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ORTHOGONAL POLYNOMIALS FOR THE SOLUTION OF SEMI LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We present an integral equation method for the solution of a class of nonlinear two-point boundary value problems. The method relies on the use of the Kumar-Sloan transformation and uses special orthogonal polynomials to efficiently implement a Galerkin method for the solution of the resulting nonlinear integral equation. Numerical examples show the rapid convergence for smooth solutions which is a consequence of approximation theorems of Jackson.

1. Introduction. In this article we present a numerical method for the approximate solution of the semi linear two-point boundary value problem

$$\left(\frac{d}{dx}\right)^2 u(x) = f(x, u(x)), \quad x \in [0,1], \tag{1.1}$$
$$u(0) = u(1) = 0.$$

When $f(x,u) = f(x)$, we refer to this problem as the Poisson equation on $[0,1]$. Instead of the homogeneous boundary conditions, we might also consider nonhomogeneous conditions, but then we can transform the problem to an equation of the form (1.1) with a new nonlinearity. In this paper we will assume that f is sufficiently regular and fulfills certain growth conditions to guarantee that (1.1) has solutions, see [**1, 7, 10**].

We use the Kumar-Sloan transformation, see [**1, 3, 5**], to solve (1.1) and therefore we investigate the numerical solution of the nonlinear integral equation

$$v(x) = f\left(x, \int_0^1 G(x,y)\, v(y)\, dy\right), \quad x \in [0,1], \tag{1.2}$$

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where G is the Green function for the Poisson equation on $[0,1]$. If $\overline{v}$ is a solution of (1.2), then a solution $\overline{u}$ of (1.1) is given by

$$\overline{u}(x) = \int_0^1 G(x,y)\,\overline{v}(y)\,dy, \quad x \in [0,1]. \tag{1.3}$$

In their recent articles [**1, 3**] Atkinson et al. studied the numerical solution of (1.2) in the two-dimensional case on certain two-dimensional domains, like the unit circle or a square. Atkinson and Sommariva used a spectral Galerkin method in [**3**] to approximate the solution of (1.2). If the trial space Π_N is given by

$$\Pi_N := \langle \phi_0, \ldots, \phi_N \rangle \tag{1.4}$$

where the functions ϕ_j are eigenfunctions of the Laplace operator with Dirichlet boundary conditions

$$\Delta\phi_j = \lambda_j \phi_j, \quad \lambda_j > 0, \tag{1.5}$$

then the integral in (1.3) is given explicitly and no numerical integration is necessary. This is one of the major advantages of the spectral methods. In our one-dimensional example the eigenfunctions are given by

$$\phi_j(x) = \sin(2\pi j x), \quad j \in \mathbf{N}. \tag{1.6}$$

Although the use of eigenfunctions for the trial space is not so important in the one-dimensional case because the Green function is not singular, the explicit formula

$$\int_0^1 G(x,y)\phi_j(y)\,dy = \frac{1}{(2\pi j)^2}\,\phi_j \tag{1.7}$$

facilitates any numerical method, for example the Galerkin method. But the eigenfunctions ϕ_j, $j \in \mathbf{N}_0$, also show a disadvantage of the spectral method: if the solution $\overline{v}$ of (1.2) is arbitrarily smooth $\overline{v} \in C^\infty[0,1]$, but not periodic, $\overline{v} \notin C_p^2[0,1]$, then the rate of convergence will be slow

$$\left\| P_N \overline{v} - \overline{v} \right\|_{L^2} = O\left(\frac{1}{N}\right), \quad N \to \infty, \tag{1.8}$$

see also [**3**]. Here P_N denotes the orthogonal projection of $L^2[0,1]$ onto Π_N and the function space $C_p^k[0,1]$ is defined as all k times differentiable functions on $[0,1]$ where the function and its first k derivatives are periodic.

Atkinson proposed now the question if one can find trial functions $\hat{q}_j$ which still need no numerical integration for the evaluation of the integral in (1.2) but for which the convergence would be faster than (1.8) for nonperiodic functions.

