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APPROXIMATE SOLUTION OF MULTIVARIABLE INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. For a multidimensional integral equation of the second kind with a smooth kernel, using the orthogonal projection onto a space of discontinuous piecewise polynomials of degree r, Atkinson has established an order $r+1$ convergence for the Galerkin solution and an order $2r+2$ convergence for the iterated Galerkin solution. In a recent paper [**15**], a new method based on projections has been shown to give a $4r+4$ convergence for one-dimensional second kind integral equations. The size of the system of equations that must be solved in implementing this method remains the same as for the Galerkin method. In this paper, this method is extended to multi-dimensional second kind equations and is shown to have convergence of order $4r+4$. For interpolatory projections onto a space of piecewise polynomials, it is shown that the order of convergence of the new method improves on the previously established orders of convergence for the collocation and the iterated collocation methods. A two-grid norm convergent method based on the new method is also defined.

1. Introduction. Over the years approximate solution of one dimensional Fredholm integral equations of the second kind has been extensively studied. (See [**2, 5, 7, 8, 11, 14, 20**].) The classical methods are the Galerkin method based on a sequence of projections converging pointwise to the identity operator and the Nyström method based on a numerical quadrature. The improvement of the Galerkin solution by using an iteration technique was first proposed by Sloan in [**18**]. Chandler [**7**], in his Thesis, proved that if the kernel and the righthand side are smooth, then, in the case of the orthogonal projection onto a space of piecewise polynomials, the order of convergence in the iterated Galerkin solution is twice that of the Galerkin solution. Chatelin and Lebbar [**9**] proved similar results for the iterated collocation at Gauss points. For situations where the righthand side of the operator equation is less smooth than the kernel of the integral operator, the higher

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order of convergence of the Kantorovich solution compared with that for the Galerkin solution is discussed in Schock [**17**] and Sloan [**19**]. An interpolation post-processing technique as an alternative to the iteration technique for improving the collocation solution has been proposed by Lin et al. [**16**]. In [**12**], Hu discusses interpolation post-processing technique for Fredholm integro-differential equations.

Recently, in [**15**], the author has proposed a method based on projections for approximate solutions of compact operator equations. For both the orthogonal projections and the interpolatory projections at Gauss points with the range as a space of piecewise polynomials, it is shown that, if the kernel and the righthand side of the second kind one-dimensional integral equation are smooth, then the order of convergence in the iterated version of the proposed method is twice that of the iterated Galerkin solution and four times that of the Galerkin solution.

In this paper, the results of [**15**] are extended to multi-dimensional second kind equations. The results in this paper depend heavily on Atkinson [**2**]. Even though, for the sake of simplicity, only two-dimensional integral equations are considered here, the results extend to the multi-dimensional integral equations. It is established that if the kernel and the righthand side are sufficiently smooth, then for the orthogonal projections onto a space of piecewise polynomials of degree less than or equal to r, the orders of convergence of the new method and its iterated version are respectively $3r + 3$ and $4r + 4$. This is an improvement over the orders of convergence $r + 1$ and $2r + 2$ in the Galerkin and the iterated Galerkin methods, respectively, proved in Atkinson [**2**]. In the case of the interpolatory projections, if r is even, then the orders of convergence in the proposed method and its iterated version are shown to be respectively $2r + 3$ and $2r + 4$. These orders of convergence are to be compared with the orders of convergence $r+1$ and $r+2$ in the collocation and the iterated collocation methods, respectively. (See [**2**].) It is to be noted that the size of the system of equations that needs to be solved remains the same as in the collocation/Galerkin method. It is shown in Atkinson-Chandler [**3**] that an appropriate choice of interpolation nodes in the case of piecewise linear interpolation gives higher order of convergence for the collocation and the iterated collocation methods. Similar improvement is observed in the proposed method and its iterated version.

Two-grid methods based on the Nyström method were introduced in Brakhage [**6**] and subsequently generalized in Atkinson [**1**]. Two-grid methods for collocation methods and degenerate kernel methods are discussed in Atkinson [**2**]. In [**13**], Kelley has suggested a modification to improve the Nyström iteration method. In [**10**] and [**11**], Hackbusch has discussed multi-grid methods. In this paper, a two-grid norm convergent method based on the new method is given. The performance of this two-grid method is compared with Nyström iteration methods 1 and 2 by applying it to a univariate integral equation. It is seen that, while the costs of the proposed two grid method and the Nyström iteration methods 1 and 2 are comparable, the proposed two grid method requires much less number of iterates.

The outline of the paper is as follows. In Section 2 notation is set and a new approximation method is defined. The orders of convergence for the proposed method, both for the interpolatory projection and the orthogonal projection, are derived in Section 3. In this section a discrete version of the proposed method is also discussed. In Section 4, a two-grid method is defined along with its implementation and assessment of computational cost. Section 5 is devoted to the numerical results.

2. Method, notation and definitions. Consider the integral equation

$$\lambda\,\rho(x,y) - \int_R k(x,y,\xi,\eta)\,\rho(\xi,\eta)\,d\xi\,d\eta = \psi(x,y), \quad (x,y)\in R, \tag{2.1}$$

where R is a polygonal region in $\mathbf{R}^2$.

The integral operator

$$K\,\rho(x,y) = \int_R k(x,y,\xi,\eta)\,\rho(\xi,\eta)\,d\xi\,d\eta$$

is assumed to be compact from $L^\infty(R)$ into $C(R)$. The equation (2.1) is written as

$$(\lambda - K)\,\rho = \psi. \tag{2.2}$$

Let $\mathcal{T}_n = \{\Delta_1,\dots,\Delta_n\}$ be a triangulation of R. It is assumed that the triangles Δ_j and Δ_k intersect only at vertices or along all

of a common edge. The vertices of the triangle Δ_k are denoted by $\{v_{k,1},\ v_{k,2},\ v_{k,3}\}$.
Consider the unit simplex

$$\sigma = \{(s,t) \,|\, s,t \geq 0,\ s+t \leq 1\}.$$

The map $T_k : \sigma \to \Delta_k$ defined by

$$T_k(s,t) = (1-s-t)\, v_{k,1} + t\, v_{k,2} + s\, v_{k,3} \tag{2.3}$$

is affine, one-to-one and onto.

A piecewise polynomial interpolation is defined as follows.

Piecewise constant interpolation. For $g \in C(R)$, let

$$(P_n g)(T_k(s,t)) \,=\, g\left(\frac{v_{k,1}+v_{k,2}+v_{k,3}}{3}\right), \quad (s,t) \in \sigma, \quad k = 1,\dots,n. \tag{2.4}$$

It is shown in Atkinson et al. [**4**] that P_n can be extended to $L^\infty(R)$ and that P_n is a bounded projection on $L^\infty(R)$, with $\|P_n\| = 1$.

Piecewise polynomial interpolation of degree $r \geq 1$. Let

$$(s_i, t_j) = \left(\frac{i}{r}, \frac{j}{r}\right), \quad i,j \geq 0, \quad i+j \leq r. \tag{2.5}$$

These $f_r = [(r+1)(r+2)]/2$ nodes in σ are sequentially ordered as $\{d_1, \dots, d_{f_r}\}$. Let $l_i(s,t)$ denote the Lagrange polynomial of degree r such that

$$l_i(d_j) = \delta_{ij}, \quad i,j = 1,\dots,f_r.$$

For $g \in C(R)$, let

$$(P_n g)(T_k(s,t)) = \sum_{j=1}^{f_r} g(T_k(d_j))\, l_j(s,t), \quad (s,t) \in \sigma, \quad k = 1,\dots,n. \tag{2.6}$$

Then $P_n g$ is a continuous function on R and satisfies the interpolation conditions:

$$(P_n g)(T_k(d_j)) = g(T_k(d_j)), \quad j = 1, \dots, f_r, \;\; k = 1, \dots, n.$$

Also, P_n defines a bounded projection on $C(R)$ and

$$\|P_n\| = \max_{(s,t)\in\sigma} \sum_{j=1}^{f_r} |\, l_j(s,t)\,| \,.$$

The following notation is used throughout the paper.

