

**BOUNDEDNESS IN NONLINEAR FUNCTIONAL  
DIFFERENTIAL EQUATIONS WITH APPLICATIONS  
TO VOLTERRA INTEGRODIFFERENTIAL EQUATIONS**

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ABSTRACT. Non-negative definite Lyapunov functions are employed to obtain sufficient conditions that guarantee boundedness of solutions of nonlinear functional differential systems. The theory is illustrated with several examples.

**1. Introduction.** In this paper, we make use of non-negative definite Lyapunov functions and obtain sufficient conditions that guarantee the boundedness of all solutions of the system of functional differential equations

$$(1.1) \quad x'(t) = G(t, x(s); 0 \leq s \leq t) \stackrel{\text{def}}{=} G(t, x(\cdot))$$

where  $x \in \mathbf{R}^n$ ,  $G : \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a given nonlinear continuous function in  $t$  and  $x$ . For a vector  $x \in \mathbf{R}^n$  we take  $\|x\|$  to be the Euclidean norm of  $x$ . Let  $t_0 \geq 0$ , then for each continuous function  $\phi : [0, t_0] \rightarrow \mathbf{R}^n$ , there is at least one continuous function  $x(t) = x(t, t_0, \phi)$  on an interval  $[t_0, I]$  satisfying (1.1) for  $t_0 \leq t \leq I$  and such that  $x(t, t_0, \phi) = \phi(t)$  for  $0 \leq t_0 \leq I$ . It is assumed that at  $t = t_0$ ,  $x'(t)$  is the right hand derivative of  $x(t)$ . For conditions ensuring existence, uniqueness and continuability of solutions of (1.1), we refer the reader to [2] and [5].

In [10], the author studied the boundedness of solutions of the initial value problem

$$\begin{aligned} x'(t) &= G(t, x(t)); \quad t \geq 0 \\ x(t_0) &= x_0 \end{aligned}$$

by making use of non-negative definite Lyapunov functions.

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A stereotype of equation (1.1) is the Volterra integrodifferential equation

$$(1.2) \quad x'(t) = Ax(t) + \int_0^t B(t,s)f(x(s)) ds.$$

We are mainly interested in applying our results to Volterra integrodifferential equations of the forms of (1.2) with  $f(x) = x^n$  where  $n$  is positive and rational. We emphasize that the results of [10] do not apply to equations similar to (1.2). In [1] Burton et al. proved general theorems using Lyapunov functionals of convolution types and obtained conditions for boundedness of solutions and stability of the zero solution of (1.1). However, in this paper our conditions are different and offer a new perspective at looking at the notion of boundedness. As application, we will apply our obtained results to nonlinear Volterra integrodifferential equations. At the end of the paper we will compare our theorems to those obtained in [11] and show that our results are different when it comes to applications. For more on the boundedness and stability of solutions of (1.1), we refer the interested reader to [3, 4, 6–9, 12].

**2. Boundedness of solutions.** In this section we use non-negative Lyapunov type functionals and establish sufficient conditions to obtain boundedness results on all solutions  $x(t)$  of (1.1). From this point forward, if a function is written without its argument, then the argument is assumed to be  $t$ .

**Definition 2.1.** We say that solutions of system (1.1) are bounded, if any solution  $x(t, t_0, \phi)$  of (1.1) satisfies

$$\|x(t, t_0, \phi)\| \leq C(\|\phi\|, t_0), \quad \text{for all } t \geq t_0,$$

where  $C : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is a constant that depends on  $t_0$  and  $\phi$  is a given continuous and bounded initial function. We say that solutions of system (1.1) are uniformly bounded if  $C$  is independent of  $t_0$ .

If  $x(t)$  is any solution of system (1.1), then for a continuously differentiable function

$$V : \mathbf{R}^+ \times \mathbf{R}^n \longrightarrow \mathbf{R}^+,$$

we define the derivative  $V'$  of  $V$  by

$$V'(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^n \frac{\partial V(t, x)}{\partial x_i} f_i(t, x).$$

A continuous function  $W : [0, \infty) \rightarrow [0, \infty)$  with  $W(0) = 0$ ,  $W(s) > 0$  if  $s > 0$  and  $W$  strictly increasing is called a wedge. (In this paper wedges are always defined by  $W$  or  $W_i$  where  $i$  is a positive integer).

