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SOME VOLTERRA-TYPE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH A MULTIVARIABLE CONFLUENT HYPERGEOMETRIC FUNCTION AS THEIR KERNEL

H.M. SRIVASTAVA AND R.K. SAXENA

ABSTRACT. Motivated essentially by several recent works on interesting generalizations of the first-order Volterra-type integro-differential equation governing the unsaturated behavior of the free electron laser (FEL) by making use of fractional calculus, that is, calculus of integrals and derivatives of an arbitrary real or complex order, the authors investigate the solutions of several Cauchy-type and Cauchy problems associated with some general fractional Volterra-type integro-differential equations in which the kernel involves the confluent hypergeometric function $\Phi_2^{(n)}$ in n complex variables. The closed-form solution of each of these general Cauchy-type problems is derived in terms of the function $\Phi_2^{(n)}$ itself. Several special cases of the main results are also shown to yield generalizations of the results investigated in the aforementioned and other earlier works.

1. Introduction. The unsaturated behavior of the free electron laser (FEL), when no field mode structures are taken into consideration, is governed by the following first-order integro-differential equation of Volterra type, cf. [**6, 8**]:

$$\frac{d}{d\tau}\, h(\tau) = -i\pi g_0 \int_0^\tau \xi \exp(i\nu\xi)\, h(\tau-\xi)\, d\xi, \tag{1.1}$$

where τ is a dimensionless time variable ($0 \leqq \tau \leqq 1$), g_0 is a positive constant called the small signal gain, and ν is a real constant referred

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to as the detuning parameter. The function $h(\tau)$ is the complex-field amplitude, which is assumed to be dimensionless, satisfying the initial condition $h(0) = 1$. The exact closed-form solution of the integro-differential equation (1.1) under this initial condition, which is valid in the whole range of practical interest and suitable for numerical calculations, was given by Dattoli et al. [**7**].

Recently, by employing the (Riemann-Liouville) operator D_z^μ of *fractional calculus*, defined by, cf., e.g., [**10**, Vol. II, p. 181 *et seq.*]; see also [**22**],

$$D_z^\mu\{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-\zeta)^{-\mu-1}\, f(\zeta)\, d\zeta & \Re(\mu) < 0 \\ \dfrac{d^m}{dz^m}\, D_z^{\mu-m}\{f(z)\} & m-1 \leqq \Re(\mu) < m; \\ & m \in \mathbf{N}, \end{cases} \tag{1.2}$$

provided that the integral exists, a number of workers (including, possibly among others, Boyadjiev et al. [**5**], Al-Shammery et al. [**2, 3**] and Saxena and Kalla [**23**]) introduced and investigated several generalizations of the first-order Volterra-type integro-differential equation (1.1), $\mathbf{N}$ being, as usual, the set of *positive* integers. In each of the aforecited recent works on *fractional* integro-differential equations of Volterra type ([**2, 3, 5, 23**]), use is also made of expansion formulas for the confluent hypergeometric ${}_1F_1$ function in series of the *entire* incomplete gamma function $\gamma^*(\alpha, z)$ which is given, in terms of the ${}_1F_1$ function, by [**1**, p. 260 *et seq.*]

$$\gamma^*(\alpha, z) = \frac{e^{-z}}{\Gamma(\alpha+1)}\, {}_1F_1(1;\alpha+1;z) = \frac{1}{\Gamma(\alpha+1)}\, {}_1F_1(\alpha;\alpha+1;-z)\,. \tag{1.3}$$

In fact, as already observed by Srivastava [**25**], these expansion formulas do not hold true as asserted and applied by Boyadjiev et al. [**5**, p. 5, equations (14) and (15)], Al-Shammery et al. [**2**, p. 504, equation (15); **3**, p. 86] and Saxena and Kalla [**23**, p. 93, equation (2.18)].

A comprehensive account of the various extensions and generalizations of the FEL equation (1.1) can be found in a survey paper by

Boyadjiev and Kalla [**4**]. More recently, Kilbas et al. [**13**] systematically investigated a Cauchy-type problem associated essentially with the following generalization of the FEL equation (1.1):

$$\begin{gathered} D_{\tau}^{\alpha}\ h(\tau) = \lambda \int_0^{\tau} (\tau-\xi)^{\mu-1}\ E_{\kappa,\mu}^{\rho}\big(\omega(\tau-\xi)^{\kappa}\big) h(\xi)\ d\xi + f(\tau) \\ \tau \in [a,b] \subset \mathbf{R};\ \kappa,\lambda,\mu,\rho \in \mathbf{C};\ \Re(\alpha) > 0;\ \omega \in \mathbf{R}, \end{gathered} \tag{1.4}$$

where f is assumed to be (Lebesgue) integrable over the interval (a,b) and the function $E_{\kappa,\mu}^{\rho}(z)$, defined by, cf. [**20**],

$$E_{\kappa,\mu}^{\rho}(z) := \sum_{l=0}^{\infty} \frac{(\rho)_l}{\Gamma(\kappa l+\mu)}\,\frac{z^l}{l!} \qquad \Re(\kappa) > 0;\ (\rho)_l := \frac{\Gamma(\rho+l)}{\Gamma(\rho)}, \tag{1.5}$$

generalizes the classical Mittag-Leffler functions $E_{\kappa}(z)$, for $\rho = \mu = 1$, and $E_{\kappa,\mu}(z)$, for $\rho = 1$, and, ultimately, the exponential function e^z, for $\rho = \kappa = \mu = 1$. Indeed, in terms of the Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ with p numerator and q denominator parameter-pairs, cf., e.g., [**11, 30**]; see also [**27**, p. 21, equation 1.2 (38)] and [**28**, p. 42], we have

$$E_{\kappa,\mu}^{\rho}(z) = \frac{1}{\Gamma(\rho)}\ {}_1\Psi_1\left[(\rho,1);(\mu,\kappa);z\right]. \tag{1.6}$$

