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ALMOST AUTOMORPHIC MILD SOLUTIONS TO SOME SEMI-LINEAR ABSTRACT DIFFERENTIAL EQUATIONS WITH DEVIATED ARGUMENT

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ABSTRACT. In this paper we consider the semi-linear differential equation with deviated argument $x'(t) = Ax(t) + f(t, x(t), x[\alpha(x(t), t)])$, $t \in \mathbf{R}$, in a Banach space $(X, \|\cdot\|)$, where A is the infinitesimal generator of a C_0-semigroup satisfying some conditions of exponential stability. Under suitable conditions on the functions f and α we prove the existence and uniqueness of an almost automorphic mild solution to the equation.

1. Introduction. Everywhere in the paper, $(X, \|\cdot\|)$ will be a Banach space.

The concept of almost automorphy is a generalization of almost periodicity and it has been introduced in the literature by Bochner, as follows.

Definition 1.1. We say that a continuous function $f : \mathbf{R} \to X$, is almost automorphic, if every sequence of real numbers $(r_n)_n$, contains a subsequence $(s_n)_n$, such that for each $t \in \mathbf{R}$, there exists $g(t) \in X$ with the property

$$\lim_{n\to+\infty} d(g(t), f(t+s_n)) = \lim_{n\to+\infty} d(g(t-s_n), f(t)) = 0.$$

(The above convergence on $\mathbf{R}$ is pointwise.) The set of all almost automorphic functions with values in X is denoted by $AA(X)$.

In a very recent paper [**3**] the existence and uniqueness of almost automorphic mild solutions with values in Banach spaces, for the

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differential equation

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbf{R},$$

is proved, where A is the infinitesimal (bounded) generator of a C_0 semi-group of operators $(T(t))_{t\geq 0}$ on a Banach space, satisfying some exponential-type conditions of stability.

The goal of the present note is to prove the existence and uniqueness of almost automorphic mild solutions for the more general differential equation, with deviated argument, $x'(t)=Ax(t)+f(t, x(t), x[\alpha(x(t), t)])$, $t \in \mathbf{R}$.

2. Basic result. We consider the following abstract differential equation with deviated argument in the Banach space $(X, \|\cdot\|)$,

$$x'(t) = Ax(t) + f\left(t, x(t), x\left[\alpha(x(t), t)\right]\right), \quad t \in \mathbf{R}.$$

It is easy to prove that any solution of this problem has the form

$$x(t) = T(t-a)[x(a)] + \int_a^t T(t-s)\left[f\left(s, x(s), x(\alpha(x(s), s))\right)\right] ds,$$

for every $a \in \mathbf{R}$, every $t \geq a$, and we refer to any continuous $x \in C(\mathbf{R}, X)$ satisfying the above relation as a mild solution of the above problem. Obviously, because of the absence, in general, of its differentiability, a mild solution is not a strong solution of the problem.

This section is concerned with almost automorphic mild solutions of the above differential equation with deviated argument. The almost automorphic property of the deviation function $\alpha(s, t)$ with respect to t and a Lipschitz condition in s, uniformly with respect to t, turn out to be enough to show this fundamental existence and uniqueness result. This permits us to slightly generalize some of the results found in the literature for the semi-linear ordinary differential equations with deviated arguments. We assume in the sequel that A is a bounded operator in X that generates a C_0 semi-group $(T(t))_{t\geq 0}$ on X such that there is $K > 0$, $\omega < 0$, with

$$\|T(t)\| \leq Ke^{\omega t}, \quad \text{for all} \quad t \geq 0.$$

The main result is the following.

