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HOW TO SOLVE HAMMERSTEIN EQUATIONS

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To Ken, with friendship and admiration

What is a good method for finding solutions $u : \Omega \to \mathbf{R}$ *of the nonlinear integral equation of Hammerstein type*

$$u(x) = v(x) + \lambda \int_\Omega k(x,y) f(y, u(y))\, dy, \quad \lambda \in \mathbf{R} \tag{1}$$

with given functions $v : \Omega \to \mathbf{R}$*,* $k : \Omega \times \Omega \to \mathbf{R}$*, and* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *?* This is the question which provides the main focus of this brief note. More precisely, we will discuss a variety of methods (topological degree, fixed point methods, spectral theory, variational approach, monotonicity methods, positivity methods, etc.) which turn out to be useful tools for solving (1). We point out that the presentation is *quite elementary*, so this note may be considered as a stimulation for exercises for students attending courses in nonlinear analysis, operator theory, or integral equations, rather than a sophisticated research contribution.

Usually, equation (1) is written as an operator equation

$$u - \lambda A u = v, \tag{2}$$

where the Hammerstein operator A may be represented as composition $A = KF$ of the linear Fredholm operator

$$Ku(x) = \int_\Omega k(x,y) u(y)\, dy \tag{3}$$

generated by the kernel function k, and the nonlinear Nemytskij operator

$$Fu(x) = f(x, u(x)) \tag{4}$$

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generated by the nonlinearity f. For simplicity, we assume that Ω is a compact domain in the Euclidean space $\mathbf{R}^n$ in what follows.

1. Topological methods. The simplest topological method is fixed point theory or, more generally, degree theory. Of course, the right function space for studying (1) is essentially determined by the properties of the kernel function k and the nonlinearity f. If both k and f are continuous, one could try to use Schauder's theorem in the space C of continuous functions. If f is (even locally) Lipschitz continuous or smooth, it is worthwhile to try (a local version of) the Banach contraction mapping principle in the space C^k. On the other hand, if k is only measurable, a useful device is using the Lebesgue space L_p (in case f has polynomial growth), or an Orlicz space L_M (in case f has nonpolynomial, e.g., exponential growth, see [**10**]).

Choosing the "right" space depends not only on the properties of the data k and f, but also on the result we are interested in. So, one should prove *existence* of solutions in a possibly narrow space, but *uniqueness* of solutions in a possibly large space. If one wants to prove both existence and uniqueness, one has to find some kind of "compromise." Moreover, sometimes it is necessary (or just useful) to pass from the operator equation (2) to an equivalent operator equation in a "better" space.

We illustrate this by means of the following *equivalence principle* in Lebesgue spaces. Suppose that the operator (4) maps L_p into L_q, while the operator (3) maps L_q into L_p, where $(1p+1)/q = 1$, so one may study (2) as an operator equation (in particular, a fixed point equation for $v = 0$) in L_p. However, if we succeed in decomposing the kernel function k in the form

$$k(x,y) = \int_\Omega m(x,z)m(z,y)\,dz, \tag{5}$$

where the integral operator M generated by m maps L_2 into L_p, then its adjoint M^* maps L_q into L_2, where $MM^* = K$, and equation (2) is equivalent to the operator equation

$$h - \lambda G h = w, \tag{6}$$

where $u = Mh$, $G = M^*FM$, and $v = Mw$, and the equivalence of equations (2) and (6) follows from the invertibility of M. Now, what we have gained in this way is that the operator G in (5) acts in the *Hilbert space* L_2 rather than in the Banach space L_p. This makes life easier in many applications (see below).

Using Schauder's fixed point theorem (or one of its numerous generalizations), one still has the problem of finding bounded, closed, and convex subsets (typically closed balls) that are invariant under A. One may overcome this difficulty by applying Vignoli's fixed point theorem [**13**] which states that a compact operator A has a fixed point if its *quasinorm*

$$[A]_\infty := \limsup_{||u||\to\infty} \frac{||Au||}{||u||}$$

is less than 1. We illustrate the advantage of this condition by means of a very simple example.