This article shows that this is indeed possible for the one-dimensional case, and the two-dimensional case will be studied in a future article. Because we like to apply the ideas of the present article for the two-dimensional case we consider the equation on the interval $[0,1]$. For the circle in $\mathbf{R}^2$ the variable x will be replaced by the radial variable r where it is natural to consider $[0,1]$ and not $[-1,1]$.

In the next section we present the Galerkin method for the approximate solution of (1.2) and in Section 3 we will construct the trial functions $\hat{q}_j$, $j \in \mathbf{N}$. We prove that we get a three term recurrence relation for the functions $\hat{q}_j$ which allows fast numerical evaluation. In Section 4 we present numerical examples which illustrate the rapid convergence of our method for smooth nonperiodic functions $\overline{v}$.

2. The Galerkin method. To approximate the solution $\overline{v}$ of (1.2) we use a Galerkin method with a trial space $\Pi_N := \langle \hat{q}_0, \dots, \hat{q}_N \rangle$, where the functions $\hat{q}_j$ are polynomial functions of degree j which will be constructed in Section 3.

In general the Galerkin method for (1.2) leads to the following system of equations

$$\sum_{j=0}^{N} \alpha_j (\hat{q}_j, \hat{q}_k)_2 = \left(f\left(\cdot\,, \int_0^1 G(\cdot\,, y) \sum_{j=0}^{N} \alpha_j \hat{q}_j(y)\, dy \right), \hat{q}_k \right)_2, \quad k = 0, \dots, N. \tag{2.1}$$

By $(\cdot,\cdot)_2$ we denote the standard scalar product in $L^2[0,1]$. Once the coefficients $(\alpha_0, \dots, \alpha_N)$ which solve (2.1) are known, the approximate solution $\overline{v}_N$ for $\overline{v}$ is given by

$$\overline{v}_N := \sum_{j=0}^{N} \alpha_j\, \hat{q}_j. \tag{2.2}$$

In [**1**] an existence result for the finite-dimensional equation (2.1) and an error estimate for $\|\overline{v}_N - \overline{v}\|_2$ has been proved. Our first goal is to choose $\hat{q}_j$ in such a way that

$$\int_0^1 G(x,y)\,\hat{q}_j(y)\,dy \tag{2.3}$$

can be calculated explicitly (here G is again the Green function). If we define

$$\hat{q}_j := A\,\hat{p}_j \tag{2.4}$$

with a polynomial $\hat{p}_j$ of degree j and A the linear operator

$$(Au)(x) := \left(\frac{d}{dx}\right)^2 (x(1-x)u(x)), \quad u \in C^2[0,1], \tag{2.5}$$

we get

$$\int_0^1 G(x,y)\,\hat{q}_j(y)\,dy = x(1-x)\,\hat{p}_j(x), \quad j \in \mathbf{N}_0. \tag{2.6}$$

This already shows that the $\hat{q}_j$, $j \in \mathbf{N}_0$, are a system of linearly independent functions because their images under the linear mapping $\hat{q} \mapsto \int_0^1 G(\cdot,y)\hat{q}(y)\,dy$ are linearly independent. In the next section, we show that one can choose $\hat{p}_j$ so that

$$(\hat{q}_j, \hat{q}_k)_2 = c_k \delta_{j,k}, \quad j,k \in \mathbf{N}_0, \tag{2.7}$$

where c_k is a positive constant and $\delta_{j,k}$ denotes the Kronecker symbol. Consequently the equation (2.1) is simplified to

$$\begin{aligned} 0 &= \alpha_k - \frac{1}{c_k}\left(f\left(x, \sum_{j=0}^N \alpha_j x(1-x)\,\hat{p}_j(x)\right), \hat{q}_k(x)\right)_2 \\ &= \alpha_k - \frac{1}{c_k}\int_0^1 f\left(x, \sum_{j=0}^N \alpha_j x(1-x)\,\hat{p}_j(x)\right)\hat{q}_k(x)\,dx \\ &\qquad k = 0, \dots, N. \end{aligned} \tag{2.8}$$

Once we have constructed polynomials $\hat{p}_j$ which satisfy (2.7), we need to solve (2.8) with a solver for nonlinear equation systems, in this paper we choose Newton's method. The only integrals that we compute numerically are those required in (2.8) by the scalar product $(\cdot,\cdot)_2$. First we construct the polynomials $\hat{p}_j$.