The subject of fractional calculus, that is, calculus of derivatives and integrals of any arbitrary real or complex order, has gained importance and popularity during the past three decades or so (see, for details, [**12, 16–19, 22**]; see also [**29**]). With this point in view, we propose here to investigate the solutions of several Cauchy-type and Cauchy problems associated with some general Volterra-type integro-differential equations of fractional order in which the kernel involves the confluent hypergeometric function $\Phi_2^{(n)}$ of n complex variables. The method used here is based, in part, upon the classical Laplace transform and the closed-form solution derived in each case is suitable for numerical computations.

2. Definitions and preliminaries. Some important and useful *multivariable* extensions of the classical (Gaussian) hypergeometric ${}_2F_1$ function include the Lauricella function $F_D^{(n)}$ and its confluent case $\Phi_2^{(n)}$ in n variables. In terms of the Pochhammer symbol $(\lambda)_n$ used already in the definition (1.5), we have [**27**, p. 33, equation 1.4 (4)]

(2.1)

$$F_D^{(n)}\left[a, b_1, \dots, b_n; c; z_1, \dots, z_n\right] := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!}$$

$$\max\left\{|z_1|, \dots, |z_n|\right\} < 1; \; c \notin \mathbf{Z}_0^- := \{0, -1, -2, \dots\}$$

and [**27**, p. 34, equation 1.4 (8) and 1.4 (10)]

$$\Phi_2^{(n)}\left[b_1, \dots, b_n; c; z_1, \dots, z_n\right] := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!}$$

$$= \lim_{|a| \to \infty} \left\{ F_D^{(n)}\left[a, b_1, \dots, b_n; c; \frac{z_1}{a}, \dots, \frac{z_n}{a}\right] \right\} \tag{2.2}$$

$$\max\left\{|z_1|, \dots, |z_n|\right\} < \infty; \; c \notin \mathbf{Z}_0^-.$$

Each of these multivariable hypergeometric functions can be represented by a *multiple* Mellin-Barnes contour integral which, in case of the function $\Phi_2^{(n)}$, can be written as follows, cf. [**9**, p. 232]; see also [**27**, pp. 284–285]:

(2.3)

$$\Phi_2^{(n)}\left[b_1, \dots, b_n; c; z_1, \dots, z_n\right] = \frac{\Gamma(c)}{\Gamma(b_1) \cdots \Gamma(b_n)} \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \frac{\Gamma(b_1 + \zeta_1) \cdots \Gamma(b_n + \zeta_n)}{\Gamma(c + \zeta_1 + \cdots + \zeta_n)}$$

$$\cdot \Gamma(-\zeta_1) \cdots \Gamma(-\zeta_n) (-z_1)^{\zeta_1} \cdots (-z_n)^{\zeta_n} \, d\zeta_1 \cdots d\zeta_n$$

$$i = \sqrt{-1};$$

$$\max\left\{|\arg(-z_1)|, \dots, |\arg(-z_n)|\right\} \leqq \pi - \varepsilon \quad (0 < \varepsilon < \pi);$$

$$c \notin \mathbf{Z}_0^-.$$

It is also readily seen from the definition (2.2) that

$$\begin{aligned} \Phi_2^{(n)}\left[b_1,\dots,b_n;c;z_1,0,\dots,0\right] &= \Phi_2^{(n)}\left[b_1,0,\dots,0;c;z_1,\dots,z_n\right] \\ &= {}_1F_1\left(b_1;c;z_1\right) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \lim_{\min\{|b_1|,\dots,|b_n|\}\to\infty} &\left\{\Phi_2^{(n)}\left[b_1,\dots,b_n;c;\frac{z_1}{b_1},\dots,\frac{z_n}{b_n}\right]\right\} \\ &= {}_0F_1\left(\text{---};c;z_1+\cdots+z_n\right), \end{aligned} \tag{2.5}$$

where the familiar confluent hypergeometric functions ${}_1F_1$ and ${}_0F_1$ are connected by Kummer's second formula [**27**, p. 322, equation 9.4 (184)]:

$$e^{-z}\,{}_1F_1\left(c;2c;2z\right) = {}_0F_1\left(\text{---};c+\frac{1}{2};\frac{1}{4}\,z^2\right). \tag{2.6}$$

