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DISSIPATIVE BOUNDARY CONDITIONS FOR ONE-DIMENSIONAL WAVE PROPAGATION

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To John Nohel for his sixty-fifth birthday

1. Introduction. There has been a great deal of work in recent years on evolution equations which contain memory terms. The most interesting situation occurs for wave propagation in elastic materials. One starts with a model which conserves energy and then modifies it by including a memory term which produces damping (*viscoelasticity*). John Nohel has been a major figure in these studies and the results are summarized in his book with Hrusa and Renardy [**7**].

The present paper is concerned with a closely related but slightly different idea. Here we maintain an energy conserving equation but produce damping through boundary conditions. Let us describe the problem and then we will indicate why it is of interest.

We deal with one-dimensional longitudinal motions of a bar which has uniform cross section but may be inhomogeneous. The basic balance law, in the absence of body forces, is

$$\rho u_{tt} = \sigma_x, \tag{1.1}$$

where ρ is density, u displacement and σ stress. The specific problem we consider is this:

$$(P(\varphi,\psi)) \qquad \begin{gathered} \rho(x)u_{tt}(x,t) = (\mu(x)u_x(x,t))_x, \quad 0 < x < L, \\ u(x,0) \equiv u_t(x,0) \equiv 0 \\ u(0,t) = \varphi(t), \quad \mu(L)u_x(L,t) = \mathcal{F}\,[u^t(L,\cdot)] + \psi(t). \end{gathered}$$

Here, φ and ψ are given and $\mathcal{F}$ denotes a functional of the history

$$u^t(L,\tau) = u(L,t-\tau).$$

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The differential equation in $(P(\varphi,\psi))$ is energy conserving. We seek conditions on $\mathcal{F}$ so that the boundary conditions at $x = L$ produce dissipation, that is damping. This means, roughly, that if φ and ψ tend to zero as t tends to infinity, so should u. (See Remark 2.3.) These ideas are also described in [**2**] and [**8**].

Let us motivate the above problem. Suppose the bar is actually semi-infinite, $0 < x < \infty$, but is inhomogeneous with $\sigma(x,t) = \mu(x)u_x(x,t)$. It starts from rest (for ease of exposition) with a prescribed displacement at the left end, $x = 0$. The total problem is then like $(P(\varphi,\psi))$ on $0 < x$. There is no second boundary condition but the solution needs to be outgoing. Suppose we are only interested in a finite interval $0 < x < L$. Then one could (in theory) solve the equation on $x > L$ and obtain a relation between the stress at $x = L$ and the displacement at $x = L$. This relation will have the history form in $(P(\varphi,\psi))$. We carry out this calculation in Section 4. Since energy is flowing off to infinity we expect to have damping on $(0, L)$, and this can come only from the boundary condition at $x = L$.

A second idea is this. Suppose the bar is composite with an abrupt change at $x = L$. Let $\sigma(x,t) = \mu(x)u_x(x,t), \quad 0 < x < L$, and suppose the portion $x > L$ is homogeneous but viscoelastic. Once again one could solve on $x > L$ to obtain a relation between $\sigma(L,t)$ and the history $u^t(L,\cdot)$. Since both u and σ are continuous across $x = L$ this yields a problem of the form $(P(\varphi,\psi))$. This case is also treated in Section 4.

The final notion is what really prompted this study, the idea of *approximate boundary conditions*. This is a numerical device. Even if one knew what the functional $\mathcal{F}$ was, $(P(\varphi,\psi))$ would be difficult to handle numerically because of the time non-locality. What we seek are approximate functionals which are more localized in time to use instead of $\mathcal{F}$. This idea has been pursued for wave scattering problems in exterior regions by many authors, starting with Engquist and Majda [**4, 5**].

In [**1**] the authors studied the application of this method to the semi-infinite bar problem. It serves as a very simple model problem. The main difficulty is to devise approximate conditions which preserve the dissipativity. We discuss this in Section 5, examining some possible approximations and giving some partial results on their validity.

In Section 2 we discuss dissipativity of the boundary function $\mathcal{F}$. We first give conditions in the time domain and show how they produce damping. In Section 3 we give alternate conditions in the frequency domain. These are conditions which are familiar in viscoelastic theory, and they are the ones most useful in the applications.

2. Dissipative boundary conditions. The functionals $\mathcal{F}$ in $(P(\varphi,\psi))$ will be assumed to have the form

$$\mathcal{F}[\zeta^t] = -\frac{d}{dt}(\alpha\zeta(t) + (k*\zeta)(t)), \tag{2.1}$$

where $k*\zeta$ denotes convolution. We make the following hypotheses:

$$\alpha > 0, \quad k = k_\infty + K, \quad k_\infty \geq 0, \quad K \in L_1(0,\infty). \tag{H$_1$}$$

There is a $\gamma > 0$ such that, for any $T > 0$ and any ζ,

$$\int_0^T (\alpha\zeta(t) + (k*\zeta)(t))\zeta(t)\,dt \geq \gamma\int_0^T \zeta(t)^2\,dt. \tag{H$_2$}$$

Hypotheses (H_1) and (H_2) yield the following estimate if $\zeta(0) = 0$:

$$\begin{aligned}\int_0^T \dot\zeta(t)\mathcal{F}[\zeta^t]\,dt = &-\alpha\int_0^T \dot\zeta(t)^2\,dt - \int_0^T k_\infty\dot\zeta(t)\zeta(t)\,dt\\ &-\int_0^T \dot\zeta(t)(k*\dot\zeta)(t)\,dt\\ \leq &-\gamma\int_0^T \dot\zeta(t)^2\,dt - \frac{1}{2}k_\infty\zeta^2(T).\end{aligned} \tag{2.2}$$

For technical convenience we will assume that the functions φ and ψ have derivatives of all orders, continuous on $t \geq 0$, and vanishing when $t = 0$. It is easier to state our results if we make a preliminary transformation. Let $v_0(x) = (1 - x/L)^2$ and, for any solution u of $(P(\varphi,\psi))$, put $w = u - \varphi v_0$. Then w satisfies the problem

$$\begin{gathered}\rho w_{tt} = (\mu w_x)_x + f, \quad 0 < x < L,\\ w(x,0) = w_t(x,0) = 0\\ w(0,t) = 0, \quad \mu(L)w_x(L,t) = \mathcal{F}[w^t(L,\cdot)] + \psi(t),\end{gathered} \tag{$P(f,\psi)$}$$

where

$$f = \varphi(\mu v_0')' - \rho\ddot{\varphi} v_0. \tag{2.3}$$

We will need the following hypotheses:

$$(\mathrm{H}_3)_k \qquad \frac{\partial^j f}{\partial t^j} \in L_1((0,\infty) : L_2(0,L)) \cap L_2((0,\infty) : L_2(0,L))$$
$$\psi^{(j)} \in L_2(0,\infty), \quad j \leq k.$$

For (f,ψ) satisfying $(\mathrm{H}_3)_k$, we set

(2.4)
$$||(f,\psi)||_k^2 = \sum_{j=0}^{k} \bigg\{ \bigg|\bigg|\frac{\partial^j f}{\partial t^j}\bigg|\bigg|^2_{L_1((0,\infty):L_2(0,L))} + \bigg|\bigg|\frac{\partial^j f}{\partial t^j}\bigg|\bigg|^2_{L_2((0,\infty):L_2(0,L)} + ||\psi^{(j)}||^2_{L_2(0,\infty)} \bigg\}.$$

For the functions w, we introduce the norms

$$\begin{aligned} \mathcal{E}^{\,k}[w](t) &= \sum_{j=0}^{k+1} \bigg|\bigg|\frac{\partial^j w}{\partial t^j}\bigg|\bigg|^2_{L_2(0,L)} + \sum_{j=0}^{k} \bigg|\bigg|\frac{\partial^j w}{\partial t^j}\bigg|\bigg|^2_{H_1(0,L)} \\ \mathcal{E}\,{}^{k}_{T}[w] &= \int_0^T (\mathcal{E}^{\,k}[w](t))^2\,dt. \end{aligned} \tag{2.5}$$