3. The orthogonal polynomials. We define a new scalar product $(\cdot,\cdot)_A$ on $C^2[0,1]$ by

$$(f,g)_A := (Af, Ag)_2, \quad f,g \in C^2[0,1]. \tag{3.1}$$

where the operator A is given by (2.5). That $(\cdot,\cdot)_A$ is a scalar product relies on the fact that the operator A is linear and that the Dirichlet problem on $[0,1]$ is uniquely solvable.

Our first goal is to find a sequence of polynomials $p_n \in \Pi_n$ which are A-orthogonal to Π_{n-1}: $p_n \perp_A \Pi_{n-1}$. Such a sequence could be determined by the Gram-Schmidt procedure; but, in view of its numerical instability one typically applies methods which are described in [**4**, Section 2.1]. The basis for these methods is the three-term recurrence relation. But the shift property, see [**4**, (1.3.1)],

$$(x \cdot p(x), q(x))_A = (p(x), x \cdot q(x))_A \quad \text{for all polynomials} \quad p, q,$$

which guarantees the three-term recurrence relation does not hold for the scalar product $(\cdot,\cdot)_A$:

$$\begin{gathered}(x \cdot x^0, x^1)_A = 4 \neq 1 = (x^0, x \cdot x^1)_A, \\ \text{here} \quad p(x) := x^0, \quad q(x) := x^1.\end{gathered} \tag{3.2}$$

On the other hand, we know that the Legendre polynomials are orthogonal with respect to the scalar product $(\cdot,\cdot)_2$, so it easy to see that the functions

$$p_n(x) := \frac{1}{x(1-x)} \left(\frac{d}{dx}\right)^{n-2} (x^n(1-x)^n) \tag{3.3}$$

are A-orthogonal: $p_n \perp_A \Pi_{n-1}$, $n \geq 2$. Here it is important that the function $x^n(1-x)^n$ has a zero of multiplicity n at 0 and 1, so the function p_n is a polynomial. Notice that to complete our discussion we

must define p_0 and p_1 so that $(p_n)_{n\in\mathbf{N}_0}$ is a sequence of A-orthogonal polynomials. As we will show later, this task can be easily established.

Our next goal is to find a three-term recurrence relation for the p_n, even if it is not clear why such a relation should exist. The first step is to find the leading coefficient of the polynomial p_n:

$$\begin{aligned}\left(\frac{d}{dx}\right)^{n-2}(x^n(1-x)^n) &= \left(\frac{d}{dx}\right)^{n-2}\sum_{j=0}^{n}\binom{n}{j}(-1)^j x^{n+j}\\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^j\,\frac{(j+n)!}{(j+2)!}\,x^{j+2}\\ \Longrightarrow p_n(x) &= \frac{1}{x(1-x)}\left(\frac{d}{dx}\right)^{n-2}(x^n(1-x)^n)\\ &= -\sum_{j=0}^{n}\binom{n}{j}(-1)^j\,\frac{(j+n)!}{(j+2)!}\,\frac{x^{j+1}}{x-1}\\ &= (-1)^{n+1}\,\frac{(2n)!}{(n+2)!}\,x^n+\cdots\end{aligned} \tag{3.4}$$

where we have calculated the first step of synthetic division in the last equality. By using the Leibniz rule and the binomial theorem, one can show

$$p_n(x) = x(1-x)\sum_{k=0}^{n-2} C_k^{[n]}\,x^k, \tag{3.5}$$

where

$$C_k^{[n]} = (-1)^k\frac{n!}{(n+1)(n+2)}\sum_{j=n-2-k}^{n-2}\binom{n-2}{j}\binom{n+2}{j+2}\binom{j}{j-n+2+k}.$$

