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AN ABSTRACT GRONWALL LEMMA AND APPLICATIONS TO GLOBAL EXISTENCE RESULTS FOR FUNCTIONAL DIFFERENTIAL AND INTEGRAL EQUATIONS OF FRACTIONAL ORDER

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1. Introduction. The aim of this paper is two-fold. On the one hand, we prove an abstract generalization of a Gronwall lemma which gives *a priori* estimates for various (functional) differential and integral equations, of Volterra type, under a linear growth condition on the nonlinearity. We believe that this result is of independent interest and discuss it in a rather general setting. On the other hand, we apply a simple special case of this abstract result to obtain the existence of *global* solutions of the functional differential equation of fractional type

(1)
$$D^{\alpha}x(t) = f\big(t, x(t-c_1), \dots , x(t-c_n), D^{\alpha_1}x(t-a_1), \dots , D^{\alpha_k}x(t-a_k), I^{\beta_1}x(t-b_1), \dots , I^{\beta_m}x(t-b_m)\big)$$

under a linear growth condition on f. Here, $a_j, b_j, c_j \geq 0$, and $\alpha > \alpha_j > 0$ denote the, not necessarily integer, order of the corresponding (either Riemann-Liouville or Caputo) differential operators while $\beta_j > 0$ denote the, not necessarily integer, order of the (Abel) integral operators. We also consider inclusion problems of the type (1).

For $n = m = 0$, i.e., if the righthand side depends only on $(t, D^{\alpha_1}x(t-a_1), \dots , D^{\alpha_k}x(t-a_k))$, equation (1) has provoked some interest in the literature [**1, 2, 7, 9–12, 14, 25**]. In comparison with the existence results in these references, our assumptions are more natural. In contrast to these references, we only require that f has a linear growth and need not assume that this linear growth is sufficiently small. Of course, we can do this only because we obtain the required *a priori* estimate for the solution by means of our Gronwall lemma. We also drop the requirement that f is real-valued and consider the general

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case of a vector-valued Carathéodory function f. In [**1, 2, 11, 14**], the function f is assumed to be continuous while in [**7**] it is assumed that f is a real-valued Carathéodory function and in [**9, 10, 25**] and [**12**] the case of a Carathéodory function with a monotonicity condition, respectively, a function of bounded variation, was studied.

The plan of the paper is as follows. In Section 2 we prove the abstract Gronwall lemma which will be our main tool. After recalling some properties of the fractional (Abel) integral operator in Section 3, we prove our main existence result in Section 4. The particular case of equation (1) is treated in Section 5. Finally, the multi-valued case is discussed in Section 6.

2. The abstract Gronwall lemma. The Gronwall lemma can be interpreted as a result which gives *a priori* bounds for the norm of a solution of an implicit inequality under the assumption of a linear growth estimate.

In order to obtain such a result for fractional differential equations, we formulate the Gronwall lemma in an abstract form. We consider the following setting.

Let $X_{\mathbf{R}}$ be a (real) Banach lattice, and $K\colon X_{\mathbf{R}} \to X_{\mathbf{R}}$ be a positive bounded linear operator. We recall that the spectral radius of K is defined as the number

$$r(K) := \inf_n \sqrt[n]{\|K^n\|} = \lim_{n\to\infty} \sqrt[n]{\|K^n\|}\,. \tag{2}$$

It is well known that this limit always exists and that $r(K) < 1$, or, equivalently, $\|K^n\| < 1$ for some n, if and only if the Neumann series $\sum K^n$ converges in operator norm; in this case the series converges to the inverse of $I - K$ (which is then an isomorphism onto $X_{\mathbf{R}}$).

Proposition 2.1. *If K is linear and positive with $r(K) < 1$, then we have for each $b \in X_{\mathbf{R}}$ that*

$$0 \le y \le Ky + b \Longrightarrow \|y\| \le \|(I-K)^{-1}b\|.$$

Proof. Since K^n is positive, a trivial induction shows that $0 \le y \le Ky + b$ implies $0 \le y \le K^n y + \sum_{k=0}^{n-1} K^k b$, and so

$$\|y\| \le \|K^n y\| + \left\| \sum_{k=0}^{n-1} K^k b \right\| \to 0 + \|(I-K)^{-1} b\|, \quad n \to \infty. \qquad \square$$

This result is particularly useful in view of the fact that a linear θ-Volterra operator K, see below, has, if it is compact, in all interesting situations spectral radius 0, see [**30**]. However, we intend to generalize Proposition 2.1 to the nonlinear situation and for more general spaces than Banach lattices, in particular for spaces of vector-valued functions.

Let X be a linear vector space, let I be a linearly ordered set with a smallest element θ, and let $P_i : X \to X$, $i \in I$, be a family of linear projections with the property that $P_i P_j = P_j P_i = P_i$ for $i \le j$, i.e., the ranges of the projections P_i are nondecreasing and the null spaces are nonincreasing with respect to the index $i \in I$; assume also that $P_\theta = 0$. According to [**30**], we call a, not necessarily linear, operator $V : X \to X$ a *θ-Volterra operator* if $P_i V x$ depends only on $P_i x$, i.e., if

$$P_i V = P_i V P_i, \quad i \in I.$$

With the particular choice

$$I := \{\theta, 1\}, \quad P_\theta := 0 \quad \text{and} \quad P_1 := \mathrm{id} \tag{3}$$

each map $V : X \to X$ becomes a θ-Volterra operator, so the following result is actually not a result restricted to a particular class of operators. However, for the applications we have in mind the statement about the uniformity in $i \in I$ plays an important role.

Theorem 2.1 (Abstract Gronwall lemma). *Let in the above situation X be a normed space, and let $V : X \to X$ be a θ-Volterra operator, with respect to a family $(P_i)_{i \in I}$ of projections. Let $X_{\mathbf{R}}$ be a Banach lattice with positive cone C, and assume that there is a map $]\cdot[: X \to C$ such that for some constant $N < \infty$ the estimate*

$$\|x\| \le N \| \,]x[\, \|, \quad x \in X$$

holds. Assume in addition that there is a map $A\colon C\to X_{\mathbf{R}}$ *with*

$$]P_iVx[\,\le A]x[\,,\quad x\in X,\quad i\in I.$$

1. *If the implication*

$$0\le y\le Ay\Longrightarrow \|y\|\le M \tag{4}$$

holds, then we have the a priori *estimate*

$$]P_ix[\,\le\,]P_iVx[\,\Longrightarrow \|P_ix\|\le NM,\quad x\in X \tag{5}$$

uniformly for all $i\in I$.

2. *If* $Ax=Kx+b$ *where* $b\in X_{\mathbf{R}}$ *and* $K\colon X_{\mathbf{R}}\to X_{\mathbf{R}}$ *is positive and linear with* $r(K)<1$, *then* (4) *and* (5) *hold with* $M:=\|(I-K)^{-1}b\|$.

Proof. If

$$]P_ix[\,\le\,]P_iVx[\,=\,]P_iVP_ix[\,\le A]P_ix[\,,$$

then (4) implies $\|\,]P_ix[\,\|\le M$, which gives the estimate $\|P_ix\|\le NM$. The second claim follows from the first and Proposition 2.1. □

Let us first observe that the Gronwall lemma is indeed a special case of this result.

Let $X_{\mathbf{R}}$ be an *ideal space* of measurable functions $x\colon[0,T]\to\mathbf{R}$, i.e., a Banach space of measurable functions with the property that for each $x\in X_{\mathbf{R}}$ and each measurable $y\colon[0,T]\to\mathbf{R}$ with $|y(s)|\le|x(s)|$ almost everywhere we have $y\in X_{\mathbf{R}}$ and $\|y\|\le\|x\|$.

Let $(Y,|\cdot|)$ be a Banach space, and let X_Y be the space of all measurable functions $x\colon[0,T]\to Y$ with $|x|\in X_{\mathbf{R}}$, endowed with the norm $\|x\|:=\||x|\|$. Then X_Y is an ideal space (in particular complete, see [**26**]).