All of our theorems hold under hypotheses (H_1) and (H_2). They are stated for solutions w of $(P(f,\psi))$ but, from (2.3), they are easily translated into theorems about solutions u of $(P(\varphi,\psi))$. The first two results are energy estimates:

THEOREM 2.1. *There is a constant $M > 0$ such that, for any (f,ψ) satisfying* $(\mathrm{H}_3)_k$,

(2.6)
$$\mathcal{E}^{\,k}[w](T) + \sum_{j=0}^{k} \bigg|\bigg|\frac{\partial^j w}{\partial t^j}(L,\cdot)\bigg|\bigg|^2_{L_2(0,T)} \leq M||(f,\psi)||_k^2, \quad \forall\, T \in (0,\infty).$$

THEOREM 2.2. *If $k_\infty = 0$, there is a constant $N > 0$ such that, for any (f, φ) satisfying* $(\mathrm{H}_3)_k$,

$$\mathcal{E}^{\,k}_{T}[w] \le N||(f,\psi)||_k^2 \quad \forall\, T \in (0,\infty). \tag{2.7}$$

From these theorems we obtain immediately two decay results:

THEOREM 2.3. *For any (f, ψ) satisfying* $(\mathrm{H}_3)_1$,

$$w_t(L,t) \to 0 \quad \text{as } t \to \infty. \tag{2.8}$$

THEOREM 2.4. *If $k_\infty = 0$ then, for any (f, ψ) satisfying* $(\mathrm{H}_3)_1$,

$$w(x,t) \to 0 \quad \text{ as } t \to \infty \quad \forall\, x \in (0,L). \tag{2.9}$$

PROOF OF THEOREM 2.3. By Theorem 2.1, $w_t(L,t)$ and $w_{tt}(L,t)$ both belong to $L_2(0,\infty)$ from which (2.8) follows. □

PROOF OF THEOREM 2.4. By Theorem 2.2, the maps $t \to ||w(\cdot,t)||_{H_1(0,L)}$ and $t \to ||w_t(\cdot,t)||_{H_1(0,L)}$ are both in $L_2(0,\infty)$. Hence $||w(\cdot,t)||_{H_1(0,L)} \to 0$ as $t \to \infty$ and (2.9) follows since $w(0,t) = 0$. □

REMARK 2.1. We are not sure of the status of decay of solutions if $k_\infty > 0$. Notice that, in this case, Theorem 2.3 says only that the velocity at $x = L$ goes to zero as t tends to infinity. In the examples of Section 4 we will have $k_\infty = 0$ so that we get the strong decay result (2.9). When we deal with the approximate condition in Section 5, however, it will not always be true that $k_\infty = 0$. We comment further there. We suspect there is decay even if $k_\infty \neq 0$.

REMARK 2.2. Theorem 2.4 admits an extension to the case of approach to steady state. Suppose $k_\infty = 0$ and consider $P(\varphi, 0)$ when

$$\varphi(t) = \varphi_\infty + \Phi(t). \tag{2.10}$$

Let $\overline{u}(x)$ be the solution of the problem

$$(2.11)\qquad \begin{gathered} (\mu\overline{u}')' = 0, \quad 0 < x < L, \\ \overline{u}(0) = \varphi_\infty, \quad \mu(L)\overline{u}'(L) = 0. \end{gathered}$$

If u is the corresponding solution of $P(\varphi, 0)$, put $v = u - \overline{u}$. Then one readily checks that v is a solution of $P(\Phi, \psi)$, where

$$(2.12)\qquad \psi(t) = \mathcal{F}[\overline{u}^t] = -\frac{d}{dt}\bigg(\int_0^T k(t-\tau)\,d\tau\bigg)\overline{u}(L) = -K(t)\overline{u}(L).$$

It will be seen in Section 3 how to insure that k and $\ddot{k}$ are in $L_2(0,\infty)$. It is then easy to see that, if one suitably restricts Φ in (2.10), one can apply Theorem 2.4 to conclude that $v(x,t) \to 0$ as $t \to \infty$.

Proof of Theorem 2.1. We multiply the equation in $(P(f,\psi))$ by w_t and integrate by parts to obtain

$$\begin{aligned} \int_0^L [\rho w_{tt} - (\mu w_x)_x] w_t\,dx &= \int_0^L [\rho w_{tt} w_t\,dx + \mu w_x w_{xt}]\,dx \\ &\quad - \mu(L) w_x(L,t) w_t(L,t) \\ &= \frac{1}{2}\frac{d}{dt}\bigg\{\int_0^L (\rho w_t^2 + \mu w_x^2)\,dx\bigg\} \\ &\quad - \mu(L) w_t(L,t)(\mathcal{F}[w^t(L,\cdot)] + \psi(t)) \\ &= \int_0^L w_t f\,dx. \end{aligned}$$

Now integrate with respect to t from 0 to T and use (2.2):

(2.13)

$$\begin{aligned} &\frac{1}{2}\int_0^L (\rho w_t^2(x,T) + \mu w_x^2(x,T))\,dx + \gamma\int_0^T w_t^2(L,t)\,dt \\ &\quad \le \mu(L)\int_0^T w_t(L,t)\psi(t)\,dt + \int_0^T\int_0^L w_t f\,dx\,dt \le \frac{\gamma}{2}\int_0^T w_t^2(L,t)\,dt \\ &\qquad + \frac{\mu^2(L)}{2}\int_0^T \psi^2(t)\,dt \\ &\qquad + \int_0^T\bigg(\int_0^L \frac{\rho}{2} w_t^2(x,t)\,dx\bigg)^{1/2}\int_0^T\bigg(\int_0^L \frac{2f^2(x,t)}{\rho}\,dx\bigg)^{1/2} dt. \end{aligned}$$

Put

$$\zeta(t) = \frac{1}{2}\int_0^L \rho w_t^2(x,t)\,dx, \qquad \chi(t) = \left(\int_0^L \frac{2\,f^2(x,t)}{\rho}\,dx\right)^{1/2},$$
$$C = \frac{\mu^2(L)}{2}\int_0^T \psi^2(t)\,dt;$$

then (2.13) yields

$$\zeta(T) \le \frac{\gamma}{2}\int_0^T w_t^2(L,t)\,dt \le C + \int_0^T \chi(t)\zeta(t)^{1/2}\,dt. \tag{2.14}$$

From (2.14), we obtain

$$\zeta(T) + \frac{\gamma}{2}\int_0^T w_t^2(L,t)\,dt \le 2C + \frac{1}{2}\int_0^T \chi(t)\,dt,$$

and this yields (2.6) for $k = 0$.

In order to obtain the higher k estimates we simply differentiate the problem $(P(f,\psi))$ with respect to t. The vanishing of the derivatives of φ at $t = 0$ implies, by (2.3), that the t derivatives of f vanish at $t = 0$ and hence, by the differential equation in $(P(f,\psi))$, the vanishing of the derivatives of w at $t = 0$. Since the derivatives of ψ also vanish at $t = 0$ we have, by (2.1),

$$\left(\frac{\partial}{\partial t}\right)^j (\mathcal{F}\,[w^t(L,\cdot) + \psi(t)) = \mathcal{F}\,[(w^j(L,\cdot))^t] + \psi^{(j)}(t).$$

Thus $\partial^j w/\partial t^j$ is a solution of $P(\partial^j f/\partial t^j, \psi^{(j)})$ and we can apply the estimate (2.6) for $k = 0$, for $j = 1, \ldots, k$, to obtain (2.6). $\square$

Proof of Theorem 2.2. Put

$$Z[w](t) = \int_0^L g(x) w_x(x,t) w_t(x,t)\,dx,$$

where g is to be chosen, subject to $g(0) = 0$. We have

(2.15)
$$\begin{aligned}\dot{Z}[w](t) &= \int_0^L \left[g w_t w_{tx} + \frac{g}{\rho\mu}(\mu w_x)\rho w_{tt} \right] dx \\ &= \frac{1}{2}\int_0^L g(w_t^2)_x\, dx + \int_0^L \left[\frac{g}{\rho\mu}(\mu w_x)(\mu w_x)_x + \frac{g}{\rho} w_x f \right] dx \\ &= \frac{1}{2}\int_0^L \left[g(w_t^2)_x + \frac{g}{\rho\mu}((\mu w_x)^2)_x + \frac{2g}{\rho\mu}\mu w_x f \right] dx \\ &= \frac{1}{2} g(L) w_t^2(L,t) + \frac{g(L)}{\rho(L)\mu(L)}(\mu(L) w_x(L,t))^2 \\ &\quad - \frac{1}{2}\int_0^L g' w_x^2\, dx - \frac{1}{2}\int_0^L \left(\frac{g}{\rho\mu}\right)' \mu w_x^2\, dx + \int_0^L \frac{g}{\rho}\, w_x f\, dx.\end{aligned}$$