From (3.4) or (3.5), we derive

$$C_{n-2}^{[n]} = (-1)^n\,\frac{(2n)!}{(n+2)!}. \tag{3.6}$$

Because the polynomials p_j, $j \in \mathbf{N}_0$, have the precise degree j, we can write

$$p_{n+1}(x) = a_n x \cdot p_n(x) + \sum_{j=0}^{n} \alpha_j^{[n]} p_j(x) \tag{3.7}$$

$$\begin{aligned} a_n &= \frac{-C_{n-1}^{[n+1]}}{-C_{n-2}^{[n]}} \\ &= (-1)^{n+2} \frac{(2n+2)!}{(n+3)!} (-1)^{n+1} \frac{(n+2)!}{(2n)!} \\ &= -\frac{(2n+1)(2n+2)}{n+3} \end{aligned} \tag{3.8}$$

$$\alpha_j^{[n]} = -a_n \frac{(x \cdot p_n, p_j)_A}{(p_j, p_j)_A}, \quad j = 0, \dots, n. \tag{3.9}$$

But, for $j \le n-2$, we get

(3.10)

$$\begin{aligned} (x \cdot p_n, p_j)_A &= \int_0^1 \left(\frac{d}{dx}\right)^2 \left(x \left(\frac{d}{dx}\right)^{n-2} (x^n(1-x)^n)\right) \left(\frac{d}{dx}\right)^j \\ &\qquad \times (x^j(1-x)^j)\, dx \\ &= \int_0^1 \left(2\left(\frac{d}{dx}\right)^{n-1} (x^n(1-x)^n) \right. \\ &\qquad \left. + x\left(\frac{d}{dx}\right)^n (x^n(1-x)^n)\right) \left(\frac{d}{dx}\right)^j (x^j(1-x)^j)\, dx \\ &\overset{\text{by parts}}{=} 2(-1)^{n-1} \int_0^1 x^n(1-x)^n \left(\frac{d}{dx}\right)^{n+j-1} (x^j(1-x)^j)\, dx \\ &\quad + (-1)^n \int_0^1 x^n(1-x)^n \left(\frac{d}{dx}\right)^n \left(x\left(\frac{d}{dx}\right)^j (x^j(1-x)^j)\right) dx \\ &= 0 + 0 \end{aligned}$$

because $n+j-1 \ge 2j+1 > 2j$ and $n \ge j+2 > j+1$. This shows that in (3.7) $\alpha_j^{[n]} = 0$, $j = 0, \dots, n-2$, and we have indeed a three-term recurrence:

$$p_{n+1}(x) = (a_n x + b_n)\, p_n(x) + c_n p_{n-1}(x). \tag{3.11}$$

To calculate b_n and c_n according to formula (3.9), we still have to find certain scalar products which we calculate in the following:

$$(p_n,p_n)_A = \int_0^1 \left(\frac{d}{dx}\right)^n (x^n(1-x)^n)\left(\frac{d}{dx}\right)^n (x^n(1-x)^n)\,dx \tag{3.12}$$
$$\stackrel{\text{by parts}}{=} \frac{(n!)^2}{2n+1}$$

$$(x\cdot p_n,p_n)_A = \int_0^1 \left(\frac{d}{dx}\right)^2 \left(x\left(\frac{d}{dx}\right)^{n-2}(x^n(1-x)^n)\right)\left(\frac{d}{dx}\right)^n \times (x^n(1-x)^n)\,dx \tag{3.13}$$
$$\stackrel{\text{by parts}}{=} \frac{(n!)^2}{2(2n+1)}$$

(3.14)
$$(x\cdot p_n,p_{n-1})_A = \int_0^1 \left(\frac{d}{dx}\right)^2 \left(x\left(\frac{d}{dx}\right)^{n-2}(x^n(1-x)^n)\colon \left(\frac{d}{dx}\right)^{n-1} \times (x^{n-1}(1-x)^{n-1})\,dx$$
$$\stackrel{\text{by parts}}{=} \frac{(2-n)(n!)^2}{2n(2n-1)(2n+1)}$$

