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ITERATIVELY REGULARIZED GRADIENT METHOD WITH A POSTERIORI STOPPING RULE FOR 2D INVERSE GRAVIMETRY PROBLEM

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ABSTRACT. A nonlinear operator equation $F(x) = f_\delta$, $\|f - f_\delta\| \le \delta$, on a pair of real Hilbert spaces H_1 and H_2 is considered. The operator F is assumed to be Fréchet differentiable without such structural assumptions as monotonicity, invertibility of $F'(x)$, etc., i.e. the problem is ill-posed. In order to solve the above equation numerically we suggest the iteratively regularized gradient method [**3**] combined with a new generalized discrepancy principle:

$$\|F(x_N) - f_\delta\|^2 < \tau\delta \le \|F(x_n) - f_\delta\|^2, \quad 0 \le n < N,\ \tau > 1.$$

A convergence theorem is proven under a source type condition

$$\hat{x} - x_0 = F'^*(\hat{x})v, \quad v \in H_2.$$

The proposed algorithm is tested on the 2D inverse gravimetry problem [**15**] reduced to a nonlinear integral equation of the first kind. The results of numerical simulations are presented and some practical recommendations on the choice of parameters are given.

1. Introduction. Consider the following inverse problem:

$$F(x) = f, \quad F : H_1 \longrightarrow H_2, \tag{1.1}$$

where F is a nonlinear Fréchet differentiable operator on a pair of real Hilbert spaces H_1 and H_2. Assume that the element $f \in H_2$ is given by its δ-approximation:

$$\|f - f_\delta\| \le \delta \tag{1.2}$$

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By taking the standard gradient process as a basic numerical scheme, we get the following iteratively regularized gradient method [**3, 5**]:

$$x_0 \in H_1, \quad x_{n+1} = x_n - \mu_n \left\{ F'^*(x_n)(F(x_n) - f_\delta) + \alpha_n(x_n - x_0) \right\}. \tag{1.3}$$

Here μ_n is an *a priori* prescribed step size, and α_n is a regularization parameter at the nth iteration:

$$\mu_n > 0, \quad \alpha_n \geq \alpha_{n+1} > 0, \quad \lim_{n\to\infty} \alpha_n = 0. \tag{1.4}$$

In the case when $\delta > 0$, the sequence $\{x_n\}$ generated by an iteratively regularized process (1.3) does not usually converge to a minimizer $\hat{x}$ of (1.1). It does, however, allow a stable approximation to $\hat{x}$ if iterations are stopped at an appropriate step $n = N(\delta)$ such that $\lim_{\delta\to 0} \left\| x_{N(\delta)} - \hat{x} \right\| = 0$. When functional (1.1) is convex, convergence of $x_{N(\delta)}$ to $\hat{x}$ can be established by using the scheme proposed for monotone operator equations in [**4**]. Unfortunately, the convexity assumption is not satisfied for most applied nonlinear problems. For that reason, instead of convexity of $\Phi(x)$, certain restrictions on the type of nonlinearity of F have been used in the literature. For example, the condition

$$F'(y) = F'(x)R(x,y), \quad x, y \in B(\hat{x}) \tag{1.5}$$

with linear operator $R(x,y)$ satisfying a Lipschitz-type estimate

$$\|R(x,y) - I\| = \|R(x,y) - R(x,x)\| \leq C_R \|x - y\| \tag{1.6}$$

was considered in [**14**]. It is closely related to the so-called affine covariant Lipschitz condition [**9**]. Identities (1.5) and (1.6) have been verified for some parameter identification problems in PDEs, where the forward operator consisted of a nonlinear solution operator, composed with a linear operator mapping the solution to the given boundary values.

An alternative nonlinearity condition [**11–14**] takes the following form:

$$F'(y) = R(x,y)F'(x), \quad x, y \in B(\hat{x}), \tag{1.7}$$

where $R(x,y)$ are regular operators. The related Newton-Mysovskii conditions have been discussed in [**8**].