For example, if $X_{\mathbf{R}}$ is the Lebesgue-space $X_{\mathbf{R}}=L_p([0,T])$, $1\le p\le\infty$, then X_Y is the Lebesgue-Bochner space $X_Y=L_p([0,T],Y)$ with the usual norm.

Let $f\colon[0,T]\times[0,T]\times Y\to Y$ satisfy the *Carathéodory condition*, i.e.

1. $f(\cdot,\cdot,y)$ is measurable for each $y\in Y$, and

2. $f(t,s,\cdot)$ is continuous for almost all $(t,s) \in [0,T]^2$.

Let us assume that we have the linear growth condition

$$|f(t,s,y)| \leq k(t,s)|y| + b(t,s)$$

where k, b are nonnegative and measurable and

$$Ky(t) := \int_0^t k(t,s)y(s)\,ds$$

maps $X_{\mathbf{R}}$ into itself, and the function $B(t) := \int_0^t b(t,s)\,ds$ belongs to $X_{\mathbf{R}}$.

Then, with $I := [0,T]$, and

$$P_i x(t) := \begin{cases} x(t) & \text{if } t < i, \\ 0 & \text{if } t \geq i, \end{cases} \tag{6}$$

the integral operator

$$Vx(t) = \int_0^t f(t,s,x(s))\,ds + c(t)$$

defines a nonlinear θ-Volterra operator $V\colon X_Y \to X_Y$ for each $c \in X_Y$. Putting $]x[\,(t) := |x(t)|$, we obtain the following result.

Corollary 2.1 (Gronwall lemma for Volterra-Urysohn equations). *If in the above situation K is compact or $r(K) < 1$, then $I - K$ is an isomorphism and for each $i \in [0,T]$ the following holds*: *Each function $x \in X_Y$ which solves the implicit inequality*

$$|x(t)| \leq \left| \int_0^t f(t,s,x(s))\,ds + c(t) \right|$$

for almost all $t \in [0,i]$ is subject to the a priori *estimate*

$$\|P_i x\| \leq \|(I-K)^{-1}(B + |c|)\|.$$

Proof. If the linear Volterra operator K is compact, then $r(K) = 0$, see [**27**]. Hence, $r(K) < 1$ in both cases. It is no loss of generality to assume that for $s > t$ the values $f(t,s,y)$, $k(t,s)$ and $b(t,s)$ vanish. Then $|P_i V x(t)| \le K|x|(t) + B(t) + |c(t)|$ for almost all $t \in [0,T]$. Hence, the result follows from Theorem 2.1. □

If we assume that k can be chosen independent of t, which is possible if f is *sufficiently regular*, Corollary 2.1 takes a more familiar form. In fact, in this case we can calculate $(I-K)^{-1}$ explicitly:

Lemma 2.1. *Let the ideal space $X_{\mathbf{R}}$ contain the constant functions, and let k be a* (*real or complex-valued*) *measurable function such that $k \cdot x$ is integrable for each $x \in X_Y$. Then the operator*

$$Kx(t) := \int_0^t k(s)x(s)\,ds$$

is a bounded linear endomorphism of X_Y with spectral radius 0*, and the formulas*

$$K^n x(t) = \frac{1}{(n-1)!}\int_0^t \left(\int_s^t k(\sigma)\,d\sigma\right)^{n-1} k(s)x(s)\,ds, \quad n=1,2,\dots, \tag{7}$$

and

$$(I-K)^{-1}x(t) = x(t) + \int_0^t e^{\int_s^t k(\sigma)\,d\sigma} k(s)x(s)\,ds \tag{8}$$

hold.

Proof. By [**26**, Theorem 3.4.3], the function k belongs to the associate space of X, i.e., the functional $f(x) := \int_0^T |k(s)|x(s)\,ds$ is bounded on X_Y. Hence, $\|Kx\|_{L_\infty([0,T],Y)} \le \|f\|\|x\|_{X_Y}$, and since $L_\infty([0,T],Y)$ is by our assumption continuously embedded into X_Y, it follows that $K\colon X_Y \to X_Y$ is bounded. Formula (7) is proved by induction. For the induction step, observe that by Fubini-Tonelli

$$\begin{aligned} &K^{n+1}x(t) \\ &\quad = \frac{1}{(n-1)!}\int_0^T \chi_{[0,t]}(\tau)k(\tau)\int_0^\tau \left(\int_s^\tau k(\sigma)\,d\sigma\right)^{n-1} k(s)x(s)\,ds\,d\tau \\ &\quad = \int_0^t \left\{\int_s^t \frac{1}{(n-1)!}\,k(\tau)\left(\int_s^\tau k(\sigma)\,d\sigma\right)^{n-1} d\tau\right\} k(s)x(s)\,ds. \end{aligned}$$

The expression in curly braces is absolutely continuous with respect to t, and its derivative is (a.e.) the same as that of the absolutely continuous function

$$k_{n,s}(t) := \frac{1}{n!}\left(\int_s^t k(\sigma)\,d\sigma\right)^n.$$

Since the values for $t = s$ coincide, we conclude that

$$K^{n+1}x(t) = \int_0^t k_{n,s}(t)k(s)x(s)\,ds = \int_0^t \frac{1}{n!}\left(\int_s^t k(\sigma)\,d\sigma\right)^n k(s)x(s)\,ds,$$

and the induction step is complete. Hence, (7) is proved. From (7), we obtain by a straightforward estimate that $\|K^n\| \le 1/(n-1)!\|k\|_{L_1}^{n-1}\|f\|$ which implies that the spectral radius of K is 0. Finally, using the Neumann series, we obtain (8) from (7) by interchanging the order of summation and integration. The latter is justified by Lebesgue's dominated convergence theorem and Levi's monotone convergence theorem, see, e.g., [**31**, Theorem 1.18]. □

As a consequence, Corollary 2.1 takes for $X_{\mathbf{R}} = L_\infty$ a form which contains the usual Gronwall lemma as an obvious special case. The only reason why the following proof requires some reasoning concerning Lebesgue points is that we do not restrict our attention to the usual situation that x is continuous.

Corollary 2.2 (Gronwall lemma for nonsingular Volterra-Urysohn equations). *If $f\colon [0,\infty) \times [0,\infty) \times Y \to Y$ satisfies a Carathéodory condition and $|f(t,s,y)| \le k(s)|y| + b(t,s)$ with nonnegative measurable k and b, then each bounded measurable function x which solves the implicit inequality*

$$|x(t)| \le \left|\int_0^t f(t,s,x(s))\,ds + c(t)\right|$$

for almost all $t \in [0,T]$ is for almost all $t \in [0,T]$ subject to the explicit a priori *estimate*

$$|x(t)| \le \operatorname*{ess\,sup}_{s\in[0,t]}(B(s) + |c(s)|) + \int_0^t e^{\int_s^t k(\sigma)\,d\sigma} k(s)(B(s) + |c(s)|)\,ds$$

with $B(s) := \int_0^s b(s,\sigma)\,d\sigma$.

Proof. Apply Corollary 2.1 with $X_Y := L_\infty([0,T],Y)$. Then $Kx(t) = \int_0^t k(s)x(s)\,ds$, and so Lemma 2.1 implies

$$(I-K)^{-1}y(t) = y(t) + \int_0^t e^{\int_s^t k(\sigma)\,d\sigma} k(s)y(s)\,ds.$$

We obtain that

$$\|x\| \le \|B + |c|\| + \int_0^a e^{\int_s^a k(\sigma)\,d\sigma} k(s)(B(s) + |c(s)|)\,ds.$$

Hence, if a is a Lebesgue point of $|x|$, we obtain the required estimate for $t = a$. Since almost all points of $|x|$ are Lebesgue points, see, e.g., [**24**] or [**31**, Theorem 7.3], the claim follows. □

3. A mapping property of the fractional integral operator. It is a well-known consequence of an inequality of Young that the linear fractional integral operator

$$I_0^\alpha y(s) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds,$$

with $\alpha > 0$ and Γ denoting Euler's gamma-function, sends $L_q([0,T])$ continuously into $L_p([0,T])$ if $p, q \in [1,\infty]$ satisfy $q > 1/(\alpha + (1/p))$, see, e.g., [**13**] (a deep result from interpolation theory implies that even $q = 1/(\alpha + (1/p))$ is allowed if $1 < p < \infty$, but we will not make use of this fact. This stronger result was first proved by other methods in [**15**]). Since I_0^α is positive, and thus a regular integral operator, it is clear by a straightforward estimate that this acting property carries over to the vector-valued case, i.e., $I_0^\alpha\colon L_q([0,T],Y) \to L_p([0,T],Y)$ (boundedly) for each Banach space Y. The following lemma is folklore in case $Y = \mathbf{R}$, but to see that the equicontinuity also holds in the vector-valued case, we provide a proof.