Now we are able to compute the missing b_n and c_n in formula (3.11):

$$\begin{aligned} b_n &= -a_n \frac{(x\cdot p_n,p_n)_A}{(p_n,p_n)_A} \\ &= -a_n \frac{(n!)^2}{2(2n+1)} \frac{2n+1}{(n!)^2} \\ &= -\frac{1}{2}\,a_n \\ &= \frac{(2n+1)(n+1)}{n+3} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} c_n &= -a_n \frac{(xp_n,p_{n-1})_A}{(p_{n-1},p_{n-1})_A} \\ &= \frac{(2n+1)(2n+2)}{n+3} \frac{(2-n)(n!)^2}{2n(2n-1)(2n+1)} \frac{2n-1}{((n-1)!)^2} \\ &= \frac{(n+1)(2-n)n}{(n+3)}. \end{aligned} \tag{3.16}$$

Formula (3.11) holds only for $n \geq 3$ until now, because the formula (3.3) is only valid for $n \geq 2$. The formulas for a_n, b_n and c_n can be evaluated for all n. So we define p_0 and p_1 in such a way that (3.11) holds for $n \in \mathbf{N}_0$, if we set $p_{-1} = 0$:

$$\left.\begin{aligned} p_0(x) &:= \frac{1}{2} \\ p_1(x) &:= \frac{1}{3}\,x - \frac{1}{6} \end{aligned}\right\}. \tag{3.17}$$

With this definition the formula (3.11) also holds for $n = 1, 2$. The formulas (3.8), (3.15), (3.16) together with the recurrence (3.11) allow the calculation and evaluation of the polynomials p_n.

The following figure shows the polynomials p_{10} and p_{15}. These graphs also show that the polynomials are growing very fast, so another normalization is necessary.

If we define

$$\left.\begin{aligned} \hat{p}_0(x) &:= 1, \\ \hat{p}_1(x) &:= x, \\ \hat{p}_n(x) &:= -1/(C_{n-2}^{[n]})\, p_n(x), \quad n \geq 2, \end{aligned}\right\} \tag{3.18}$$

we get a sequence of monic orthogonal polynomials. These are the polynomials which we introduced in (2.4). Now we see

$$\begin{aligned} \hat{p}_2(x) &= -\,p_2(x) \\ &= \left(x - \frac{1}{2}\right)\hat{p}_1(x) - \frac{1}{4}\,\hat{p}_0(x) \end{aligned} \tag{3.19}$$

and, for $n \geq 2$,

(3.20)

$$\begin{aligned} \hat{p}_{n+1} &= -\,\frac{1}{C_{n-1}^{[n+1]}}\,p_{n+1}(x) \\ &= -\,\frac{1}{C_{n-1}^{[n+1]}}\left(a_n x - \frac{1}{2}\,a_n\right)\left(-\,C_{n-2}^{[n]}\right)\hat{p}_n(x) + \frac{C_{n-3}^{[n-1]}}{C_{n-1}^{[n+1]}}\,c_n \hat{p}_{n-1}(x) \\ &= \left(x - \frac{1}{2}\right)\hat{p}_n(x) + \hat{c}_n \hat{p}_{n-1}(x) \end{aligned}$$