For $g \in C^{r+1}(R)$, let

$$\|g\|_{r+1,\infty} \;=\; \max_{\substack{i,j\geq 0\\ i+j=r+1}} \; \max_{(x,y)\in R} \left| \frac{\partial^{r+1} g(x,y)}{\partial x^i \partial y^j} \right| .$$

For fixed integers $p,\;\; q \;\geq\; 0$, assume that $k(.,.,\xi,\eta) \in C^p(R)$, for all $(\xi,\eta) \in R$ and $k(x,y,.,.) \in C^q(R)$, for all $(x,y) \in R$, with the derivatives uniformly bounded with respect to both (x,y) and (ξ,η) in R. Let

$$\|k\|_{p,q,\infty}$$
$$= \max \left\{ \max_{\substack{i,j\geq 0\\ i+j=p}} \; \max_{\substack{(x,y)\in R\\ (\xi,\eta)\in R}} \left| \frac{\partial^p k(x,y,\xi,\eta)}{\partial x^i \partial y^j} \right|, \max_{\substack{i,j\geq 0\\ i+j=q}} \; \max_{\substack{(x,y)\in R\\ (\xi,\eta)\in R}} \left| \frac{\partial^q k(x,y,\xi,\eta)}{\partial \xi^i \partial \eta^j} \right| \right\}.$$

If $k(.,.,\xi,\eta) \in \, C^{r+1}(R)$ and

$$\frac{\partial^{r+1} k(x,y,.,.)}{\partial x^i \partial y^j} \,\in\, C^1(R)$$
$$\text{for} \quad i+j = r+1 \quad \text{and for all} \quad (x,y) \in R,$$

with the derivatives uniformly bounded with respect to both (x, y) and (ξ, η), then let

$$\|k\|_{r+2,\infty} = \max\left\{ \max_{\substack{i,j\geq 0\\ i+j=r+1}} \max_{\substack{(x,y)\in R\\ (\xi,\eta)\in R}} \left| \frac{\partial^{r+2} k(x,y,\xi,\eta)}{\partial x^i\, \partial y^j\, \partial \xi} \right|, \max_{\substack{i,j\geq 0\\ i+j=r+1}} \max_{\substack{(x,y)\in R\\ (\xi,\eta)\in R}} \left| \frac{\partial^{r+2} k(x,y,\xi,\eta)}{\partial x^i\, \partial y^j\, \partial \eta} \right| \right\}.$$

The following result is proved in Atkinson [**2**, Theorem 5.1.2, p. 167].

Theorem 2.1. *Let R be a polygonal region in $\mathbf{R}^2$, $r \geq 0$ be an integer and P_n be an interpolatory projection defined by* (2.4) *or* (2.6).

(a) *For all $g \in C(R)$, the interpolant $P_n g$ converges uniformly to g on R.*

(b) *Let $\delta_n = \max\{\text{diameter}\,(\Delta_k) \mid k = 1, \ldots, n\}$. If $g \in C^{r+1}(R)$, then*

$$\|g - P_n g\|_\infty \leq c\, \|g\|_{r+1,\infty}\, (\delta_n)^{r+1} \tag{2.7}$$

with c, a generic constant, independent of n and g.

The following form of refinement of triangles in $\mathcal{T}_n$ is referred to as *symmetric triangulations* by Atkinson [**2**, p. 173].

Each triangle $\Delta \in \mathcal{T}_n$ is divided into four congruent triangles by joining the midpoints of the three sides of Δ. Then the new triangulation $\mathcal{T}_{4n}$ has four times the triangles in $\mathcal{T}_n$ and $\delta_{4n} = (1/2)\delta_n$.

The following result is valid for symmetric triangulations and is quoted for future reference.

Theorem 2.2 [**2**, p. 180]. *If r is even, $k(x, y, ., .) \in C^1(R)$ for all $(x, y) \in R$, with the derivatives uniformly bounded with respect to both*

(x, y) and (ξ, η) and if $g \in C^{r+1}(R)$, then

$$\|K(I-P_n)g\|_\infty \leq c\, \|k\|_{0,1,\infty}\, \|g\|_{r+1,\infty}\, (\delta_n)^{r+2}. \tag{2.8}$$

A superconvergent piecewise linear interpolation. In Atkinson-Chandler [**3**] the following choice of interpolation nodes in the case of piecewise linear interpolation nodes is shown to exhibit superconvergence.

Let

$$d_1 = \left(\frac{1}{6}, \frac{1}{6}\right), \quad d_2 = \left(\frac{1}{6}, \frac{2}{3}\right), \quad d_3 = \left(\frac{2}{3}, \frac{1}{6}\right) \tag{2.9}$$

be three nodes in σ and P_n is defined as in the case of piecewise polynomial interpolation with $r = 1$. The range of P_n is not contained in $C(R)$, but following Atkinson et al. [**4**], P_n can be extended to $L^\infty(R)$ and

$$\|P_n\| = \frac{7}{3}.$$

The following result, which is valid for symmetric triangulations, follows easily from Atkinson-Chandler [**3**, Corollary 3.4].

Theorem 2.3 (Atkinson-Chandler [**3**]). *If $k(x, y, ., .) \in C^2(R)$ for all $(x, y) \in R$, with the first and the second derivatives uniformly bounded with respect to both (x, y) and (ξ, η) and if $g \in C^4(R)$, then*

$$\begin{aligned}\|K(I-P_n)g\|_\infty &\leq c\, (\|k\|_{0,0,\infty} + \|k\|_{0,1,\infty} + \|k\|_{0,2,\infty}) \\ &\quad \cdot \max(\|g\|_{2,\infty},\ \|g\|_{3,\infty},\ \|g\|_{4,\infty})\, (\delta_n)^4.\end{aligned} \tag{2.10}$$

Method. In the collocation method, (2.2) is approximated by

$$(\lambda - P_n K P_n)\, \rho_n = P_n \psi, \tag{2.11}$$

while in the iterated collocation method, it is approximated by

$$(\lambda - K P_n)\, \tilde{\rho}_n = \psi. \tag{2.12}$$

We propose to approximate (2.2) by

$$(\lambda - (P_nKP_n + P_nK(I-P_n) + (I-P_n)KP_n))\,\rho_n^M = \psi. \tag{2.13}$$

Let

$$K_n^M = P_nKP_n + P_nK(I-P_n) + (I-P_n)KP_n \tag{2.14}$$

denote the associated finite rank operator.

An iterated solution is defined by

$$\tilde{\rho}_n^M = \frac{K\,\rho_n^M + \psi}{\lambda}. \tag{2.15}$$

The approximating operator in the collocation method is $K_n^C = P_nKP_n$, while in the iterated collocation method it is $K_n^S = KP_n$. Thus

$$\begin{aligned} K - K_n^M &= (I-P_n)K(I-P_n), \\ K - K_n^C &= (I-P_n)K + P_nK(I-P_n) \end{aligned}$$

and

$$K - K_n^S = K(I-P_n).$$

The extra factor of $(I-P_n)$ in $K-K_n^M$ as compared to $K-K_n^C$ and $K-K_n^S$ is likely to make the solution obtained by the proposed method converge faster.

As the dimension of the range of P_n is nf_r, the operators K_n^C and K_n^S have rank $\leq nf_r$, while the rank of K_n^M is $\leq 2nf_r$. However it is shown below that the size of the system of equations that needs to be solved in the proposed method remains nf_r as in the case of the collocation/iterated collocation method.

Applying P_n and $I-P_n$ to (2.13) we obtain

$$\lambda\,P_n\,\rho_n^M - (P_nKP_n + P_nK(I-P_n))\,\rho_n^M = P_n\psi \tag{2.16}$$

and

$$\lambda(I-P_n)\,\rho_n^M - (I-P_n)KP_n\,\rho_n^M = (I-P_n)\psi. \tag{2.17}$$

Substituting

$$(2.18) \qquad (I-P_n)\,\rho_n^M = \frac{(I-P_n)KP_n\,\rho_n^M + (I-P_n)\psi}{\lambda}$$

in (2.16), we get

$$(2.19) \quad \lambda\,P_n\,\rho_n^M - \left(P_nKP_n + \frac{P_nK(I-P_n)KP_n}{\lambda}\right)P_n\,\rho_n^M = P_n\psi + \frac{P_nK(I-P_n)\psi}{\lambda}.$$

Let

$$v_{k,j} = T_k(d_j), \quad j = 1,\dots,f_r,\ \ k = 1,\dots,n$$

be the interpolation nodes, collectively referred to as

$$\{v_1, v_2, \dots, v_{nf_r}\}.$$

Let $w_n^M = P_n\,\rho_n^M$. Then

$$w_n^M(v_{k,j}) = \rho_n^M(v_{k,j}), \quad j = 1,\dots,f_r,\ \ k = 1,\dots,n$$

and

$$w_n^M(x,y) = \sum_{j=1}^{f_r} w_n^M(v_{k,j})\,l_j(s,t), \quad (x,y) = T_k(s,t) \in \Delta_k, \quad k = 1,\dots,n.$$