**Theorem 2.2.** *Let  $D$  be a set in  $\mathbf{R}^n$ . Suppose there exists a continuously differentiable Lyapunov functional  $V : \mathbf{R}^+ \times D \rightarrow \mathbf{R}^+$  that satisfies*

$$(2.1) \quad \lambda_1 W_1(|x|) \leq V(t, x) \leq \lambda_2 W_2(|x|) + \lambda_2 \int_0^t \varphi_1(t, s) W_3(|x(s)|) ds$$

and

$$(2.2) \quad V'(t, x) \leq -\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \varphi_2(t, s) W_5(|x(s)|) ds + L$$

for some positive constants  $\lambda_1, \lambda_2, \lambda_3$  and  $L$ , where  $\varphi_i(t, s) \geq 0$  is a scalar function continuous for  $0 \leq s \leq t < \infty$ ,  $i = 1, 2$ , such that for some constant  $\gamma \geq 0$  the inequality

$$(2.3) \quad W_2(|x|) - W_4(|x|) + \int_0^t (\varphi_1(t, s) W_3(|x(s)|) - \varphi_2(t, s) W_5(|x(s)|)) ds \leq \gamma$$

holds. Moreover, if  $\int_0^t \phi_1(t, s) ds \leq B$  for some positive constant  $B$ , then all solutions of (1.1) that stay in  $D$  are uniformly bounded.

*Proof.* Let  $M = \lambda_3/\lambda_2$ . For any initial time  $t_0 \geq 0$ , let  $x(t)$  be any solution of (1.1) with  $x(t) = \phi(t)$ , for  $0 \leq t \leq t_0$ . Then,

$$\frac{d}{dt}(V(t, x(t))e^{M(t-t_0)}) = [V'(t, x(t)) + MV(t, x(t))]e^{M(t-t_0)}.$$

For  $x(t) \in \mathbf{R}^n$ , using (2.2) we get

$$\begin{aligned}
 (2.4) \quad & \frac{d}{dt}(V(t, x(t))e^{M(t-t_0)}) \\
 & \leq \left[ -\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \varphi_2(t, s) W_5(|x(s)|) ds + L \right. \\
 & \quad \left. + M\lambda_2 W_2(|x|) + M\lambda_2 \int_0^t \varphi_1(t, s) W_3(|x(s)|) ds \right] e^{M(t-t_0)} \\
 & = \lambda_3 \left[ W_2(|x|) - W_4(|x|) \right. \\
 & \quad \left. + \int_0^t (\varphi_1(t, s) W_3(|x(s)|) - \varphi_2(t, s) W_5(|x(s)|)) ds + L \right] e^{M(t-t_0)} \\
 & \leq (\lambda_3 \gamma + L) e^{M(t-t_0)} \\
 & =: K e^{M(t-t_0)}.
 \end{aligned}$$

Integrating (2.4) from  $t_0$  to  $t$  we obtain,

$$\begin{aligned}
 V(t, x(t))e^{M(t-t_0)} & \leq V(t_0, \phi) + \frac{K}{M} e^{M(t-t_0)} - \frac{K}{M} \\
 & \leq \lambda_2 V(t_0, \phi) + \frac{K}{M} e^{M(t-t_0)}.
 \end{aligned}$$

Consequently,

$$V(t, x(t)) \leq \lambda_2 V(t_0, \phi) e^{-M(t-t_0)} + \frac{K}{M}.$$

From condition (2.1) we have  $\lambda_1 W_1(|x|) \leq V(t, x(t))$ , which implies that

$$|x| \leq W_1^{-1} \left[ \left\{ \frac{1}{\lambda_1} \right\} \left( \lambda_2 W_2(|\phi|) + \lambda_2 W_3(|\phi|) \int_0^{t_0} \varphi_1(t_0, s) ds + \frac{K}{M} \right) \right];$$

for all  $t \geq t_0$ . This completes the proof.  $\square$

**Example 2.3.** Consider the scalar nonlinear Volterra integrodifferential equation

$$\begin{aligned}
 (2.5) \quad & x' = \sigma(t)x(t) + \int_0^t B(t, s)x^{2/3}(s) ds \quad t \geq 0, \\
 & x(t) = \phi(t) \quad \text{for } 0 \leq t \leq t_0.
 \end{aligned}$$

If

$$2\sigma(t) + \int_0^t |B(t, s)| ds + \int_t^\infty |B(u, t)| du \leq -1,$$

$$\int_0^t \int_t^\infty |B(u, s)| du ds, \quad \int_0^t |B(t, s)| ds < \infty,$$

and

$$\frac{|B(t, s)|}{3} \geq \int_t^\infty |B(u, s)| du,$$

then all solutions of (2.5) are uniformly bounded.