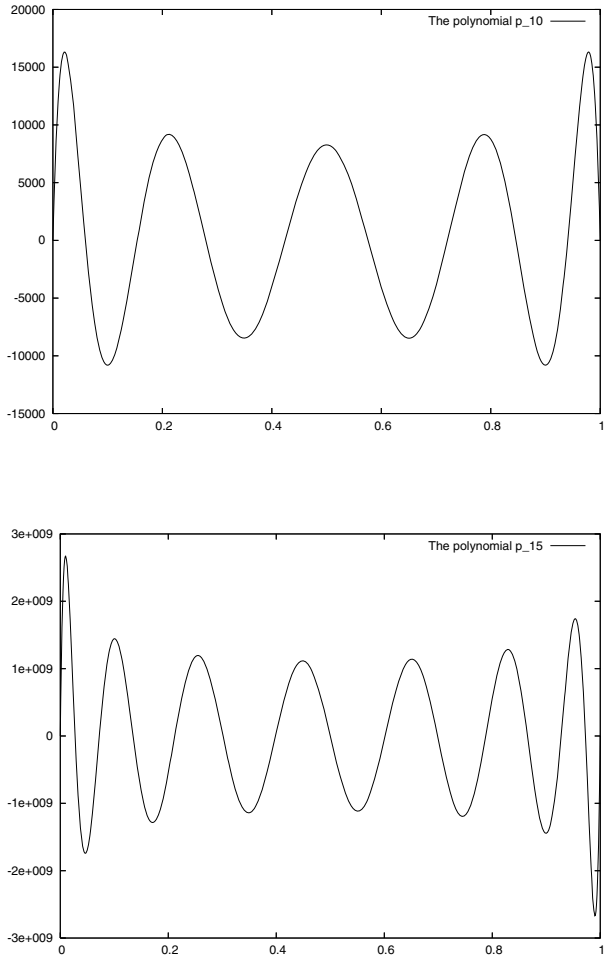

FIGURE 1. The polynomials p_{10} (top) and p_{15} (bottom).

where

$$
\begin{aligned}
\hat{c}_n &:= c_n \frac{C_{n-3}^{[n-1]}}{C_{n-1}^{[n+1]}} \\
&= \frac{(n+1)(2-n)n}{(n+3)} \frac{(-1)^n(2n-2)!}{(n+1)!} \frac{(n+3)!}{(-1)^n(2n+2)!} \\
&= \frac{(n+1)(2-n)n}{(n+3)} \frac{(n+2)(n+3)}{(2n-1)(2n)(2n+1)(2n+2)} \\
&= \frac{(2-n)(2+n)}{4(2n-1)(2n+1)} \\
&= \frac{4-n^2}{4(4n^2-1)}, \quad n \geq 2,
\end{aligned} \tag{3.21}
$$

$$
\hat{c}_1 := -\frac{1}{4}. \tag{3.22}
$$

With this definition (3.20) holds for all $n \geq 1$. The following figure shows the polynomials $\hat{p}_{30}(x)$ and $\hat{p}_{31}(x)$.

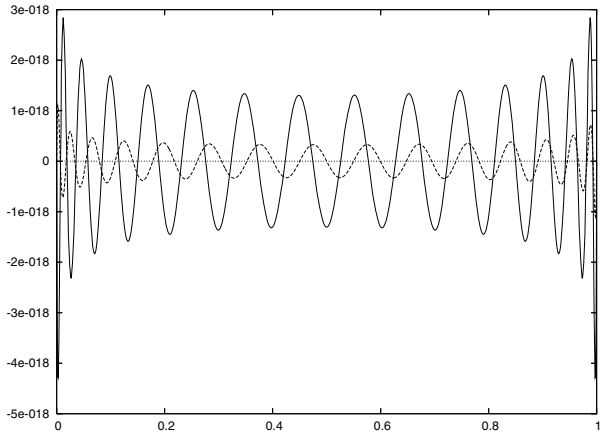

FIGURE 2. The polynomials $\hat{p}_{30}$ and $\hat{p}_{31}$.

Now we have constructed the system of polynomials which satisfies (2.7) if the polynomials $\hat{q}_n$ are given by (2.4):

$$\begin{aligned} \hat{q}_n(x) &:= (A\hat{p}_n)(x) \\ &= \frac{1}{C_{n-2}^{[n]}} \left(\frac{d}{dx}\right)^n (x^n(1-x)^n). \end{aligned} \tag{3.23}$$

So our trial functions are the shifted Legendre polynomials with a nonstandard scaling and it is not surprising that we get a three term recurrence relation for the polynomials $\hat{q}_n$:

(3.24)

$$\begin{aligned} \hat{q}_0(x) &= -2 \\ \hat{q}_1(x) &= -6\left(x - \frac{1}{2}\right) \\ \hat{q}_n(x) &= \left(x - \frac{1}{2}\right)\frac{n+2}{n}\,\hat{q}_{n-1}(x) - \frac{(n^2-1)(n+2)}{4(2n-3)(2n-1)n}\,\hat{q}_{n-2}(x), \quad n \geq 2. \end{aligned}$$

The three term recurrence relations (3.24) and (3.20) allow the fast evaluation of the polynomials in equation (2.8) without needing any Gram-Schmidt orthogonalization.