The equation (2.19) is then equivalent to the following system of equations of size nf_r.

$$(2.20) \quad \lambda\,w_n^M(v_i) - (Kw_n^M)(v_i) - \frac{(K^2w_n^M)(v_i) - (KP_nKw_n^M)(v_i)}{\lambda} = \psi(v_i) + \frac{(K\psi)(v_i) - (KP_n\psi)(v_i)}{\lambda}, \quad i = 1,\dots,nf_r.$$

Note that

$$
\begin{aligned}
(K w_n^M)(v_i) &= 2 \sum_{k=1}^{n} \text{Area}\,(\Delta_k) \sum_{j=1}^{f_r} w_n^M(v_{k,j}) \\
&\qquad \cdot \int_\sigma k(v_i, T_k(s,t))\, l_j(s,t)\, d\sigma, \\
(K^2 w_n^M)(v_i) &= 2 \sum_{k=1}^{n} \text{Area}\,(\Delta_k) \sum_{j=1}^{f_r} w_n^M(v_{k,j}) \\
&\qquad \cdot \int_\sigma k(v_i, T_k(s,t))\, (K\, l_j(s,t))\, d\sigma, \\
(K P_n K w_n^M)(v_i) &= 2 \sum_{k=1}^{n} \text{Area}\,(\Delta_k) \sum_{j=1}^{f_r} (K w_n^M)(v_{k,j}) \\
&\qquad \cdot \int_\sigma k(v_i, T_k(s,t))\, l_j(s,t)\, d\sigma, \\
(K\psi)(v_i) &= 2 \sum_{k=1}^{n} \text{Area}\,(\Delta_k) \int_\sigma k(v_i, T_k(s,t))\, \psi(T_k(s,t))\, d\sigma
\end{aligned}
$$

and

$$
\begin{aligned}
(K P_n \psi)(v_i) &= 2 \sum_{k=1}^{n} \text{Area}\,(\Delta_k) \sum_{j=1}^{f_r} \psi(v_{k,j}) \\
&\qquad \cdot \int_\sigma k(v_i, T_k(s,t))\, l_j(s,t)\, d\sigma.
\end{aligned}
$$

We obtain $w_n^M(v_{k,j})$ by solving the system of equations (2.20) and since by (2.18)

$$
\rho_n^M = w_n^M + \frac{K w_n^M - P_n K w_n^M + (I - P_n)\psi}{\lambda},
$$

we have

$$
\begin{aligned}
\rho_n^M(T_k(s,t)) = \sum_{j=1}^{f_r} w_n^M(v_{k,j})\, l_j(s,t) &+ \frac{1}{\lambda}\bigg(\sum_{j=1}^{f_r} w_n^M(v_{k,j})\, (K\, l_j)(s,t) \\
&- \sum_{j=1}^{f_r} \big((K w_n^M)(v_{k,j}) + \psi(v_{k,j}) \big)\; l_j(s,t) + \psi(T_k(s,t)) \bigg), \\
&\qquad k = 1, \dots, n.
\end{aligned}
$$

On the other hand, since

$$\tilde{\rho}_n^M = \frac{K\,\rho_n^M + \psi}{\lambda},$$

we have

$$\begin{aligned}
&\tilde{\rho}_n^M(T_k(s,t)) \\
&\quad = \sum_{j=1}^{f_r} w_n^M(v_{k,j})\,(Kl_j)(s,t) + \frac{1}{\lambda}\biggl(\sum_{j=1}^{f_r} w_n^M(v_{k,j})\,(K^2\,l_j)(s,t) \\
&\qquad\qquad - \sum_{j=1}^{f_r} \bigl((Kw_n^M)(v_{k,j}) + \psi(v_{k,j})\bigr)\,(Kl_j)(s,t) \\
&\qquad\qquad\qquad + (K\psi)(T_k(s,t)) + \Psi(T_k(s,t))\biggr), \quad k = 1, \dots, n.
\end{aligned}$$

It is seen in the next section that the solution obtained by using the new method has a higher order of convergence as compared to the solutions obtained by using collocation/iterated collocation methods.

3. Error estimates.

3.1 Interpolatory projection. The order of convergence for ρ_n^M is obtained in the following theorem.

Theorem 3.1. *Let R be a polygonal region in $\mathbf{R}^2$, and let $\mathcal{T}_n$ be a sequence of triangulations of R. Assume that $\delta_n = \max\{\text{diameter}\,(\Delta_k) \mid k = 1, \dots, n\} \to 0$ as $n \to \infty$ and that the integral equation $(\lambda - K)\,\rho = \psi$ is uniquely solvable, with $K : L^\infty(R) \to C(R)$ a compact operator. Let P_n be the interpolatory projection defined by* (2.4) *or* (2.6).

a) *For sufficiently large n, we have*

$$\rho - \rho_n^M = (\lambda - K_n^M)^{-1}(I - P_n)K(I - P_n)\,\rho$$

and $\rho_n^M \to \rho$ as $n \to \infty$.

b) *Assume that* $\psi \in C^{r+1}(R)$, $k(.,.,\xi,\eta) \in C^{r+1}(R)$ *for all* $(\xi,\eta) \in R$, *with the uniformly bounded derivatives. Then*

$$\|\rho - \rho_n^M\|_\infty \leq c\,(\delta_n)^{2r+2}. \tag{3.1}$$

Proof. a) Since P_n converges pointwise to the identity operator on $C(R)$ and $K : L^\infty(R) \to C(R)$ is compact,

$$\|K - K_n^M\| = \|(I - P_n)K(I - P_n)\| \to 0, \quad \text{as} \quad n \to \infty.$$

Hence for sufficiently large n, say $n \geq N$, (2.13) is uniquely solvable and the inverses $(\lambda - K_n^M)^{-1}$ are uniformly bounded on $C(R)$.

By the resolvent identity, we get

$$\rho - \rho_n^M = (\lambda - K_n^M)^{-1}(I - P_n)K(I - P_n)(\lambda - K)^{-1}\psi.$$

Thus

$$\|\rho - \rho_n^M\|_\infty \leq \|(\lambda - K_n^M)^{-1}\|\,\|(I - P_n)K(I - P_n)\,\rho\|_\infty \longrightarrow 0$$

as $n \to \infty$.

b) Note that since $k(.,.,\xi,\eta) \in C^{r+1}(R)$, for $u \in L^\infty(R)$ and $i, j \geq 0$, $i + j = r + 1$, we have

$$\frac{\partial^{r+1} Ku\,(x,y)}{\partial x^i \partial y^j} = \int_R \frac{\partial^{r+1} k(x,y,\xi,\eta)}{\partial x^i \partial y^j}\, u(\xi,\eta)\, d\xi\, d\eta. \tag{3.2}$$

Thus the range of K is contained in $C^{r+1}(R)$. It follows that

$$\|Ku\|_{r+1,\infty} \leq \text{area}\,(R)\,\|k\|_{r+1,0,\infty}\,\|u\|_\infty. \tag{3.3}$$

Hence, using (2.7) we get

$$\begin{aligned}\|K(I - P_n)g\|_{r+1,\infty} &\leq \text{area}\,(R)\,\|k\|_{r+1,0,\infty}\,\|(I - P_n)g\|_\infty \\ &\leq c\,\text{area}\,(R)\,\|k\|_{r+1,0,\infty}\,\|g\|_{r+1,\infty}\,(\delta_n)^{r+1}.\end{aligned}$$

As $\psi \in C^{r+1}(R)$ and

$$\rho = \frac{K\,\rho + \psi}{\lambda},$$

it follows that $\rho \in C^{r+1}(R)$. Thus

$$\begin{aligned}\|\rho - \rho_n^M\|_\infty &\le \|(\lambda - K_n^M)^{-1}\| \, \|(I - P_n) K (I - P_n)\, \rho\|_\infty \\ &\le c\, \|(\lambda - K_n^M)^{-1}\| \, \|K(I - P_n)\, \rho\|_{r+1,\infty}\, (\delta_n)^{r+1} \\ &\le c\, \|(\lambda - K_n^M)^{-1}\| \, \|k\|_{r+1,0,\infty} \|\rho\|_{r+1,\infty}\, (\delta_n)^{2r+2},\end{aligned}$$

which proves (3.1). □

Remark 3.2. The above estimate should be compared with the following estimate in the collocation method.

