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THE CAUCHY PROBLEM IN THE THEORY OF THERMOELASTIC PLATES WITH TRANSVERSE SHEAR DEFORMATION

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ABSTRACT. A uniqueness theorem is proved for the weak solution of the Cauchy problem for an infinite thermoelastic plate with transverse shear deformation. The problem for the homogeneous version of the governing system of equations is then studied by means of some special initial potentials. This is a fundamental step in the construction of a potential theory for dynamic problems for thermoelastic plates, since its results make it possible to reduce various initial boundary value problems to their analogs with homogeneous initial conditions, which, in turn, may then be solved by means of dynamic (retarded) potentials.

1. Introduction. Plate theories reduce three-dimensional initial-boundary value problems for this type of mechanical structures to ones in two dimensions. Although Kirchhoff's old mathematical model (1850) is a good approximation in many practical situations, it is in some respects not refined enough to satisfy today's increasing demand for accuracy and detail. This drawback is remedied to a large extent by transverse shear deformation models (see, for example, [**1**]), where the displacement field, the moments, and the shear force can be computed in full, thereby giving a better picture of the physical process of bending. The model considered in [**1**] was generalized to the case of a thermoelastic plate in [**2**].

In what follows we consider the time-dependent bending of a very large, that is, regarded as mathematically infinite, plate subject to external forces, moments, internal heat sources, and natural initial conditions. After studying some properties of the corresponding matrix of fundamental solutions, we define "initial" potentials of the first and

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second kinds, investigate their behavior, and show that the original Cauchy problem has a unique distributional solution, which in the case of the homogeneous equation can be represented in terms of these potentials. The analogs of the above results for plates with transverse shear deformation in the absence of thermal effects were obtained in [**3**].

2. Formulation of the problem. We consider an infinite elastic plate of thickness $h_0 = \text{const} > 0$, which occupies a region $\mathbf{R}^2 \times [-h_0/2, h_0/2]$ in $\mathbf{R}^3$. The displacement vector at a point x' in this region at $t \geq 0$ is denoted by $v(x',t) = (v_1(x',t), v_2(x',t), v_3(x',t))^{\mathrm{T}}$, where the superscript T signifies matrix transposition. The temperature in the plate is denoted by $\tau(x',t)$. Let $x' = (x, x_3)$, $x = (x_1, x_2) \in \mathbf{R}^2$. In plate models with transverse shear deformation it is assumed [**1**] that

$$v(x',t) = (x_3 u_1(x,t), x_3 u_2(x,t), u_3(x,t))^{\mathrm{T}}.$$

If thermal effects are taken into account, we also consider the "averaged" temperature across thickness defined by [**2**]

$$u_4(x,t) = \frac{1}{h^2 h_0} \int_{-h_0/2}^{h_0/2} x_3 \tau(x, x_3, t)\, dx_3, \qquad h^2 = \frac{h_0^2}{12}.$$

The factor $1/h^2$ has been introduced for reasons of convenience. Then the vector function $U(x,t) = (u(x,t)^{\mathrm{T}}, u_4(x,t))^{\mathrm{T}}$, where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))^{\mathrm{T}}$, satisfies the equation

$$(1) \qquad \mathcal{B}_0 \partial_t^2 U(x,t) + \mathcal{B}_1 \partial_t U(x,t) + \mathcal{A} U(x,t) = \mathcal{Q}(x,t), \quad (x,t) \in G,$$

where $G = \mathbf{R}^2 \times (0, \infty)$, $\mathcal{B}_0 = \operatorname{diag}\{\rho h^2, \rho h^2, \rho, 0\}$, $\partial_t = \partial/\partial t$, $\rho > 0$ is the constant density of the material,

$$\mathcal{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta\partial_1 & \eta\partial_2 & 0 & \varkappa^{-1} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} & & & h^2\gamma\partial_1 \\ & A & & h^2\gamma\partial_2 \\ & & & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix},$$

$$A = \begin{pmatrix} -h^2\mu\Delta - h^2(\lambda+\mu)\partial_1^2 + \mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1 \\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta - h^2(\lambda+\mu)\partial_2^2 + \mu & \mu\partial_2 \\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta \end{pmatrix},$$