Remark. a. Finally we would like to emphasize that the above construction shows $\hat{p}_j = \widehat{A}^{-1}\hat{q}_j$. Here $\widehat{A} : D(\widehat{A}) \subset C^2[0,1] \to L^2[0,1]$ is the Friedrich extension of the positive operator A defined in (2.5). The operator $\widehat{A}^{-1}$ is given by the integral in (2.6). This relation suggests the existence of a three term recurrence and will hopefully be a starting point for the construction of polynomials with the same kind of properties in the two-dimensional case.

b. Professor C. Schneider, Johannes Gutenberg-University Mainz, pointed out to the author that we also have the relation

$$p_n(x) = \gamma_n\, x(1-x)\, J_{n-2}^{(2,2)}(x),$$

where $J_{n-1}^{(2,2)}$ is a Jacobi polynomial and γ_n a constant, see [**6, 8**]. This is an interesting connection, especially because the Jacobi polynomials are orthogonal with respect to a scalar product which does not include

any derivatives, see [**6, 8**]. Again this relation explains the three term recurrence relation.

4. Numerical examples. In Section 3 we constructed the polynomials $\hat{p}_j$ and $\hat{q}_j$, $j \in \mathbf{N}_0$, for the Galerkin method (2.8). The next step is to calculate the integrals in (2.8). Here we choose a piecewise Gaussian quadrature rule. Once N in (2.8) is given we subdivide $[0,1]$ in N intervals and in each of these intervals we apply a Gaussian rule with $N_G \geq (N+1)/2$ points. This guarantees that the polynomials are integrated exactly and the additional subdivision improves the integration of the nonlinear terms. Because the method converges so fast we have to calculate the integrals very precisely, otherwise the error is determined by the quadrature rule and not by the approximation properties of our trial functions.

Finally, we still have to solve a nonlinear equation system for the coefficients $(\alpha_0, \ldots, \alpha_N)$ in (2.8). In our numerical examples we have not used sophisticated methods since low degree polynomials were able to approximate the solution up to machine precision. In particular we used Newton's method as a nonlinear solver and the above quadrature rule to approximate the first derivatives of the system (2.8).

We have chosen the following functions for the numerical examples

$$\begin{aligned} f_1(x,y) &:= e^y \\ f_2(x,y) &:= 1 + xe^y \\ f_3(x,y) &:= 1 + \sin(4\pi xy). \end{aligned} \tag{4.1}$$

For each example we calculated the numerical solutions $\overline{v}_N$, for $N = 1, \ldots, 30$. To estimate the errors we calculated

$$\begin{aligned} E_N &:= \max_{j=0}^{10000} \Big\{ \, |\overline{v}_N(\xi_j) - \overline{v}_{30}(\xi_j)| \, \Big\}, \\ \xi_j &:= \frac{j}{10000}, \quad j = 0, \ldots, 10000. \end{aligned} \tag{4.2}$$

Here E_N is an approximation for $\|\overline{v}_N - \overline{v}\|_\infty$. We present the examples only up to the point where the accuracy reaches 10^{-14} in the first example, because this is the accuracy up to which we calculate the Gauss abscissas and weights for the numerical integration.

$f_1(x,y)$		$f_2(x,y)$		$f_3(x,y)$	
N	E_N	N	E_N	N	E_N
1	8.24×10^{-2}	1	7.77×10^{-3}	1	5.88×10^{-1}
2	3.01×10^{-3}	2	3.70×10^{-3}	2	1.90×10^{-1}
3	2.25×10^{-3}	3	2.97×10^{-3}	3	3.66×10^{-2}
4	4.72×10^{-5}	4	1.01×10^{-3}	4	2.13×10^{-2}
5	4.04×10^{-5}	5	3.48×10^{-6}	5	4.04×10^{-3}
6	6.75×10^{-7}	6	1.84×10^{-6}	6	1.38×10^{-3}
7	6.45×10^{-7}	7	3.34×10^{-7}	7	5.99×10^{-4}
8	1.00×10^{-8}	8	1.84×10^{-7}	8	2.73×10^{-4}
9	9.94×10^{-9}	9	3.33×10^{-8}	9	1.53×10^{-4}
10	1.48×10^{-10}	10	2.21×10^{-10}	10	4.96×10^{-5}
11	1.48×10^{-10}	11	7.54×10^{-11}	11	1.31×10^{-5}
12	2.16×10^{-12}	12	4.51×10^{-12}	12	4.78×10^{-6}
13	2.16×10^{-12}	13	4.59×10^{-12}	13	9.51×10^{-7}
14	7.04×10^{-14}	14	5.73×10^{-13}	14	8.50×10^{-8}