Lemma 3.1. *The map* $I_0^\alpha\colon L_q([0,T]) \to L_p([0,T])$ *is compact for* $p, q \in [1,\infty]$ *if* $q > 1/(\alpha + (1/p))$ *and either* $q > 1$ *or* $p < \infty$. *In particular,* $I_0^\alpha\colon L_p([0,T]) \to L_p([0,T])$ *is compact for each* $p \in [1,\infty]$.

Moreover, if $q > \max\{1, (1/\alpha)\}$ *then* I_0^α *sends bounded subsets of* $L_q([0,T],Y)$ *into bounded equicontinuous subsets of* $C([0,T],Y)$ *(if we define* $I_0^\alpha x(0) := 0$*).*

Proof. Assume first $q > 1$ and $p < \infty$. Then the assumptions hold also if we replace q by $q_0 \in (1, q)$ and p by $p_0 \in (p, \infty)$. By the above remarks, $I_0^\alpha \colon L_{q_0}([0,T]) \to L_{p_0}([0,T])$. Since I_0^α is a positive, and thus regular, integral operator, it follows that $I_0^\alpha \colon L_q([0,T]) \to L_p([0,T])$ is compact, see, e.g., [**18**].

Let now $q > \max\{(1/\alpha), 1\}$. Then we have by Hölder's inequality with $1/q + 1/q' = 1$ in view of $q'(\alpha - 1) > -1$ that for each $0 \le t < \tau \le T$ (in view of $1 \le q' < \infty$) that

$$\begin{aligned}
&|I_0^\alpha x(\tau) - I_0^\alpha x(t)|\,\Gamma(\alpha) \\
&\quad \le \int_t^\tau (\tau - s)^{\alpha-1} |x(s)|\,ds + \int_0^t |(\tau - s)^{\alpha-1} - (t - s)^{\alpha-1}||x(s)|\,ds \\
&\quad \le \bigg(\int_t^\tau (\tau - s)^{(\alpha-1)q'}\,ds\bigg)^{1/q'} \|x\|_q \\
&\qquad + \bigg(\int_0^t |(\tau - s)^{\alpha-1} - (t - s)^{\alpha-1}|^{q'}\,ds\bigg)^{1/q'} \|x\|_q \\
&\quad \le \Bigg(\bigg(\frac{(\tau - t)^{1+(\alpha-1)q'}}{1 + (\alpha - 1)q'}\bigg)^{1/q'} \\
&\qquad + \bigg(\int_0^t |(\tau - s)^{(\alpha-1)q'} - (t - s)^{(\alpha-1)q'}|\,ds\bigg)^{1/q'} \|x\|_q \\
&\quad \le C(\alpha, q)\Big(|\tau - t|^{\alpha-(1/q)} \\
&\qquad + |(\tau - t)^{1+(\alpha-1)q'} - \tau^{1+(\alpha-1)q'} + t^{1+(\alpha-1)q'}|^{1/q'}\Big)\|x\|_q
\end{aligned}$$

with some finite constant $C(\alpha, q)$, depending only on α and q, from which the equicontinuity of the image of bounded sets under I_0^α is obvious; in view of $I_0^\alpha x(0) = 0$ also the boundedness follows. Note that in case $0 < \alpha < 1$ the above estimate implies

$$|I_0^\alpha x(\tau) - I_0^\alpha x(t)|\,\Gamma(\alpha) \le 2C(\alpha, q)|\tau - t|^{\alpha-(1/q)}\|x\|_q,$$

and so I_0^α is bounded from $L_q([0,T], Y)$ into the Hölder space $C_0^{\alpha-1/q}([0,T], Y)$, a result which is of course well known in case $Y = \mathbf{R}$.

By the Arzelà-Ascoli theorem, it follows that for $q > \max\{(1/\alpha), 1\}$ the map $I_0^\alpha \colon L_q([0,T]) \to C([0,T])$ is compact, and so also $I_0^\alpha \colon L_q([0,T]) \to L_p([0,T])$ is compact.

Let now $q = 1$. Then our assumption states $p < \infty$ and $\alpha + (1/p) > 1$. By Schauder's theorem, it suffices to prove that the adjoint operator $(I_0^\alpha)^*: L_{p'}([0,T]) \to L_\infty([0,T])$ $((1/p) + (1/p') = 1)$ is compact. A routine calculation of the adjoint of I_0^α (which is a regular integral operator, see, e.g., [**18**]), shows that this adjoint has, up to the composition with transformations $x \mapsto x(T - \cdot)$, the same form as I_0^α, and the latter is compact as we have seen before, because $p' > \max\{(1/\alpha), 1\}$. □

Corollary 3.1. *If $p < \infty$, then, in the operator norm of linear endomorphisms of $L_p([0,T])$,*

$$\|P_{D_n} I_0^\alpha\| \downarrow 0, \quad D_n \downarrow \varnothing \tag{9}$$

where $P_D x(t) := \chi_D(t) x(t)$. Moreover, if $1 < p < \infty$ also

$$\|I_0^\alpha P_{D_n}\| \downarrow 0, \quad D_n \downarrow \varnothing. \tag{10}$$

Proof. Since I_0^α is a compact positive integral operator, the claim follows from [**6**] (in the case $1 < p < \infty$ this is also contained in [**20**]). □

Unfortunately, (10) does not hold in case $p = 1$. Indeed, the norm of $I_0^\alpha P_{D_n}$ is the norm of the adjoint operator $P_{D_n}^*(I_0^\alpha)^* = P_{D_n}(I_0^\alpha)^*$ (in the space of linear endomorphisms of $L_\infty([0,T])$), and the latter can tend to zero for every sequence $D_n \downarrow \varnothing$ only if $(I_0^\alpha)^* = 0$ which evidently is not the case.

4. Gronwall lemma and global existence for Volterra-Urysohn functional equations of fractional type. In a Banach space Y, we consider now the nonlinear functional Volterra-Urysohn operator

$$Vx(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t, s, (V_1 x)(s), \dots, (V_N x)(s))\, ds + c(t)$$
$$t \in [0,T].$$

Here, $f\colon [0,T] \times [0,T] \times Y_1 \times \cdots \times Y_N \to Y$ (with Banach spaces $Y_1, \dots, Y_N$) is a function satisfying the Carathéodory condition, $c \in L_p([0,T],Y)$ with $1 \le p \le \infty$, and $V_k\colon L_p([0,T],Y) \to L_p([0,T],Y_k)$, $k = 1, \dots, N$. We assume that all V_k are continuous and bounded, i.e., map bounded sets onto bounded sets, and that there are linear positive operators $K_k\colon L_p([0,T]) \to L_p([0,T])$ which satisfy the Volterra property

$$x|_{[0,t]} = y|_{[0,t]} \quad \text{a.e.} \implies K_k x|_{[0,t]} = K_k y|_{[0,t]} \quad \text{a.e.} \tag{11}$$

and majorize V_k in the sense that

$$|V_k x(t)| \le K_k|x|(t) \quad \text{for almost all} \quad t \in [0,T] \tag{12}$$

for each $x \in L_p([0,T],Y)$. Moreover, we assume that $\alpha > 0$ and that f satisfies the linear growth condition

$$|f(t,s,y_1,\dots,y_n)| \le C(|y_1| + \cdots + |y_n|) + b(t,s) \tag{13}$$

with $C \in [0,\infty)$ and a, necessarily nonnegative, measurable b with the property that the function

$$B(t) := \int_0^t (t-s)^{\alpha-1} b(t,s)\, ds \tag{14}$$

belongs to $L_p([0,T])$.