In this paper, the convergence analysis is done without using any structural assumptions on either $\Phi(x)$ or $F(x)$. It covers, therefore, the case of a general nonlinear operator, for example, an integral operator with a smooth nonlinear kernel. The analysis is based on the sourcewise representation condition:

$$\hat{x} - x_0 = F'^{*}(\hat{x})\, v, \quad v \in H_2. \tag{1.8}$$

The stopping number $n = N(\delta)$ is chosen by the generalized discrepancy principle, see [**6**]:

$$\|F(x_N) - f_\delta\|^2 < \tau\delta \le \|F(x_n) - f_\delta\|^2, \quad 0 \le n < N,\ \tau > 1. \tag{1.9}$$

The paper is organized as follows. In Section 2 the main convergence result, Theorem 2.1, is established, and examples of regularization parameters are given. In Section 3 the application of algorithm (1.3), (1.9) to the 2D inverse problem of gravimetry [**1, 15**] is studied. The problem consists of finding an interface between two media of different densities. It is reduced to a nonlinear integral equation of the first kind (nonlinear ill-posed problem):

$$F(x) := g \,\triangle\sigma \int_a^b \int_c^d \left\{ \frac{1}{[\,(\xi - t)^2 + (\nu - s)^2 + x^2(\xi,\nu)\,]^{1/2}} - \frac{1}{[\,(\xi - t)^2 + (\nu - s)^2 + h^2\,]^{1/2}} \right\} d\xi\, d\nu = f(t,s),$$

where g is the gravitational constant, $\triangle\sigma$ is the density jump on the interface and $f(t,s)$ is the gravitational strength anomaly. Based on the results of numerical experiments, some practical recommendations on the choice of parameters α_n, μ_n and τ are provided.

2. Regularization procedure and convergence theorem. In order to analyze the behavior of iterations generated by method (1.3), (1.9), we state *a priori* conditions on the rate of decay for parameters $\{\alpha_n\}$ and $\{\mu_n\}$, see (2.2) below, as well as certain limitations on the initial element x_0 given by formulas (2.3)–(2.5). Conditions (2.2) are independent of a specific operator on a class described by inequalities (2.1). Thus, algorithm (1.3), (1.9) is well defined for all operators in class (2.1).

Theorem 2.1. 1. *Assume equation $F(x) = f$ is solvable, maybe non-uniquely.*

2. *The Fréchet derivative of the operator F is bounded and Lipschitz-continuous*:

$$\|F'(x)\| \le 1, \quad \|F'(x) - F'(y)\| \le L\,\|x - y\| \quad \text{for any } x, y \in H_1. \tag{2.1}$$

3. *The element f is known approximately and inequality* (1.2) *is fulfilled.*

4. *For some positive constants ε, $\xi > 0$,*

$$\mu_n = \varepsilon\alpha_n > 0, \quad \alpha_n \searrow 0 \quad \text{as} \quad n \to 0, \quad \frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2 \alpha_{n+1}} \le \varepsilon\,\xi. \tag{2.2}$$

5. *The source-wise condition holds in the form*

$$\hat{x} - x_0 = F'^{*}(\hat{x})\,v, \quad v \in H_2, \tag{2.3}$$

where $\hat{x} \in H_1$ is a solution to $F(x) = f$, and

$$\|v\| < \min\left\{\frac{2 - \varepsilon(3 + \alpha_0^2)}{2L(1 + \varepsilon\alpha_0 + 2\varepsilon\alpha_0^2 L)}, 2\right\}, \tag{2.4}$$

$$2c\,\|v\|\,(1 + \varepsilon\,\xi\alpha_0^2)\left\{\frac{L}{2} + \frac{1}{(\sqrt{\tau} - 1)^2}\right\} + \xi \le \eta. \tag{2.5}$$

Here $\tau > 1$,

$$c := \sqrt{(1 + 3\varepsilon\alpha_0)(1 + \varepsilon + \varepsilon\alpha_0(2 + L))}, \tag{2.6}$$

$$\eta := 2\left[1 - L\,\|v\|\,(1 + \varepsilon\alpha_0 + 2\varepsilon\alpha_0^2 L)\right] - \varepsilon(3 + \alpha_0^2) \tag{2.7}$$

6. *The initial element* x_0 *satisfies the condition*

$$\frac{\|x_0 - \hat{x}\|^2}{\alpha_0} \leq \frac{2\,\|v\|^2\,(1+\varepsilon+\varepsilon\alpha_0(2+L))(1+\varepsilon\,\xi\alpha_0^2)}{\eta-\xi} := l. \tag{2.8}$$

Then

1. *For* $n = 0, 1, \dots, N(\delta)$*, one has*

$$\|x_n - \hat{x}\|^2 \leq l\,\alpha_n, \tag{2.9}$$

where $N = N(\delta)$ *is chosen by the generalized discrepancy principle*

$$\|F(x_N) - f_\delta\|^2 \leq \tau\delta < \|F(x_n) - f_\delta\|^2, \quad 0 \leq n \leq N,\ \tau > 1. \tag{2.10}$$