Theorem 2.1. *Assume that $f(t,u,v)$ is almost automorphic in t for each $u \in X$ and $v \in X$, that $f : \mathbf{R} \times X \times X \to X$ satisfies the Lipschitz-type condition in u and v, uniformly with respect to t, of the form*

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le C\left(\|u_1 - u_2\| + \|v_1 - v_2\|\right),$$
$$\forall\, u_1,u_2,v_1,v_2 \in X,\ t \in \mathbf{R},$$

$\alpha : X \times \mathbf{R} \to \mathbf{R}$ is almost automorphic (hence, bounded) in t, for each $u \in X$ and satisfies the Lipschitz condition, uniformly in t,

$$|\alpha(u,t) - \alpha(v,t)| \le S\,\|u - v\|, \quad \forall\, u,v \in X,\ t \in \mathbf{R},$$

where $C(SL+2)(K/|\omega|) < 1$, with $L = M\left(1 + |||A|||(K/|\omega|)\right)$, $M = \sup_{t\in\mathbf{R},\ u,v\in X} \|f(t,u,v)\| < +\infty$ and $|||\cdot|||$ corresponds to the operator norm.

Then the equation

$$x'(t) = A\left[x(t)\right] + f\left[t, x(t), x\left(\alpha(x(t),t)\right)\right], \quad t \in \mathbf{R},$$

has a unique almost automorphic mild solution in the Banach space $AA_L(X) = \{x \in AA(X);\ \|x(t) - x(s)\| \le L\,|t-s|,\ \text{for all}\ t,s \in \mathbf{R}\}$.

Proof. Let $x(t)$ be a mild solution. It is continuous and satisfies the integral equation

$$x(t) = T(t-a)\left[x(a)\right] + \int_a^t T(t-s)\left[f\left(s, x(s), x(\alpha(x(s),s))\right)\right] ds,$$
$$\text{for all} \quad a \in \mathbf{R}, \quad \text{for all} \quad t \ge a.$$

Obviously $AA_L(X)$ is not empty, since $h : \mathbf{R} \to X$ given by $h(t) = x\sin(t)$, for all $t \in \mathbf{R}$, with fixed $x \in X$ satisfying $\|x\| = L$, is an element in $AA_L(X)$. Indeed, since h is periodic, it is almost periodic and therefore almost automorphic. The Lipschitz condition is obviously satisfied.

Since $AA(X)$ is a Banach space with respect to the uniform norm

$$\|y\|_\infty = \sup\{\|y(t)\|\,;\, t \in \mathbf{R}\},$$

and it is easy to show that $AA_L(X)$ is closed under the uniform convergence, it follows that $AA_L(X)$ also is a Banach space with respect to the uniform norm.

Consider now $\int_a^t T(t-s)[f(s,x(s),x(\alpha(x(s),s)))]\,ds$ and the nonlinear operator $G : AA_L(X) \to AA(X)$ given by

$$(G\Phi)(t) := \int_{-\infty}^{t} T(t-s)\left[f\left(s,\Phi(s),\Phi(\alpha(\Phi(s),s))\right)\right]\,ds.$$

First we show that $G\Phi \in AA_L(X)$ for $\Phi \in AA_L(X)$. Denote

$$F(s) = f\left(s,\Phi(s),\Phi\left[\alpha(\Phi(s),s)\right]\right), \quad s \in \mathbf{R},$$

with $\Phi \in AA_L(X)$. Since $\Phi \in AA(X)$ and α is continuous in Φ and almost automorphic in t, by [**2**, p. 14, Theorem 2.1.5] it follows that $\alpha(\Phi,\cdot) : \mathbf{R} \to \mathbf{R}$ is in $AA(\mathbf{R})$. Since Φ is also continuous, by the same above mentioned theorem we get that $\Phi\left[\alpha(\Phi,\cdot)\right] \in AA(X)$. Denoting $\beta(s) = \Phi\left[\alpha(\Phi(s),\cdot)\right]$, we therefore have $\beta \in AA(X)$ and $F(s) = f(s,\Phi(s),\beta(s))$, for every $s \in \mathbf{R}$.

Since $f(t,u,v)$ is almost automorphic in t for each u and v, reasoning exactly as in the proof of Theorem 2.2.6 in [**2**, p. 22], we immediately obtain that $F \in AA(X)$. Because $F(t) \in AA(X)$, then it is bounded in norm so that $M = \sup_{s\in\mathbf{R}} |F(s)|$ exists and is finite. Moreover, as in the proof of Theorem 3.2 in [**3**], we immediately obtain that $G\Phi \in AA(X)$. Thus, the map G is well defined.

