$$\lim_{\delta \to 0} F_{\delta,r}(x) = F(x), \quad \forall\, x \in B_r. \tag{5.8}$$

We prove the main result of this section:

THEOREM 5.2. *Assume* (2.2), (4.1) (*with $\omega = 0$*) *and* (5.2). *Then problem* (5.1) *has a unique mild solution u. If, moreover, $J_{\delta,r}(Q) \subset Q$ for $\delta \in]0, \delta_r[$ and $x \in Q$, then $u(t) \in Q$ for all $t \geq 0$.*

PROOF. Fix $r > 0$, let $x \in B_r$ and $\delta \in]0, \delta_r[$. Consider the approximating problem

$$\begin{cases} \frac{d}{dt}(u_\delta(t) + (\beta * u_\delta)(t)) = Au_\delta(t) + F_{\delta,r}(u_\delta(t)), \quad t > 0, \\ u_\delta(0) = x, \end{cases} \tag{5.9}$$

which is equivalent to

$$u_\delta(t) = R_\delta(t)x + \frac{1}{\delta}\int_0^t R_\delta(t-s)J_{\delta,r}(u_\delta(s))\,ds, \tag{5.10}$$

where R_δ is the resolvent operator of problem (2.1) with A replaced by $A - 1/\delta$. By standard arguments, equation (5.10) has a unique solution in a maximal interval $[0, \tau_\delta[$. By (4.3) and (5.5),

$$||u_\delta(t)|| \leq s_{\beta+1/\delta}(t)||x|| + \frac{1}{\delta}\int_0^t s_{\beta+1/\delta}(t-s)||u_\delta(s)||\,ds. \tag{5.11}$$

Then

$$||u_\delta(t)|| \leq \psi_\delta(t)||x||, \tag{5.12}$$

where ψ_δ is the solution to the integral equation

$$\psi_\delta(t) = s_{\beta+1/\delta}(t) + \frac{1}{\delta}\int_0^t s_{\beta+1/\delta}(t-s)\psi_\delta(s)\,ds. \tag{5.13}$$

As is easily checked, $\psi_\delta(t) = s_\beta(t)$, so that

$$||u_\delta(t)|| \leq s_\beta(t)||x||. \tag{5.14}$$

This implies that the solution u_δ of (5.10) is global.

Now, it remains to prove that there exists the limit $\lim_{\delta\to 0} u_\delta(t) = u(t)$ and that u is the required solution. For this purpose we consider the solution $u(t)$ of equation (5.3) in its existence maximal interval $[0, \tau[$; by (5.8) and the Contraction Principle (depending on the parameter δ), it follows that

$$\lim_{\delta\to 0} u_\delta(t) = u(t) \tag{5.15}$$

uniformly in all intervals $[0, t_1] \subset [0, \tau[$. Thus we obtain the a priori estimate

$$||u(t)|| \leq s_\beta(t)||x||, \quad \text{for all } t \in [0, \tau[, \tag{5.16}$$

and problem (3.1) has a global solution.

Let us now assume that $nJ_{\delta,r}(Q) \subset Q$; then, by (5.10), it follows that $u_\delta(t) \in Q$ for all $t \geq 0$ and $\delta > 0$. Thus, by (5.15), we have $u(t) \in Q$ for all $t \geq 0$, and the proof is complete. □

REFERENCES

1. Ph. Clément and J.A. Nohel, *Asymptotic behaviour of solutions of nonlinear Volterra equations with completely positive kernels,* SIAM J. Math. Anal. **12** (1981), 514–535.

2. M.G. Crandall and J.A. Nohel, *An abstract functional differential equation and a related nonlinear Volterra equation,* Israel J. Math. **29** (1978), 313–328.

3. G. Da Prato and M. Iannelli, *Existence and regularity for a class of integro-differential equations of parabolic type,* J. Math. Anal. Appl. **112** (1985), 36–55.

4. J.L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation,* Institut de Hautes Etudes Scientifiques, Publ. Math. **19** (1964), 5–68.

5. A. Lunardi, *On the linear heat equation for materials of fading memory type,* SIAM J. Math. Anal., to appear.

6. ———, *Laplace transform methods in integrodifferential equations,* J. Integral Equations Appl. **10** (1985), 185–211.

7. J.W. Nunziato, *On heat conduction in materials with memory,* Quart. Appl. Math. **29** (1971), 187–204.

8. J. Prüß, *On Linear Volterra equations of parabolic type in Banach spaces,* Trans. Amer. Math. Soc. **301** (1987), 691–721.

Technische Universiteit, POB 356, 2600 AJ Delft, The Netherlands

Scuola Normale Superiore, Piazza dei Cavalieri, 6, 56126 Pisa, Italy