By applying a method based, in part, upon the classical Laplace transform:

$$\begin{gathered} \mathcal{L}\left\{f(t):s\right\} := \int_0^\infty e^{-st}\,f(t)\,dt =: \mathcal{F}(s) \\ \Re(s) > 0, \end{gathered} \tag{2.7}$$

which may be written symbolically as follows:

$$\mathcal{F}(s) = \mathcal{L}\left\{f(t):s\right\} \quad \text{or} \quad f(t) = \mathcal{L}^{-1}\left\{\mathcal{F}(s):t\right\}, \tag{2.8}$$

provided that the function $f(t)$ is continuous for $t \geqq 0$, it being tacitly assumed that the integral in (2.7) exists, Srivastava [**24**] derived an explicit solution of the Volterra integral equation:

$$\begin{aligned} \int_0^\tau \frac{(\tau-\xi)^{\gamma-1}}{\Gamma(\gamma)}\,e^{\varepsilon(\tau-\xi)}\,\Phi_2^{(n)}&\left[\alpha_1,\dots,\alpha_n;\gamma;\right. \\ &\left.\lambda_1(\tau-\xi),\dots,\lambda_n(\tau-\xi)\right]h(\xi)\,d\xi = f(\tau) \end{aligned} \tag{2.9}$$

in the following explicit form, see also [**26**, p. 64 *et seq.*]:

$$
\begin{aligned}
h(\tau) = \int_0^\tau & \frac{(\tau-\xi)^{m-\gamma-1}}{\Gamma(m-\gamma)} e^{\varepsilon(\tau-\xi)} \\
& \cdot \Phi_2^{(n)} [-\alpha_1, \dots, -\alpha_n; m-\gamma; \lambda_1(\tau-\xi), \dots, \lambda_n(\tau-\xi)] \\
& \cdot \left\{ \left(\frac{d}{d\xi} - \varepsilon \right)^m f(\xi) \right\} d\xi \\
& \qquad 0 < \Re(\gamma) < m; \quad f \in C^m [0,\infty); \\
& f^{(j)}(0) = 0 \quad (j = 0, 1, \dots, m-1); \ m \in \mathbf{N}.
\end{aligned}
\tag{2.10}
$$

The confluent hypergeometric function $\Phi_2^{(n)}$ occurs naturally in the derivation of moments and density of the trace of a non-central Wishart matrix, see, for details, [**15**]. It is useful also in various other practical problems such as, for example, in storage capacity of a dam, queuing models, geometric probabilities, and time series analysis, cf. [**15, 28**]. For this confluent hypergeometric function $\Phi_2^{(n)}$ in n variables, the following Laplace-transform formula is known [**10**, Vol. I, p. 222]:

$$
\begin{aligned}
& \mathcal{L}\left\{ \frac{t^{\gamma-1}}{\Gamma(\gamma)} \Phi_2^{(n)} [\alpha_1, \dots, \alpha_n; \gamma; \lambda_1 t, \dots, \lambda_n t] : s \right\} \\
& \qquad = s^{-\gamma} \prod_{j=1}^n \left\{ \left(1 - \frac{\lambda_j}{s} \right)^{-\alpha_j} \right\} \\
& \Re(\gamma) > 0; \quad \Re(s) > \max_{j \in \{1, \dots, n\}} \{0, \Re(\lambda_j)\},
\end{aligned}
\tag{2.11}
$$

so that, by appealing to the familiar convolution theorem for the Laplace transform, it is not difficult to derive the following *integral addition formula* for $\Phi_2^{(n)}$:

(2.12)

$$
\begin{aligned}
& \int_0^t u^{\gamma-1} (t-u)^{\delta-1} \ \Phi_2^{(n)} [\alpha_1, \dots, \alpha_n; \gamma; \lambda_1 u, \dots, \lambda_n u] \\
& \qquad \cdot \Phi_2^{(n)} [\beta_1, \dots, \beta_n; \delta; \lambda_1 (t-u), \dots, \lambda_n (t-u)] \ du \\
& = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} t^{\gamma+\delta-1} \ \Phi_2^{(n)} [\alpha_1+\beta_1, \dots, \alpha_n+\beta_n; \gamma+\delta; \lambda_1 t, \dots, \lambda_n t] \\
& \qquad \min\{\Re(\gamma), \Re(\delta)\} > 0; \ \max\{|\lambda_1 t|, \dots, |\lambda_n t|\} < \infty.
\end{aligned}
$$

It is a special case when

$$\beta_1 = \cdots = \beta_n = 0,$$

we find from (2.12) that

$$\begin{aligned} &\int_0^t u^{\gamma-1} (t-u)^{\delta-1} \; \Phi_2^{(n)} [\alpha_1, \ldots, \alpha_n; \gamma; \lambda_1 u, \ldots, \lambda_n u] \, du \\ &= \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \, t^{\gamma+\delta-1} \; \Phi_2^{(n)} [\alpha_1, \ldots, \alpha_n; \gamma+\delta; \lambda_1 t, \ldots, \lambda_n t] \\ &\quad \min\{\Re(\gamma), \Re(\delta)\} > 0; \; \max\{|\lambda_1 t|, \ldots, |\lambda_n t|\} < \infty. \end{aligned} \tag{2.13}$$

Moreover, by virtue of the reduction formula (2.4), (2.12) readily yields the following known special case [**9**, p. 271, equation 6.10 (15)]:

$$\begin{aligned} &\int_0^t u^{\gamma-1} (t-u)^{\delta-1} \; {}_1F_1(\alpha; \gamma; \lambda u) \; {}_1F_1\big(\beta; \delta; \lambda(t-u)\big) \, du \\ &\qquad = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \, t^{\gamma+\delta-1} \, {}_1F_1(\alpha+\beta; \gamma+\delta; \lambda t) \\ &\qquad\quad \min\{\Re(\gamma), \Re(\delta)\} > 0; \; |\lambda t| < \infty. \end{aligned} \tag{2.14}$$