Note that by our above assumptions and the remarks at the beginning of Section 3, the operator V sends $L_p([0,T],Y)$ into itself.

Theorem 4.1 (Gronwall lemma for Volterra-Urysohn functional equations of fractional type). *Let $1 \le p \le \infty$. There is a constant $M < \infty$ depending only on the above data such that each $x \in L_p([0,T],Y)$ which satisfies*

$$|x(t)| \le |Vx(t)| \tag{15}$$

for almost all $t \in [0,T]$ is subject to the norm estimate $\|x\|_{L_p([0,T])} < M$.

Moreover, if all V_k, and thus also V, satisfy also the Volterra-property (11)*, then each $x \in L_p([0,T],Y)$ which satisfies* (15) *only for almost all $t \in [0,i]$ with $i \in [0,T]$ must satisfy the norm estimate $\|x\|_{L_p([0,i])} < M$.*

Proof. Putting $K_0 := K_1 + \cdots + K_n$, we have, since K_k are positive, that

$$|Vx(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(CK_0|x|(s) + b(s))\, ds + |c(t)|.$$

Hence, putting

$$Ax = CI_0^\alpha K_0 x + B + |c|,$$

we have the required estimate $|Vx(t)| \leq A|x|(t)$. The claim thus follows from Theorem 2.1 (with either the trivial family of projections (3) or the canonical projections (6)) if we can prove that the spectral radius of $K := CI_0^\alpha K_0$ in $L_p([0,T])$ is less than 1. We show that actually $r(K) = 0$.

Assume first that $p < \infty$. Since K is clearly a linear compact θ-Volterra operator with respect to the canonical projections $P_i x := \chi_{[0,i)} \cdot x$, then [**29**, Theorem 2] implies that $r(K) = 0$ if K is partially additive, which is trivially satisfied by the linearity, and the range of K is regular in the sense that for each $i \in [0,T)$ and each function x in the range we have

$$\inf_{j>i} \|P_{i,j}x\| = 0 \tag{16}$$

where $P_{i,j} := P_j - P_i$. Condition (16) means that $\inf_{j>i} \|\chi_{[i,j)}x\|_p = 0$ which holds for each $x \in L_p([0,T])$ by Lebesgue's dominated convergence theorem (in the case $p < \infty$).

Assume now that $p = \infty$ or at least $p > \max\{1,(1/\alpha)\}$. In this case, Lemma 3.1 implies that the range of K is contained in the space $C_0([0,T])$ of continuous functions vanishing at 0, and so we have in view of the Gel'fand formula (2)

$$r(K) = \inf_n \sup_{\substack{y=Kx \\ \|x\|_p \leq 1}} \sqrt[n+1]{\|K^n y\|} \;\leq\; \inf_n \sup_{\substack{y\in C_0([0,T]) \\ \|y\|\leq\|K\|}} \sqrt[n+1]{\|K^n y\|}\,.$$

Hence, it suffices to prove that K has spectral radius 0 in the space $C_0([0,T])$. In this space, we consider the canonical projections

$$P_i x(t) := \begin{cases} x(t) & \text{if } t \le i, \\ x(i) & \text{if } t \ge i, \end{cases}$$

and it is clear that (16) holds for these projections. Since K is compact by Lemma 3.1 and the Arzelà-Ascoli theorem, the same argument as above shows that the spectral radius of K in $C_0([0,T])$ vanishes, as required. ◻

One could try to obtain a global existence result by Theorem 4.1 analogously as one does for ordinary differential equations: One could first prove that each solution on some interval $[0,i)$, $i=0$ not excluded, can be extended to a solution on a slightly larger interval $[0,i+\varepsilon_i)$. Then one could look for a maximal solution and apply the *a priori* bound of Theorem 4.1, see, e.g., [**30**, Proposition 6.7] for an abstract formulation of the principle and [**30**, Section 7.1] for an example. However, we proceed in a different way, making use of the Leray-Schauder principle, see e.g., [**8**]:

Theorem 4.2 (Leray-Schauder). *Let Ω be an open subset of a Banach space X with $0 \in \Omega$, and $V\colon \overline{\Omega} \to X$ continuous with compact $\overline{V(\Omega)}$. If λV has for no $\lambda \in [0,1]$ a fixed point on $\partial\Omega$, then V has a fixed point in Ω.*

To apply the Leray-Schauder principle, we only need the following special case of Theorem 4.1.

Corollary 4.1. *With $R := M$ as in Theorem* 4.1 *the following holds*: *If there is a solution $x \in L_p([0,T],Y)$ of the equation $\lambda x = Vx$ with $\|x\|_p \ge R$ and a scalar λ, then $|\lambda| < 1$.*

In order to employ Corollary 4.1, we need a result on complete continuity of the involved operators. For the extremal cases $p=1$ and $p=\infty$, we get such a result if we require in addition that f is actually independent of t and that we consider functions with values in a finite-dimensional space:

Lemma 4.1. *Assume, in addition to our general hypotheses,* (13)–(14)*, that the Carathéodory function f is independent of t and that the space Y has finite dimension. Then the Volterra-Hammerstein operator*

$$A(x_1,\dots,x_N)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x_1(s),\dots,x_N(s))\,ds$$

is continuous and compact from $L_p([0,T],Y_1\times\cdots\times Y_N)$ into $L_p([0,T], Y)$ for $1\le p<\infty$. The same holds for $p=\infty$ if the family $\{f(s,\cdot) : s\in[0,T]\}$ is equicontinuous on bounded subsets of $Y_1\times\cdots\times Y_n$.

Proof. Clearly, A is the composition of the superposition operator

$$F(x_1,\dots,x_N)(s) := f(s,x_1(s),\dots,x_N(s))$$

and the partial integral operator I_0^α. The growth condition (13) implies that F acts from $X := L_p([0,T],Y_1\times\cdots\times Y_N)$ into $Z := L_p([0,T],Y)$ and is bounded. Moreover, if $p<\infty$, then, since f is a Carathéodory function, X is an ideal space and Z is a regular ideal space, $F\colon X\to Z$ is automatically continuous, see [**26**, Theorem 5.2.1]. If $p=\infty$, the continuity of F is obtained by a straightforward calculation from the equicontinuity assumption. Since I_0^α is compact in $L_p([0,T],Y)$ by Lemma 3.1 (because Y has finite dimension, one can consider the coordinates separately), the claim follows. □

If Y has infinite dimension or f depends also on t, we can treat only the case $1<p<\infty$, and we have to require the following additional assumptions:

(A) f is a strict Carathéodory function in the sense of [**30**], i.e., the function $g(t,s) := f(t,s,\cdot,\dots,\cdot)$ is measurable as a function from $[0,T]\times[0,T]$ into the space $C(Y_1\times\cdots\times Y_n,Y)$ of continuous functions, endowed with the topology of uniform convergence on bounded sets.

(B) For almost all $(t,s)\in[0,T]\times[0,T]$ the map $g(t,s)$ is compact, i.e., for each bounded $M_k\subseteq Y_k$ the image $g(t,s)(M_1\times\cdots\times M_n)$ is a precompact subset of Y.