When $k_\infty = 0$, we have, by (2.1) and (H_1),

(2.16)
$$\begin{aligned}\int_0^T (\mu(L) w_x(L,t))^2\, dt &= \int_0^T (\alpha w_t(L,t) + K w_t(L,\cdot)(t) + \psi(t))^2\, dt \\ &\le C\big\{\alpha + ||K||_{L_1(0,\infty)} ||w_t(L,\cdot)||^2_{L_2(0,T)} \\ &\qquad + ||\psi||^2_{L_2(0,T)}\big\}.\end{aligned}$$

We have also, for any $\varepsilon > 0$,

$$\int_0^L \frac{g}{\rho} w_x(x,t) f(x,t)\, dx \le \varepsilon ||w(\cdot,t)||^2_{H_1(0,L)} + \frac{1}{4\varepsilon} ||gf/\rho||^2_{L_2(0,L)}. \tag{2.17}$$

We choose g so that $g'(x) > 0$ and $(g/(\rho\mu))'(x) > 0$ on $0 \le x \le L$, and we choose ε so small that

$$\frac{1}{2}\int_0^L \left(\frac{g}{\rho\mu}\right)' \mu w_x^2\, dx - \varepsilon ||w(\cdot,t)||^2_{H_1(0,L)} \ge \delta ||w(\cdot,t)||^2_{H_1(0,L)}, \tag{2.18}$$

with $\delta > 0$. Now integrate (2.13) from 0 to T and use (2.16)–(2.18) to obtain the estimate

$$\begin{aligned}Z[w](T) + \mathcal{E}_T[w] \le C'\big\{&\alpha + ||K||_{L_1(0,\infty)} ||w_t(L,\cdot)||^2_{L_2(0,T)} \\ &+ ||\psi||^2_{L_2(0,T)} + ||f||_{L_2(0,T:L_2(0,L))}\big\}.\end{aligned} \tag{2.19}$$

From Theorem 2.1 and the hypotheses on f, the right side of (2.19) is bounded independently of T. We see also that Theorem 2.1 implies that $Z[w](T)$ is similarly bounded. Hence we obtain (2.7) for $k = 0$. Once again we can obtain the higher order estimates by successive differentiation with respect to t.

REMARK 2.3. The use of the functional $Z[w]$ was suggested by Professor J. Lagnese.

3. Frequency domain methods. For any function φ we write $\hat{\varphi}(s) = \mathcal{L}[\varphi](s), s = \xi + i\eta$, for its Laplace transform whenever it exists. We put $\Pi = \{s : \xi > 0\}$. The functions we study will have transforms which belong to a space we call A. $A =$ set of all functions $\hat{\varphi} : \overline{\Pi} \to \mathbf{C}$ such that

$$\begin{gathered} \hat{\varphi} \in C^{(2)}(\overline{\Pi}), \quad \hat{\varphi} \text{ analytic in } \Pi, \quad \hat{\varphi} \text{ real on } \eta = 0, \\ \hat{\varphi}(s) = \varphi_0 s^{-1} + \varphi_1 s^{-2} + 0(s^{-3}), \qquad \hat{\varphi}'(s) = -\varphi_1 s^{-2} + 0(s^{-3}) \\ \hat{\varphi}''(s) = 0(s^{-3}) \quad \text{as } s \to \infty. \end{gathered} \tag{3.1}$$

LEMMA 3.1. *Suppose φ satisfies the following conditions*

$$\varphi \in C^{(2)}[0,\infty), \qquad t^j \varphi^{(k)} \in L_1(0,\infty), \quad j = 0,1,2, \quad k = 0,1,2.$$

Then φ has a transform $\hat{\varphi} \in A$.

PROOF. We have $\hat{\varphi}(s) = \int_0^\infty e^{-st}\varphi(t)\,dt$. This is well defined and continuous in $\overline{\Pi}$ and is analytic in Π. We have

$$\hat{\varphi}(s) = \varphi(0)s^{-1} + \dot{\varphi}(0)s^{-2} + s^{-2}\int_0^\infty e^{-st}\ddot{\varphi}(t)\,dt.$$

We have, further, $\hat{\varphi}^{(j)}(s) = \int_0^\infty e^{-st}(-1)^j t^j \varphi(t)\,dt$, and the estimates in (3.1) follow. □

LEMMA 3.2. *Suppose $\hat{\varphi} \in A$ and*

$$\varphi(t) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{i\eta t}\hat{\varphi}(i\eta)\,d\eta. \tag{3.2}$$

Then $\varphi \in C^{(1)}[0,\infty) \cap L_1(0,\infty)$ with $\varphi(0) = \varphi_0, \dot{\varphi}(0) = \varphi_1$.

PROOF. Let $\chi_k(t) = t^k e^{-t}$. Then $\hat{\chi}_k(s) = (k-1)!(s+1)^{-k}$. We then have $\hat{\varphi}(s) = \varphi_0\hat{\chi}_0 + (\varphi_1 - \varphi_0)\hat{\chi}_1 + \hat{\psi}$ where $\hat{\psi}(s) = 0(s^{-3})$. Then (3.2) yields

$$\varphi(t) = \varphi_0 e^{-t} + (\varphi_1 - \varphi_0)te^{-t} + \psi(t)$$
$$\psi(t) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{i\eta t}\hat{\psi}(s)\,ds,$$

with $\psi \in C^1[0,\infty)$. Thus $\varphi \in C^{(1)}[0,\infty), \varphi(0) = \varphi_0, \dot{\varphi}(0) = \varphi_1$. Next we integrate (3.2) twice by parts to obtain,

$$\psi(t) = -\frac{1}{2\pi t^2}\int_{-\infty}^{+\infty} e^{i\eta t}\hat{\varphi}''(i\eta)\,d\eta.$$

It follows that $\psi(t) = 0(t^{-2})$ as $t \to \infty$, hence $\psi \in L_1(0,\infty)$. □

The following result is established in [**10**].

LEMMA 3.3. *Suppose k has a transform $\hat{k} \in A$. Given any $T > 0$ and any $\zeta \in C[0,T]$, put $\zeta_T(t) = \zeta(t),\ 0 \le t < T, \zeta_T(t) = 0, t > T$. Then*

$$\int_0^T \zeta(t)(k*\zeta)(t)\,dt = \frac{2}{\pi}\int_0^\infty \mathrm{Re}\,\hat{k}(i\eta)|\tilde{\zeta}_T(\eta)|^2\,d\eta \tag{3.3}$$

where $\tilde{\zeta}(\eta)$ is the Fourier transform of ζ_T.

The non-local boundary condition in $(P(\varphi,\psi))$ can be formally transformed to yield the relation

$$\mu(L)\hat{u}_x(x,s) = \hat{\mathcal{F}}(s)\hat{u}(L,s) + \hat{\psi}(s). \tag{3.4}$$

$\mathcal{F}$ is actually a distribution, and if it has the form (2.1),

$$\hat{\mathcal{F}}(s) = -s(\alpha + \hat{k}(s)) = -s(\alpha + k_\infty s^{-1} + \hat{K}(s)). \tag{3.5}$$

We want to establish conditions on the transform which will guarantee that $\mathcal{F}$ has the form (2.1) and satisfies (H_1) and (H_2). Let us do the formal calculations first and then we will state the theorem.