In our present investigation, we shall also make use of the following results for the Laplace transforms of fractional integrals and fractional derivatives, cf. [**16, 17, 19, 22**]:

$$\begin{aligned} &\mathcal{L}\{D_t^{\mu} \, f(t) : s\} \\ &= \begin{cases} s^{\mu} \; \mathcal{F}(s) & \Re(\mu) < 0 \\ s^{\mu} \; \mathcal{F}(s) - \sum_{k=0}^{m-1} s^k \; D_t^{\mu-k-1} \; f(t) \Big|_{t=0} & m-1 < \Re(\mu) \leqq m, \end{cases} \end{aligned} \tag{2.15}$$

where $m \in \mathbf{N}$ and $\mathcal{F}(s)$ is given by (2.7).

3. Solution of a generalized FEL equation. We begin by stating our solution of the following Cauchy-type problem involving a general fractional integro-differential equation of Volterra type.

Theorem 1. *Consider the following fractional integro-differential equation of Volterra type*:

$$\begin{aligned} D_{\tau}^{\alpha}\, h(\tau) = \frac{\kappa}{\Gamma(\gamma)} \int_0^{\tau} \xi^{\gamma-1}\, h(\tau-\xi) \\ \cdot\, \Phi_2^{(n)}\left[\alpha_1,\ldots,\alpha_n;\gamma;\lambda_1\xi,\ldots,\lambda_n\xi\right] d\xi + \mu f(\tau) \\ 0 \leqq \tau \leqq 1;\ \kappa,\mu \in \mathbf{C};\ \min\left\{\Re(\alpha),\Re(\gamma)\right\} > 0 \end{aligned} \tag{3.1}$$

together with the initial conditions:

$$\begin{aligned} D_{\tau}^{\alpha-k}\, h(\tau)\Big|_{\tau=0} = A_k, \quad k = 1,\ldots,N \\ N := -\left[-\Re(\alpha)\right];\ N-1 < \Re(\alpha) \leqq N;\ N \in \mathbf{N}, \end{aligned} \tag{3.2}$$

where $A_1,\ldots,A_N$ *are prescribed constants and* $f(\tau)$ *is assumed to be continuous on every finite closed interval* $[0,T]$, $0 < T < \infty$, *and of the exponential order* $e^{\eta\tau}$ *when* $\tau \to \infty$.

Then there exists a unique continuous solution of the Cauchy-type problem (3.1) *and* (3.2) *given by*

$$h(\tau) = \sum_{k=1}^{N} A_k\, \Lambda_k(\tau) + \mu \int_0^{\tau} \Omega(\tau-\xi)\, f(\xi)\, d\xi, \tag{3.3}$$

where

(3.4)

$$\begin{aligned} \Lambda_k(\tau) := \tau^{\alpha-k} \sum_{r=0}^{\infty} \frac{\kappa^r\, \tau^{(\alpha+\gamma)r}}{\Gamma\big(\alpha+(\alpha+\gamma)r-k+1\big)} \\ \cdot\, \Phi_2^{(n)}\left[\alpha_1 r,\ldots,\alpha_n r;\ \alpha+(\alpha+\gamma)r-k+1;\ \lambda_1\tau,\ldots,\lambda_n\tau\right] \\ k = 1,\ldots,N \end{aligned}$$

and

$$\begin{aligned} \Omega(\tau) := \tau^{\alpha-1} \sum_{r=0}^{\infty} \frac{\kappa^r\, \tau^{(\alpha+\gamma)r}}{\Gamma\big(\alpha+(\alpha+\gamma)r\big)} \\ \cdot\, \Phi_2^{(n)}\left[\alpha_1 r,\ldots,\alpha_n r;\alpha+(\alpha+\gamma)r;\lambda_1\tau,\ldots,\lambda_n\tau\right]. \end{aligned} \tag{3.5}$$

Proof. If we take the Laplace transforms of both sides of the integro-differential equation (3.1), using the known formulas (2.11) and (2.15), we obtain

$$\begin{aligned} & s^{\alpha}\ \mathcal{H}(s) - \sum_{k=1}^{N} s^{k-1}\ D_{\tau}^{\alpha-k}\ h(\tau)\Big|_{\tau=0} \\ & \qquad = \kappa s^{-\gamma}\ \mathcal{H}(s) \prod_{j=1}^{n} \left\{ \left(1-\frac{\lambda_j}{s}\right)^{-\alpha_j} \right\} + \mu\mathcal{F}(s) \\ & \min\{\Re(\alpha), \Re(\gamma)\} > 0; \quad \Re(s) > \max_{j\in\{1,\dots,n\}} \{0, \Re(\lambda_j), \Re(\eta)\}, \end{aligned} \tag{3.6}$$

where $\mathcal{F}(s)$ is given by (2.7), η is involved in the hypothesis of Theorem 1 concerning the exponential order of $f(\tau)$ for *large* τ, and, as usual, $\mathcal{H}(s)$ denotes the Laplace transform of the unknown function $h(\tau)$.