$\partial_\alpha = \partial/\partial_\alpha$, $\alpha = 1, 2$, $\eta, \varkappa$, and γ are positive constants, λ and μ are the Lamé constants of the material satisfying $\lambda + \mu > 0$, $\mu > 0$, and $\mathcal{Q}(x,t) = (q(x,t)^{\mathrm{T}}, q_4(x,t))^{\mathrm{T}}$, where $q(x,t) = (q_1(x,t), q_2(x,t), q_3(x,t))^{\mathrm{T}}$ is a combination of the forces and moments acting on the plate and its faces and $q_4(x,t)$ is a combination of the averaged heat source density and the temperature and heat flux on the faces.

The classical Cauchy problem for (1) consists in finding $U(x,t) \in \mathrm{C}^2(G)$, $u \in \mathrm{C}^1(\bar{G})$, $u_4 \in \mathrm{C}(\bar{G})$, such that

$$\begin{aligned} \mathcal{B}_0 \partial_t^2 U(x,t) + \mathcal{B}_1 \partial_t U(x,t) + \mathcal{A} U(x,t) = \mathcal{Q}(x,t), \quad (x,t) \in G, \\ U(x,0) = U_0(x), \qquad \partial_t u(x,0) = \psi(x), \quad x \in \mathbf{R}^2, \end{aligned} \tag{2}$$

where $U_0(x) = (\varphi(x)^{\mathrm{T}}, \theta(x))^{\mathrm{T}}$, $\varphi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x))^{\mathrm{T}}$, and the initial "velocity" $\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x))^{\mathrm{T}}$ are prescribed.

To formulate this problem variationally, we introduce weighted Sobolev spaces of vector functions defined on G. For every $\kappa > 0$, consider the space $\mathbf{H}_{1,\kappa}(G)$ of all four-component distributions $U(x,t)$ on G with finite norm defined by

$$\|U\|_{1,\kappa;G}^2 = \int_G e^{-2\kappa t} \Big\{ |U(x,t)|^2 + |\partial_t U(x,t)|^2 + \sum_{i=1}^4 |\nabla u_i(x,t)|^2 \Big\} \, dx \, dt. \tag{3}$$

We remark that (3) is equivalent to the norm

$$\Big\{ \int_G e^{-2\kappa t} \big[(1 + |\xi|)^2 |\tilde{U}(\xi,t)|^2 + |\partial_t \tilde{U}(\xi,t)|^2 \big] \, d\xi \, dt \Big\}^{1/2},$$

where $\tilde{U}(\xi,t) = (\tilde{u}(\xi,t)^{\mathrm{T}}, \tilde{u}_4(\xi,t))^{\mathrm{T}}$, $\tilde{u}(\xi,t) = (\tilde{u}_1(\xi,t), \tilde{u}_2(\xi,t), \tilde{u}_3(\xi,t))^{\mathrm{T}}$, is the Fourier transform of $U(x,t)$ with respect to x. In what follows we do not distinguish between equivalent norms and denote them by the same symbol.

Let $W(x,t) = (w(x,t)^{\mathrm{T}}, w_4(x,t))^{\mathrm{T}} \in \mathrm{C}_0^\infty(\bar{G})$; that is, each component of $w = (w_1, w_2, w_3)^{\mathrm{T}}$ and w_4 are functions of class $\mathrm{C}^\infty(\bar{G})$ such that $\operatorname{supp} w_i \subset \bar{G}$, $i = 1, 2, 3, 4$. We multiply the ith component, $i = 1, 2, 3$, of the vector differential equation in (2) by the complex conjugate $\bar{w}_i$ of w_i, and the complex conjugate form of the forth component in (2) by $h^2 \gamma \eta^{-1} w_4$, integrate the new equalities over G, and add them together. As a result, we obtain

$$\begin{aligned}(4)\quad \int_G \big[&(B_0\partial_t^2 u, w) + (Au, w) + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t u_4)\\ &- h^2\gamma\eta^{-1}(w_4, \Delta u_4) + h^2\gamma(w_4, \partial_t \operatorname{div} u) + h^2\gamma(\nabla u_4, w)\big]\,dx\,dt\\ &= \int_G \big[(q, w) + h^2\gamma\eta^{-1}(w_4, q_4)\big]\,dx\,dt,\end{aligned}$$