We assume that $\hat{\mathcal{F}}(s)$ has the form

$$\hat{\mathcal{F}}(s) = -\alpha s + \beta + \hat{r}(s), \quad \alpha > 0, \tag{3.6}$$

where $\hat{r} \in A$. We rewrite this formula as

$$\begin{gathered} \hat{\mathcal{F}}(s) = -s(\alpha + k_\infty s^{-1} + \hat{K}(s)) \\ k_\infty = -(\beta + \hat{r}(0)) = -\hat{\mathcal{F}}(0) \\ \hat{K}(s) = \frac{\hat{r}(0) - \hat{r}(s)}{s} = \frac{\hat{\mathcal{F}}(0) - \hat{\mathcal{F}}(s)}{s} - \alpha. \end{gathered} \tag{3.7}$$

Let us study $\hat{K}(s)$. If $\hat{\mathcal{F}} \in C^2(\overline{\Pi})$ then $\hat{K} \in C^2(\overline{\Pi}\backslash\{0\})$. Since $\hat{\mathcal{F}}(0) - \hat{\mathcal{F}}(s)$ vanishes at $s = 0$, $\hat{K}$ will still be once differentiable at $s = 0$ if $\hat{K}'(s)$ is defined by its limit at $s = 0$. If, in addition, $\hat{\mathcal{F}} \in C^{(3)}(\overline{\pi})$, then $\hat{K} \in C^{(2)}(\overline{\pi})$. If $\hat{r} \in A$, one can readily check that $\hat{K}$ satisfies the appropriate behavior at infinity so that $\hat{K} \in A$. Thus, for (3.6) with $\hat{\mathcal{F}} \in C^{(3)}(\overline{\pi})$, $\hat{r} \in A$ and $\hat{\mathcal{F}}(0) \leq 0$, we will have $\mathcal{F}[\zeta^t] = -\partial[\alpha + (k * \zeta)(t)]/\partial t$ with $k(t) = k_\infty + K(t), k_\infty \geq 0, K \in L_1(0, \infty)$.

We have, from (3.7),

$$\alpha + \operatorname{Re}\hat{k}(i\eta) = -\frac{\operatorname{Im}\hat{\mathcal{F}}(i\eta)}{\eta}. \tag{3.8}$$

For large η, (3.8) and (3.6) yield

$$\alpha + \operatorname{Re}\hat{k}(i\eta) = \alpha + 0\left(\frac{1}{\eta}\right) \quad \text{as } \eta \to \infty. \tag{3.9}$$

For small η, (3.8) yields

$$\alpha + \operatorname{Re}\hat{k}(i\eta) = -\hat{\mathcal{F}}'(0) + 0(\eta) \quad \text{as } \eta \to 0. \tag{3.10}$$

Suppose we impose the conditions

$$-\operatorname{sign}\eta \operatorname{Im}\hat{\mathcal{F}}(i\eta) > 0 \quad \text{for all } \eta \neq 0, \quad \hat{\mathcal{F}}'(0) < 0. \tag{3.11}$$

Then from (3.8)–(3.11) we see that there is a $\gamma > 0$ such that

$$\alpha + \operatorname{Re}\hat{k}(i\eta) \geq \gamma \quad \text{for all } \eta. \tag{3.12}$$

By Lemma (3.3), (3.12) implies (H_2). We summarize our result.

THEOREM 3.1. *Suppose the transform $\hat{\mathcal{F}}$ of the non-local boundary conditions has the form* (3.6) *with $\hat{\mathcal{F}} \in C^{(3)}(\overline{\pi})$ and $\hat{\mathcal{F}}$ satisfies the conditions*

(3.13)

$$\hat{\mathcal{F}}(0) < 0, \quad \hat{\mathcal{F}}'(0) < 0, \quad -\operatorname{sign}\eta \operatorname{Im}\hat{\mathcal{F}}(i\eta) > 0 \quad \textit{for all}\ \eta \neq 0.$$

Then $\mathcal{F}$ has the form (2.1) *with* (H_1) *and* (H_2) *satisfied.*

REMARK 3.1. The importance of the behavior of $\hat{\mathcal{F}}(s)$ when s is small was suggested by the work in [**3**].

4. Exact dissipative boundary conditions.

EXAMPLE 1. *Composite elastic-viscoelastic bar.* This example is suggested by [**9**]. We suppose our bar is elastic, but inhomogeneous, on $0 < x < L$ so that u satisfies the differential equation in $(P(\varphi, \psi))$. Suppose that the portion $x > L$ is viscoelastic but homogeneous. This means that, for $x > L$,

$$\begin{aligned} \sigma(x,t) &= \frac{\partial}{\partial t} a * u_x(x,\cdot)(t) \\ \rho_0 u_{tt}(x,t) &= \sigma_x(x,t). \end{aligned} \tag{4.1}$$

If we use the Laplace transform, we obtain

$$\begin{aligned} \hat{\sigma}(x,s) &= s\hat{a}(s)(x,s) \\ \rho_0 s^2 \hat{u}(x,s) &= s\hat{a}(s)\hat{u}_{xx}(x,s). \end{aligned} \tag{4.2}$$

We want the solution u to be outgoing in $x > L$, which means we want $\hat{u}(x,s)$ to tend to zero as $s \to \infty$. From (4.2) we obtain the relations

$$\begin{aligned} \hat{u}(x,s) &= e^{-\gamma(s)(x-L)}\hat{u}(L,s) \\ \hat{\sigma}(L,s) &= -\, s\hat{a}(s)\hat{\gamma}(s)\hat{u}(L,s) :\equiv \hat{\mathcal{F}}(s)\hat{u}(L,s), \end{aligned} \tag{4.3}$$

where $\hat{\gamma}(s) = \sqrt{\rho_0 s/\hat{a}(s)}$. We give conditions which guarantee that $\hat{\mathcal{F}}$ in (4.3) satisfies the hypotheses of Theorem 3.1.

The conditions on a in (4.1) are dictated by viscoelasticity theory. Typical hypotheses, which we will adopt, are

$$\begin{gathered} a(t) \in C^{(2)}[0,\infty), \quad (-1)^j a^{(j)}(t) > 0, \quad j = 0,1,2 \\ a(t) = a_\infty + b(t), \quad a_\infty > 0, \quad b^{(j)} \in L_1(0,\infty). \end{gathered} \tag{4.4}$$

We will assume, in addition, that $t^j b^{(j)} \in L_1(0,\infty)$ so that, according to Lemma 3.1 $\hat{b} \in A$. We also assume $t^3 b \in L_1(0,\infty)$ which means that $\hat{b} \in C^{(3)}(\overline{\Pi})$.

Clearly $\hat{\mathcal{F}} \in C^3(\overline{\pi}\backslash\{0\})$. We have

$$\begin{gathered} s\hat{a}(s) = a(0) + \dot{a}(0)s + 0(s^{-2}), \\ \hat{\gamma}(s) = \frac{\sqrt{\rho_0}s}{\sqrt{a(0)}}\left(1 - \frac{1}{2}\frac{\dot{a}(0)}{a(0)s} + 0(s^{-2})\right) \quad \text{as } s \to \infty, \\ \hat{\mathcal{F}}(s) = -\sqrt{a(0)}s + \frac{1}{2}\frac{\dot{a}(0)}{\sqrt{a(0)}} + 0\left(\frac{1}{s}\right); \end{gathered} \tag{4.5}$$

$$\begin{gathered} s\hat{a}(s) = a_\infty + s\hat{b}(0) + 0(s^2) \\ \hat{\gamma}(s) = \frac{\sqrt{\rho_0}s}{\sqrt{a_\infty}} + 0(s^2) \quad \text{as } s \to 0. \\ \hat{\mathcal{F}}(s) = -\sqrt{a_\infty}s + 0(s^2). \end{gathered} \tag{4.6}$$

Equation (4.5) gives the relation (2.1) with $\alpha = -\sqrt{a(0)}$ and $\beta = -\dot{a}(0)/(2\sqrt{a(0)})$. Equation (4.6) shows that $\hat{\mathcal{F}}(0) = 0$. If the expansion in (4.6) is continued, it will show that $\hat{\mathcal{F}}$ is three times differentiable at $s = 0$.