2. *The sequence* $\{N(\delta)\}$ *is admissible, i.e.,*

$$\lim_{\delta\to 0} \left\|x_{N(\delta)} - x^*\right\| = 0, \tag{2.11}$$

x^* *is a solution to* $F(x) = f$*. If* $N(\delta) \to \infty$ *as* $\delta \to 0$*, then* $x^* = \hat{x}$*.*

Remark 2.2. One of the main assumptions of the paper is that, for all $x, y \in H_1$, $\|F'(x)\| \leq 1$, $\|F'(x) - F'(y)\| \leq L\,\|x - y\|$. The second condition is often fulfilled in practical applications. But if one considers, e.g., the operator of autoconvolution, then the first estimate holds only in a bounded region. The structure of the proofs for convergence and stability of the proposed method is always that it is shown

$$\frac{\|x_n - \hat{x}\|}{\sqrt{\alpha_n}} \leq \sqrt{l}$$

holds and, because α_n is monotone decreasing, all iterates for $n \leq N$ will stay in a bounded neighborhood of the solution $\hat{x}$,

$$\|x_n - \hat{x}\| \leq \sqrt{\alpha_0\, l}.$$

Therefore it is actually sufficient for the first estimate $\|F'(x)\| \leq 1$ to hold in a sufficiently large neighborhood of $\hat{x}$.

Proof of Theorem 2.1. Take arbitrary $n < N(\delta)$ and suppose that, for any k, $0 < k \le n < N(\delta)$, the induction assumption

$$\frac{\|x_k - \hat{x}\|^2}{\alpha_k} \le l \tag{2.12}$$

holds. By condition (2.1) it follows that

$$0 = F(\hat{x}) - f = F(x_n) - f + F'(x_n)(\hat{x} - x_n) + G(x_n, \hat{x}), \tag{2.13}$$

$$\|G(x_n, \hat{x})\| \le \frac{L}{2}\, \|x_n - \hat{x}\|^2 \tag{2.14}$$

Thus source-wise condition (2.3) yields

$$\begin{aligned} x_{n+1} - \hat{x} = {} & (1 - \mu_n\alpha_n)(x_n - \hat{x}) - \mu_n F'^*(x_n)F'(x_n)(x_n - \hat{x}) \\ & + \mu_n F'^*(x_n)(G(x_n, \hat{x}) - f + f_\delta) \\ & - \mu_n\alpha_n (F'(\hat{x}) - F'(x_n))^* v \\ & - \mu_n\alpha_n F'^*(x_n)v. \end{aligned} \tag{2.15}$$

Since H_1 is a real Hilbert space

(2.16)

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 = {} & \|(1 - \mu_n\alpha_n)(x_n - \hat{x}) - \mu_n F'^*(x_n)F'(x_n)(x_n - \hat{x}) \\ & \qquad - \mu_n\alpha_n F'^*(x_n)v\|^2 \\ & + 2\mu_n \big((1 - \mu_n\alpha_n)(x_n - \hat{x}) - \mu_n F'^*(x_n) \\ & \quad \times F'(x_n)(x_n - \hat{x}) - \mu_n\alpha_n F'^*(x_n)v, \\ & \quad F'^*(x_n)(G(x_n, \hat{x}) - f + f_\delta) - \alpha_n (F'(\hat{x}) - F'(x_n))^* v\big) \\ & + \mu_n^2 \|F'^*(x_n)(G(x_n, \hat{x}) - f + f_\delta) \\ & \qquad - \alpha_n (F'(\hat{x}) - F'(x_n))^* v\|^2. \end{aligned}$$

Estimate the first term in (2.16)

$$\begin{aligned} A_n := {} & \|(1 - \mu_n\alpha_n)(x_n - \hat{x}) - \mu_n F'^*(x_n)F'(x_n)(x_n - \hat{x}) \\ & \qquad - \mu_n\alpha_n F'^*(x_n)v\|^2 \\ = {} & (1 - \mu_n\alpha_n)\|x_n - \hat{x}\|^2 + \mu_n^2 \|F'^*(x_n)F'(x_n)(x_n - \hat{x})\|^2 \\ & + \mu_n^2\alpha_n^2 \|F'^*(x_n)v\|^2 \\ & - 2\mu_n(1 - \mu_n\alpha_n)\big(F'^*(x_n)F'(x_n)(x_n - \hat{x}), x_n - \hat{x}\big) \\ & - 2\mu_n\alpha_n(1 - \mu_n\alpha_n)\big(F'^*(x_n)v, x_n - \hat{x}\big) \\ & + 2\mu_n^2\alpha_n\big(F'^*(x_n)F'(x_n)(x_n - \hat{x}), F'^*(x_n)v\big). \end{aligned} \tag{2.17}$$