These are no additional assumptions, i.e., they are automatically satisfied for a Carathéodory f, if the spaces $Y_1,\dots,Y_n$ have finite dimension, see [**5**]. In general, we have only:

Proposition 4.1. *The Carathéodory function f is a strict Carathéodory function if and only if there is a set $S \subseteq [0,T] \times [0,T]$ of full measure such that for the function g above the image $g(S)$ is separable in $C(Y_1 \times \cdots \times Y_n, Y)$.*

Proof. In view of [**30**, Proposition 8.8] this is an immediate consequence of [**30**, Theorem 8.5]. □

Unfortunately, if one of the spaces Y_k has infinite dimension, then the space $C(Y_1 \times \cdots \times Y_n, Y)$ is never separable (for $Y \neq \{0\}$ of course) even if $Y = \mathbf{R}$, see [**28**]. Nevertheless the assumption that f be a strict Carathéodory function is not so severe for applications as one might guess from this observation at a first glance: For many applications, compact operators like $g(t,s)$, for fixed t, s, come from Urysohn integral operators as, e.g., when $Y_1 = \cdots = Y_n$ is an ideal space (over some measure space S) and also Y is an ideal space and

$$g(t,s)(x)(\tau) = \int_S h_{t,s}(\tau, \sigma, x(\sigma))\, d\sigma.$$

In this situation, if $(t, s, \tau, \sigma, u) \mapsto h_{t,s}(\tau, \sigma, u)$ is a strict Carathéodory function also f is a strict Carathéodory function under very mild growth conditions; for more details, we refer to [**30**, Section 10].

Lemma 4.2. *Let conditions* (A) *and* (B) *hold, in addition to our general hypotheses* (13)–(14). *If $1 < p < \infty$, then the Volterra-Urysohn operator*

$$A(x_1, \dots, x_N)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t, s, x_1(s), \dots, x_N(s))\, ds$$

is continuous and compact from $L_p([0,T], Y_1 \times \cdots \times Y_N)$ into $L_p([0,T], Y)$.

Proof. Endowing $Y_1 \times \cdots \times Y_N$ with the sum-norm $|(y_1, \dots, y_k)| := |y_1| + \cdots + |y_n|$ and putting $P_D x(t) := \chi_D(t)x(t)$, we have the estimate

$$\int_{D_n\cap[0,t]} |(t-s)^{\alpha-1} f(t,s,x_1(s),\dots,x_N(s))|\,ds \\ \le \Gamma(\alpha)(I_0^\alpha P_{D_n})(C|(x_1,\dots,x_N)|)(t) + \int_{D_n\cap[0,t]} (t-s)^{\alpha-1} b(t,s)\,ds.$$

As $D_n \downarrow \varnothing$, the norm, in $L_p([0,T])$, of the righthand side converges to 0 by (10) uniformly for $x = (x_1, \dots, x_N)$ on bounded subsets of $L_p([0,T], Y_1 \times \cdots \times Y_N)$. Similarly, (9) implies that

$$\chi_{D_n}(t) \int_0^t |(t-s)^{\alpha-1} f(t,s,x_1(s),\dots,x_N(s))|\,ds$$

converges in norm to 0 as $D_n \downarrow \varnothing$, uniformly for $x = (x_1, \dots, x_N)$ on bounded subsets of $L_p([0,T], Y_1 \times \cdots \times Y_N)$. Also, we obtain immediately from (13) that

$$\int_0^t \sup_{|(y_1,\dots,y_N)|\le n} |(t-s)^{\alpha-1} f(t,s,y_1,\dots,y_N)|\,ds < \infty$$

for each $n \in \mathbf{N}$ and almost all t. In view of [**30**, Proposition 9.13], in particular the formula (9.11) in the subsequent remarks, it follows that A is uniformly regular in the sense of [**30**, Definition 9.2]. Hence, the claim follows from [**30**, Theorem 9.10] (the assumption (9.8) in that theorem is satisfied with $r_n \equiv 0$ by [**30**, Proposition 9.11]). □

Lemma 4.3. *Let conditions* (A) *and* (B) *hold, in addition to our general hypotheses,* (11)–(14). *If* $1 < p < \infty$, *then the Volterra-Urysohn functional equation*

(17)
$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t,s,(V_1x)(s),\dots,(V_Nx)(s))\,ds + c(t) \\ \text{for almost all} \quad t \in [0,T]$$

has a global solution $x \in L_p([0,T], Y)$ *for each* $c \in L_p([0,T], Y)$.

If f is not a strict Carathéodory function but independent of t, and if Y has finite dimension, then an analogous result holds for $1 \le p < \infty$ and also for $p = \infty$ if the family $\{f(s,\cdot) : s \in [0,T]\}$ is equicontinuous.

Proof. Up to the constant function c, the operator V on the righthand side is the composition of the continuous operator $L\colon L_p([0,T],Y) \to L_p([0,T],Y_1 \times \cdots \times Y_N)$ defined by

$$Lx(t) := (V_1x(t), \dots, V_nx(t)) \tag{18}$$

and the continuous compact operator A from Lemma 4.2, respectively from Lemma 4.1. Hence, $V\colon L_p([0,T],Y) \to L_p([0,T],Y)$ is continuous and compact, and its fixed points are precisely the solutions $x \in L_p([0,T],Y)$ of (17).

To see that V has a fixed point, we apply the Leray-Schauder principle (Theorem 4.2) with $X := L_p([0,T],Y)$ and $\Omega := \{x \in X : \|x\| < R\}$ with R as in Corollary 4.1. We have to verify that the equation $x = \lambda Vx$ has no solution with $\|x\|_p = R$ and $0 < \lambda < 1$. But this follows immediately from Corollary 4.1. Hence, the Leray-Schauder principle implies that V has a fixed point in Ω which is a required solution. □

The assumption $p < \infty$ can be dropped if the operators V_k are "sufficiently well behaved" for at least *some* finite $p > 1$. This is our main existence result.

Theorem 4.3 (Global existence for Volterra-Urysohn functional equations of fractional type). *Let $1 < p \le \infty$ and $Y_1, \dots, Y_N$ and Y be Banach spaces. The Volterra-Urysohn equation*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t,s,(V_1x)(s),\dots,(V_Nx)(s))\,ds + c(t) \quad \text{for almost all} \quad t \in [0,T] \tag{19}$$

has a (global) solution $x \in L_p([0,T],Y)$ under the following assumptions:

1. $c \in L_p([0,T],Y)$.
2. *f is a strict Carathéodory function, i.e.,* (A) *holds, and*

3. *$f(t,s,\cdot,\dots,\cdot): Y_1\times\cdots\times Y_N$ is a compact map for almost all (t,s).*

4. *f satisfies the linear growth estimate* (13) *with $C\in[0,\infty)$ and a measurable function b such that the function* (14) *belongs to $L_p([0,T])$.*

5. *$V_k: L_p([0,T],Y)\to L_p([0,T],Y_k)$ are linear and bounded for $k=1,\dots,N$.*

6. *There is some $q\le p$ with $1<q<\infty$ such that also $V_k: L_q([0,T],Y)\to L_q([0,T],Y_k)$ are linear and dominated in the sense* (12) *by positive linear operators $K_k: L_q([0,T])\to L_q([0,T])$ which satisfy the Volterra-property* (11).

If Y has finite dimension and f is independent of t, then it suffices that f is a Carathéodory function, not necessarily a strict Carathéodory function, and also the choices $q=1$ or even $p=1$ are admissible. If additionally the family $\{f(s,\cdot): s\in[0,T]\}$ is equicontinuous, then also the choice $q=p=\infty$ is admissible.

Proof. Observe first that a variant of the Riesz-Thorin interpolation theorem for Lebesgue-Bochner spaces implies that, for each $r\in[q,p]$ the operator V_k is also bounded from $L_r([0,T],Y)$ into $L_r([0,T],Y_k)$ (this follows, e.g., from slight variations of the proofs in [**17**, Chapter IV]). Hence, the estimate

$$|Vx(t)|\le (C\cdot I_0^\alpha|(V_1x,\dots,V_Nx)|)(t)+B(t)+|c(t)|$$

shows that V maps $L_r([0,T],Y)$ into $L_\rho([0,T],Y)$ whenever $q\le r\le\rho\le p$ and $1/\rho>(1/r)-\alpha$ (because, as remarked earlier, $I_0^\alpha: L_r([0,T])\to L_\rho([0,T])$).