It follows from results in [**8**] that conditions (4.4) imply

$$\operatorname{Re}\hat{a}(i\eta) > 0 \quad \text{for all } \eta, \qquad -\operatorname{sign}\eta \operatorname{Im}\hat{a}(i\eta) > 0, \quad \eta \neq 0. \tag{4.7}$$

We have $\hat{\mathcal{F}}(i\eta) = -i\eta\sqrt{i\eta\hat{a}(i\eta)}$. It follows from (4.7) that $\operatorname{Re}\sqrt{i\eta\hat{a}(i\eta)} > 0$, hence $\operatorname{sign}\eta \operatorname{Im}\hat{\mathcal{F}}(i\eta) > 0$ for $\eta \neq 0$. From $(4.6)_3$ we have

$\hat{\mathcal{F}}'(0) = -\sqrt{a_\infty} < 0$. Thus, all the conditions of Theorem 3.1 are satisfied.

REMARK 4.1. An interesting question is what happens if the composite bar is finite and viscoelastic on $L < x < \overline{L}$. We conjecture, but have not yet proved, that the resulting nonlocal condition is dissipative.

REMARK 4.2. A special case of the viscoelastic problem is that in which $a(t) \equiv a_\infty$, that is, the bar is elastic and homogeneous. This case was considered in [**6**] when the bar on $0 < x < L$ is nonlinearly elastic.

EXAMPLE 2. *Inhomogeneous semi-infinite bar.* This was the problem studied in [**1**]. We assume that the bar is ultimately homogeneous, i.e., $\rho(x) \equiv \rho_0, \sigma(x,t) = \mu(x)u_x(x,t)$ for $x \geq \overline{L}$. The outgoing condition is that $\mu_0 u_x(x,t) \equiv -\sqrt{u_0\rho_0}u_x(x,t)$ for $x \geq \overline{L}$, in particular, at $x = \overline{L}$. Thus the problem is

$$\begin{gathered} \rho(x)u_{tt}(x,t) = (\mu(x)u_x(x,t))_x, \quad 0 < x < L, \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = \varphi(t), \qquad \mu_0 u_x(\overline{L},t) = -\sqrt{\mu_0\rho_0}\, u_t(\overline{L},t). \end{gathered} \tag{4.8}$$

We want to reduce this to problem $(P(\varphi,\psi))$. Define $\hat{U}(x,s)$ by

$$\begin{gathered} \rho(x)s^2\hat{U}(x,s) = (\mu(x)\hat{U}_x(x,s))_x, \quad L < x < \overline{L}, \\ \hat{U}(L,s) = 1, \qquad \mu_0\hat{U}_x(\overline{L},s) = \sqrt{\mu_0\rho_0}\, s\, \hat{U}(\overline{L},s). \end{gathered} \tag{4.9}$$

Then the transform $\hat{u}$ of the solution of (4.8) satisfies $\hat{u}(x,s) = \hat{U}(x,s)\hat{u}(L,s)$ on $L \leq x \leq \overline{L}$. Thus,

$$\mu(L)\hat{u}_x(L,s) = \mu(L)\hat{U}_x(L,s)\hat{u}(L,s) = \hat{\mathcal{F}}(s)\hat{u}(L,s). \tag{4.10}$$

We will establish the following result.

THEOREM 4.1. *$\hat{\mathcal{F}}$, as defined by* (4.9)*, satisfies the hypotheses of Theorem* 3.1.

The proof is more complicated here since we do not have an explicit formula for $\hat{\mathcal{F}}$. We begin with

LEMMA 4.1. *Problem* (4.9) *has a unique solution for any* $s \in \overline{\Pi}$.

PROOF. To show that (4.9) has a solution for a given s, it suffices to show that the only solution if $\hat{U}(L,s) = 1$ is replaced by $\hat{U}(L,s) = 0$ is $\hat{U}(x,s) \equiv 0$ (see Appendix). Suppose $\hat{U}$ is such a solution. Then we have

$$s^2 \int_L^{\overline{L}} \rho |\hat{U}|^2 \, dx + \int_L^{\overline{L}} \mu |\hat{U}_x|^2 \, dx + \sqrt{\mu_0 \rho_0} \, s \, |\hat{U}(\overline{L}, s)^2| = 0. \tag{4.11}$$

If $s = \xi, \xi \geq 0$, one sees immediately from (4.11) that $\hat{U}(x,s) \equiv 0$. If $s = \xi + i\eta, \xi > 0$ we take the imaginary part of (4.11) and conclude $\int_L^{\overline{L}} \rho |\hat{U}|^2 dx = 0$, hence $\hat{U}(x,s) \equiv 0$. If $s = i\eta$, taking the imaginary part of (4.11) yields $\hat{U}(L, i\eta) = 0$. Hence $\hat{U}_x(L,s) = 0$ also. But $\rho s^2 \hat{U}(x,s) = (\mu \hat{U}_x(x,s)_x$, hence $\hat{U}(x,s)) \equiv 0$. □

LEMMA 4.2. $\hat{\mathcal{F}}$, *defined by* (4.10), *is in* $C^{(m)}(\overline{\Pi})$ *for any* m, *is analytic in* Π *and real for* s *real.*

PROOF. Suppose that one formally takes the derivative of $\hat{U}$ in (4.9) with respect to $\overline{s}$. Then one sees that $\hat{U}_{\overline{s}}$ satisfies the homogeneous problem, hence is zero, meaning $\hat{U}$ is analytic. This argument can be made rigorous by taking different quotients. Similarly, suppose one differentiates in (4.9) with respect to s. Then $\hat{V} = \hat{U}_s$ satisfies

$$\begin{gathered} \rho(x) s^2 \hat{V} - (\mu(x) \hat{V}_x)_x = -2s\rho \hat{U}(x,s) \\ \hat{V}(L,s) = 0, \quad \mu_0 \hat{V}_X(L,s) = -\sqrt{\mu_0\rho_0}\, s \, \hat{V}(L,s) - \sqrt{\mu_0\rho_0}\, \hat{U}(L,s). \end{gathered} \tag{4.12}$$

Once again the fact that the homogeneous problem has only the zero solution guarantees a solution of (4.12) in $\overline{\Pi}$. Again, one makes this rigorous with difference quotients and one can continue differentiating to obtain all derivatives. The reality on s real is immediate. □

LEMMA 4.3. $\hat{\mathcal{F}}$ *satisfies* $-\operatorname{sign} \eta \hat{\mathcal{F}}(i\eta) > 0$ *for* $\eta \neq 0$.

PROOF. We multiply the equation in (4.9) by $\overline{\hat{U}}(x,s)$ and integrate by parts, using (4.10) at $x = L$. This yields

$$
\begin{aligned}
&-\eta^2 \int_L^{\overline{L}} \rho |\hat{U}(x,i\eta)|^2 \, dx + \int_L^{\overline{L}} \mu(x) |\hat{U}_x(x,i\eta)|^2 + \sqrt{\mu_0\rho_0}\, i\eta |\hat{U}(\overline{L},i\eta)|^2 \\
&\qquad + \hat{\mathcal{F}}(i\eta)|\hat{U}(L,i\eta)|^2 = 0.
\end{aligned} \tag{4.13}
$$

We cannot have $\hat{U}(\overline{L},i\eta) = 0$, for then, as above, we would also have $\hat{U}_x(\overline{L},i\eta) = 0$ and, hence, $\hat{U}(x,s) \equiv 0$. The same argument shows that $\hat{U}(L,i\eta) \neq 0$, and the conclusion follows by taking the imaginary part of (4.13).

What remains is to study $\hat{\mathcal{F}}$ for large and small s. The large s situation was considered in [**1**]. What was found was that the solution of (4.9) has a formal asymptotic expansion,

$$
\hat{U}(x,s) \sim e^{-s\phi(x)} \sum_{k=0}^{\infty} U_k(x) s^{-k}, \quad \phi'(x) = \sqrt{\rho(x)\mu(x)}. \tag{4.14}
$$

Formulas were given to compute the coefficients U_k, recursively. Then formal differentiation of (4.14) yields

$$
\hat{\mathcal{F}}(s) = \mu(L)\hat{U}_x(L,s) \sim -\alpha s - \beta - \sum_{k=0}^{\infty} \alpha_k s^{-k}. \tag{4.15}
$$

The coefficients α, β and α_k are determined by values of ρ and μ and their derivatives at $x = L$. In particular, $\alpha = \sqrt{\rho_0\mu_0} > 0$. □

We will review this procedure briefly in the Appendix and will also establish its validity by the following results.