From the inequality $\|F'(x)\| \leq 1$, one concludes

$$
\begin{aligned}
A_n \leq & \left[(1-\mu_n\alpha_n)^2 + \mu_n^2\right] \|x_n - \hat{x}\|^2 + \mu_n^2\alpha_n^2 \|v\|^2 \\
& - 2\mu_n (1-\mu_n\alpha_n) \|F'(x_n)(x_n - \hat{x})\|^2 \\
& + 2\mu_n\alpha_n (1-\mu_n\alpha_n) \|v\| \|F'(x_n)(x_n - \hat{x})\| \\
& + 2\mu_n^2\alpha_n \|x_n - \hat{x}\| \|v\| .
\end{aligned}
\tag{2.18}
$$

Clearly,

$$
2\alpha_n \|v\| \|F'(x_n)(x_n - \hat{x})\| \leq \|F'(x_n)(x_n - \hat{x})\|^2 + \alpha_n^2 \|v\|^2
$$

and

$$
2\alpha_n \|x_n - \hat{x}\| \|v\| \leq \|x_n - \hat{x}\|^2 + \alpha_n^2 \|v\|^2 ,
$$

which implies together with (2.18) that

$$
\begin{aligned}
A_n \leq & \left[1 - 2\mu_n\alpha_n + \mu_n^2(\alpha_n^2 + 2)\right] \|x_n - \hat{x}\|^2 \\
& - \mu_n(1-\mu_n\alpha_n) \|F'(x_n)(x_n - \hat{x})\|^2 + \mu_n\alpha_n^2 \left[1 - \mu_n\alpha_n + 2\mu_n\right] \|v\|^2 .
\end{aligned}
\tag{2.19}
$$

Now estimate the second term in (2.16)

$$
\begin{aligned}
B_n := {} & 2\mu_n\big((1-\mu_n\alpha_n)(x_n - \hat{x}) - \mu_n F'^*(x_n)F'(x_n)(x_n - \hat{x}) \\
& \qquad - \mu_n\alpha_n F'^*(x_n)v, F'^*(x_n)\left(G(x_n, \hat{x}) - f + f_\delta\right) \\
& \qquad\qquad - \alpha_n \left(F'(\hat{x}) - F'(x_n)\right)^* v\big) \\
\leq {} & \mu_n (1-\mu_n\alpha_n) \left\{ \left[\|G(x_n, \hat{x})\| + \delta\right]^2 + \|F'(x_n)(x_n - \hat{x})\|^2 \right\} \\
& + 2\mu_n\alpha_n(1-\mu_n\alpha_n) L \|v\| \|x_n - \hat{x}\|^2 \\
& + \mu_n^2 \left\{ \left[\|G(x_n, \hat{x})\| + \delta\right]^2 + \|x_n - \hat{x}\|^2 \right\} + 2\mu_n^2\alpha_n L \|v\| \|x_n - \hat{x}\|^2 \\
& + \mu_n^2\alpha_n \left\{ \left[\|G(x_n, \hat{x})\| + \delta\right]^2 + \|v\|^2 \right\} \\
& + \mu_n^2\alpha_n^2 L \|v\|^2 \left\{ \|x_n - \hat{x}\|^2 + 1 \right\}.
\end{aligned}
\tag{2.20}
$$

From (2.20), (2.4) and (2.14) one derives

$$
\begin{aligned}
B_n \leq & \mu_n(1+\mu_n) \left[\frac{L}{2} \|x_n - \hat{x}\|^2 + \delta\right]^2 \\
& + \mu_n(1-\mu_n\alpha_n) \|F'(x_n)(x_n - \hat{x})\|^2 \\
& + \mu_n \left[2\,\alpha_n L \|v\| (\mu_n + 1) + \mu_n\right] \|x_n - \hat{x}\|^2 \\
& + \mu_n^2\alpha_n (1 + \alpha_n L) \|v\|^2 .
\end{aligned}
\tag{2.21}
$$

Finally,

(2.22)

$$\begin{aligned} C_n &:= \mu_n^2 \left\| F'^*(x_n)\,(G(x_n,\hat{x}) - f + f_\delta) - \alpha_n \left(F'\left(\hat{x}\right) - F'(x_n)\right)^* v \right\|^2 \\ &\le 2\mu_n^2 \left\{ \left[\frac{L}{2} \left\| x_n - \hat{x} \right\|^2 + \delta \right]^2 + \alpha_n^2 L^2 \left\| v \right\|^2 \left\| x_n - \hat{x} \right\|^2 \right\}. \end{aligned}$$