If $\rho \in C^{r+1}(R)$, then it is proved in Atkinson [**2**, p. 178] that

$$\|\rho - \rho_n\|_\infty \le c\, (\delta_n)^{r+1}. \tag{3.4}$$

Note that, in general, the condition $\rho \in C^{r+1}(R)$ is deduced from the assumption in part (b) of the above theorem, that is, by assuming that the righthand side $\psi \in C^{r+1}(R)$ and that kernel $k(.,.,\xi,\eta) \in C^{r+1}(R)$ for all $(\xi, \eta) \in R$.

As the rank of P_n is nf_r, the size of the system of equations in the collocation method that needs to be solved is nf_r. In Section 2 it is seen that the computation of ρ_n^M also involves solution of a system of equations (2.20) of size nf_r.

Thus, essentially under the same assumptions as in the collocation method and by solving a system of equations of the same size as in the collocation method, the order of convergence is improved from $r+1$ to $2(r+1)$.

Remark 3.3. It can be seen that the iterated solutions $\tilde{\rho}_n^M$ and $\tilde{\rho}_n$ have the same orders of convergence as ρ_n^M and ρ_n, respectively.

$$\|\rho - \tilde{\rho}_n^M\|_\infty \le c\, (\delta_n)^{2r+2}, \tag{3.5}$$

$$\|\rho - \tilde{\rho}_n\|_\infty \le c\, (\delta_n)^{r+1}, \tag{3.6}$$

When r is even and the triangulation is symmetric, we obtain the following improved error bounds.

Theorem 3.4. *Let R be a polygonal region in $\mathbf{R}^2$, and let $\mathcal{T}_n$ be a sequence of symmetric triangulations of R such that $\delta_n =$*

$\max\{\text{diameter}\,(\Delta_k)\,|\,k=1,\dots,n\} \to 0$ *as* $n\to\infty$. *Let* K *be a compact operator on* $C(R)$ *and assume that the integral equation* $(\lambda - K)\,\rho = \psi$ *is uniquely solvable. Assume that for* r *even,* $\psi \in C^{r+1}(R)$*,* $k(.,.,\xi,\eta) \in C^{r+1}(R)$ *for all* $(\xi,\eta)\in R$*, and*

$$\frac{\partial^{r+1}k}{\partial x^i \partial y^j}\,(x,y,.,.) \,\in\, C^1(R)$$
$$\text{for all}\quad i,j\geq 0,\quad i+j=r+1\quad\text{and}\quad (x,y)\in R.$$

Then

$$\|\rho-\rho_n^M\|_\infty \,\leq c\,(\delta_n)^{2r+3}, \tag{3.7}$$
$$\|\rho-\tilde\rho_n^M\|_\infty \,\leq c\,(\delta_n)^{2r+4}. \tag{3.8}$$

Proof. If $g\in C^{r+1}(R)$, then using (2.8) and (3.2) it can be shown that

$$\|K(I-P_n)g\|_{r+1,\infty} \,\leq c\,\|k\|_{r+2,\infty}\,\|g\|_{r+1,\infty}\,(\delta_n)^{r+2}. \tag{3.9}$$

Hence

$$\begin{aligned}\|\rho-\rho_n^M\|_\infty &\leq c\,\|(\lambda-K_n^M)^{-1}\|\,\|K(I-P_n)\,\rho\|_{r+1,\infty}\,(\delta_n)^{r+1}\\ &\leq c\,\|(\lambda-K_n^M)^{-1}\|\,\|k\|_{r+2,\infty}\,\|\rho\|_{r+1,\infty}\,(\delta_n)^{2r+3},\end{aligned}$$

which proves (3.7).

Note that

$$\begin{aligned}\rho-\tilde\rho_n^M &= \frac{1}{\lambda}\,K(\rho-\rho_n^M)\\ &= \frac{1}{\lambda}\,K(\lambda-K)^{-1}(K-K_n^M)(\lambda-K_n^M)^{-1}\psi\\ &= \frac{1}{\lambda}\,(\lambda-K)^{-1}K(I-P_n)K(I-P_n)\,\rho_n^M\\ &= \frac{1}{\lambda}\,(\lambda-K)^{-1}K(I-P_n)K(IP_n)(\rho+\rho_n^M-\rho).\end{aligned}$$

Hence

$$\begin{aligned}\|\rho-\tilde\rho_n^M\|_\infty \,\leq \frac{1}{|\lambda|}\,\|(\lambda-K)^{-1}\|\,\big(&\|K(I-P_n)K(I-P_n)\,\rho\|_\infty\\ &+\|K(I-P_n)K(I-P_n)\|\,\|\rho_n^M-\rho\|_\infty\big).\end{aligned} \tag{3.10}$$

Using the estimates (2.8) and (3.9), it follows that

(3.11)

$$\begin{aligned}\|K(I-P_n)K(I-P_n)\,\rho\|_\infty &\le c\,\|k\|_{0,1,\infty}\,\|K(I-P_n)\,\rho\|_{r+1,\infty}\,(\delta_n)^{r+2}\\ &\le c\,\|k\|_{0,1,\infty}\,\|k\|_{r+2,\infty}\,\|\rho\|_{r+1,\infty}\,(\delta_n)^{2r+4}.\end{aligned}$$

For $u \in C(R)$, using (2.7) and (3.3), we obtain

$$\begin{aligned}\|(I-P_n)Ku\|_\infty &\le c\,\|Ku\|_{r+1,\infty}\,(\delta_n)^{r+1}\\ &\le c\,\mathrm{area}\,(R)\,\|k\|_{r+1,0,\infty}\,\|u\|_\infty(\delta_n)^{r+1}.\end{aligned}$$

Hence

$$\|(I-P_n)K\| \le c\,(\delta_n)^{r+1}. \tag{3.12}$$

Since P_ns are uniformly bounded, the estimate (3.8) follows by combining (3.7), (3.10), (3.11) and (3.12). □

Remark 3.5. The above estimates should be compared with the estimate (3.4) in the collocation method and the following estimate in the iterated collocation method proved in [**2**, p. 180].

$$\|\rho-\tilde{\rho}_n\|_\infty \le c\,(\delta_n)^{r+2}. \tag{3.13}$$

The additional condition that the kernel needs to satisfy in Theorem 3.4 is

$$\frac{\partial^{r+1}k}{\partial x^i \partial y^j}\,(x,y,.,.) \in C^1(R)$$

for all $i,j \ge 0$, $i+j=r+1$ and $(x,y) \in R$.

3.1.1 *Superconvergent piecewise linear interpolation.*

Theorem 3.6. *Let R be a polygonal region in $\mathbf{R}^2$, $\mathcal{T}_n$ be a sequence of symmetric triangulations of R and $K: L^\infty(R) \to C(R)$ be a compact operator. Assume that $\delta_n = \max\{\mathrm{diameter}\,(\Delta_k)\,|\,k=1,\ldots,n\} \to 0$ as $n \to \infty$ and that the integral equation $(\lambda - K)\,\rho = \psi$ is uniquely solvable. Let P_n be a sequence of projections based on the nodes defined* *by* (2.9). *Assume that* $\psi \in C^4(R)$, $k(.,.,\xi,\eta) \in C^2(R)$ *for all* $(\xi,\eta) \in R$, *and* $k(x,y,.,.) \in C^4(R)$ *for all* $(x,y) \in R$, *Then*

$$\|\rho - \rho_n^M\|_\infty \le c\,(\delta_n)^6, \tag{3.14}$$

$$\|\rho - \tilde{\rho}_n^M\|_\infty \le c\,(\delta_n)^8. \tag{3.15}$$

The proof of the theorem above uses the estimates (2.7) and (2.10) and is similar to that of Theorem 3.4.

Remark 3.7. The following estimates in the collocation and the iterated collocation methods follow from the results in Atkinson-Chandler [**3**].

$$\|\rho - \rho_n\|_\infty \le c\,(\delta_n)^2, \tag{3.16}$$

$$\|\rho - \tilde{\rho}_n\|_\infty \le c\,(\delta_n)^4. \tag{3.17}$$

The estimates above and the estimates in Theorem 3.6 should be compared with the estimates (3.1), (3.4), (3.5) and (3.6) with $r = 1$.