To see this we let

$$V(t, x) = x^2 + \int_0^t \int_t^\infty |B(u, s)| du x^2(s) ds.$$

Then along solutions of (2.5) we have

$$\begin{aligned} V'(t, x) &= 2xx' + \int_t^\infty |B(u, t)| x^2(t) du - \int_0^t |B(t, s)| x^2(s) ds \\ &\leq 2\sigma(t)x^2 + 2 \int_0^t |B(t, s)| |x(t)| x^{2/3}(s) ds \\ &\quad + \int_t^\infty |B(u, t)| x^2(t) du - \int_0^t |B(t, s)| x^2(s) ds. \end{aligned}$$

Using the fact that  $ab \leq a^2/2 + b^2/2$ , the above inequality simplifies to

$$(2.6) \quad \begin{aligned} V'(t, x) &\leq 2\sigma(t)x^2 + \int_0^t |B(t, s)| (x^2(t) + x^{4/3}(s)) ds \\ &\quad + \int_t^\infty |B(u, t)| x^2(t) du - \int_0^t |B(t, s)| x^2(s) ds. \end{aligned}$$

To further simplify (2.6) we make use of Young's inequality, which says for any two nonnegative real numbers  $w$  and  $z$ , we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } 1/e + 1/f = 1.$$

Thus, for  $e = 3/2$  and  $f = 3$ , we get

$$\begin{aligned} \int_0^t |B(t, s)|x^{4/3}(s)ds &= \int_0^t |B(t, s)|^{1/3}|B(t, s)|^{2/3}x^{4/3}(s) ds \\ &\leq \int_0^t \left( \frac{|B(t, s)|}{3} + \frac{2}{3}|B(t, s)|x^2(s) \right) ds. \end{aligned}$$

By substituting the above inequality into (2.6), we arrive at

$$\begin{aligned} V'(t, x) &\leq \left( 2\sigma(t) + \int_0^t |B(t, s)| ds + \int_t^\infty |B(u, t)| du \right) x^2(t) \\ &\quad - \int_0^t \left( |B(t, s)| - \frac{2}{3}|B(t, s)| \right) x^2(s) ds + L \\ &\leq -x^2(t) - \int_0^t \frac{|B(t, s)|}{3} x^2(s) ds + L, \end{aligned}$$

where  $L = (1/3) \int_0^t |B(t, s)| ds$ . By taking  $W_1 = W_2 = W_4 = x^2(t)$ ,  $W_3 = W_5 = x^2(s)$ ,  $\lambda_1 = \lambda_2 = \lambda_3 = 1$  and  $\varphi_1(t, s) = \int_t^\infty |B(u, s)| du$ ,  $\varphi_2(t, s) = (|B(t, s)|)/3$ , we see that all the conditions of Theorem 2.2 are satisfied. Hence all solutions of (2.5) are uniformly bounded.  $\square$

Note that  $B(t, s) = e^{-k(t-s)}$ ,  $k = 3$ , will satisfy all requirements of Example 2.3. Also, we assert that Example 2.3 can be easily generalized to handle nonlinear Volterra equations of the form

$$x' = \sigma(t)x(t) + \int_0^t B(t, s)f(s, x(s)) ds,$$

where  $|f(t, x(t))| \leq x^{2/3}(t) + M$ , for some positive constant  $M$ . Condition (2.3) did not come into play, which was due to the fact that  $r = q = 2$ . In the next example, we consider a nonlinear system in which condition (2.3) naturally comes into play.