In particular, each solution of $x=Vx$ in $L_r([0,T],Y)$ automatically belongs to $L_\rho([0,T],Y)$ if $q\le r\le\rho\le p$ and $1/\rho>(1/r)-\alpha$. A trivial induction by n thus shows that each solution of $x=Vx$ in $L_q([0,T],Y)$ automatically belongs to $L_r([0,T],Y)$ if $r\in[q,p]$ and $1/r>(1/p)-n\alpha$. Hence, the solution automatically belongs to $L_p([0,T],Y)$. Since Lemma 4.3 implies the existence of a solution in $L_q([0,T],Y)$, the claim follows. □

Remark 4.1. The proof shows that Theorem 4.3 remains valid for nonlinear operators V_k, provided one assumes in addition that $V_k: L_r([0,T],Y)\to L_r([0,T],Y_k)$ is continuous and bounded for each $r\in[q,p]$.

5. Application to functional differential equations of fractional order. We recall that the fractional integral operator of order $\alpha > 0$ with lefthand point a is defined by

$$I_a^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds.$$

Using the known relations between the Beta- and Gamma-function, a well-known calculation with the Fubini-Tonelli theorem shows that $I_a^{\alpha+\beta} x = I_a^\alpha I_a^\beta x$ for each $x \in L_1([a,b])$ and each $\alpha, \beta > 0$. In particular, I_a^n is the nth iterate of the usual integral operator, and so I_a^α may indeed be considered as a corresponding fractional integral. We define the corresponding (Riemann-Liouville) differential operator

$$D_a^{n+\alpha} := D^{n+1} I_a^{1-\alpha} \tag{20}$$

when $n \in \mathbf{N}_0 := \{0, 1, 2, \dots\}$ and $\alpha \in [0, 1)$. Here, D denotes the usual differential operator. This is an appropriate definition of a fractional derivative by the following observation.

Lemma 5.1. *For $0 < \alpha \le \beta$ we have*

$$D_a^0 x = x \ \textit{a.e.}, \quad x \in L_1([a,b], Y),$$

and for each $x \in L_1([a,b], Y)$ we have

$$D_a^\beta I_a^\alpha x = D_a^{\beta-\alpha} x, \quad \textit{a.e.}, \tag{21}$$

and either both sides of (21) *are defined or none. In particular, when $\alpha = \beta$,* (21) *means that the operator $D_a^\alpha I_a^\alpha$ is defined on $L_1([a,b], Y)$ and that D_a^α is the left-inverse of I_a^α. For an integer $\alpha = \beta = n$ and $x \in I_a^n(L_1([a,b], Y))$, we have $D_a^n x = D^n x$.*

Proof. The first claim, i.e., $D I_a^1 x = x$, follows from the fact that almost all points of $x \in L_1([a,b], Y)$ are Lebesgue points, see, e.g., [**31**, Corollary 7.1]. Let $\beta = n + \gamma$ and $\beta - \alpha = m + \delta$ with $n, m \in \mathbf{N}_0$ and $\gamma, \delta \in [0, 1)$. Then we have in view of $D I_a^1 x = x$ that

$$\begin{aligned} D_a^\beta I_a^\alpha x &= D^{n+1} I_a^{1-\gamma} I_a^\alpha x = D^{n+1} I_a^{1+\alpha-\gamma} x = D^{n+1} I_a^{1+n-m-\delta} x \\ &= D^{m+1} D^{n-m} I_a^{n-m} I_a^{1-\delta} x = D^{m+1} I_a^{1-\delta} x = D_a^{\beta-\alpha} x. \qquad \square \end{aligned}$$

We consider the functional differential equation of fractional type

$$D_0^\alpha x(t) = f\big(t, x(t-c_1), \dots, x(t-c_n), D_{-A_1}^{\alpha_1} x(t-a_1), \dots, D_{-A_k}^{\alpha_k} x(t-a_k), I_{-B_1}^{\beta_1} x(t-b_1), \dots, I_{-B_m}^{\beta_m} x(t-b_m)\big) \tag{22}$$

for $t \in [0,T]$ when $0 < \alpha_j < \alpha$, $\beta_1, \dots, \beta_m > 0$, $0 \le a_j \le A_j$, $0 \le b_j \le B_j$, and $c_j \ge 0$.

In order for problem (22) to make sense, we have to assume that the initial values $x_{[-A,0]} := X_0$ are given where $A := \max\{A_1, \dots, A_k, B_1, \dots, B_m\}$.

Proposition 5.1. *Assume that $0 < \beta < \alpha$ and that $x\colon [-A,T] \to Y$ is a solution of the Volterra-Urysohn functional equation of fractional type*

$$x = I_0^{\alpha-\beta} Gx \tag{23}$$

where $Gx \in L_1([0,T], Y)$ with

$$Gx(s) := f\big(s, I_0^\beta x(s-c_1), \dots, I_0^\beta x(s-c_n), D_{-A_1}^{\alpha_1} I_0^\beta x(s-a_1), \dots, D_{-A_k}^{\alpha_k} I_0^\beta x(s-a_k), I_{-B_1}^{\beta_1} I_0^\beta x(s-b_1), \dots, I_{-B_m}^{\beta_m} I_0^\beta x(s-b_m)\big) \tag{24}$$

Then $y := I_0^\beta x$ solves (22) almost everywhere on $[0,T]$.

Proof. We have $y = I_0^\beta (I_0^{\alpha-\beta} Gx) = I_0^\alpha Gx$ which, in view of Lemma 5.1, implies that $D_0^\alpha y$ is defined almost everywhere and equal to Gx which gives the claim. □

Theorem 5.1. *Let $n, k, m \in \mathbf{N} \cup \{0\}$ and $T \in (0, \infty)$ and suppose*

1. *$f\colon [0,T] \times Y^{n+k+m} \to Y$ is a strict Carathéodory function.*
2. *$f(s, \cdot)\colon Y^{n+k+m} \to Y$ is a compact map for almost all $s \in [0,T]$.*
3. *f satisfies the linear growth estimate*

$$|f(s, y_1, \dots, y_{n+k+m})| \le C(|y_1| + \dots + |y_n|) + b(s) \tag{25}$$

with $C \in [0, \infty)$ and some $b \in L_1([0,T], Y)$.

Let $\beta := \max\{\alpha_1, \ldots, \alpha_k\} < \alpha$ *with* $\alpha_j > 0$, $\beta_1, \ldots, \beta_m > 0$, $0 \le a_j \le A_j$, $0 \le b_j \le B_j$, $c_j \ge 0$ *and* $A := \max\{A_1, \ldots, A_k, B_1, \ldots, B_m\}$.

Then for each initial data $X_0 \colon [-A, 0] \to Y$ *which is contained in the range* $I_0^\beta(L_1([-A,0], Y))$ *there is a function* $y \colon [-A, T] \to Y$ *with* $y|_{[-A,0]} = X_0$ *which solves* (22) *almost everywhere.*

Proof. Note first that by the remarks in Section 3 we have for any $b \in L_1([0,T], Y)$ that the function (14) belongs to $L_p([0,T], Y)$ for $1 \le p \le 1/(1-\alpha)$ (if $\alpha < 1$, in case $\alpha \ge 1$ the function belongs even to $C([0,T], Y)$). In particular, the function belongs to $L_p([0,T], Y)$ for some $p \in (1, \infty)$.