3.2 Orthogonal projection. Let $r \ge 0$ be an integer and X_n be the set of all $\phi \in L^\infty(R)$ such that $\phi|_{\Delta_k}$ is a polynomial of degree $\le r$, for $k = 1, \dots, n$. The dimension of X_n is $nf_r = (n(r+1)(r+2))/2$. Let $\tilde{P}_n : L^2(R) \to X_n$ be the orthogonal projection. As in the case of the interpolatory projection, let

$$(\lambda - (\tilde{P}_n K \tilde{P}_n + \tilde{P}_n K (I - \tilde{P}_n) + (I - \tilde{P}_n) K \tilde{P}_n))\,\rho_n^M = \psi \tag{3.18}$$

and

$$\tilde{\rho}_n^M = \frac{K\,\rho_n^M + \psi}{\lambda}. \tag{3.19}$$

Define

$$K_n^M = \tilde{P}_n K \tilde{P}_n + \tilde{P}_n K (I - \tilde{P}_n) + (I - \tilde{P}_n) K \tilde{P}_n. \tag{3.20}$$

We first prove some preliminary results.

Proposition 3.8 a) *If* $k(x,y,.,.) \in C^{r+1}(R)$ *for all* $(x,y) \in R$*, then*

$$\|K(I-\tilde{P}_n)\| \le c\,(\delta_n)^{r+1}, \tag{3.21}$$

where the operator norm is either the L^2 *norm or the* L^∞ *norm.*

b) *If* $k(x,y,.,.) \in C^{r+1}(R)$ *for all* $(x,y) \in R$*,* $k(.,.,\xi,\eta) \in C^{r+1}(R)$ *for all* $(\xi,\eta) \in R$*, and* $g \in C^{r+1}(R)$*, then*

$$\|K(I-\tilde{P}_n)K(I-\tilde{P}_n)g\|_\infty \le c\,(\delta_n)^{4r+4}. \tag{3.22}$$

Proof. Let $P_n : L^\infty(R) \to X_n$ be the piecewise constant interpolatory projection defined by (2.4) or $P_n : C(R) \to X_n$ be the piecewise polynomial interpolatory projection, with $r \ge 1$, defined by (2.6). We have

$$\begin{aligned} \|(I-\tilde{P}_n)g\|_2 &= \|g - P_n g + \tilde{P}_n P_n g - \tilde{P}_n g\|_2 \\ &\le 2\,\|g - P_n g\|_2 \\ &\le 2\,\sqrt{\operatorname{area}(R)}\,\|g - P_n g\|_\infty. \end{aligned} \tag{3.23}$$

If $g \in C^{r+1}(R)$, then using (2.7) we get

$$\|(I-\tilde{P}_n)g\|_2 \le 2\,\sqrt{\operatorname{area}(R)}\;c\,\|g\|_{r+1,\infty}(\delta_n)^{r+1}. \tag{3.24}$$

Similarly,

$$\begin{aligned} \|(I-\tilde{P}_n)g\|_\infty &\le (1 + \|\tilde{P}_n\|_{L^\infty})\,\|g - P_n g\|_\infty \\ &\le (1 + \sup_n \|\tilde{P}_n\|_{L^\infty})\;c\,\|g\|_{r+1,\infty}(\delta_n)^{r+1}. \end{aligned} \tag{3.25}$$

For a fixed $(x,y) \in R$, define $k_{(x,y)}(\xi,\eta) = k(x,y,\xi,\eta)$, $(\xi,\eta) \in R$ and let $g \in L^2(R)$.

Consider

$$\begin{aligned} \left|K(I-\tilde{P}_n)g(x,y)\right| &= \left|\int_R k(x,y,\xi,\eta)\,(I-\tilde{P}_n)\,g(\xi,\eta)\,d\xi\,d\eta\right| \\ &= \left|\left\langle (I-\tilde{P}_n)\,g, \overline{k}_{(x,y)}\right\rangle\right| \\ &= \left|\left\langle g, (I-\tilde{P}_n)\,\overline{k}_{(x,y)}\right\rangle\right| \\ &\le \|(I-\tilde{P}_n)\,\overline{k}_{(x,y)}\|_2\,\|g\|_2. \end{aligned}$$

Hence using (3.24) we get

$$\|K(I-\tilde{P}_n)g\|_\infty \le 2\sqrt{\text{area}\,(R)}\; c\, \|k\|_{0,r+1,\infty}\, \|g\|_2\, (\delta_n)^{r+1}.$$

It follows that

$$\|K(I-\tilde{P}_n)\|_{L^2} \le 2\,\text{area}\,(R)\; c\, \|k\|_{0,r+1,\infty}\, (\delta_n)^{r+1}$$

and

$$\|K(I-\tilde{P}_n)\|_{L^\infty} \le 2\,\text{area}\,(R)\; c\, \|k\|_{0,r+1,\infty}\, (\delta_n)^{r+1},$$

which completes the proof of (3.21).

b) Since

$$\left|K(I-\tilde{P}_n)g(x,y)\right| = \left|\left\langle (I-\tilde{P}_n)\, g, (I-\tilde{P}_n)\, \overline{k}_{(x,y)}\right\rangle\right|,$$

if $g \in C^{r+1}(R)$, then by using (3.25)

$$\|K(I-\tilde{P}_n)g\|_\infty \le c\, \|k\|_{0,r+1,\infty}\, \|g\|_{r+1,\infty}(\delta_n)^{2r+2}.$$

In a similar fashion, using (3.2) it can be seen that

$$\|K(I-\tilde{P}_n)g\|_{r+1,\infty} \le c\, \|k\|_{r+1,r+1,\infty}\, \|g\|_{r+1,\infty}(\delta_n)^{2r+2}. \tag{3.26}$$

Thus for $g \in C^{r+1}(R)$

$$\begin{aligned}\|K(I-\tilde{P}_n)K(I-\tilde{P}_n)g\|_\infty &\\ &\le c\, \|k\|_{0,r+1,\infty}\, \|K(I-\tilde{P}_n)g\|_{r+1,\infty}\, (\delta_n)^{2r+2}\\ &\le c\, \|k\|_{0,r+1,\infty}\, \|k\|_{r+1,r+1,\infty} \|g\|_{r+1,\infty}\, (\delta_n)^{4r+4},\end{aligned}$$

which completes the proof. $\square$

The estimates for $\|\rho-\rho_n^M\|_\infty$ and $\|\rho-\tilde{\rho}_n^M\|_\infty$ are obtained below.

Theorem 3.9. *Let R be a polygonal region in $\mathbf{R}^2$, and let $\mathcal{T}_n$ be a sequence of triangulations of R such that $\delta_n = \max\{\text{diameter}\,(\Delta_k) \mid k = 1, \ldots, n\} \to 0$ as $n \to \infty$. Assume that the integral equation*

$(\lambda - K)\,\rho = \psi$ is uniquely solvable, with $K : L^2(R) \to L^2(R)$ or $K : L^\infty(R) \to C(R)$ a compact operator.

a) *For sufficiently large n, say $n \geq N$, (2.13) is uniquely solvable, and the inverses $(\lambda - K_n^M)^{-1}$ are uniformly bounded. We have*

$$\rho - \rho_n^M = (\lambda - K_n^M)^{-1}(I - \tilde{P}_n)K(I - \tilde{P}_n)\,\rho \tag{3.27}$$

and $\rho_n^M \to \rho$ as $n \to \infty$.

b) *Assume that $\psi \in C^{r+1}(R)$, $k(x,y,.,.) \in C^{r+1}(R)$ for all $(x,y) \in R$, and $k(.,.,\xi,\eta) \in C^{r+1}(R)$ for all $(\xi,\eta) \in R$. Then*

$$\|\rho - \rho_n^M\|_\infty \leq c\,(\delta_n)^{3r+3}, \tag{3.28}$$

$$\|\rho - \tilde{\rho}_n^M\|_\infty \leq c\,(\delta_n)^{4r+4}. \tag{3.29}$$

Proof. a) Since $\widetilde{P}_n$ converges pointwise to the identity operator on $L^2(R)$ and K is compact,

$$\|K - K_n^M\| = \|(I - \tilde{P}_n)K(I - \tilde{P}_n)\| \longrightarrow 0, \quad \text{as} \quad n \to \infty,$$

where the operator norm is the L^2 norm.