Since $n < N(\delta)$, according to (2.10)

$$\tau\delta \le \left\| F(x_n) - f_\delta \right\|^2.$$

Thus by (1.2)

$$\sqrt{\tau\delta} \le \left\| F(x_n) - f \right\| + \left\| f - f_\delta \right\| \le \left\| x_n - \hat{x} \right\| + \delta, \tag{2.23}$$

and

$$\sqrt{\tau\delta} - \delta \le \left\| x_n - \hat{x} \right\|. \tag{2.24}$$

Without loss of generality one can assume $\delta < 1$, therefore

$$\delta \le \frac{\left\| x_n - \hat{x} \right\|^2}{\left(\sqrt{\tau} - 1\right)^2}. \tag{2.25}$$

Combining (2.2), (2.19), (2.21), (2.22) and (2.25), one gets

$$\left\| x_{n+1} - \hat{x} \right\|^2 \le (1 - \eta\,\varepsilon\alpha_n^2) \left\| x_n - \hat{x} \right\|^2 + \lambda\alpha_n \left\| x_n - \hat{x} \right\|^4 + \rho\,\alpha_n^3, \tag{2.26}$$

where η is defined by (2.7) and

$$\lambda := \varepsilon\,(1 + 3\,\varepsilon\alpha_0) \left\{ \frac{L}{2} + \frac{1}{(\sqrt{\tau} - 1)^2} \right\}^2, \tag{2.27}$$

$$\rho = \varepsilon \left\| v \right\|^2 \left[1 + \varepsilon\alpha_0(2 + L) + \varepsilon \right]. \tag{2.28}$$

Let $\gamma_n := \left\| x_n - \hat{x} \right\|^2 / \alpha_n$. Then one has

$$\gamma_{n+1} \le (1 - \eta\,\varepsilon\alpha_n^2)\,\frac{\alpha_n}{\alpha_{n+1}}\,\gamma_n + \lambda\,\frac{\alpha_n^3}{\alpha_{n+1}}\,\gamma_n^2 + \rho\,\frac{\alpha_n^3}{\alpha_{n+1}}. \tag{2.29}$$

The sequence $\{\alpha_n\}$ was chosen to satisfy the condition $\alpha_n - \alpha_{n+1}/\alpha_n^2\alpha_{n+1} \le \varepsilon\,\xi$, and by induction assumption $\gamma_n \le l$. So by (2.5) and (2.8)

(2.30)

$$\gamma_{n+1} - l \le \alpha_n^2 \left\{-\varepsilon(\eta-\xi)\,l + \lambda\left(1+\varepsilon\,\xi\alpha_0^2\right)\,l^2 + \rho\left(1+\varepsilon\,\xi\alpha_0^2\right)\right\} \le 0.$$

The sequence $N = N(\delta)$ is nondecreasing as $\delta \to 0$. Two cases are possible:

1. $N(\delta) = N_0$ for any $\delta \le \delta_0$. Then $\lim_{\delta\to 0} x_{N_0}(f_\delta)$ is a solution to $F(x) = f$.

2. $N(\delta) \to \infty$ as $\delta \to 0$. Then

$$\left\|x_{N(\delta)} - \hat{x}\right\| \le \sqrt{l\,\alpha_{N(\delta)}} \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

In any case $\lim_{\delta\to 0}\left\|x_{N(\delta)} - x^*\right\| = 0$, x^* is a solution to $F(x) = f$. This completes the proof. □

Remark 2.3. The sequence $\alpha_n = \alpha_0/(1+n)^p$ satisfies assumption (2.2) if

$$0 < p \le \frac{1}{2} \quad \text{and} \quad \alpha_0 \ge \sqrt{\frac{p}{\varepsilon\,\xi}}.$$

As another example one can take $\alpha_n = \alpha_0/\ln(1+n)$ with $\alpha_0 \ge \sqrt{1/\varepsilon\,\xi}$.