Our assumption on the initial data implies in view of Lemma 5.1 that there exists a function $Z_0 \in L_1([-A, 0], Y)$ for which $D_0^\beta Z_0 = X_0$, almost everywhere, where the lefthand side is defined. In order to apply Proposition 5.1, we restrict our attention to functions x which satisfy $x|_{[-A,0]} = Z_0$. With this additional requirement, we can rewrite the arguments of f in (24) for $s \ge 0$ as the sum of a fixed function $h_{i,j}$ (depending on Z_0 but not on x) and a linear operator of Volterra type $V_{i,j}$ which depends only on $x|_{[0,T]}$. More precisely, we substitute in (24)

$$\begin{aligned} x(s - c_j) &= h_{1,j}(s) + V_{1,j}x(s) \\ D_{-A_j}^{\alpha_j} I_0^\beta x(s - a_j) &= h_{2,j}(s) + V_{2,j}x(s) \\ I_{-B_j}^{\beta_j} x(t - b_j) &= h_{3,j}(s) + V_{3,j}x(s) \end{aligned}$$

where we put

$$\begin{aligned} h_{1,j}(s) &:= \begin{cases} Z_0(s - c_j) & \text{if } s \le c_j, \\ 0 & \text{if } s > c_j, \end{cases} \\ V_{1,j}x(s) &:= \begin{cases} 0 & \text{if } s \le c_j, \\ x(s - c_j) & \text{if } s > c_j, \end{cases} \\ h_{2,j}(s) &:= \begin{cases} D_{-A_j}^{\alpha_j} I_0^\beta Z_0(s - a_j) & \text{if } s \le a_j, \\ D_{-A_j}^{\alpha_j} I_0^\beta Z_0(0) & \text{if } s > a_j, \end{cases} \end{aligned}$$

$$V_{2,j}x(s) := \begin{cases} 0 & \text{if } s \le a_j, \\ D_0^{\alpha_j} I_0^\beta x(s-a_j) & \text{if } s > a_j, \end{cases} = \begin{cases} 0 & \text{if } s \le a_j, \\ I_0^{\beta-\alpha_j} x(s-a_j) & \text{if } s > a_j, \end{cases}$$

$$h_{3,j}(s) := \begin{cases} I_{-B_j}^{\beta_j} I_0^\beta Z_0(s-b_j) & \text{if } s \le b_j, \\ I_{-B_j}^{\beta_j} I_0^\beta Z_0(0) & \text{if } s > b_j, \end{cases}$$

$$V_{3,j}x(s) := \begin{cases} 0 & \text{if } s \le b_j, \\ I_0^{\beta_j} I_0^\beta x(s-b_j) & \text{if } s > b_j. \end{cases} = \begin{cases} 0 & \text{if } s \le b_j, \\ I_0^{\beta_j+\beta} x(s-b_j) & \text{if } s > b_j. \end{cases}$$

The crucial point is that the operators $V_{i,j}$ are linear operators, of Volterra type, and depend only on $x|_{[0,T]}$. Moreover, in view of $\beta \ge \alpha_j$, all the operators $V_{i,j}$ are bounded in $L_p([0,T],Y)$. The function

$$\begin{aligned} &\tilde{f}(s, u_1, \dots, u_n, v_1, \dots, v_k, w_1, \dots, w_m) \\ &\quad := f\big(s, h_{1,1}(s) + u_1, \dots, h_{1,n}(s) + u_n, h_{2,1}(s) + v_1, \dots, h_{2,k}(s) \\ &\qquad\qquad + v_k, h_{3,1}(s) + w_1, \dots, h_{3,m}(s) + w_m\big) \end{aligned}$$

satisfies (25) (with $\tilde{b} := b + C\sum_{i,j} h_{i,j}$), and now it is evident that (24) has the form (19) where all assumptions of Theorem 4.3 are satisfied (with $q = p$ and the Volterra operators $V_{i,j}$). Hence, Theorem 4.3 implies that we find a solution $x \in L_p([0,T],Y)$ of the rewritten equation. Putting $y(t) := I_0^\beta x(t)$ for $t > 0$, the claim now follows by Proposition 5.1. □

Besides the fractional Riemann-Louville derivative (20) also the *Caputo* derivative

$$\frac{d_a^{n+\alpha} x}{dt^{n+\alpha}} := D^n I_a^{1-\alpha} Dx, \quad n = 0, 1, \dots, \ 0 \le \alpha < 1 \tag{26}$$

is of interest for applications. The connection with the Riemann-Liouville derivative is well-known and easy to see:

Proposition 5.2. *If $x\colon [a,b] \to Y$ is absolutely continuous and the derivative Dx exists almost everywhere with $Dx \in L_1([a,b],Y)$, then*

$$\frac{d_a^\alpha x}{dt^\alpha} = D_a^{n+\alpha}(x - x_a), \quad \alpha > 0$$

where x_a denotes the constant function with value $x(a)$ (and either both sides exist or none if $\alpha > 1$).

Proof. We note first that under our assumption on Dx the absolute continuity of x is equivalent to the fundamental theorem of calculus, see, e.g., [**31**, Theorem 7.6], i.e., equivalent to $I_a^1 Dx = x - x_a$. Hence, in view of Lemma 5.1, we have for each $\alpha \in [0,1)$ and each $n = 0, 1, \dots$ that

$$\begin{aligned}\frac{d_a^{n+\alpha}x}{dt^{n+\alpha}} &= D^n(D_a^\alpha I_a^\alpha)I_a^{1-\alpha}Dx = (D^n D_a^\alpha)(I_a^\alpha I_a^{1-\alpha})Dx \\ &= (D^{n+1}I_a^{1-\alpha})I_a^{\alpha+1-\alpha}Dx = D_a^{n+\alpha}(I_a^1 Dx) = D_a^{n+\alpha}(x - x_a). \quad \square\end{aligned}$$

Remark 5.1. It is well known that, if Y has finite dimension, then the absolute continuity of x already implies that Dx exists almost everywhere and is integrable, see, e.g., [**31**, Theorem 7.5]. However, in general, e.g., for $Y = L_1([a,b], \mathbf{R})$, there are Y-valued absolutely continuous functions which are nowhere differentiable. The question of whether absolute continuity of a Y-valued function implies the existence of an integrable derivative is related to the geometry of Y.

The above definition (26) has the disadvantage that it completely loses its meaning if x fails to be (almost everywhere) differentiable. For this reason, we use the property of Proposition 5.2 to define the Caputo derivative in general, i.e., we put

$$\frac{d_a^\alpha x}{dt^\alpha}(t) := D_a^\alpha x(t) - D_a^\alpha x_a(t) = D_a^\alpha x(t) - \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}x(a). \tag{27}$$

Proposition 5.2 implies that, for absolutely continuous functions with an integrable derivative, this definition coincides with the usual definition (26) of the Caputo derivative.

For more remarks concerning the connection of the Riemann-Liouville and the Caputo derivative, we refer to [**22, 23**].

Theorem 5.2. *Theorem* 5.1 *remains true if we replace in* (22) *in some, or all, occurrences the Riemann-Liouville derivative by the Caputo derivative* (*in the sense* (27)).

Proof. We insert (27) into all occurrences of the Caputo derivative, in equation (22), using always the corresponding initial value X_0 for the second term in (27). Thus, we can rewrite the given initial value problem into an equivalent problem where only the Riemann-Liouville derivative occurs. This new problem can also be written in the form (22) with a slightly modified function f (similar as in the proof of Theorem 5.1). This modified function evidently satisfies all requirements of Theorem 5.1, and so the existence of a solution follows from Theorem 5.1. □

6. The multi-valued case. In this section, we extend some results of the previous sections to the case of inclusions. For the corresponding Gronwall lemma we need no hypothesis concerning the continuity or measurability of the involved multi-valued maps. Concerning existence results, we consider only lower semi-continuous maps. As usual in this case, the philosophy is to apply an appropriate selection theorem. However, under our assumptions, it is not clear whether one can find corresponding Carathéodory selections so that the existence results of this section do not immediately follow from our previous results.

Let us first discuss the results of Section 4 in the multi-valued setting. We consider the inclusion

$$x(t) \in \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(t,s,(\mathcal{V}_1 x)(s),\dots,(\mathcal{V}_N x)(s))\,ds + c(t), \tag{28}$$

where $\mathcal{V}_k \colon L_p([0,T],Y) \multimap L_p([0,T],Y_k)$, $k=1,\dots,N$, and $F\colon Y_1 \times \cdots \times Y_N \multimap Y$ are multi-valued (with nonempty values) and $c \in L_p([0,T],Y)$. Here, the righthand side is understood as the set

$$\mathcal{V}x(t) := \bigcup \bigg\{ \int_0^t (t-s)^{\alpha-1} F(t,s,y_1(s),\dots,y_N(s))\,ds + c(t) \\ : y_k \in \mathcal{V}_k x \text{ for } k=1,\dots,N \bigg\},$$

where, as usual, the integral of a multi-valued function is understood as the family of integrals of all measurable selections of the integrand.