Also, since $\widetilde{P}_n$ converges pointwise to the identity operator on $C(R)$ and the range of K is contained in $C(R)$, it follows that

$$\|K - K_n^M\| \longrightarrow 0, \quad \text{as} \quad n \to \infty,$$

in the L^∞ norm. The result now follows as in the proof of Theorem 3.1 (a).

b) Note that since $\rho \in C^{r+1}(R)$, by (3.25) and (3.26)

$$\begin{aligned}
&\|(I - \widetilde{P}_n)K(I - \widetilde{P}_n)\,\rho\|_\infty \\
&\qquad \leq (1 + \sup_n \|\widetilde{P}_n\|_{L^\infty})\; c\; \|K(I - \widetilde{P}_n)\,\rho\|_{r+1,\infty}(\delta_n)^{r+1} \\
&\qquad \leq (1 + \sup_n \|\widetilde{P}_n\|_{L^\infty})\; c\; \|k\|_{r+1,r+1,\infty}\; \|\rho\|_{r+1,\infty}(\delta_n)^{3r+3}.
\end{aligned}$$

The estimate (3.28) follows from (3.27) and the above result.

Since

$$\rho - \tilde{\rho}_n^M = \frac{1}{\lambda}(\lambda - K)^{-1} K(I - \widetilde{P}_n) K(I - \widetilde{P}_n)(\rho + \rho_n^M - \rho),$$

we get

$$\|\rho - \tilde{\rho}_n^M\|_\infty \le \frac{1}{|\lambda|}\,\|(\lambda - K)^{-1}\|\,\big(\|K(I - \widetilde{P}_n)K(I - \widetilde{P}_n)\,\rho\|_\infty \\ + \|K(I - \widetilde{P}_n)K(I - \widetilde{P}_n)\|\,\|\rho_n^M - \rho\|_\infty\big).$$

Since $\widetilde{P}_n$'s are uniformly bounded, the estimate (3.29) follows from (3.21), (3.22) and (3.28). □

Remark 3.10. We list below the estimates in the Galerkin and the iterated Galerkin methods for comparison.

In the Galerkin method (2.2) is approximated by

$$(\lambda - \widetilde{P}_n K)\,\rho_n = \tilde{P}_n \psi,$$

while in the iterated Galerkin method it is approximated by

$$(\lambda - K\widetilde{P}_n)\,\tilde{\rho}_n = \psi.$$

If $\rho \in C^{r+1}(R)$, then it is proved in Atkinson [**2**, pp. 183–184] that

$$\|\rho - \rho_n\|_\infty \le c\,(\delta_n)^{r+1}.$$

and if, in addition, the kernel function $k(x, y, ., .) \in C^{r+1}(R)$, then

$$\|\rho - \tilde{\rho}_n\|_\infty \le c\,(\delta_n)^{2r+2}.$$

Thus, in the proof of Theorem 3.9, the conditions imposed on the kernel are essentially the same as in the iterated Galerkin method.

3.3 Discrete methods. We first consider the case of interpolatory projections. In practice, the integrals appearing in the system of equations (2.20) need to be evaluated numerically.

Consider a composite formula based on

$$\int_\sigma g(s,t)\,d\sigma \approx \sum_{i=1}^{M} w_i\, g(\mu_i)$$

with degree of precision d and $M \geq f_r$. The integral operator K is approximated by

$$(K^N \rho)(x,y) = 2 \sum_{k=1}^{n} \operatorname{Area}(\Delta_k) \sum_{i=1}^{M} w_i\, k(x,y,T_k(\mu_i))\, \rho(T_k(\mu_i)),$$
$$(x,y) \in R.$$

Thus, in the discretized version of the proposed method, the operator K_n^M defined by (2.14) is replaced by

$$K_n^D = P_n K^N P_n + P_n K^N (I - P_n) + (I - P_n) K^N P_n.$$

Let

$$(\lambda - K_n^D)\, \rho_n^D = \psi$$

and

$$\tilde{\rho}_n^D = \frac{K^N \rho_n^D + \psi}{\lambda}.$$

The estimates (2.8) and (2.10) are valid with K replaced by K^N. Hence, under the assumption of Theorem 3.1, we have

$$\|\rho - \rho_n^D\|_\infty \leq c\,(\delta_n)^{\min\{d+1,2r+2\}}.$$

Thus, in order to retain the orders of convergence of ρ_n^M, it is necessary to choose $d \geq 2r+1$.

Also, under the assumption of Theorem 3.4, it can be shown that

$$\|\rho - \rho_n^D\|_\infty \leq c\,(\delta_n)^{\min\{d+1,2r+3\}}$$

and

$$\|\rho - \tilde{\rho}_n^D\|_\infty \leq c\,(\delta_n)^{\min\{d+1,2r+4\}}.$$

Similarly, under the assumption of Theorem 3.6, it can be shown that

$$\|\rho - \rho_n^D\|_\infty \le c\,(\delta_n)^{\min\{d+1,6\}}$$

and

$$\|\rho - \tilde{\rho}_n^D\|_\infty \le c\,(\delta_n)^{\min\{d+1,8\}}.$$

Thus, if $d \ge 5$, respectively $d \ge 7$, then the order of convergence in (3.14), respectively in (3.15), is retained.

In the case of orthogonal projections, integrals in the integral operator as well as in the inner product need to be evaluated numerically. Thus, following Atkinson [**2**, pp. 143–145], K is replaced by K^N, the inner product is replaced by the discrete inner product

$$\langle f, g\rangle_M = 2\sum_{k=1}^{n} \operatorname{Area}(\Delta_k) \sum_{i=1}^{M} w_i\, f(T_k(\mu_i))\, g(T_k(\mu_i))$$

and the orthogonal projection $\tilde{P}_n$ is replaced by the discrete projection Q_n. The discretized version of K_n^M defined by (3.20) is

$$K_n^D = Q_n K^N Q_n + Q_n K^N (I - Q_n) + (I - Q_n) K^N Q_n.$$

We have

$$\|K^N (I - Q_n)\| \le c\,(\delta_n)^{r+1}$$

and

$$\|K^N (I - Q_n) K^N (I - Q_n) g\|_\infty \le c\,(\delta_n)^{4r+4}.$$

Using these estimates it can be seen that, under the assumptions of Theorem 3.9,

$$\|\rho - \rho_n^D\|_\infty \le c\,(\delta_n)^{\min\{d+1,3r+3\}}$$

and

$$\|\rho - \tilde{\rho}_n^D\|_\infty \le c\,(\delta_n)^{\min\{d+1,4r+4\}}.$$

4. A two grid method. Consider the following integral equation

$$\lambda\, u(t) - \int_R k(t,s) u(s)\, ds = v(t), \quad t \in R, \tag{4.1}$$

that is,

$$(\lambda - K)u = v.$$

It is assumed that $k(t, s)$ is continuous in t and s and that $(\lambda - K)$ is invertible.

Let $r \geq 0$ and X_n denote the set of piecewise polynomials of degree $\leq r$ with respect to a triangulation $\mathcal{T}_n$ of R. Let $q_n = nf_r$ and $\{t_{n,1}, \dots, t_{n,q_n}\}$ be the interpolation points defined by (2.4) or (2.6). Let $\{l_{n,1}, \dots, l_{n,q_n}\}$ be the basis of Lagrange functions for X_n, that is, $l_{n,i} \in X_n$ and

$$l_{n,i}(t_{n,j}) = \delta_{i,j}, \quad i, j = 1, \dots, q_n.$$

The interpolatory projection is defined by

$$P_n g = \sum_{j=1}^{q_n} g(t_{n,j})\, l_{n,j}, \quad g \in C(R).$$

Consider a convergent quadrature formula is defined as follows.

$$\int_R g(t)\, dt \approx \sum_{j=1}^{q_n} w_{n,j}\, g(t_{n,j}), \tag{4.2}$$

which has the same nodes as the interpolation scheme.

Let

$$(K_n u)(t) = \sum_{j=1}^{q_n} w_{n,j}\, k(t, t_{n,j})\, u(t_{n,j}), \quad t \in R$$

be the Nyström approximation of K. The equation (4.1) is approximated by

$$\lambda\, u_n(t) - \sum_{j=1}^{q_n} w_{n,j}\, k(t, t_{n,j})\, u_n(t_{n,j}) = v(t), \quad t \in R, \tag{4.3}$$

that is,

$$(\lambda - K_n)u_n = v.$$

The system of equations (4.3) is equivalent to

(4.4)

$$\lambda\, u_n(t_{n,i}) - \sum_{j=1}^{q_n} w_{n,j}\, k(t_{n,i}, t_{n,j})\, u_n(t_{n,j}) = v(t_{n,i}), \quad i = 1, \dots, q_n$$

and

$$u_n(t) = \frac{1}{\lambda}\left[v(t) + \sum_{j=1}^{q_n} w_{n,j}\, k(t, t_{n,j})\, u_n(t_{n,j})\right], \quad t \in R. \tag{4.5}$$

Since K_n converges to K in a collectively compact fashion, it follows that, for all n large enough, $(\lambda - K_n)$ is invertible.