Example 1. For $\alpha > 0$, consider the equation

$$u(x) = v(x) + \lambda \int_0^1 e^{x-y} \log(1 + |u(y)|^\alpha)\, dy, \quad \lambda \in \mathbf{R} \tag{7}$$

which is (1) with $\Omega = [0,1]$, $k(x,y) = e^{x-y}$ and $f(x,u) = \log(1 + |u|^\alpha)$. It follows from standard calculations that the operator (3) generated by k has norm $||K|| = e - 1$ in the space $C[0,1]$. On the other hand, both the Lipschitz continuity and the growth of the operator (4) generated by f depend on α. In case $\alpha < 1$ we have $f'(u) \to \infty$ as $u \to 0^+$, and so f cannot satisfy a Lipschitz condition. In case $\alpha = 1$ we have $|f'(u)| \le 1$, and this is the best possible Lipschitz constant. In case $\alpha > 1$, an easy calculation shows that the best possible Lipschitz constant for f is $(\alpha-1)^{(\alpha-1)/\alpha}$. So the Banach contraction mapping principle guarantees *existence and uniqueness* for

$$|\lambda| < \frac{1}{(e-1)(\alpha-1)^{(\alpha-1)/\alpha}}. \tag{8}$$

If we are merely interested in existence and want to apply Schauder's theorem, we have to find invariant balls, which usually requires the imposition of growth restrictions on the nonlinearity f. In this example these restrictions are rather mild, since $|f(u)| = \log(1 + |u|^\alpha) \le c|u|^\beta$

for *any* $\beta > 0$. So the closed ball in $C[0,1]$ of radius R is left invariant for all values of λ satisfying

$$||v|| + |\lambda|(e-1)\, cR^{\beta} \leq R, \tag{9}$$

and this may always be achieved by taking $\beta < 1$ and choosing R sufficiently large.

However, we can do much better by employing Vignoli's fixed point theorem. In fact, the Nemytskij operator (4) generated by f has quasinorm $[F]_\infty = 0$ for every value of $\alpha > 0$, and so Vignoli's theorem applies. Consequently, we are led to the following scheme concerning existence and uniqueness of solutions of (7). Of course, one may also apply more sophisticated methods like Darbo's [**4**] or Sadovskij's [**11**] fixed point principle for maps which are condensing with respect to some measure of noncompactness. □

TABLE 1. Solvability of equation (7).

	Banach	*Schauder*	*Vignoli*
$\alpha < 1$	not applicable	existence for λ as in (9)	existence for all λ
$\alpha = 1$	existence and uniqueness for $\lvert\lambda\rvert < (e-1)^{-1} \approx 0.582$	existence for λ as in (9)	existence for all λ
$\alpha > 1$	existence and uniqueness for λ as in (8)	existence for λ as in (9)	existence for all λ

2. Monotonicity methods. Recall that an operator A from a Banach space X into its dual X^* is called *monotone* if $\langle Au - Av, u - v\rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ denotes the duality between X and X^*. The most important existence result for monotone operators is Minty's celebrated theorem [**8**] which states that, in case of a reflexive space X, every monotone continuous operator $A : X \to X^*$ which is *coercive*, in the sense that

$$\lim_{||u|| \to \infty} \frac{\langle Au, u\rangle}{||u||} = \infty$$

is *onto*, i.e., satisfies $A(X) = X^*$.

In order to apply this theorem to equation (2) in the space L_p for $1 < p < \infty$, say, we have to impose suitable sign and growth conditions on k and f. For example, the following is true:

Theorem 1. *Suppose that k satisfies*

$$\langle Ku, u\rangle = \int_\Omega \left\{ \int_\Omega k(x,y)u(y)\,dy \right\} u(x)\,dx \geq 0,$$

and f satisfies

$$|f(x,u)| \leq a(x) + b|u|^{p-1}, \quad f(x,u)u \geq d|u|^p. \tag{10}$$

Assume, in addition, that either $f(x,\cdot)$ is increasing or K is compact. Then equation (1) *has a solution $u \in L_p$ for $v = 0$.*

We remark that the nonlinearity $f(u) = \log(1+|u|^\alpha)$ from Example 1 satisfies the first estimate in (10), but not the second one. More general hypotheses on k and f which ensure the applicability of Minty's theorem to equation (1) may be found in Section 28.4 of the monograph [**14**].