LEMMA 4.4. *Put* $\hat{\mathcal{F}}_N(s) = -\alpha_s - \beta - \sum_{k=0}^{N} \alpha_k s^{-k}$. *Then*

$$
\hat{\mathcal{F}}(s) - \hat{\mathcal{F}}_N(s) = 0(s^{-N}) \quad \text{for large } s. \tag{4.16}
$$

This result shows that $\hat{\mathcal{F}}$ has the correct behavior, for large s, to belong to A.

Let us consider the small s situation. This was not done in [**1**]. We seek a formal expansion of the solution of (4.9) as a power series in s:

$$\hat{U}(x,s) = \sum_{k=1}^{\infty} U_k(x) s^k. \tag{4.17}$$

The U_k can again be determined recursively. We write down the expressions for U_0 and U_1:

$$\begin{gathered} (\mu(x)U_0'(x))' = 0, \quad U_0(L) = 1, \quad \mu_0 U_0'(\overline{L}) = 0 \\ \mu(x)U_1'(x) = 0, \quad U_1(L) = 0, \quad \mu_0 U_1'(\overline{L}) = -\sqrt{\rho_0\mu_0}\, U_0(\overline{L}). \end{gathered} \tag{4.18}$$

One sees that $U_0(x) \equiv 1$, $U_1(x) = -\sqrt{\rho_0\mu_0}\int_L^x \mu(\xi)^{-1}\, d\xi$. Thus

$$\hat{\mathcal{F}}(s) = \mu(L)\hat{U}_x(L,x) = -\sqrt{\rho_0\mu_0}s + 0(s^2). \tag{4.19}$$

We will again establish the validity of (4.19) in the Appendix.

Equation (4.19) shows that

$$\hat{\mathcal{F}}(0) = 0, \quad \hat{\mathcal{F}}'(0) = -\sqrt{\rho_0\mu_0} < 0.$$

Thus we have established all the hypotheses of Theorem 3.1, and we have a dissipative boundary condition.

5. Approximate boundary conditions. The idea discussed in [**1**] is based on the formula (4.15). This idea is to truncate the series by using the $\hat{\mathcal{F}}_N$ of Lemma (4.4). If we translate back to the time domain, these correspond to nonlocal boundary conditions of the form

$$\begin{gathered} \mathcal{F}_N[\zeta^t] = -\frac{\partial}{\partial t}(\alpha + k_N * \zeta), \\ \alpha = \sqrt{\rho_0\mu_0}, \quad k_N(t) = \beta + \sum_{k=1}^{\infty} \frac{1}{k!}\, t^{k+1}. \end{gathered} \tag{5.1}$$

It is clear that (5.1) does not fit the form (2.1) since k_N is not in $L_1(0,\infty)$. It was shown in [**1**] that the use of $\mathcal{F}_N$ numerically produces exponential error growth. What was suggested in [**1**] was a stabilization procedure. This amounts to replacing $\hat{\mathcal{F}}_N$ by a function $\hat{\mathcal{L}}_N$ such that

$$\hat{\mathcal{L}}_N(s) - \hat{\mathcal{F}}_N(s) = 0(s^{-N-1}) \quad \text{as } s \to \infty, \tag{5.2}$$

but with the associated operator $\mathcal{L}_N$ stable. In the language of the present paper this means we want $\hat{\mathcal{L}}_N$ to satisfy the conditions of Theorem 3.1.

We illustrate the idea of [**1**] in the case $N = 1$. In order for our idea to work it is essential that the constant β in (4.15) be positive. The calculations in [**1**] show that

$$\beta = \frac{1}{4\rho(L)}(\rho\mu)'(L).$$

Thus we must assume that the bar is such that the product $\rho\mu$ is increasing. We set

$$\hat{\mathcal{L}}_1(s) = -\alpha s - \beta - \frac{\alpha_1}{s+\delta}. \tag{5.3}$$

Thus $\hat{\mathcal{L}}_1(s) - \hat{\mathcal{F}}_1(s) = 0(s^{-2})$ as $s \to \infty$ for any δ. We have

$$\hat{\mathcal{L}}_1(0) = -\beta - \frac{\alpha_1}{\delta}, \qquad \mathcal{L}_1'(0) = -\alpha + \frac{\alpha_1}{\delta^2}.$$

If we choose $\delta > 0$ so that

$$\delta > \max\left(\frac{|\alpha_1|}{\beta}, \frac{|\alpha_1|^{1/2}}{\alpha^{1/2}}\right), \tag{5.4}$$

then $\hat{\mathcal{L}}_1(0)$ and $\hat{\mathcal{L}}_1'(0)$ will both be negative. Furthermore, for $\eta > 0$,

$$\operatorname{Im}\hat{\mathcal{L}}_1(i\eta) = -i\eta\left(\alpha - \frac{\alpha_1}{\eta^2+\delta^2}\right) < -\eta\left(\alpha - \frac{\alpha_1}{\delta^2}\right).$$

Thus (5.4) also insures that $-\operatorname{sign}\eta \operatorname{Im}\hat{\mathcal{L}}_1(i\eta) > 0$ for $\eta \neq 0$.

The choice (5.4) thus guarantees that the operator $\mathcal{L}_1$ associated with $\hat{\mathcal{L}}_1$ satisfies (H_1) and (H_2) so that Theorems 2.1 and 2.3 apply. We see that the operator $\mathcal{L}_1$ is given by

$$\mathcal{L}_1[\zeta^t] = -\alpha\,\dot{\zeta}(t) - \beta\zeta(t) - \alpha_1\int_0^t e^{-\delta(t-\tau)}\zeta(\tau)\,d\tau. \tag{5.5}$$

It was observed in [**1**] that the boundary condition at $x = L$ can be localized. In the transform domain,

$$\mu(L)\hat{u}_x(L,s) = -\left(\alpha s + \beta + \frac{\alpha_1}{s+\delta}\right)\hat{u}(L,s) + \hat{\chi}(s)$$

or

$$(s+\delta)\mu(L)\hat{u}_x(L,s) = [-(\alpha s+\beta)(s+\delta)+\alpha_1]\hat{u}(L,s) + (s+\delta)\hat{\chi}(s).$$

Thus,

(5.6)
$$\begin{aligned}\mu(L)u_{xt}(L,t) + \delta\mu(L)u_x(L,t) = &-\alpha u_{tt}(L,t) - (\alpha\delta+\beta)u_t(L,t)\\ &+ (\alpha_1+\beta\delta)u(L,t) + \dot{\chi}(t) + \delta_{\chi}(t).\end{aligned}$$

It is shown in [**1**] how to implement (5.6) in a Galerkin method procedure.

We note that, although $\mathcal{L}_1$ satisfies (H_1) and (H_2), we have $\hat{\mathcal{L}}_1(0) = -\beta - \alpha_1/\delta$ which is not, in general, zero. Thus Theorems 2.3 and 2.4 do not apply. We give here an alternative approximation. Define $\hat{\mathcal{H}}_1$ by

$$\hat{\mathcal{H}}_1(s) = s\left\{-\alpha - \frac{\beta}{s+\delta} + \frac{\alpha_1-\delta\beta}{(s+\delta)^2}\right\}. \tag{5.7}$$

We have $\mathcal{H}_1(s) - \hat{\mathcal{F}}_1(s) = 0(s^{-2})$ as $s\to\infty$. We also have $\hat{\mathcal{H}}_1(0) = 0$ and $\mathcal{H}_1'(0) = -\alpha < 0$. We assert that, if δ is chosen sufficiently large, we will have

$$-\operatorname{sign}\eta\; \hat{\mathcal{H}}_1(i\eta) > 0 \quad \text{for } \eta \neq 0. \tag{5.8}$$

We have

$$\begin{aligned}\operatorname{Im}\hat{\mathcal{H}}_1(i\eta) &= \eta\left\{-\alpha - \frac{\beta\delta}{\delta^2+\eta^2} + \frac{(\delta_1-\delta\beta)(\delta^2-\eta^2)}{(\delta^2-\eta^2)^2+4\delta^2\eta^2}\right\}\\ &= \eta\left\{-\alpha - \frac{\beta\delta}{\delta^2+\eta^2} + \frac{(\alpha_1-\delta\beta)}{\delta^2}\left[\frac{1-(\eta/\delta)^2}{((1-\eta/\delta)^2)^2+4(\eta/\delta)^2}\right]\right\}.\end{aligned}$$

The quantity in square brackets is bounded for all η/δ and our conclusion follows. Relation (5.7) translates into a condition like (5.6) in the time domain. Both $\mathcal{L}$ and $\mathcal{H}$ have generalizations to larger values of N, but we will not write these explicitly.