Solving (3.6) for $\mathcal{H}(s)$ under the initial conditions (3.2), we find that

$$\begin{aligned} \mathcal{H}(s) &= \left(\sum_{k=1}^{N} A_k\ s^{k-1} + \mu\mathcal{F}(s)\right) \left[s^{\alpha} - \kappa s^{-\gamma} \prod_{j=1}^{n} \left\{ \left(1-\frac{\lambda_j}{s}\right)^{-\alpha_j} \right\} \right]^{-1} \\ &= \sum_{k=1}^{N} A_k \sum_{r=0}^{\infty} \kappa^r\ s^{k-\alpha-(\alpha+\gamma)r-1} \prod_{j=1}^{n} \left\{ \left(1-\frac{\lambda_j}{s}\right)^{-\alpha_j r} \right\} \\ &\quad + \mu\mathcal{F}(s) \sum_{r=0}^{\infty} \kappa^r\ s^{-\alpha-(\alpha+\gamma)r} \prod_{j=1}^{n} \left\{ \left(1-\frac{\lambda_j}{s}\right)^{-\alpha_j r} \right\}, \end{aligned} \tag{3.7}$$

where we have tacitly assumed that

$$\left| \kappa s^{-\alpha-\gamma} \prod_{j=1}^{n} \left\{ \left(1-\frac{\lambda_j}{s}\right)^{-\alpha_j} \right\} \right| < 1.$$

By appealing to the formula (2.11) once again, we find from (3.7) that

(3.8)

$$\begin{aligned}h(\tau) = {} & \sum_{k=1}^{N} A_k \sum_{r=0}^{\infty} \frac{\kappa^r \ \tau^{\alpha+(\alpha+\gamma)r-k}}{\Gamma\bigl(\alpha+(\alpha+\gamma)r-k+1\bigr)} \\ & \cdot \Phi_2^{(n)}\left[\alpha_1 r, \ldots, \alpha_n r; \alpha+(\alpha+\gamma)r-k+1; \lambda_1\tau, \ldots, \lambda_n\tau\right] \\ & + \mu \sum_{r=0}^{\infty} \kappa^r \int_0^{\tau} \frac{(\tau-\xi)^{\alpha+(\alpha+\gamma)r-1}}{\Gamma\bigl(\alpha+(\alpha+\gamma)r\bigr)} \\ & \cdot \Phi_2^{(n)}\left[\alpha_1 r, \ldots, \alpha_n r;\ \alpha+(\alpha+\gamma)r; \lambda_1(\tau-\xi), \ldots, \lambda_n(\tau-\xi)\right] \\ & \cdot f(\xi)\ d\xi,\end{aligned}$$

which, in view of the definitions (3.4) and (3.5), is precisely the solution (3.3) asserted by Theorem 1.

In order to establish the uniqueness of the solution (3.3), we set

$$\xi \longmapsto \tau - \xi$$

in (3.1) and operate upon both sides by $D_\tau^{-\alpha}$, $\Re(\alpha) > 0$. Then, after some calculations using the Eulerian integral for the beta function, (3.1) is seen to be transformed into the following form:

$$\begin{aligned}h(\tau) = {} & \mu D_\tau^{-\alpha}\ f(\tau) + \frac{\kappa}{\Gamma(\alpha+\gamma)} \int_0^{\tau} h(\xi)(\tau-\xi)^{\alpha+\gamma-1} \\ & \cdot \Phi_2^{(n)}\left[\alpha_1, \ldots, \alpha_n; \alpha+\gamma; \lambda_1(\tau-\xi), \ldots, \lambda_n(\tau-\xi)\right] d\xi.\end{aligned} \tag{3.9}$$

Since (3.9) is a Volterra integral equation with a *continuous* kernel, it does admit a unique continuous solution, see [**14**].

Remark 1. The solution of the Cauchy-type problem (3.1) and (3.2) can also be developed by the method of successive approximations (see, for details, [**13, 21**]). Furthermore, such a seemingly unnecessary parameter as μ, which occurs in (3.1) and elsewhere in this paper, is being retained in this paper for the sake of later convenience in considering various specialized or limit cases of the problems and solutions investigated systematically by us.

4. Special cases and consequences of Theorem 1. First of all, we set

$$\lambda_1 = \lambda, \quad \lambda_2 = \cdots = \lambda_n = 0, \quad \text{and} \quad \alpha_1 = \beta \tag{4.1}$$

in Theorem 1. Then, in light of the hypergeometric identity (2.4), we obtain the following result.

Corollary 1. *Under the various relevant hypotheses of Theorem* 1, *a unique continuous solution of the Cauchy-type problem involving the Volterra-type integro-differential equation*:

$$D_\tau^\alpha \ h(\tau) = \frac{\kappa}{\Gamma(\gamma)} \int_0^\tau \xi^{\gamma-1} \ h(\tau - \xi) \ {}_1F_1(\beta; \gamma; \lambda\xi) \ d\xi + \mu f(\tau) \tag{4.2}$$
$$0 \leqq \tau \leqq 1; \ \kappa, \mu \in \mathbf{C}; \ \min\{\Re(\alpha), \Re(\gamma)\} > 0$$

and the initial conditions (3.2) *are given by*

$$h(\tau) = \sum_{k=1}^{N} A_k \ \Theta_k(\tau) + \mu \int_0^\tau \Xi(\tau - \xi) f(\xi) \ d\xi, \tag{4.3}$$