where $B_0 = \operatorname{diag}\{\rho h^2, \rho h^2, \rho\}$ and $(\cdot\,,\cdot)$ is the inner product in the corresponding vector space $\mathbf{C}^m$. For simplicity, we also use the generic notation $(\cdot\,,\cdot)_0$ for the inner product in $\big[L^2(\mathbf{R}^2)\big]^m$ for all $m \in \mathbf{N}$. Integrating by parts in (4) and making use of the initial conditions in (2), we arrive at

$$\begin{aligned}(5)\quad \int_0^\infty \big[&a(u, w) - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t u_4)_0\\ &+ h^2\gamma\eta^{-1}(\nabla w_4, \nabla u_4)_0 - h^2\gamma(\nabla w_4, \partial_t u)_0 + h^2\gamma(\nabla u_4, w)_0\big]\,dt\\ &= (B_0\prime, \gamma_0 w)_0 + \int_0^\infty \big[(q, w)_0 + h^2\gamma\eta^{-1}(w_4, q_4)_0\big]\,dt,\end{aligned}$$

where γ_0 is the continuous trace operator from the Sobolev space of index $m \in \mathbf{N}$ and with weight $\exp(-2\kappa t)$, $t > 0$, of functions (or vector functions) defined in G to the corresponding standard Sobolev space of index $m-1/2$ of functions (vector functions) defined in $\mathbf{R}^2$, and $a(u, w)$ is the sesquilinear form defined by

$$a(u, w) = 2\int_{\mathbf{R}^2} E(u, w)\,dx,$$

where

$$\begin{aligned}2E(u, w) &= h^2 E_0(u, w) + h^2\mu(\partial_2 u_1 + \partial_1 u_2)(\partial_2 \bar{w}_1 + \partial_1 \bar{w}_2)\\ &\quad + \mu[(u_1 + \partial_1 u_3)(\bar{w}_1 + \partial_1 \bar{w}_3) + (u_2 + \partial_2 u_3)(\bar{w}_2 + \partial_2 \bar{w}_3)],\\ E_0(u, w) &= (\lambda + 2\mu)\big[(\partial_1 u_1)(\partial_1 \bar{w}_1) + (\partial_2 u_2)(\partial_2 \bar{w}_2)\big]\\ &\quad + \lambda\big[(\partial_1 u_1)(\partial_2 \bar{w}_2) + (\partial_2 u_2)(\partial_1 \bar{w}_1)\big].\end{aligned}$$

The form $E(u, u)$ is the potential energy density of the plate [**1**]. We remark that if $f \in \mathrm{C}^2(\mathbf{R}^2)$ and $g \in \mathrm{C}_0^\infty(\mathbf{R}^2)$, then $(Af, g)_0 =$

$a(f,g)$. Equation (5) suggests a variational formulation of the Cauchy problem (2). Thus, we say that $U(x,t)$ is a *weak solution* of (2) if $U \in \mathbf{H}_{1,\kappa}(G)$ for some $\kappa > 0$, U satisfies (5) for any $W \in \mathrm{C}_0^\infty(\bar{G})$, and $\gamma_0 U = U_0(x)$.

3. Uniqueness theorem. We examine the question of uniqueness of the solution to the variational Cauchy problem for (2).

Theorem 1. *The Cauchy problem* (2) *has at most one weak solution of class* $\mathbf{H}_{1,\kappa}(G)$.