3. Variational methods. Loosely speaking, the idea of the classical calculus of variations consists in reducing the problem of solving the operator equation $Au = 0$ to the search for critical points of a corresponding *potential* J of $A = \nabla J$. A crucial condition on J is the *Palais-Smale condition* which may be formulated as follows. Any sequence $(h_n)_n$ with the property that

$$\limsup_{n\to\infty} |J(h_n)| < \infty, \quad \lim_{n\to\infty} \nabla J(h_n) = 0$$

is *compact*, i.e., admits a convergent subsequence. The famous *Mountain Pass lemma* [**1**], see also [**9**], states that every functional J on a Hilbert space H satisfying the Palais-Smale condition and $J(0) = 0$ has a critical point, provided that

$$\inf_{||h||=r} J(h) > 0, \quad \inf_{||h||>r} J(h) \leq 0$$

for some $r > 0$. In order to apply this to equation (2) in the Lebesgue space L_p, we have to consider first a potential for the Nemytskij operator (4), viz. the *Golomb functional*

$$\Gamma(h) := \int_\Omega \left\{ \int_0^{Mh(x)} f(x,u)\,du \right\} dx$$

in the Lebesgue space L_2, where M denotes again the integral operator defined by the factorization (5). With this notation, the solutions of equation (2) may then be searched for as critical points of the functional

$$J(h) = \frac{1}{2}\,||h||^2 - \Gamma(h), \quad h \in L_2. \tag{11}$$

A sample result in this spirit due to Tersian and Zabrejko [**12**] reads as follows.

Theorem 2. *Suppose that f satisfies the condition*

$$\int_0^v f(x,u)\,du \le \mu v f(x,v) + c(x), \quad c \in L_1, \quad \frac{1}{p} \le \mu < \frac{1}{2}$$

and admits a representation of the form $f(x,u) = b(x)u + \omega(x,u)$, where

$$|\omega(x,u)| \le u^2 g(x,u) + \delta|u|, \tag{12}$$

and the Nemytskij operator $Gu(x) = g(x,u(x))$ generated by g maps L_p into L_{p-2}. Assume, in addition, that the spectrum $\sigma(KB)$, with $Bu(x) = b(x)u(x)$ being the multiplication operator defined by b, does not meet the interval $[1,\infty)$, and that $\delta||K||\,||(I - M^*BM)^{-1}|| < 1$, where δ is the constant from (12)*. Then the functional J defined by* (11) *satisfies a Palais-Smale condition and has a critical point.*

4. Positivity methods. Sometimes it is interesting to find *positive solutions* of the integral equation (1). In this situation one usually studies the operator equation (2) in some *space with cone*, i.e., with a convex subset C such that $tu \in C$ for $u \in C$ and $t > 0$. We restrict ourselves to a classical result on *cone-compressing operators* in L_p due to Krasnosel'skij [**6**]. Again, we consider the factorization (5) of the kernel function k.