where

$$\begin{aligned} \Theta_k(\tau) := \tau^{\alpha-k} \sum_{r=0}^{\infty} & \frac{\kappa^r \ \tau^{(\alpha+\gamma)r}}{\Gamma\big(\alpha + (\alpha+\gamma) r - k + 1\big)} \\ & \cdot \ {}_1F_1\big(\beta r; \alpha + (\alpha+\gamma) r - k + 1; \lambda\tau\big) \\ & k = 1, \ldots, N \end{aligned} \tag{4.4}$$

and

$$\Xi(\tau) := \tau^{\alpha-1} \sum_{r=0}^{\infty} \frac{\kappa^r \ \tau^{(\alpha+\gamma)r}}{\Gamma\big(\alpha + (\alpha+\gamma) r\big)} \ {}_1F_1\big(\beta r; \alpha + (\alpha+\gamma) r; \lambda\tau\big). \tag{4.5}$$

Remark 2. In its further special case when

$$f(\tau) = \frac{\tau^{\delta-1}}{\Gamma(\delta)} \ {}_1F_1(\rho; \delta; \omega\tau), \tag{4.6}$$

if we apply the integral addition formula (2.14), Corollary 1 would yield the main result of Saxena and Kalla [**23**, p. 91, Theorem 1], which itself is a generalization of the result given earlier by Al-Shammery et al. [**2**, p. 504, equation (14)].

Next, if we set

$$f(\tau) = \frac{\tau^{\delta-1}}{\Gamma(\delta)}\ \Phi_2^{(n)}[\beta_1, \ldots, \beta_n; \delta; \lambda_1\tau, \ldots, \lambda_n\tau] \tag{4.7}$$

and apply the general (*multivariable*) integral addition formula (2.12), Theorem 1 would give us the following result.

Corollary 2. *Under the various relevant hypotheses of Theorem* 1*, a unique continuous solution of the Cauchy-type problem involving the Volterra-type integro-differential equation*:

$$\begin{aligned} D_\tau^\alpha\ h(\tau) = &\ \frac{\kappa}{\Gamma(\gamma)} \int_0^\tau \xi^{\gamma-1}\ h(\tau-\xi) \\ &\cdot \Phi_2^{(n)}[\alpha_1, \ldots, \alpha_n; \gamma; \lambda_1\xi, \ldots, \lambda_n\xi]\ d\xi \\ &+ \mu\ \frac{\tau^{\delta-1}}{\Gamma(\delta)}\ \Phi_2^{(n)}[\beta_1, \ldots, \beta_n; \delta; \lambda_1\tau, \ldots, \lambda_n\tau] \\ 0 \leqq \tau \leqq 1;\ &\kappa, \mu \in \mathbf{C};\ \min\{\Re(\alpha), \Re(\gamma), \Re(\delta)\} > 0 \end{aligned} \tag{4.8}$$

and the initial conditions (3.2) are given by

$$h(\tau) = \sum_{k=1}^{N} A_k\ \Psi_k(\tau) + \mu\Delta(\tau), \tag{4.9}$$

where

(4.10)

$$\begin{aligned} \Psi_k(\tau) := &\ \tau^{\alpha-k} \sum_{r=0}^{\infty} \frac{\kappa^r\ \tau^{(\alpha+\gamma)r}}{\Gamma\big(\alpha+(\alpha+\gamma)r-k+1\big)} \\ &\cdot \Phi_2^{(n)}[\alpha_1 r, \ldots, \alpha_n r; \alpha+(\alpha+\gamma)r-k+1; \lambda_1\tau, \ldots, \lambda_n\tau] \end{aligned}$$

and

(4.11)

$$\begin{aligned} \Delta(\tau) := &\ \tau^{\alpha+\delta-1} \sum_{r=0}^{\infty} \frac{\kappa^r\ \tau^{(\alpha+\gamma)r}}{\Gamma\big(\alpha+\delta+(\alpha+\gamma)r\big)} \\ &\cdot \Phi_2^{(n)}[\alpha_1 r+\beta_1, \ldots, \alpha_n r+\beta_n; \alpha+\delta+(\alpha+\gamma)r; \lambda_1\tau, \ldots, \lambda_r\tau]. \end{aligned}$$

Remark 3. By setting

$$\alpha_1 = \beta, \quad \beta_1 = \rho, \quad \text{and} \quad \alpha_2 = \cdots = \alpha_n = \beta_2 = \cdots = \beta_n = 0, \tag{4.12}$$

and applying the hypergeometric identity (2.4) once again, it is not difficult to deduce the aforementioned *main* result of Saxena and Kalla [**23**, p. 91, Theorem 1] as a *further* special case of Corollary 2 as well, see Remark 2 above.

5. A Cauchy problem involving the Caputo fractional derivatives. Such initial values as those occurring in (2.15) and (3.2) are usually not interpretable physically in a given initial-value problem. This situation is overcome at least partially by making use of the so-called *Caputo fractional derivative* which arose in several important works, dated 1969 onwards, by M. Caputo (see, for details, [**19**, p. 78 *et seq.*]).