3. Numerical simulations. To test iteratively regularized gradient method (1.3), (1.9) numerically, the two-dimensional inverse gravimetry problem was considered. It was assumed that the lower half-space is formed by two media with constant densities separated by a surface $x = x(\xi,\nu)$. The gravitational anomaly, $f = f(t,s)$, is caused by the deviation of the above surface from the horizontal plane $x(\xi,\nu) = h$. As it is shown in [**15**], in the Descartes coordinate system the inverse gravimetry problem can be reduced to the 2D nonlinear integral equation of the first kind:

(3.1)

$$\begin{aligned} F(x) := g\,\triangle\sigma \int_a^b\!\int_c^d \Bigg\{ & \frac{1}{[\,(\xi-t)^2 + (\nu-s)^2 + x^2(\xi,\nu)\,]^{1/2}} \\ & - \frac{1}{[\,(\xi-t)^2 + (\nu-s)^2 + h^2\,]^{1/2}} \Bigg\}\, d\xi\, d\nu \\ = f(t,s), \end{aligned}$$

and it consists of finding the unknown function $x = x(\xi, \nu)$, which describes the interface, from the measured data $f = f(t, s)$. Here g is the gravitational constant and $\triangle\sigma$ is the density jump on the interface. Choose $H_1 = H^1(\Omega)$ and $H_2 = L_2(\Omega)$, $\Omega = [a, b] \times [c, d]$. If one denotes the kernel of the nonlinear operator $F(x)$ by $K(t, s, \xi, \nu, x(\xi, \nu))$:

$$K(t,s,\xi,\nu,x(\xi,\nu)) := \left\{ \frac{1}{[\,(\xi-t)^2+(\nu-s)^2+x^2(\xi,\nu)\,]^{1/2}} \right. \\ \left. \frac{1}{[\,(\xi-t)^2+(\nu-s)^2+h^2\,]^{1/2}} \right\}, \tag{3.2}$$

then the Fréchet derivative of $F(x)$ is given by

$$F'(x)y = g\,\triangle\sigma \int_\Omega\!\!\int K'_x(t,s,\xi,\nu,x(\xi,\nu))y(\xi,\nu)\,d\xi\,d\nu, \tag{3.3}$$

where

$$K'_x(t,s,\xi,\nu,x(\xi,\nu)) = \frac{-x(\xi,\nu)}{[\,(\xi-t)^2+(\nu-s)^2+x^2(\xi,\nu)\,]^{3/2}}. \tag{3.4}$$

To evaluate the corresponding adjoint operator, one needs to solve an auxiliary integro-differential equation. Hence, direct numerical realization of (1.3) is very complicated in our case. It was shown in [**3**] that one can find an approximate solution to (3.1) by replacing the integral in (3.1) with a quadrature formula. Thus, from now on, we consider a finite-dimensional analog of equation (3.1) and construct iterations (1.3) in a suitable Euclidian space.

The numerical experiments were performed on a Pentium IV desktop using Matlab 6.5.1. Discretization was done by taking an evenly spaced grid over the domain $\Omega = [2.8, 20.0] \times [0.0, 8.0]\,(\mathrm{km}^2)$ with 43×20 node points, respectively. This resolution gave the corresponding mesh width of $w = 0.4\,(\mathrm{km})$ in both directions. The two-dimensional analog of the midpoint quadrature rule was used to approximate the integral operator in (3.1). The ground surface height was taken to be $h = 2.0\,(\mathrm{km})$. The gravitational anomaly was calculated by solving the direct problem with $x = x_{\mathrm{mod}}(\xi, \nu)$, and the identity was rescaled by dropping the constants g and $\triangle\sigma$ from the original formula. The constant horizontal plane $\mathbf{x_0}(\xi, \nu) = \mathbf{0.1}\,(\mathrm{km})$ was taken as the initial guess for all the simulations below. In order to select the regularization parameter α_n, the step size parameter μ_n, and the discrepancy parameter τ, the following representative, asymmetric model solution, obtained by translating and scaling Gaussian distributions, was considered (see 'Built-in Peaks Surface Function' in Matlab manual):

(3.5)

$$x^{(1)}_{\mathrm{mod}}(\tilde{\xi},\tilde{\nu}) = \frac{1}{14}\left\{3(1-\tilde{\xi})^2 e^{-\tilde{\xi}^2-(\tilde{\nu}+1)^2} - 10\left(\frac{1}{5}\,\tilde{\xi} - \tilde{\xi}^3 - \tilde{\nu}^5\right) e^{-\tilde{\xi}^2-\tilde{\nu}^2} - \frac{1}{3}\, e^{-(\tilde{\xi}+1)^2-\tilde{\nu}^2}\right\} + 1,$$

where

$$\tilde{\xi} = 6\left(\frac{\xi - c}{d-c}\right) - 3, \quad \text{and} \quad \tilde{\nu} = 6\left(\frac{\nu - a}{b-a}\right) - 3$$