Theorem 3. *With* m *as in* (5), *let*

$$\alpha := \inf_x \int_\Omega m(x,y)\,dy, \quad \beta := \sup_x \int_\Omega |m(x,y)|^2\,dy, \quad \gamma := \sup_{x,y} m(x,y). \tag{13}$$

Suppose that the nonlinearity f *satisfies the asymptotic conditions*

$$\lim_{u\to 0} \frac{f(x,u)}{u} < \frac{1}{\beta}, \quad \lim_{u\to\infty} \frac{f(x,u)}{u} > \frac{\gamma^2}{\alpha^4}. \tag{14}$$

Then equation (1) *has a positive solution.*

Example 2. Consider the same equation as in Example 1, but now with $f(u) = \log(1+|u|^\alpha)$ replaced by $g(u) = u\log(1+|u|^\alpha)$ (and $v = 0$). The factorization (5) becomes trivial for $k(x,y) = e^{x-y}$, since $m(x,y) = k(x,y)$. So for the constants in (13) we get here

$$\alpha = 1 - \frac{1}{e}, \quad \beta = \frac{1}{2}\,(e^2-1), \quad \gamma = e.$$

Since $\log(1+|u|^\alpha) \to 0$, respectively ∞, as $u \to 0$, respectively ∞, the estimates (14) are trivially satisfied. So we deduce from Theorem 3 that the equation

$$u(x) = \lambda \int_0^1 e^{x-y} u(y) \log(1+|u(y)|^\alpha)\,dy, \quad \lambda \in \mathbf{R}$$

has a positive solution. $\square$

TABLE 2. How to solve Hammerstein equations.

Properties of k	*Properties of* f	*Space*	*Method*
continuous	continuous	C	Schauder theorem
continuous	Lip-continuous	C	Banach theorem
measurable	Carathéodory polynomial growth	L_p $(1 \leq p \leq \infty)$	Schauder theorem Darbo theorem
measurable	Carathéodory nonpolynomial growth	L_M	Schauder theorem Darbo theorem
positive	monotone, coercive polynomial growth	L_p $(1 < p < \infty)$	Minty theorem
positive	monotone, coercive nonpolynomial growth	L_M $(M \in \Delta_2)$	Minty theorem
measurable spectral condition	potential sign condition	L_p $(2 < p < \infty)$	Mountain Pass lemma
positive	strictly positive asymptotic condition	L_p $(1 \leq p < \infty)$	Krasnosel'skij theorem

5. Other methods. It goes without saying that there are many more methods to derive existence, uniqueness, and multiplicity results for solutions of equation (1). For example, in case of a more complicated dependence of equation (1) on the real parameter λ, one could try to apply the classical *implicit function theorem* or its numerous generalizations; for some nonclassical results in this spirit we refer to the survey [**3**]. Moreover, in recent years *spectral methods* for nonlinear operators have found increasing attention in view of their applicability to integral equations and boundary value problems, see Chapter 12 of the recent book [**2**].

We summarize our discussion in the following table which describes how to find the right method (last column) in an appropriate function space (last but one) for solving a Hammerstein equation with given kernel and nonlinearity.

The theorems stated above may be applied in the usual way to get existence results for nontrivial solutions to boundary value problems for elliptic equations or systems. For example, this refers to the classical semi-linear equation

$$-\Delta u(x) + a(x)u(x) = f(x, u(x)), \quad x \in \Omega, \tag{15}$$

subject to Dirichlet boundary conditions, where f is a Carathéodory function satisfying $f(x,0) \equiv 0$. It is well-known, see, e.g., [**5, 7**], that the linear problem

$$-\Delta u(x) + a(x)u(x) = f(x), \quad x \in \Omega,$$

with Dirichlet boundary condition has, for nonnegative $a \in C(\Omega)$, a unique generalized solution $u = Kf$, where the integral operator K maps the Sobolev space $H^{-1}(\Omega)$ into the Sobolev space $H_0^1(\Omega)$ and is bounded. By the classical Sobolev imbedding lemma, the operator K is then also bounded between $L_q(\Omega)$ and $L_p(\Omega)$, where we may choose $2 < p \leq \infty$ if $n = 1$, $2 < p < \infty$ if $n = 2$, and $2 < p < 2n/(n-2)$ if $n \geq 3$. Consequently, one may apply all results formulated above in the setting of Lebesgue spaces to get existence of generalized solutions of equation (15). Moreover, one may also find additional smoothness conditions on the function f in (4) to ensure that all generalized solutions are actually classical solutions.

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