In many recent works, especially in the theory of viscoelasticity and hereditary solid mechanics, the following (Caputo's) definition is adopted for the fractional derivative of order $\alpha > 0$ of a *causal* function $f(t)$, that is, $f(t) = 0$ for $t < 0$:

$$\frac{d^\alpha}{dt^\alpha} f(t) := \begin{cases} f^{(m)}(t) & \alpha = m \in \mathbf{N}_0 := \mathbf{N} \cup \{0\} \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau & m-1 < \alpha < m; \quad m \in \mathbf{N}, \end{cases} \tag{5.1}$$

where $f^{(m)}(t)$ denotes the usual (ordinary) derivative of $f(t)$ of order m $(m \in \mathbf{N}_0)$. As a matter of fact, it follows easily from the definitions (2.7) and (5.1) that

$$\mathcal{L}\left\{\frac{d^\alpha}{dt^\alpha} f(t) : s\right\} = s^\alpha\, \mathcal{F}(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}\, f^{(k)}(0)$$
$$m-1 < \alpha \leqq m; \; m \in \mathbf{N},$$

which obviously is *more* suited for initial-value problems than (2.15), $\mathcal{F}(s)$ being given, as before, by (2.7).

The method of proof of Theorem 1 can now be applied *mutatis mutandis* in order to solve the following Cauchy problem involving the Caputo fractional derivatives defined by (5.1).

Theorem 2. *Consider the following fractional integro-differential equation of Volterra type*:

$$
\begin{aligned}
\frac{d^\alpha}{d\tau^\alpha}\, h(\tau) = \frac{\kappa}{\Gamma(\gamma)} \int_0^\tau \xi^{\gamma-1}\, h(\tau-\xi) \\
\cdot\, \Phi_2^{(n)}\left[\alpha_1, \ldots, \alpha_n; \gamma; \lambda_1\xi, \ldots, \lambda_n\xi\right] d\xi + \mu f(\tau) \\
0 \leqq \tau \leqq 1;\ \kappa, \mu \in \mathbf{C};\ \alpha > 0;\ \Re(\gamma) > 0
\end{aligned} \tag{5.3}
$$

together with the initial conditions:

$$
\begin{aligned}
\left.\frac{d^k}{d\tau^k}\, h(\tau)\right|_{\tau=0} = B_k, \quad k = 0, \ldots, N-1 \\
N := -\left[-\alpha\right]; \quad N-1 < \alpha \leqq N; \quad N \in \mathbf{N},
\end{aligned}
$$

where $B_0, \ldots, B_{N-1}$ *are prescribed constants and* f *is constrained as in Theorem* 1.

Then there exists a unique continuous solution of the Cauchy problem (5.3) *and* (5.4) *given by*

$$
h(\tau) = \sum_{k=0}^{N-1} B_k\, \Lambda_k^*(\tau) + \mu \int_0^\tau \Omega(\tau-\xi)\, f(\xi)\, d\xi, \tag{5.5}
$$

where

$$
\begin{aligned}
\Lambda_k^*(\tau) := \tau^k \sum_{r=0}^\infty \frac{\kappa^r\ \tau^{(\alpha+\gamma)r}}{\Gamma\big(k + (\alpha+\gamma)\,r + 1\big)} \\
\cdot\, \Phi_2^{(n)}\left[\alpha_1 r, \ldots, \alpha_n r;\ k + (\alpha+\gamma)\,r + 1;\ \lambda_1\tau, \ldots, \lambda_n\tau\right] \\
k = 0, \ldots, N-1
\end{aligned} \tag{5.6}
$$

and $\Omega(t)$ *is defined, as in Theorem* 1, *by* (3.5).

Clearly, the assertions of Theorem 1 and Theorem 2 would coincide when we set

$$
\alpha = N, \quad N \in \mathbf{N}.
$$

More interestingly, it is fairly straightforward to apply Theorem 2 in order to deduce analogues of Corollary 1 and Corollary 2 for the corresponding Cauchy problems involving the Caputo fractional derivatives defined by (5.1). For example, by making the specializations listed in (4.1), we obtain the following analogue of Corollary 1 dealing with a more general Cauchy problem than that associated with the FEL equation (1.1).

Corollary 3. *Under the various relevant hypotheses of Theorem* 2*, a unique continuous solution of the Cauchy problem involving the Volterra-type integro-differential equation*:

$$\frac{d^{\alpha}}{d\tau^{\alpha}}\, h(\tau) = \frac{\kappa}{\Gamma(\gamma)} \int_0^{\tau} \xi^{\gamma-1}\, h(\tau-\xi)\, {}_1F_1(\beta;\gamma;\lambda\xi)\, d\xi + \mu f(\tau) \tag{5.7}$$

$$0 \leqq \tau \leqq 1; \quad \kappa,\mu \in \mathbf{C}; \quad \alpha > 0; \quad \Re(\gamma) > 0$$

and the initial conditions (5.4) *are given by*

$$h(\tau) = \sum_{k=0}^{N-1} B_k\, \Theta_k^{*}(\tau) + \mu \int_0^{\tau} \Xi(\tau-\xi)\, f(\xi)\, d\xi, \tag{5.8}$$

where

(5.9)

$$\Theta_k^{*}(\tau) := \tau^k \sum_{r=0}^{\infty} \frac{\kappa^r\, \tau^{(\alpha+\gamma)r}}{\Gamma\big(k+(\alpha+\gamma)r+1\big)}\, {}_1F_1\big(\beta r; k+(\alpha+\gamma)r+1; \lambda\tau\big)$$

$$k = 1, \dots, N-1$$

and $\Xi(\tau)$ *is defined, as in Corollary* 1*, by* (4.5).