Proof. Let $U_1(x,t)$ and $U_2(x,t)$ be two such solutions. Then $U = U_1 - U_2$ satisfies $\gamma_0 U = 0$ and the homogeneous equation (5); that is,

$$\int_0^\infty \big[a(u,w) - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t u_4)_0 + h^2\gamma\eta^{-1}(\nabla w_4, \nabla u_4)_0 - h^2\gamma(\nabla w_4, \partial_t u)_0 + h^2\gamma(\nabla u_4, w)_0\big]\, dt = 0 \quad \forall\, W \in \mathrm{C}_0^\infty(\bar{G}).$$

We define

$$Z(x,t) = (z(x,t)^{\mathrm{T}}, z_4(x,t))^{\mathrm{T}}, \qquad z(x,t) = (z_1(x,t), z_2(x,t), z_3(x,t))^{\mathrm{T}},$$

by writing

$$Z(x,t) = \int_0^t \int_0^\tau U(x,\zeta)\, d\zeta\, d\tau.$$

Clearly, $U(\cdot,t)$ belongs to $L^2((0,T); H_1(\mathbf{R}^2)) \cap H_1((0,T); L^2(\mathbf{R}^2))$ for any $T > 0$; that is, $U(\cdot,t)$ is square integrable as a mapping from $(0,T)$ to the standard Sobolev space $H_1(\mathbf{R}^2)$, and $U(\cdot,t)$ and $\partial_t U(\cdot,t)$ are square integrable as mappings from $(0,T)$ to $L^2(\mathbf{R}^2)$. Consequently, $Z(\cdot,t) \in H_2((0,T); H_1(\mathbf{R}^2)) \cap H_3((0,T); L^2(\mathbf{R}^2))$. In particular, $\partial_t^2 Z(\cdot,t)$ is absolutely continuous as a mapping from $[0,T]$ to $L^2(\mathbf{R}^2)$ for any $T > 0$. In addition, $Z(\cdot,0) = \partial_t Z(\cdot,0) = \partial_t^2 Z(\cdot,0) = 0$.

Let $\omega(t)$ be an "averaging kernel"; that is, a function such that

(i) $\omega \in \mathrm{C}_0^\infty(\mathbf{R})$, $\operatorname{supp}\omega \subset [-1,1]$, $\omega(t) \geq 0$;

(ii) $\int_{-\infty}^\infty \omega(t)\, dt = 1$.

We remark that, for any $T > 0$, the sequence $\omega_n(t) = n\omega(n(t-T))$, $n \in \mathbf{N}$, converges, as $n \to \infty$, to the Dirac delta $\delta(t-T)$ in the distributional sense, in other words, in the Schwartz space $\mathcal{S}'(\mathbf{R})$.

We choose an arbitrary $V(x,t) = (v(x,t)^{\mathrm{T}}, v_4(x,t))^{\mathrm{T}} \in \mathrm{C}_0^\infty(\bar{G})$ and construct the sequence $V_n(x,t) = \omega_n(t)V(x,t) = (v_n(x,t)^{\mathrm{T}}, v_{4,n}(x,t))^{\mathrm{T}}$. Also, taking

$$W_n(x,t) = \int_t^\infty \int_\tau^\infty V_n(x,\zeta)\,d\zeta\,d\tau = \big(w_n(x,t)^{\mathrm{T}}, w_{4,n}(x,t)\big)^{\mathrm{T}} \in \mathrm{C}_0^\infty(\bar{G}),$$

we deduce the equality

$$\begin{aligned}
&\int_0^\infty \big[a(z,v_n) - (B_0^{1/2}\partial_t z, B_0^{1/2}\partial_t v_n)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(v_{4,n},\partial_t z_4)_0 \\
&\qquad + h^2\gamma\eta^{-1}(\nabla v_{4,n},\nabla z_4)_0 - h^2\gamma(\nabla v_{4,n},\partial_t z)_0 + h^2\gamma(\nabla z_4, v_n)_0\big]\,dt \\
&\quad = \int_0^\infty \big[a(z,\partial_t^2 w_n) - (B_0^{1/2}\partial_t z, B_0^{1/2}\partial_t^3 w_n)_0 \\
&\qquad\qquad + h^2\gamma\eta^{-1}\varkappa^{-1}(\partial_t^2 w_{4,n},\partial_t z_4)_0 + h^2\gamma\eta^{-1}(\nabla\partial_t^2 w_{4,n},\nabla z_4)_0 \\
&\qquad\qquad\qquad - h^2\gamma(\nabla\partial_t^2 w_{4,n},\partial_t z)_0 + h^2\gamma(\nabla z_4,\partial_t^2 w_n)_0\big]\,dt \\
&\quad = \int_0^\infty \big[a(u,w_n) - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w_n)_0 \\
&\qquad\qquad + h^2\gamma\eta^{-1}\varkappa^{-1}(w_{4,n},\partial_t u_4)_0 + h^2\gamma\eta^{-1}(\nabla w_{4,n},\nabla u_4)_0 \\
&\qquad\qquad\qquad - h^2\gamma(\nabla w_{4,n},\partial_t u)_0 + h^2\gamma(\nabla u_4, w_n)_0\big]\,dt = 0.
\end{aligned}$$