are appropriate domain transformations. For the above model solution, the sequence μ_n was numerically determined so as to give the most aggressive convergence rate for the experiments conducted in a noise-free case and in the presence of noise at the level from 2% to 5%. The best convergence rate was obtained for $\mu_n = \mu_0 = 0.015$. The regularization parameter sequence $\alpha_n = \alpha_0(1+n)^{-0.25}$ with $\alpha_0 = 0.001$ was chosen to ensure the best stability in the presence of noise. For the noise-free case there was no apparent need for regularization (we could use $\alpha_n = 0$). An attractive feature of scheme (1.3) is that it is not very sensitive to the values of parameters $\{\mu_n\}$ and $\{\alpha_n\}$. For the relative noise 5% we were able to take α_0 from the interval $\left[10^{-7}, 10^{-2}\right]$ and $\alpha_n = \alpha_0/(1+n)^p$ with $0 < p \leq 1$. One can see that for our particular problem the interval for p is bigger than the interval guaranteed by the convergence theorem. As for the choice of μ_n, if one takes $\mu_n = \mu_0$ then one can use any $\mu_0 \in [0.005, 0.03]$. The sequence $\mu_n = \mu_0/(1+n)^q$, $0 < q \leq 1$ and $\mu_0 \in [0.005, 0.03]$, also works. For $q > 1$ convergence becomes rather slow.

The following algorithm was used to determine τ. Iterations (1.3) were performed with random noise functions added to the right-hand side of equation (3.1). The relative discrepancy:

$$\triangle_\delta := \frac{\|F(x_{\mathrm{mod}}) - f_\delta\|}{\|F(x_{\mathrm{mod}})\|} \tag{3.6}$$

ranged from 0.02 to 0.2. Altogether 10 different noise functions were investigated with $\triangle_{\delta(k)} = 0.02k$, $k = 1, 2, \ldots, 10$. For every noise function, iterative process (1.3) was stopped at the first number $n = n(k)$ such that

$$\frac{\|x_n - x_{\text{mod}}\|}{\|x_{\text{mod}}\|} > \frac{\|x_{n-1} - x_{\text{mod}}\|}{\|x_{\text{mod}}\|}$$

and $\tau = \tau(k)$ was calculated from the identity

$$\|F(x_n) - f_\delta\|^2 = \tau\delta.$$

It was discovered that, for $k = 1, 2, \ldots, 10$, $\tau(k) \in [8.02, 11.46]$. As the result, the value $\tau = 11.46$ was taken to reconstruct two other model solutions

$$x^{(2)}_{\text{mod}}(\tilde{\xi}, \tilde{\nu}) = \sin\left(\left|\tilde{\xi}\right| - |\tilde{\nu}|\right)/3 + 1,$$
$$\tilde{\xi} = 10\left(\frac{\xi - c}{d - c}\right) - 5, \quad \tilde{\nu} = 10\left(\frac{\nu - a}{b - a}\right) - 5$$

and

$$x^{(3)}_{\text{mod}}(\tilde{\xi}, \tilde{\nu}) = \exp(-\tilde{\xi}^2 - \tilde{\nu}^2) + 0.5,$$
$$\tilde{\xi} = 3\left(\frac{\xi - c}{d - c}\right) - 1.5, \quad \tilde{\nu} = 3\left(\frac{\nu - a}{b - a}\right) - 1.5$$

using method (1.3) and *a posteriori* stopping rule (1.9) (with parameters $\mu_n = 0.015$ and $\alpha_n = 0.001(1+n)^{-0.25}$).

In Figure 1, one can see the graph of $x^{(2)}_{\mathrm{mod}}(t,s)$, $(t,s) \in \Omega$, as well as the graphs of approximate solutions for the noise-free case, for the case when $\triangle_\delta := 0.02$ and $\triangle_\delta := 0.05$, see formula (3.6). The cross-sectional comparison for $s = 10$ is presented in Figure 2. The same results for $x^{(3)}_{\mathrm{mod}}(t,s)$ are illustrated in Figures 3 and 4, respectively. The number of iterations in the noise-free case for both model solutions was 50. When the relative level of noise was 2% and the experiment was conducted with $x^{(3)}_{\mathrm{mod}}(t,s)$, the iterations were stopped by generalized discrepancy principle (1.9) at $n = 31$. When the level of noise was 5%, for the same model solution, the iterations were stopped at $n = 22$.