**Example 2.4.** Let  $D = \{x \in \mathbf{R} : \|x\| \geq 1\}$ . Let  $\phi(t)$  be a given bounded continuous initial function such that  $\|\phi(t)\| = 1$ , for  $0 \leq t \leq t_0$ . Consider the scalar nonlinear Volterra integrodifferential equation

$$(2.7) \quad \begin{aligned} x' &= \sigma(t)x^3(t) + \int_0^t B(t, s)x^{1/3}(s) ds, \quad t \geq 0, \\ x(t) &= \phi(t) \quad \text{for } 0 \leq t \leq t_0. \end{aligned}$$

If

$$2\sigma(t) + \frac{1}{2} \int_0^t |B(t, s)|^{1/2} ds + \int_t^\infty |B(u, t)| du \leq -1,$$

$$\int_0^t \int_t^\infty |B(u, s)| du ds,$$

$$\int_0^t (|B(t, s)| + |B(t, s)|^{3/2}) ds < \infty,$$

and

$$\frac{5|B(t, s)|}{6} \geq \int_t^\infty |B(u, s)| du,$$

then all solutions of (2.7) that are in the set  $D$  are uniformly bounded.

To see this, we consider the Lyapunov functional  $V(t, x) : \mathbf{R}^+ \times D \rightarrow \mathbf{R}^+$ ,

$$V(t, x) = x^2 + \int_0^t \int_t^\infty |B(u, s)| du x^4(s) ds.$$

Then along solutions of (2.7) we have

$$V'(t, x) = 2xx' + \int_t^\infty |B(u, t)|x^4(t) du - \int_0^t |B(t, s)|x^4(s) ds$$

$$\leq 2\sigma(t)x^4 + 2 \int_0^t |B(t, s)| |x(t)||x(s)|^{1/3} ds$$

$$+ \int_t^\infty |B(u, t)|x^4(t) du - \int_0^t |B(t, s)|x^4(s) ds.$$

By noting that  $2|x(t)||x(s)|^{1/3} \leq x^2(t) + x^{2/3}(s)$  we have from the above inequality that

$$V'(t, x) \leq 2\sigma(t)x^4 + \int_0^t |B(t, s)| (x^2(t) + |x(s)|^{2/3}) ds$$

$$+ \int_t^\infty |B(u, t)|x^4(t) du - \int_0^t |B(t, s)|x^4(s) ds.$$

Next we note that

$$\int_0^t |B(t, s)| x^2(t) dt = \int_0^t |B(t, s)|^{1/2} |B(t, s)|^{1/2} x^2(t) ds$$

$$\leq \int_0^t |B(t, s)|^{1/2} \left[ \frac{|B(t, s)|}{2} + \frac{x^4(t)}{2} \right] ds.$$

Also, using Young's inequality with  $e = 6$  and  $f = 6/5$ , we get

$$\begin{aligned} x(s)^{2/3}|B(t,s)| &= x(s)^{2/3}|B(t,s)|^{1/6}|B(t,s)|^{5/6} \\ &\leq \frac{x^4(s)|B(t,s)|}{6} + \frac{5}{6}|B(t,s)|. \end{aligned}$$

$$\begin{aligned} V'(t,x) &\leq \left( 2\sigma(t) + \frac{1}{2} \int_0^t |B(t,s)|^{1/2} ds + \int_t^\infty |B(u,t)| du \right) x^4(t) \\ &\quad - \int_0^t \left( |B(t,s)| - \frac{|B(t,s)|}{6} \right) x^4(s) ds + L \\ &\leq -x^4(t) - \int_0^t \frac{5|B(t,s)|}{6} x^4(s) ds + L, \end{aligned}$$

where

$$L = \frac{5}{6} \int_0^t |B(t,s)| ds + \frac{1}{2} \int_0^t |B(t,s)|^{3/2} ds.$$

By taking  $W_1 = W_2 = x^2(t)$ ,  $W_3 = W_4 = W_5 = x^4(s)$ ,  $\lambda_1 = \lambda_2 = \lambda_3 = 1$  and  $\varphi_1(t,s) = \int_t^\infty |B(u,s)| du$ ,  $\varphi_2(t,s) = (5|B(t,s)|)/6$ , we see that conditions (2.1) and (2.2) of Theorem 2.2 are satisfied. It is left to show that condition (2.3) holds. Since

$$\frac{5|B(t,s)|}{6} \geq \int_t^\infty |B(u,s)| du,$$

we have, for  $x \in D$  that

$$\begin{aligned} W_2(|x|) - W_4(|x|) &+ \int_0^t (\varphi_1(t,s)W_3(|x(s)|) - \varphi_2(t,s)W_5(|x(s)|)) ds \\ &= x^2(t) - x^4(t) + \int_0^t \left( \int_t^\infty |B(u,s)| du - \frac{5|B(t,s)|}{6} \right) x^4(s) ds \\ &\leq x^2(1 - x^2) \leq 0. \end{aligned}$$