We understand the hypotheses (12) and (13) correspondingly for each value of the set $\mathcal{V}_k x$, respectively of the set $F(t,s,y_1,\dots,y_n)$, at the lefthand side.

Theorem 6.1 (Multi-valued a priori estimate). *With the above interpretation of the formulae, Theorem* 4.1 *continues to hold with the above operator* $\mathcal{V}$ *where we replace* (15) *by the hypothesis that there is a measurable selection* y *of* $\mathcal{V}x$ *such that*

$$|x(t)| \le |y(t)| \quad \text{for almost all} \quad t \in [0,T]. \tag{29}$$

Proof. As in the proof of Theorem 4.1, one obtains for each measurable selection y of $\mathcal{V}x$ the estimate $|y(t)| \le A|x|(t)$ with $Ax := Kx + B + |c|$ where K is a linear positive operator in $L_p([0,T])$ with spectral radius 0. Choosing here y corresponding to x with (29), we obtain with $z(t) := |x(t)|$ and the usual order in $X_{\mathbf{R}} := L_p([0,T])$ that $0 \le z \le Kz + \tilde{b}$ with $\tilde{b} := B + |c|$. Proposition 2.1 implies $\|x\| = \|z\| \le \|(I-K)^{-1}\tilde{b}\|$. □

For the rest of this section, we assume that F is independent of t. Recall that F is called a *lower Carathéodory function* if $F(\cdot, u_1, \dots, u_N)$ is measurable and $F(s, \cdot, \dots, \cdot)$ is lower semi-continuous. Unfortunately, even if $N = 1$, this does not imply that for each measurable function $y_1, \dots, y_N$ the multi-valued map $G(s) := F(s, y_1(s), \dots, y_N(s))$ admits a measurable selection, see [**3**, Example 7.2]. Therefore we have to require slightly more.

We restrict our attention to the case that Y has finite dimension. In this case, the above multi-valued function G is measurable if and only if its graph is measurable or, equivalently, if $\{s : G(s) \subseteq M\}$ is measurable for each open set $M \subseteq Y$ or, equivalently, for each closed set $M \subseteq Y$, see [**16**]. According to [**3**], we call F *sup-measurable* if the above function G is measurable for each choice of measurable functions $y_1, \dots, y_N$. If we require in addition that the values of F, i.e., of G, are closed sets, then the Kuratowski-Ryll-Nardzewski selection theorem [**19**] implies that G has a measurable selection.

Now we are in a position to formulate a multi-valued version of Theorem 4.3.

Theorem 6.2 (Global existence for lower semi-continuous Volterra-Urysohn functional inclusions of fractional type). *Let* $Y_1, \dots, Y_N$ *and* Y *be finite-dimensional spaces,* $1 \le p < \infty$*, and let* F *be independent of* t*. Suppose in addition*:

1. $c \in L_p([0,T], Y)$.

2. *F assumes only nonempty closed convex values and is a lower Carathéodory function which is sup-measurable and satisfies the linear growth estimate*

$$\sup\{|y| : y \in F(s, y_1, \dots, y_n)\} \le C(|y_1| + \dots + |y_n|) + b(s) \tag{30}$$

with $C \in [0, \infty)$ *and* $b \in L_p([0,T])$.

3. *$\mathcal{V}_k: L_p([0,T], Y) \multimap L_p([0,T], Y_k)$ are lower semi-continuous for $k = 1, \dots, N$ with nonempty closed convex values and dominated in the sense that*

$$|y(t)| \le K_k|x|(t), \quad \text{for almost all} \quad t \in [0,T]$$

for each $y \in \mathcal{V}_k x$ where $K_k: L_p([0,T]) \to L_p([0,T])$ are positive linear operators satisfying the Volterra-property (11).

Then the inclusion (28) *has a solution in* $L_p([0,T])$.

Proof. Put $Z := L_p([0,T], Y_1 \times \dots \times Y_N)$ and $X := L_p([0,T], Y)$, and define $\mathcal{L}: X \multimap Z$ by the formula (18). By Michael's selection theorem [**21**], each $\mathcal{V}_k$ has a continuous selection, and so also $\mathcal{L}$ has a continuous selection $L_0: X \to Z$. As explained above, F induces a nonlinear superposition operator $\mathcal{F}: Z \multimap X$ with nonempty values, i.e., the value $\mathcal{F}z$ consists of all measurable selections of $s \mapsto F(s, z(s)))$. In view of (30), F assumes only compact values, and $\mathcal{F}$ sends order-bounded sets into order-bounded sets. Since in addition X is regular and F is sup-measurable and lower Carathéodory, it follows from [**4**] that $\mathcal{F}$ is lower semi-continuous. Clearly, $\mathcal{F}z$ is closed and convex for each $z \in Z$. Applying Michael's selection theorem once more, we find a continuous selection $F_0: Z \to X$ of $\mathcal{F}$. We do not know whether F_0 is a superposition operator induced by a Carathéodory selection of F. However, since $\mathcal{V} = I_0^\alpha \mathcal{F} \mathcal{L}$, the operator $V_0 := I_0^\alpha F_0 L_0$ is a continuous and compact selection of $\mathcal{V}$. Choosing the constant M as in Theorem 6.1, we apply the Leray-Schauder theorem (Theorem 4.2) with the operator V_0 and $\Omega := \{x \in X : \|x\| < M\}$. For each $x \in X$ and each $\lambda \in [0,1]$ with $x = \lambda V_0 x$ we have with $y := V_0 x \in \mathcal{V}y$ that (29) holds, and so $x \in \Omega$. Hence, the Leray-Schauder theorem implies that $x = V_0 x$ has a solution which thus satisfies $x \in \mathcal{V}x$. $\square$

As a consequence of this theorem, we obtain the following "lower semi-continuous" variant of Theorems 5.1 and 5.2 for functional differential inclusions of fractional type. Since the argument is only a slight variation of the arguments used in Section 5, we skip the proof.

Theorem 6.3. *Let Y be a finite-dimensional space, $n, k, m \in \mathbf{N} \cup \{0\}$ and $T \in (0,\infty)$. Suppose that $F\colon [0,T] \times Y^{n+k+m} \multimap Y$ is a lower Carathéodory and sup-measurable function which assumes only nonempty closed convex values and satisfies the linear growth estimate* (30) *with $C \in [0,\infty)$ and $b \in L_1([0,T])$.*

Let $\beta := \max\{\alpha_1, \dots, \alpha_k\} < \alpha$ with $\alpha_j > 0$, $\beta_1, \dots, \beta_m > 0$, $0 \le a_j \le A_j$, $0 \le b_j \le B_j$, $c_j \ge 0$, and $A := \max\{A_1, \dots, A_k, B_1, \dots, B_m\}$.

Then for each initial data $X_0\colon [-A,0] \to Y$ which is contained in the range $I_0^\beta(L_1([-A,0],Y))$, there is a function $x\colon [-A,T] \to Y$ with $x|_{[-A,0]} = X_0$ which satisfies

(31)

$$D_0^\alpha x(t) \in F\big(t, x(t-c_1), \dots, x(t-c_n), D_{-A_1}^{\alpha_1} x(t-a_1), \dots, D_{-A_k}^{\alpha_k} x(t-a_k), \\ I_{-B_1}^{\beta_1} x(t-b_1), \dots, I_{-B_m}^{\beta_m} x(t-b_m)\big)$$

almost everywhere. An analogous result holds if we replace in (31) *in some, or all, occurrences the Riemann-Liouville derivative by the Caputo derivative in the sense* (27).

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