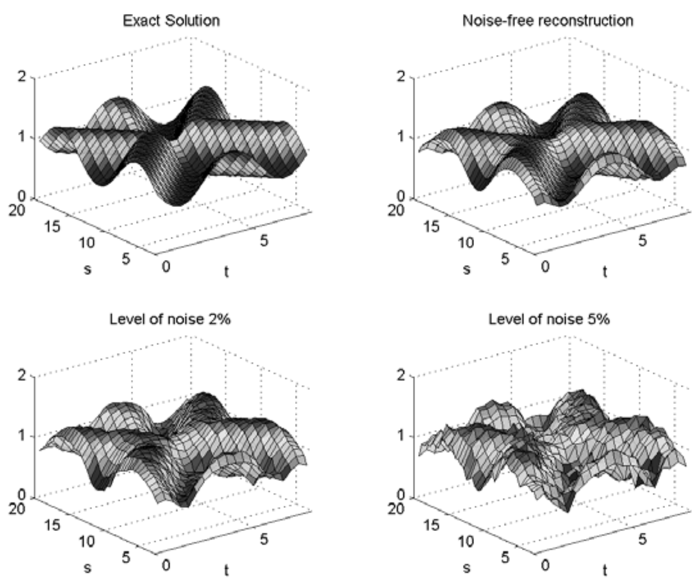

FIGURE 1.

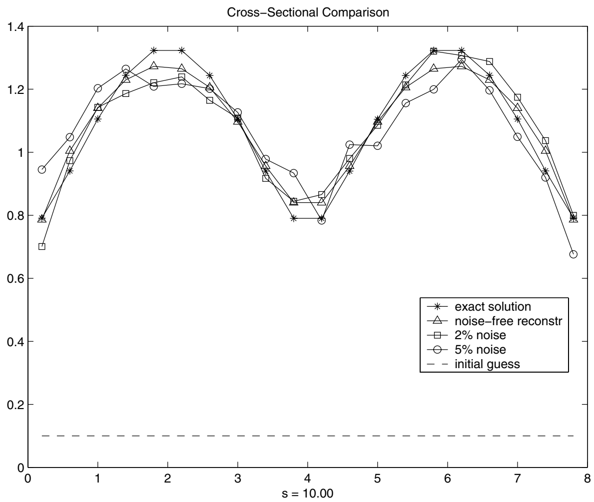

FIGURE 2.

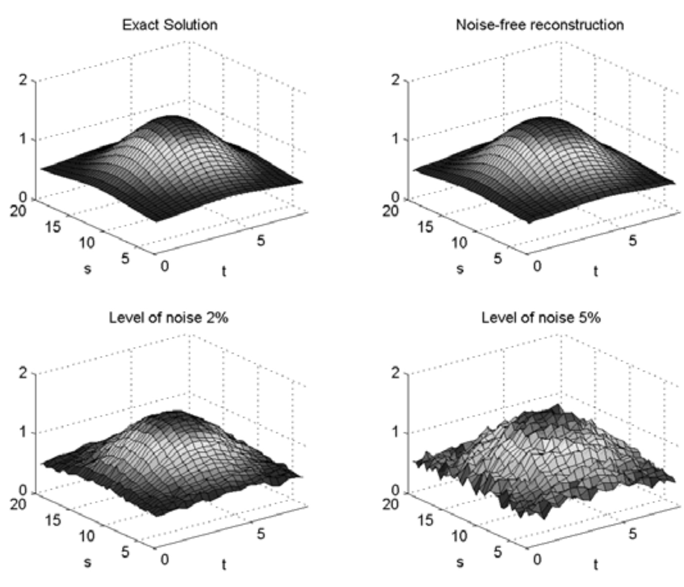

FIGURE 3.

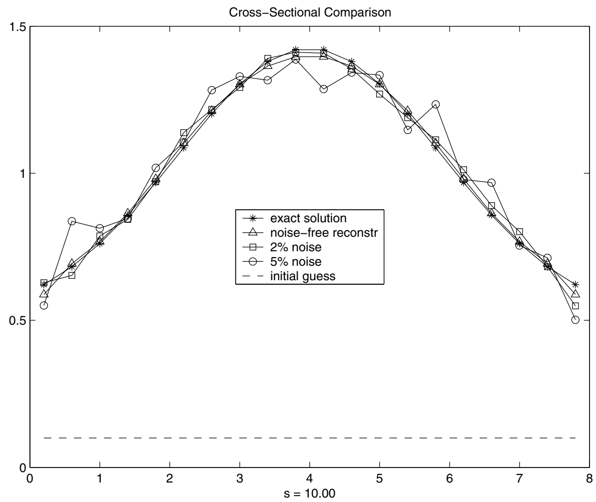

FIGURE 4.

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