Thus, condition (2.3) is satisfied for  $\gamma = 0$ . An application of Theorem 2.2 yields

$$\begin{aligned} |x(t)| &\leq \left[ |\phi^2(t_0)| + W_3(|\phi|) \int_0^{t_0} \varphi_1(t_0,s) ds + \frac{K}{M} \right]^{1/2} \\ &\leq \left[ 1 + \int_0^{t_0} \int_{t_0}^\infty |B(u,s)| du + \frac{K}{M} \right]^{1/2}; \quad \text{for all } t \geq t_0. \end{aligned}$$



Hence, every solution  $x$  with  $x(t) \in D$  satisfies

$$1 \leq |x(t)| \leq \left[ 1 + \int_0^{t_0} \int_{t_0}^{\infty} |B(u, s)| du + \frac{K}{M} \right]^{1/2}; \quad \text{for } t \geq 0. \quad \square$$

Note that  $B(t, s) = e^{-k(t-s)}$ ,  $k = 6/5$  will satisfy all requirements of Example 2.4.

In the next theorem we show that solutions are bounded.

**Theorem 2.5.** *Let  $D$  be a set in  $\mathbf{R}^n$ . Suppose there exists a continuously differentiable Lyapunov function  $V : \mathbf{R}^+ \times D \rightarrow \mathbf{R}^+$  that satisfies*

(2.8)

$$\lambda_1(t)W_1(|x|) \leq V(t, x) \leq \lambda_2(t)W_2(|x|) + \lambda_2(t) \int_0^t \varphi_1(t, s)W_3(|x(s)|) ds$$

and

$$(2.9) \quad V'(t, x) \leq -\lambda_3(t)W_4(|x|) - \lambda_3(t) \int_0^t \varphi_2(t, s)W_5(|x(s)|) ds + L$$

for some positive continuous functions  $\lambda_1(t), \lambda_2(t), \lambda_3(t)$  and positive constant  $L$ , where  $\lambda_1(t)$  is nondecreasing and  $\varphi_i(t, s) \geq 0$  is a scalar function continuous for  $0 \leq s \leq t < \infty$ ,  $i = 1, 2$ , such that for some constant  $\gamma \geq 0$  the inequality

(2.10)

$$W_2(|x|) - W_4(|x|) + \int_0^t (\varphi_1(t, s)W_3(|x(s)|) - \varphi_2(t, s)W_5(|x(s)|)) ds \leq \gamma$$

holds. Moreover, if  $\int_0^t \varphi_1(t, s) ds \leq B$  and  $\lambda_3(t) \leq N$  for some positive constants  $B$  and  $N$ , then all solutions of (1.1) that stay in  $D$  are bounded.

*Proof.* Let

$$M = \inf_{t \in \mathbf{R}^+} \frac{\lambda_3(t)}{\lambda_2(t)}.$$

For any initial time  $t_0$ , let  $x(t)$  be any solution of (1.1) with  $x(t_0) = \phi(t_0)$ . By calculating

$$\frac{d}{dt}(V(t, x(t))e^{M(t-t_0)})$$

and then by a similar argument as in Theorem 2.2 we obtain

$$(2.11) \quad V(t, x(t)) \leq \lambda_2(t_0)V(t_0, \phi) + \frac{K}{M}e^{-M(t-t_0)} + \frac{K}{M}.$$

where  $K = N\gamma + L$ . Consequently,

$$V(t, x(t)) \leq \lambda_2(t_0)V(t_0, \phi)e^{-M(t-t_0)} + \frac{K}{M}.$$

Since  $\lambda_1(t)$  is nondecreasing we have for  $t \geq t_0 \geq 0$

$$\begin{aligned} W_1(|x|) &\leq \frac{1}{\lambda_1(t)} \left( \lambda_2(t_0)V(t_0, \phi)e^{-M(t-t_0)} + \frac{K}{M} \right) \\ &\leq \frac{1}{\lambda_1(t_0)} \left( \lambda_2(t_0)V(t_0, \phi)e^{-M(t-t_0)} + \frac{K}{M} \right). \end{aligned}$$

Hence,

$$\|x\| \leq W_1^{-1} \left[ \frac{1}{\lambda_1(t_0)} \left( \lambda_2(t_0)V(t_0, \phi)e^{-M(t-t_0)} + \frac{K}{M} \right) \right].$$

This completes the proof.  $\square$

The proof of the next theorem can be found in [10].