In what follows, we also prove that $G\Phi$ actually belongs to $AA_L(X)$. Indeed, reasoning exactly as in the book [**1**, p. 85], we get

$$(G\Phi)'(t) = T(0)\left[F(t)\right] + \int_{-\infty}^{t} \frac{\partial}{\partial t}\, T(t-s)\left[F(s)\right]\,ds,$$

therefore

$$\begin{aligned}\|(G\Phi)'(t)\| &\le \|F(t)\| + |||A||| \cdot \|F\|_\infty\, K \int_{-\infty}^{t} \|T(t-s)\|\,ds \\ &\le \|F\|_\infty \left(1 + \frac{K}{|\omega|}\,|||A|||\right) \\ &\le M\left(1 + \frac{K}{|\omega|}\,|||A|||\right) = L.\end{aligned}$$

Therefore, $\|(G\Phi)'\|_\infty \leq L$, and by the mean value theorem in vector spaces, it follows $\|(G\Phi)(t) - (G\Phi)(s)\| \leq L\,|t-s|$, for all $t, s \in \mathbf{R}$, therefore $G\Phi \in AA_L(X)$.

As a conclusion, $G : AA_L(X) \to AA_L(X)$, with L given by the statement.

It remains to check that G is a contraction. Let $\Phi_1, \Phi_2 \in AA_L(X)$. We have

$$\begin{aligned}
&\|(G\Phi_1)(t) - (G\phi_2)(t)\| \\
&\qquad \leq \int_{-\infty}^{t} \|T(t-s)\|\,\|f(s,\Phi_1(s),\Phi_1(\alpha(\Phi_1(s),s))) \\
&\qquad\qquad\qquad - f\,(s,\Phi_2(s),\Phi_2(\alpha(\Phi_2(s),s)))\|\,ds \\
&\qquad \leq C\int_{-\infty}^{t} \|T(t-s)\|\,(\|\Phi_1(s)-\Phi_2(s)\| + \|\Phi_1(\alpha(\Phi_1(s),s)) \\
&\qquad\qquad\qquad - \Phi_2(\alpha(\Phi_2(s),s))\|)\,ds.
\end{aligned}$$

But

$$\begin{aligned}
&\|\Phi_1\,(\alpha(\Phi_1(s),s)) - \Phi_2\,(\alpha(\Phi_2(s),s))\| \\
&\qquad \leq \|\Phi_1\,(\alpha(\Phi_1(s),s)) - \Phi_1\,(\alpha(\Phi_2(s),s))\| \\
&\qquad\quad + \|\Phi_1\,(\alpha(\Phi_2(s),s)) - \Phi_2\,(\alpha(\Phi_2(s),s))\| \\
&\qquad \leq SL\,\|\Phi_1-\Phi_2\|_\infty + \|\Phi_1-\Phi_2\|_\infty\,.
\end{aligned}$$

This implies

$$\begin{aligned}
\|(G\Phi_1)(t) - (G\phi_2)(t)\| &\leq C(SL+2)\,\|\Phi_1-\Phi_2\|_\infty \int_{-\infty}^{t} \|T(t-s)\|\,ds \\
&\leq C(SL+2)\,\frac{K}{|\omega|}\cdot\|\Phi_1-\Phi_2\|_\infty\,,
\end{aligned}$$

which by hypothesis implies that G is a contraction on the Banach space $AA_L(X)$. In conclusion, there exists a unique $u \in AA_L(X)$ such that $Gu = u$, that is, $u(t) = \int_{-\infty}^{t} T(t-s)\,[f\,(s,u(s),u(\alpha(u(s),s)))]\,ds$ and reasoning exactly as in the proof of Theorem 3.2 in [**3**], the proof is complete. We omit the details. $\square$

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