Let us summarize our conclusions. The method of [**1**] yields a sequence of approximate boundary conditions $\mathcal{F}_N$ satisfying the estimate (5.3). These satisfy the hypotheses of Theorem 2.1 so that, if they are used in $(P(\varphi,\psi))$, one has stability. They do not satisfy $\hat{\mathcal{F}}_N(0)=0$, so one does not have the dissipation result of Theorem 2.3. The idea used for $\hat{\mathcal{H}}_1$ can be extended to yield a sequence also satisfying (5.3) but with $\hat{\mathcal{H}}_N(0)=0$, so that the hypotheses of Theorem 2.3 are satisfied and one obtains the decay result. All of these in the time domain can be expanded in differential forms like (5.6) but with higher order time derivatives.

We can use our results to give some indications of the validity of the approximate boundary conditions. Suppose u_N and v_N are solutions of $(P(\varphi,\psi))$ with conditions $\mathcal{L}_N$ and $\mathcal{H}_N$, respectively, at $x=L$, and let u be the solution of $(P(\varphi,\psi))$ with the exact condition $\mathcal{F}$ at $x=L$. Let $U^N=u-u_N$ and $V^N=u-v_N$ represent errors. Thus we will have

$$
\begin{gathered}
\rho U^N_{tt}=(\mu U^N_x)_x,\quad U^N(x,0)=U^N_t(x,0)=0,\quad 0<x<L,\\
U^N(0,t)=0,\quad \mu(L)U^N_x(L,t)=\mathcal{L}_N[U^N(L,\cdot)^t]+\psi^N(t)\\
\psi^N(t)=(\mathcal{F}-\mathcal{L}_N)(u^t(L,\cdot))
\end{gathered}
\tag{5.9}
$$

$$
\begin{gathered}
\rho V^N_{tt}=(\mu V^N_x)_x,\quad 0<x<L,\quad V^N(x,0)=V^N_t(x,0)=0\\
V^N(0,t)=0,\quad \mu^N(L)V^N_v(L,t)=\mathcal{H}_N[V^N(L,\cdot)^t]+\Psi^N(t)\\
\Psi^N(t)=(\mathcal{F}-\mathcal{H}_N)(u^t(L,\cdot)).
\end{gathered}
\tag{5.10}
$$

The problems for U^N and V^N are both of the form $P(0,\psi)$ but with different functionals at $x=L$. We can accordingly use Theorem 2.1 for U^N and Theorems 2.2, 2.3 for V^N.

We will establish the following result in the Appendix.

LEMMA 5.1. *For each integer N, there is a constant M_N such that, for any*

$$
\begin{gathered}
\zeta\in C^{(1)}([0,\infty)),\quad \text{with } \zeta(0)=0 \quad\text{and}\quad \dot{\zeta}\in L_2(0,\infty),\\
\|\mathcal{F}[\zeta^t]-\mathcal{H}_N[\zeta^t]\|_{H^{N+1}(0,\infty)}\le M_N\|\dot{\zeta}\|_{L_2(0,\infty)}.
\end{gathered}
\tag{5.11}
$$

Suppose now that u is a solution of (4.8). Then it will be a solution of $P(\varphi, 0)$ with $\mathcal{F}$ defined by (4.10). We assume $||(\varphi, 0)||_0$ exists, then Theorem 2.1 yields $||u_t(L, \cdot)||_{L_2(0,\infty)} \leq M||(\varphi, 0)||_0$. It follows from (5.9)–(5.10) that there is a constant M'_N such that

$$||\Psi^N||_{H^{N+1}(0,\infty)} \leq M'_N||(\varphi, 0)||_0.$$

We can now apply Theorem 2.2, with the device of differentiating with respect to t, to obtain the result we want.

THEOREM 5.1. *Suppose* $||(\varphi, 0)||_0$ *exists. Then, for any integer* N, *there is a constant* L_N *such that*

$$||u - v^N||_{H^{N+2}((0,\infty):L_2(0,L))} + ||u - v^N||_{H^{N+1}((0,\infty):H_1(0,L))} \leq L_N||(\varphi, 0)||_0. \tag{5.12}$$

REMARK 5.1. We want to comment on (5.12). It is an error estimate but not of a usual type. It does not say that the errors, $u - v^N$, become small as N becomes large. What it does say is that these errors become small as t becomes large. Equation (5.12) implies that

$$\begin{gathered}
\left\|\frac{\partial^j}{\partial t^j}(u(\cdot, t) - v^N(0, t))\right\|_{L_2(0,L)} \to 0 \quad \text{for } j = 0, 1, N{+}1 \quad \text{as } t \to \infty, \\
\left\|\frac{\partial^j}{\partial t^j}(u(\cdot, t) - v^N(\cdot, t))\right\|_{H_1(0,L)} \to 0 \quad \text{for } j = 0, \dots, N \quad \text{as } t \to \infty, \\
\frac{\partial^j}{\partial t^j}(u(x, t) - v^N(x, t)) \to 0 \quad \text{for each } x, y = 0, \dots, N \quad \text{as } t \to \infty.
\end{gathered} \tag{5.13}$$

Thus, increasing N makes an increasing number of time derivatives go to zero.

The above result is not too striking since both u and v^n are going to zero anyway. What makes it more striking is the result in Remark 2.3. Thus, if $\varphi(t)$ is tending to φ_∞ so that u tends to a steady state, the estimates (5.11) will still hold. This is the crucial role of the condition $\hat{\mathcal{F}}(0) = 0$. We cannot draw the same conclusions for the approximations $\mathcal{L}_N$.

Appendix

ASYMPTOTIC EXPANSIONS

We consider the problem in Section 3, which was

$$\text{(A.1)}\qquad \begin{gathered} \rho s^2 \hat{U} = (\mu \hat{U}_x)_x, \quad L < x < \overline{L}, \\ \overline{U}(L,s) = 1, \quad \hat{U}_x(L,s) = -\sqrt{\frac{\rho_0}{\mu_0}}\, s\hat{U}(L,s), \end{gathered}$$

where $\rho(L) = \rho_0, \mu(L) = \mu_0$, and we assume all derivatives of ρ and μ are zero at $x = L$. In [**1**] we derived an asymptotic expansion for large s. It has the form
(A.2)

$$\hat{U}(x,s) \sim e^{-s\phi(x)} \sum_{k=0}^{\infty} U_k(x) s^{-k}, \quad \phi'(x) = \sqrt{\rho(x)\mu(x)}, \quad \phi(L) = 0.$$

The functions U_k are determined recursively by the formulas

$$\text{(A.3)}\qquad \begin{gathered} 2\mu\phi' U_0' + (\mu\phi')' U_0 = 0, \quad U_0(L) = 1 \\ 2\mu\phi' U_{k+1}' + (\mu\phi')' U_{k+1} = (\mu U_k')', \quad U_{k+1}(L) = 0, \quad k \geq 0. \end{gathered}$$

Recall that the quantity we want is $\hat{\mathcal{F}}(s) = \hat{U}_x(L,s)$. From (A.2) and (A.3) one obtains, formally,

$$\text{(A.4)}\quad \hat{\mathcal{F}}(s) \sim \mu(L)\left\{-s\phi'(L) + \sum_{h=0}^{\infty} U_k'(L) s^{-k}\right\} = -\alpha s - \beta - \sum_{k=1}^{\infty} \alpha_k s^{-k}.$$

We see that $\alpha = \phi'(L) = \sqrt{\rho(L)\mu(L)}$. The other coefficients can be computed by using (A.3) recursively to determine $U_k'(L)$. In particular,

$$\beta = -\mu(L) U_0'(L) = \frac{1}{2}\,\frac{(\mu\phi')'}{\phi'} = \frac{1}{4\rho(L)}(\rho'(L)\,\mu(L) + \rho(L)\,\mu'(L)).$$

In order to prove Lemma (4.4) we need first to show that (A.2) is a valid asymptotic expansion. Put

$$\hat{U}^N(x,s) = e^{-s\phi(x)} \sum_{k=0}^{N} U_k(x) s^{-k}.$$

Then one verifies that

$$s^2\rho\hat{U}^N - (\mu\,\hat{U}_x^N)_x = -\,(\mu\,U_N')' s^{-N}. \tag{A.5}$$

We assert that, for any N, one has $\hat{U}_x^N(\overline{L}, s) = -\sqrt{(\rho_0 s/\mu_0)}U^N(\overline{L}, s)$. The reason for this is that all derivatives of any of the U^N vanish at $x = \overline{L}$. To see this consider the equation for $\hat{U}_0$ in (A.2). Since $(\mu\phi')' = 0$ at $x = \overline{L}$ we have $U_0'(\overline{L}) = 0$. Differentiating that equation repeatedly, we see that $U_0^{(j)}(\overline{L}) = 0$ for all j. The equations (A.3), for $k > 0$, then show that all derivatives vanish at $x = \overline{L}$.