4.1 Description of the method. For $m < n$, let

$$P_m g = \sum_{j=1}^{q_m} g(t_{m,j})\, l_{m,j} \tag{4.6}$$

be the interpolatory projection corresponding to a coarse grid. Define

$$T_m = P_m K_n P_m + P_m K_n (I - P_m) + (I - P_m) K_n P_m. \tag{4.7}$$

Then

$$K_n - T_m = (I - P_m) K_n (I - P_m).$$

Since

$$U = \{K_n u \,|\, n \geq 1,\ \|u\|_\infty \leq 1\}$$

has a compact closure in $C(R)$, it follows that

$$\begin{aligned}\sup_{n\geq m} \|(I - P_m)K_n\| &= \sup_{n\geq m} \sup_{\|u\|_\infty \leq 1} \|(I - P_m)K_n u\| \\ &\leq \sup_{y\in U} \|(I - P_m)y\| \longrightarrow 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Thus, for m big enough,

$$\|K_n - T_m\| \leq \|(I - P_m)K_n\| \quad \|(I - P_m)\| < 1.$$

Hence, for m big enough, $(\lambda - T_m)$ is invertible and

$$\|(\lambda - T_m)^{-1}\| \leq 2\, \|(\lambda - K_n)^{-1}\|.$$

A two-grid iteration is defined below.

Assume that $u_n^{(0)}$ is an initial estimate of the solution u_n of (4.3). Let

$$\begin{aligned} r^{(k)} &= v-(\lambda-K_n)u_n^{(k)} \\ u_n^{(k+1)} &= u_n^{(k)}+(\lambda-T_m)^{-1}r^{(k)}, \quad k=0,1,2,\dots . \end{aligned} \tag{4.8}$$

Note that

$$r^{(k)} = (\lambda-K_n)(u_n-u_n^{(k)}). \tag{4.9}$$

Then

$$u_n^{(k+1)} = u_n^{(k)}+(\lambda-T_m)^{-1}(\lambda-K_n)(u_n-u_n^{(k)})$$

and

$$\begin{aligned} u_n-u_n^{(k+1)} &= (I-(\lambda-T_m)^{-1}(\lambda-K_n))(u_n-u_n^{(k)}) \\ &= (\lambda-T_m)^{-1}(K_n-T_m)(u_n-u_n^{(k)}) \\ &= (\lambda-T_m)^{-1}(I-P_m)K_n(I-P_m)(u_n-u_n^{(k)}) \\ &= M_{n,m}(u_n-u_n^{(k)}) \end{aligned} \tag{4.10}$$

with

$$M_{n,m}=(\lambda-T_m)^{-1}(I-P_m)K_n(I-P_m). \tag{4.11}$$

Since

$$\sup_{n\geq m}\|M_{n,m}\| \leq \|(\lambda-T_m)^{-1}\|\,\|(I-P_m)\|\sup_{n\geq m}\|(I-P_m)K_n\|\longrightarrow 0, \quad \text{as} \quad m\to\infty,$$

for m large enough

$$\tau_m=\sup_{n\geq m}\|M_{n,m}\|<1.$$

Thus

$$\|u_n-u_n^{(k+1)}\|_\infty \leq \tau_m\|u_n-u_n^{(k)}\|_\infty$$

and

$$\|u_n-u_n^{(k+1)}\|_\infty \leq \frac{\tau_m}{1-\tau_m}\|u_n^{(k+1)}-u_n^{(k)}\|_\infty.$$

Remark 4.1. Analogous to the Nyström iteration method 2 defined in Atkinson [**2**, Section 6.2.2], an iterated version of the two-grid method described by (4.8) is defined below.

Let $\tilde{u}_n^{(0)}$ be an initial estimate of the solution u_n of (4.3) and let

(4.12)
$$\begin{aligned} \tilde{r}^{(k)} \; &= \; v - (\lambda - K_n)\tilde{u}_n^{(k)} \\ \tilde{u}_n^{(k+1)} \; &= \; \tilde{u}_n^{(k)} + \frac{1}{\lambda}\left[\tilde{r}^{(k)} + (\lambda - T_m)^{-1}K_n\tilde{r}^{(k)}\right], \quad k = 0, 1, 2, \ldots . \end{aligned}$$

In this case

$$u_n - \tilde{u}_n^{(k+1)} \; = \; \frac{1}{\lambda}(\lambda - T_m)^{-1}(I - P_m)K_n(I - P_m)K_n(u_n - \tilde{u}_n^{(k)}).$$

Let

$$\tilde{\tau}_m = \frac{1}{|\lambda|} \sup_{n \geq m} \|(\lambda - T_m)^{-1}(I - P_m)K_n(I - P_m)K_n\|.$$

For m large enough, $\tilde{\tau}_m < 1$,

$$\|u_n - \tilde{u}_n^{(k+1)}\|_\infty \; \leq \; \tilde{\tau}_m \, \|u_n - \tilde{u}_n^{(k)}\|_\infty$$

and

$$\|u_n - \tilde{u}_n^{(k+1)}\|_\infty \; \leq \; \frac{\tilde{\tau}_m}{1 - \tilde{\tau}_m} \, \|\tilde{u}_n^{(k+1)} - \tilde{u}_n^{(k)}\|_\infty.$$

Note that

$$\tilde{\tau}_m \; = \; O(\sup_{n \geq m} \|(I - P_m)K_n\|^2),$$

whereas

$$\tau_m \; = \; O(\sup_{n \geq m} \|(I - P_m)K_n\|).$$

Since $\tilde{\tau}_m \to 0$ as $m \to \infty$ at double the rate of that of τ_m, the iterates $\tilde{u}_n^{(k)}$ in this modified version of the two-grid method converge to u_n faster than the iterates $u_n^{(k)}$ in the two-grid method defined by (4.8). The extra computational effort in the modified version is in the computation of $K_n\tilde{r}^{(k)}$.

4.2 Implementation. The aim is to solve the linear system (4.4) given by

$$\lambda\, u_n(t_{n,i}) - \sum_{j=1}^{q_n} w_{n,j}\, k(t_{n,i}, t_{n,j})\, u_n(t_{n,j}) = v(t_{n,i}), \quad i = 1, \ldots, q_n$$

with the unknown

$$\underline{u}_n = [u_n(t_{n,1}), \ldots, u_n(t_{n,q_n})]\,. \tag{4.13}$$

In the iteration formula (4.8), we need to solve the system

$$(\lambda - T_m)\, e \;=\; r^{(k)} \tag{4.14}$$

with

$$T_m = P_m K_n P_m + P_m K_n (I - P_m) + (I - P_m) K_n P_m.$$

Applying P_m and $I - P_m$ to (4.14), we obtain

$$\lambda\, P_m\, e - (P_m K_n P_m + P_m K_n (I - P_m))\, e = P_m r^{(k)} \tag{4.15}$$

and

$$\lambda\, (I - P_m)\, e - (I - P_m) K_n P_m\, e = (I - P_m) r^{(k)}. \tag{4.16}$$

Substitution for $(I - P_m)\, e$ from (4.16) in (4.15) gives

$$\lambda\, P_m\, e - \left(P_m K_n P_m + \frac{P_m K_n (I - P_m) K_n P_m}{\lambda} \right) e = P_m r^{(k)} + \frac{P_m K_n (I - P_m) r^{(k)}}{\lambda}. \tag{4.17}$$

Recall that

$$(P_m g)(t_{m,i}) = g(t_{m,i}), \quad i = 1, \ldots, q_m.$$

Hence (4.17) is equivalent to

$$\lambda\, e(t_{m,i}) - (K_n P_m\, e)(t_{m,i}) - \frac{(K_n (I - P_m) K_n P_m\, e)(t_{m,i})}{\lambda} = r^{(k)}(t_{m,i}) + \frac{K_n (I - P_m) r^{(k)}(t_{m,i})}{\lambda}, \quad i = 1, \ldots, q_m. \tag{4.18}$$

The values of e at the collocation points corresponding to the fine grid are then obtained by the following formula.