The errors in the third example are larger than in the first two examples because the solution of third example seems to have a slightly more complicated behavior than the other two functions, see the following figure. In the third example the error becomes $\approx 10^{-13}$ for $N = 24$. The error for the first example shows that coefficients for all $\hat{p}_j$, j odd, are close to zero, because the function $\overline{v}$ is even with respect to $x = 1/2$. Notice that in all cases we seem to get exponential convergence.

One reason for the rapid convergence is that a solution to (1.2) is in $C^\infty[0,1]$. This follows from equation (1.2) once we have a continuous solution $\overline{v}$ with the help of a bootstrap argument. The Lebesgue constant $L_n(x)$ for the Legendre polynomials is $O(n^2)$, see [**6**, p. 290]. The theorems of Jackson, see [**6**, p. 81] or [**9**, p. 261], imply that the best approximation $\widehat{P}_N\overline{v}$, $\widehat{P}_N\overline{v}$ a polynomial of degree N, to a function $\overline{v} \in C^k[0,1]$ satisfies

$$\left\|\widehat{P}_N\overline{v} - \overline{v}\right\|_\infty = O\left(\frac{1}{N^k}\right), \quad N \to \infty, \tag{4.3}$$

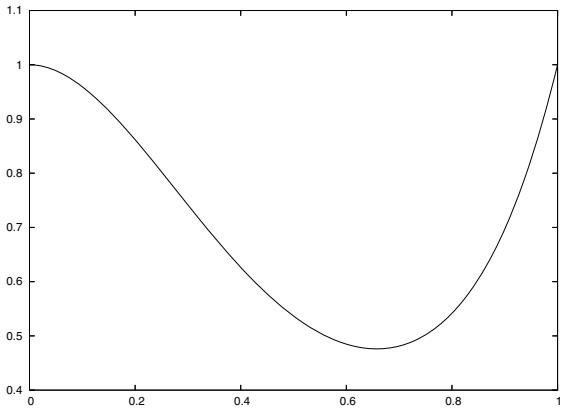

FIGURE 3. The approximation $\overline{v}_{10}$ for the third example.

and this implies

$$\left\|P_N\overline{v}-\overline{v}\right\|_\infty = O\left(\frac{1}{N^{k-2}}\right), \quad N\to\infty, \tag{4.4}$$

where P_N is the orthogonal projection. From (4.4) we derive easily

$$\left\|P_N\overline{v}-\overline{v}\right\|_2 = O\left(\frac{1}{N^{k-2}}\right), \quad N\to\infty. \tag{4.5}$$

In their article [**1**] the authors also assume that the Frechet derivative DF of the mapping

$$v \longmapsto f\left(\cdot\,, \int_0^1 G(\cdot\,,x)v(x)\,dx\right)$$

fulfills

$$1 \notin \sigma_P\left(DF\left(\overline{v}\right)\right), \tag{4.6}$$

here σ_P denotes the point spectrum of a linear operator. Now $\overline{v}\in C^\infty[0,1]$, (4.5), (4.6), and the error analysis for the Galerkin method in [**1**] show

$$\left\|\overline{v}_N-\overline{v}\right\|_2 = O\left(\frac{1}{N^{k-2}}\right), \quad N\to\infty, \tag{4.7}$$

for all $k \in \mathbf{N}$. In [**2**] one can find error estimates for the maximum norm for the Galerkin method. Again we need (4.6), the smoothness of the nonlinearity f, and (4.4).

The above results show that our method converges for solutions $\overline{v}$ of (1.2) which are in $C^\infty[0,1]$ and it is not necessary that the solution $\overline{v}$ is periodic with all its derivatives, see Figure 3.

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