**Theorem 2.6.** *Suppose there exists a continuously differentiable Lyapunov functional  $V : \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}^+$  that satisfies*

$$(2.12) \quad \lambda_1 \|x\|^p \leq V(t, x), \quad V(t, x) \neq 0 \quad \text{if } x \neq 0$$

and

$$(2.13) \quad V'(t, x) \leq -\lambda_2(t)V^q(t, x)$$

for some positive constants  $\lambda_1, p, q > 1$  where  $\lambda_2(t)$  is a positive continuous function such that

$$(2.14) \quad c_1 = \inf_{t \geq t_0 \geq 0} \lambda_2(t) > 0.$$

Then all solutions  $x(t)$  of (1.1) satisfy

$$\|x\| \leq 1/\lambda_1^{1/p} \{ [V^{1-q}(t_0, \phi) + c_1(q-1)(t-t_0)]^{-1/(q-1)} \}^{1/p}.$$

As an application of the previous theorem, we furnish the following example.

**Example 2.7.** To illustrate the application of Theorem 2.6, we consider the following two-dimensional system of nonlinear Volterra integrodifferential equations

$$y_1' = y_2 - y_1 |y_1| - y_1 y_2^2 \int_0^t |B(t, s)| f(y_1(s), y_2(s)) ds$$

$$y_2' = -y_1 - y_2 |y_2| + y_1^2 y_2 \int_0^t C(t, s) g(y_1(s), y_2(s)) ds$$

$$(y_1(t), y_2(t)) = (\varphi_1(t), \varphi_2(t)),$$

for some given initial continuous and bounded functions  $\varphi_1(t), \varphi_2(t)$ ,  $0 \leq t \leq t_0$ . The scalar functions  $|B(t, s)|, C(t, s)$  are continuous in  $t$  and  $s$  and  $|B(t, s)| \geq |C(t, s)|$ . Also, the scalars  $f(y_1(s), y_2(s))$  and  $g(y_1(s), y_2(s))$  are continuous in  $y_1$  and  $y_2$ . We assume that

$$f(y_1(s), y_2(s)) \geq 0,$$

$$|g(y_1(s), y_2(s))| \leq f(y_1(s), y_2(s)), \quad \text{for all } y_1, y_2 \in \mathbf{R}.$$

Let us take  $V(y_1, y_2) = (y_1^2 + y_2^2)/2$ . Then

$$\begin{aligned}
 V'(y_1, y_2) &= -y_1^2|y_1| - y_2^2|y_2| - y_1^2y_2^2 \left( \int_0^t |B(t, s)|f(y_1(s), y_2(s)) ds \right. \\
 &\quad \left. - \int_0^t C(t, s)g(y_1(s), y_2(s)) ds \right) \\
 &\leq -(|y_1|^3 + |y_2|^3) \\
 &\quad + y_1^2y_2^2 \int_0^t (|C(t, s)| - |B(t, s)|)f(y_1(s), y_2(s)) ds \\
 &\leq -2 \left[ \frac{|y_1|^3}{2} + \frac{|y_2|^3}{2} \right] \\
 &= -2 \left[ \frac{(|y_1|^2)^{3/2}}{2} + \frac{(|y_2|^2)^{3/2}}{2} \right] \\
 &\leq -2(|y_1|^2 + |y_2|^2)^{3/2} 2^{-3/2} \\
 &= -2V^{3/2}(y_1, y_2)
 \end{aligned}$$

where we have used the inequality

$$\left( \frac{a+b}{2} \right)^l \leq \frac{a^l}{2} + \frac{b^l}{2}, \quad a, b > 0, \quad l > 1.$$

Hence, by Theorem 2.6 all solutions of the above two-dimensional system are uniformly bounded.  $\square$

**3. Comparison.** In [11] the author considered the scalar Volterra integrodifferential equation

$$(3.1) \quad x'(t) = Af(x(t)) + \int_0^t B(t, s)g(x(s)) ds + h(t),$$

where  $f, g$  and  $h$  are continuous in their respective arguments and proved the following theorem.