(4.19)
$$e(t_{n,i}) = (P_m\, e)(t_{n,i}) + \frac{r^{(k)}(t_{n,i}) - (P_m r^{(k)})(t_{n,i}) + (K_n P_m\, e)(t_{n,i}) - (P_m K_n P_m\, e)(t_{n,i})}{\lambda},$$
$$i = 1, \ldots, q_n.$$

4.3 Computational cost.

• The LU-factorization of the matrix in (4.18) requires $(1/3)q_m^3$ flops.

• The calculation of the residuals $\{r^{(k)}(t_{n,i})\}$ and $\{r^{(k)}(t_{m,i})\}$ requires approximately $q_n(q_n + q_m)$ flops.

• The calculation of the righthand side of (4.18) requires $q_m(q_n + q_m)$ flops.

• The solution for $\{e(t_{m,i})\}$ requires approximately q_m^2 flops.

• The calculation of $\{e(t_{n,i})\}$ requires approximately $4q_n q_m$ flops.

Thus the total cost in operations per iteration is approximately $q_n(q_n + 6q_m) + 2q_m^2$ flops.

Remark 4.2. As compared to Nyström approximation method 1 described in Atkinson [**2**, Section 6.2] the additional cost involved is in generating the matrices on the lefthand side of (4.18).

The total cost in operations per iteration in Nyström iteration method 1 is approximately $q_n(q_n + 2q_m) + q_m^2$ flops, whereas in Nyström iteration method 2, it is approximately $2q_n(q_n + q_m) + q_m^2$ flops. (See Atkinson [**2**, Section 6.2].)

Thus for $q_n \gg q_m$, the total costs per iteration in Nyström iteration method 1 and the method (4.8) proposed here are comparable, whereas each iteration in Nyström iteration method 2 is approximately twice as expensive.

It can be shown that the total cost in operations per iteration in the modified multi-grid method defined by (4.12) is approximately $2q_n(q_n + 2q_m) + 2q_m^2 + q_n q_m$ flops, which is approximately the same as in Nyström iteration method 2.

Numerical results given in the next section show that the two grid method proposed here requires significantly less number of iterations as compared to Nyström iteration methods 1 and 2.

5. Numerical results. Even though the results in the previous section are valid for two variable integral equations, in this section we compare the performance of the two grid method (4.8) with the Nyström iteration methods 1 and 2 by applying it to a one variable integral equation. Consider the following integral equation from Atkinson [**2**, p. 254].

$$\lambda\, u(t) - \int_0^1 k_\gamma(s+t)u(s)\,ds \;=\; v(t), \quad t \in [0,1] \tag{5.1}$$

with

$$k_\gamma(\tau) = \frac{1-\gamma^2}{1+\gamma^2 - 2\gamma\cos(2\pi\tau)}, \quad 0 \le \gamma < 1.$$

An approximate quadrature formula is chosen to be the composite midpoint formula:

$$\int_0^1 g(t)\,dt \;\approx\; \frac{1}{n}\sum_{j=1}^{n} g\left(\frac{2j-1}{2n}\right), \quad g \in C[0,1].$$

Consider the uniform partition

$$0 < \frac{1}{m} < \frac{2}{m} < \cdots < \frac{m-1}{m} < 1$$

of $[0,1]$. The interpolatory projection P_m is chosen to be the piecewise constant interpolation with respect to the above partition with collocation points chosen as $(2j-1)/(2m)$, $j = 1, \ldots m$.

The integral equation (5.1) is solved with $\gamma = 0.8$ and the unknown functions are

$$u_1(t) \;\equiv\; 1, \qquad u_2(t) \;=\; \sin(2\pi t).$$

Let

$$\underline{u}_n^{(k)} = [u_n^{(k)}(t_{n,1}), \ldots, u_n^{(k)}(t_{n,q_n})]^T,$$

the values of the iterate $u_n^{(k)}$ at fine grid points.

In Table 5.1 we give numerical results. The initial guess is taken to be $\underline{u}_n^{(0)} = 0$, and the iteration was performed until $\|\underline{u}_n^{(k)} - \underline{u}_n^{(k-1)}\|_\infty$ was less than 10^{-13}. The columns M1, Nys 1 and Nys 2 give, respectively, the number of iterates in the proposed method (4.8), Nyström iteration method 1 and Nyström iteration method 2. The results in columns Nys 1 and Nys 2 are quoted from Atkinson [**2**, Tables 6.2 and 6.3].

TABLE 5.1.

Unknown	λ	m	n	M1	Nys 1	Nys 2
u_1	-1.00	16	32	5	18	10
u_1	-1.00	16	64	3	19	11
u_1	-1.00	16	128	2	19	11
u_1	-1.00	32	64	3	10	6
u_1	-1.00	32	128	2	10	6
u_1	0.99	32	64	3	29	17
u_1	0.99	32	128	2	29	17
u_2	-1.00	16	64	11	divergent	27
u_2	-1.00	16	128	11	divergent	27
u_2	-1.00	32	64	6	14	7
u_2	-1.00	32	128	6	14	7

Remark 5.1. It is seen from the above table that in all the cases considered here, the proposed two grid method requires significantly less number of iterates as compared to the Nyström iteration method 1. For the unknown function u_1 and for $\lambda = -1.00$, the number of iterates in the proposed method is half as compared to the Nyström iteration method 2. Note that 1 is one of the eigenvalues of the integral operator in (5.1). The performance of the proposed method is much better as compared to the Nyström iteration method 2 when $\lambda = 0.99$. In the case of the unknown function u_2 and $\lambda = -1.00$, if $m = 16$, then the number of iterates in the proposed method is less than half as compared to the Nyström iteration method 2, whereas if $m = 32$, then the number of iterates in the proposed method and the Nyström iteration method 2 are about the same.

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REFERENCES

1. K.E. Atkinson, *Iterative variants of the Nyström method for the numerical solution of integral equations*, Numer. Math. **22** (1973), 17–31.

2. ———, *The numerical solution of integral equations of the second kind*, Cambridge Univ. Press, New York, 1997.

3. K.E. Atkinson and G. Chandler, *The collocation method for solving the radiosity equation for unoccluded surfaces*, J. Integral Equations Appl. **10** (1998), 253–290.

4. K.E. Atkinson, I. Graham and I. Sloan, *Piecewise continuous collocation for integral equations*, SIAM J. Numer. Anal. **20** (1983), 172–186.

5. C.T.H. Baker, *The numerical treatment of integral equations*, Oxford Univ. Press, Oxford, 1977.

6. H. Brakhage, *Uber die numerische Behandlung von Integralgleichungen nach der Quadraturformelmethode*, Numer. Math. **2** (1960), 183–196.

7. G.A. Chandler, *Superconvergence of numerical solutions of second kind integral equations*, Ph. D. Thesis, Australian National University, 1979.

8. F. Chatelin, *Spectral approximation of linear operators*, Academic Press, New York, 1983.

9. F. Chatelin and R. Lebbar, *The iterated projection solution for the Fredholm integral equation of second kind*, J. Austral. Math. Soc. **22** (1981), 439–451.

10. W. Hackbusch, *Multi-grid methods and applications*, Springer-Verlag, Berlin, 1985.

11. ———, *Integral equations, theory and numerical treatment*, Birkhauser-Verlag, Basel, 1994.

12. Q. Hu, *Interpolation correction for collocation solutions of Fredholm integro-differential equations*, Math. Comp. **67** (1998), 987–999.

13. C. Kelley, *A fast multilevel algorithm for integral equations*, SIAM J. Numer. Anal. **32** (1995), 501–513.

14. M.A. Krasnoselskii, G.M. Vainikko, P.P. Zabreiko, Ya.B. Rutitskii and V.Ya. Stetsenko, *Approximate solution of operator equations*, Wolters-Noordhoff, Groningen, 1972.

15. R.P. Kulkarni, *A superconvergence result for solutions of compact operator equations*, Bull. Austral. Math. Soc. **68** (2003), 517–528.

16. Q. Lin, S. Zhang and N. Yan, *An acceleration method for integral equations by using interpolation post-processing*, Adv. Comput. Math. **9** (1998), 117–129.

17. E. Schock, *Galerkin like methods for equations of the second kind*, J. Integral Equations Appl. **4** (1982), 361–364.

18. I.H. Sloan, *Improvement by iteration for compact operator equations*, Math. Comp. **30** (1976), 758–764.

19. ———, *Four variants of the Galerkin method for integral equations of the second kind*, IMA J. Numer. Anal. **4** (1984), 9–17.

20. I.H. Sloan, *Superconvergence*, in *Numerical solution of integral equations* (M. Golberg, ed.), Plenum Press, New York, 1990.

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