6. A fractional-integral generalization of the Volterra integral equation (2.9). By setting $\mu = -\nu$ $(\nu \geqq 0)$ in the first assertion of (2.15), it is easily observed that

$$\mathcal{L}\left\{D_t^{-\nu}\, f(t) : s\right\} = s^{-\nu}\, \mathcal{F}(s), \quad \nu \geqq 0, \tag{6.1}$$

where $\mathcal{F}(s)$ is given, as before, by (2.7).

By means of the Laplace-transform formula (6.1), we can produce a fractional-integral generalization of the Volterra integral equation (2.9), whose solution is given by the following result.

Theorem 3. *The Volterra-type integral equation*:

(6.2)
$$D_\tau^{-\nu}\ h(\tau) = \kappa \int_0^\tau \frac{(\tau-\xi)^{\gamma-1}}{\Gamma(\gamma)}\ h(\xi) \cdot \Phi_2^{(n)}\left[\alpha_1,\ldots,\alpha_n;\gamma;\lambda_1(\tau-\xi),\ldots,\lambda_n(\tau-\xi)\right] d\zeta + \mu f(\tau)$$

has its solution given explicitly by

(6.3)
$$h(\tau) = -\mu \sum_{r=0}^{\infty} \kappa^{-r-1} \int_0^\tau \frac{(\tau-\xi)^{m+(\nu-\gamma)r-\gamma-1}}{\Gamma\big(m+(\nu-\gamma)r-\gamma\big)} \left(\frac{d^m}{d\xi^m}\ f(\xi)\right) \cdot \Phi_2^{(n)}\left[-\alpha_1(r+1),\ldots,-\alpha_n(r+1);\ m+(\nu-\gamma)r-\gamma;\ \lambda_1(\tau-\xi),\ldots,\lambda_n(\tau-\xi)\right] d\xi$$
$$0 < \Re(\gamma) < \min\{m,\nu\};$$
$$f \in C^m[0,\infty);\ f^{(j)}(0) = 0, \quad j = 0,1,\ldots,m-1;$$
$$m \in \mathbf{N}; \quad \kappa,\mu \in \mathbf{C}; \quad \nu \geqq 0.$$

Proof. In view of (2.11) and (6.1), we apply the Laplace-transform operator $\mathcal{L}$ on both sides of the Volterra integral equation (6.2), so that we readily have

(6.4)
$$\mathcal{H}(s) = \mu \mathcal{F}(s)\left[s^{-\nu} - \kappa s^{-\gamma} \prod_{j=1}^{n} \left\{\left(1-\frac{\lambda_j}{s}\right)^{-\alpha_j}\right\}\right]^{-1}$$
$$\nu \geqq 0; \quad \Re(\gamma) > 0; \quad \Re(s) > \max_{j\in\{1,\ldots,n\}} \{0, \Re(\lambda_j)\},$$

where $\mathcal{F}(s)$ and $\mathcal{H}(s)$ denote the Laplace transforms of $f(\tau)$ and $h(\tau)$, respectively.

Now, if we assume that

(6.5)
$$\left|\kappa^{-1}\ s^{\gamma-\nu} \prod_{j=1}^{n} \left\{\left(1-\frac{\lambda_j}{s}\right)^{\alpha_j}\right\}\right| < 1,$$

we find from (6.4) that

$$\begin{aligned}\mathcal{H}(s) = -\mu \sum_{r=0}^{\infty} \kappa^{-r-1}\ s^{\gamma-m-(\nu-\gamma)r} \prod_{j=1}^{n} \left\{ \left(1-\frac{\lambda_j}{s}\right)^{\alpha_j(r+1)} \right\} \\ \cdot \left[s^m\ \mathcal{F}(s) \right] \\ 0 < \Re(\gamma) < \min\{m,\nu\}; \quad m \in \mathbf{N}; \quad \nu \geqq 0,\end{aligned} \tag{6.6}$$

which leads us easily to the assertion (6.3) under the constraints stated already in Theorem 3.

This evidently completes the proof of Theorem 3. □

If, in the Volterra-type integral equation (6.2) and its solution (6.3), we set $\kappa = -\mu$, divide both sides of the resulting equation by μ, and then proceed to the limit as $|\mu| \to \infty$. Under this limit process, (6.2) and (6.3) would reduce immediately to the Volterra integral equation (2.9) and its solution (2.10), respectively, with, of course, $\varepsilon = 0$. Since $e^{\varepsilon(\tau-\xi)}$ can simply be introduced in the kernels of (2.9) and (2.10), with $\varepsilon = 0$, by means of the following notational changes:

$$h(\tau) \longmapsto e^{-\varepsilon\tau}\ h(\tau) \quad \text{and} \quad f(\tau) \longmapsto e^{-\varepsilon\tau}\ f(\tau),$$

Theorem 3 does indeed provide a generalization of Srivastava's results, cf. [**24**], involving (2.9) and (2.10).

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Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada
E-mail address: harimsri@math.uvic.ca

Department of Mathematics and Statistics, Jai Narain Vyas University Jodhpur 342005, Rajasthan, India
E-mail address: ravisax@sancharnet.in