We set $V^N = \hat{U} - \hat{U}^N$ and have

$$\begin{aligned} &\rho s^2 V^N - (\mu V_x^N)_x = f^N, \quad f^N = -\,(\mu U_n')' s^{-N} \\ &V^N(L, s) = 0, \quad V_x^N(\overline{L}, s) = -\sqrt{\frac{\rho_0}{\mu_0}}\, s\, V^N(L, s). \end{aligned} \tag{A.6}$$

We need estimates for solutions of (A.6). These are easier if we first make a Louisville transformation

$$t = \int_L^x \sqrt{\frac{\rho(\xi)}{\mu(\xi)}}\, d\xi, \quad V^N(x, s) = \frac{1}{(\rho\mu)^{1/4}} w^N(t(x), s). \tag{A.7}$$

It is not difficult to see that this transforms (A.6) into

$$s^2 w^N - w_{tt}^N + q(t)w^N = F^N \tag{A.8}$$

$$w^N(0, s) = 0, \quad w_x^N(\overline{T}, s) = -\,s w^N(\overline{T}, s), \quad \overline{T} = \int_L^{\overline{L}} \sqrt{\frac{\rho(\xi)}{\mu(\xi)}}\, d\xi.$$

Consider the problem

$$\begin{gathered} Z_{tt} - s^2 Z = h, \quad 0 < t < \overline{T}, \\ Z(0) = 0, \quad Z'(\overline{L}) = -\,sZ(L). \end{gathered}$$

One verifies that the solution is

$$\begin{aligned} Z(t) &= \frac{1}{2}\int_0^{\overline{T}} G(t, \tau, s) h(\tau)\, d\tau \\ G(t, \tau, s) &= \begin{cases} -\,\frac{1}{2}\, e^{-s(t-\tau)} + \frac{1}{2}\, e^{-s(t+\tau)}, & 0 < \tau < t, \\ -\,\frac{1}{2}\, e^{-s(t-\tau)} - \frac{1}{2}\, e^{-s(T+\tau)}, & t < \tau < \overline{T}. \end{cases} \end{aligned} \tag{A.9}$$

Note that we have, for some $M > 0$,

$$|G(t,\tau,s)| \le M, \quad |G_t(t,\tau,s)| \le M|s| \quad \text{for any } s \in \overline{\Pi}. \tag{A.10}$$

Now (A.8) is equivalent to the integral equation

(A.11)

$$W^N(t,s) = \frac{1}{s}\int_0^{\overline{T}} g(t,\tau : s)q(\tau)W^N(\tau,s)\,d\tau + \frac{1}{2}\int_0^{\overline{T}} G(t,\tau : s)F^N(t)\,d\tau.$$

Remark A.1. The equivalence of (A.8) and the integral equation (A.11) confirm the statement in the proof of Lemma 4.1 that uniqueness implies existence.

In view of the bound (A.10) we see that (A.11) can be solved by successive approximatives for $|s|$ sufficiently large. This shows that there is an $s_0 > 0$ and $P > 0$ such that

$$||W^N(\cdot,s)||_{L_\infty(0,\overline{T})} \le \frac{P}{|s|}||F^N(\cdot)||_{L_\infty(0,\overline{T})} \quad \text{for } |s| \ge s_0. \tag{A.12}$$

We can obtain an estimate for W_t^N by differentiating (A.11) and using (A.12). This yields

$$||W_t^N(\cdot,s)||_{L_\infty(0,\overline{T})} \le Q||F^N(\cdot)||_{L_\infty(0,\overline{T})}. \tag{A.13}$$

It we translate (A.13) back to the original variables, one obtains the result (4.16).

Proof of Lemma 5.1. Since $\hat{\mathcal{H}}_N$ differs from $\hat{\mathcal{F}}_N$ by terms of order s^{-N-1} as $s \to \infty$, we have, from (A.13),

$$|\hat{\mathcal{F}}(s) - \hat{\mathcal{H}}_N(s)| \le Q|s|^{-N} \quad \text{for large } s.$$

We also have $\hat{\mathcal{F}}(s) - \hat{\mathcal{H}}_N(s) = 0(s)$ for small s. Since the difference $\hat{\mathcal{F}} - \hat{\mathcal{H}}_N$ is bounded on any compact set in $\overline{\Pi}$, we conclude that, for some constant M_N,

$$|\hat{\mathcal{F}}(s) - \hat{\mathcal{H}}_N(s)| \le \frac{M_N|s|}{(1+|s|)^{N+1}} \quad \text{for all } s \in \overline{\Pi}. \tag{A.14}$$

It follows that, for any $\zeta \in C^1[0,\infty), \zeta(0) = 0$ and $\dot{\zeta} \in L_2(0,\infty)$,

$$||\mathcal{F}[\zeta^t] - \mathcal{H}_N[\zeta^t]||_{H^{N+1}(0,\infty)} = \int_{-\infty}^{+\infty} |\hat{\mathcal{F}}(i\eta) - \hat{\mathcal{H}}_N(i\eta)|^2 \, d\eta$$
$$\leq M_N \int_{-\infty}^{+\infty} |\hat{\zeta}(i\eta)|^2 \, d\eta = M_N ||\zeta||_{L_2(0,\infty)},$$

which is (5.9).

We also comment on the small s approximation. One proceeds in a way quite similar to the above proof. Set up the formal series and truncate to get an approximate solution $\hat{U}_N$. Then the error $\hat{U} - \hat{U}^N$ will satisfy a problem which, after Louisville transformation, is equivalent to an integral equation which can be solved by successive approximations for s small.

REFERENCES

1. A. Barry, J. Bielak and R.C. MacCamy, *On absorbing boundary conditions for wave propagation,* J. Comput. Phys. **79** (1988), 449–468.

2. G. Chen, *Control and stabilization for the wave equations in a bounded domain,* SIAM J. Control Optim. **17** (1979), 66–81.

3. B. Engquist and L. Halpern, *Far field boundary conditions for computation over a long long time,* Appl. Numer. Math. **79** (1988), 21–45.

4. B. Engquist and A. Majda, *Absorbing boundary conditions for the numerical simulation of waves,* Math. Comp. **31** (1979), 629–651.

5. ———, *Radiation boundary conditions for acoustic and elastic wave computation,* Comm. Pure Appl. Math. **32** (1979), 313–357.

6. J.M. Greenberg and L.A. Tsien, *The effect of boundary damping for the quasilinear wave equation,* J. Differential Equations **52** (1984), 66–75.

7. W.J. Hrusa, J.A. Nohel and M. Renardy, *Mathematical problems in viscoelasticity,* John Wiley & Sons, New York, 1987.

8. J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation,* J. Differential Equations **50** (1983), 163–182.

9. R.C. MacCamy, *Absorbing boundaries for viscoelasticity*, in *Viscoelasticity and rheology* (A.S. Lodge, M. Renardy, and J. Nohel, eds.), Academic Press, 1985.

10. R.C. MacCamy and J.S.W. Wong, *Stability theorems for some functional equation,* Trans. Amer. Math. Soc. **164** (1972), 1–37.

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