**Theorem 3.1** [11]. *Assume  $xf(x) > 0$  for all  $x \neq 0$ . Suppose there is a constant  $m > 0$  such that*

$$(3.2) \quad g^2(x) \leq m^2 f^2(x) \quad \text{for all } x \in \mathbf{R}.$$

If

$$A(t) + k \int_0^t |B(t, s)| ds + \frac{1}{2} \int_t^\infty |B(u, t)| du \leq -\rho, \quad t \geq 0$$

for some positive constant  $\rho$  and  $k$  such that  $m^2 < 2k$ ,

$$\int_0^x f(x) dx \longrightarrow \infty \quad \text{as } |x| \rightarrow \infty,$$

and

$$h(\cdot) \in L^2[0, \infty),$$

then all solutions of (3.1) are bounded.

With Example 2.3 in mind we consider the scalar nonlinear Volterra integrodifferential equation

$$(3.3) \quad \begin{aligned} x' &= \sigma(t)x(t) + \int_0^t B(t, s)x^{2/3}(s) ds + h(t), \quad t \geq 0, \\ x(t) &= \phi(t) \quad \text{for } 0 \leq t \leq t_0, \end{aligned}$$

where  $h$  is continuous in  $t$ . If

$$2\sigma(t) + \int_0^t |B(t, s)| ds + \int_t^\infty |B(u, t)| du \leq -2,$$

there exists a positive constant  $R$  such that  $|h(t)| \leq R$ , for all  $t \in \mathbf{R}$ ,

$$\int_0^t \int_t^\infty |B(u, s)| du ds, \quad \int_0^t |B(t, s)| ds < \infty,$$

and

$$\frac{|B(t, s)|}{3} \geq \int_t^\infty |B(u, s)| du$$

then all solutions of (2.5) are uniformly bounded.

The proof follows along the lines of the proof of Example 2.3 by considering the same  $V$ . On the other hand, Theorem 3 of [11] cannot be applied to (3.3) since condition (3.2) cannot hold for a positive constant  $m$  and for all  $x \in \mathbf{R}$ . Moreover, we have only required that

$h$  be uniformly bounded, while in [11] it was required that  $h$  be an  $L^2[0, \infty)$  function.

## REFERENCES

1. A. Burton, H. Quichang and W.E. Mahfoud, *Liapunov functionals of convolution type*, J. Math. Anal. Appl. **106** (1985), 249–272.
2. T. Burton, *Volterra integral and differential equations*, Academic Press, New York, 1983.
3. D. Caraballo, *On the decay rate of solutions of non-autonomous differential systems*, Electron. J. Differential Equations **2001** (2001), 1–17.
4. D. Cheban, *Uniform exponential stability of linear periodic systems in a Banach space*, Electron. J. Differential Equations **2001** (2001), 1–12.
5. D. Driver, *Ordinary and delay differential equations*, Springer Publ., New York, 1977.
6. J.K. Hale, *Theory of functional differential equations*, Springer-Verlag, Berlin, 1977.
7. S. Kato, *Existence, uniqueness and continuous dependence of solutions of delay-differential equations with infinite delay in a Banach space*, J. Math. Anal. Appl. **195** (1995).
8. V. Lakshmikantham, S. Leela and A. Martynyuk, *Stability analysis of nonlinear systems*, Marcel Dekker, New York, 1989.
9. N. Linh and V. Phat, *Exponential stability of nonlinear time-varying differential equations and applications*, Electron. J. Differential Equations **2001** (2001), 1–13.
10. Y. Raffoul, *Boundedness in nonlinear differential equations*, Nonlinear Stud. **10** (2003), 343–350.
11. J. Vanualilai, *Some stability and boundedness criteria for a class of Volterra integro-differential systems*, Electron. J. Qual. Theory Differential Equations **12** (2002), 1–20.
12. T. Yoshizawa, *Stability theory by Lyapunov second method*, The Math. Soc. Japan, Tokyo, 1966.

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