Hence, after integrating by parts, we obtain

(6)
$$\begin{aligned}
&\int_0^\infty \big[a(z,v_n) + (B_0^{1/2}\partial_t^2 z, B_0^{1/2} v_n)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(v_{4,n},\partial_t z_4)_0 \\
&\qquad + h^2\gamma\eta^{-1}(\nabla v_{4,n},\nabla z_4)_0 - h^2\gamma(\nabla v_{4,n},\partial_t z)_0 + h^2\gamma(\nabla z_4, v_n)_0\big]\,dt = 0.
\end{aligned}$$

Letting $n \to \infty$ in (6), we arrive at

(7)
$$\begin{aligned}
&a(z(x,T), v(x,T)) + (B_0^{1/2}\partial_T^2 z(x,T), B_0^{1/2} v(x,T))_0 \\
&\; + h^2\gamma\eta^{-1}\varkappa^{-1}(v_4(x,T),\partial_T z_4(x,T))_0 + h^2\gamma\eta^{-1}(\nabla v_4(x,T),\nabla z_4(x,T))_0 \\
&\qquad - h^2\gamma(\nabla v_4(x,T),\partial_T z(x,T))_0 + h^2\gamma(\nabla z_4(x,T), v(x,T))_0 = 0.
\end{aligned}$$

Since (7) holds for all $T > 0$, we may replace T by t and integrate (7) over $(0,T)$ with respect to t:

(8)
$$\int_0^T \big[a(z,v) + (B_0^{1/2}\partial_t^2 z, B_0^{1/2} v)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(v_4, \partial_t z_4)_0 + h^2\gamma\eta^{-1}(\nabla v_4, \nabla z_4)_0 - h^2\gamma(\nabla v_4, \partial_t z)_0 + h^2\gamma(\nabla z_4, v)_0\big]\, dt = 0.$$

Approximating $\partial_t z(x,t)$ and $z_4(x,t)$ by infinitely smooth, with respect to x, functions with compact support, we deduce that we can take $v = \partial_t z$ and $v_4 = z_4$ in (8) and, thus, write

$$\int_0^T \big[a(z,\partial_t z) + (B_0^{1/2}\partial_t^2 z, B_0^{1/2}\partial_t z)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(z_4, \partial_t z_4)_0 + h^2\gamma\eta^{-1}\|\nabla z_4\|_0^2\big]\, dt = 0,$$

where $\|\cdot\|_0$ is the norm in $\big[L^2(\mathbf{R}^2)\big]^m$. Consequently,

$$\int_0^T \frac{d}{dt}\big[a(z,z) + \|B_0^{1/2}\partial_t z\|_0^2 + h^2\gamma\eta^{-1}\varkappa^{-1}\|z_4\|_0^2\big]\, dt = \big[a(z,z) + \|B_0^{1/2}\partial_t z\|_0^2 + h^2\gamma\eta^{-1}\varkappa^{-1}\|z_4\|_0^2\big]_{t=T} \le 0$$

and $Z(x,T) = 0$ for all $T > 0$. Since $U = \partial_t^2 Z$, we conclude that $U(x,t) \equiv 0$ for all $t \ge 0$, which proves the theorem. □

Obviously, the solution of (2) is the sum of the solutions of two simpler problems, namely, the Cauchy problem for the homogeneous system (1) with the given initial data and the Cauchy problem for the nonhomogeneous system (1) with zero initial data. Below, we restrict our attention to the first of these problems.

4. The Cauchy problem for the homogeneous system. Let $\mathcal{Q}(x,t) \equiv 0$. Then (5) takes the form

(9)
$$\int_0^\infty \big[a(u,w) - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t u_4)_0 + h^2\gamma\eta^{-1}(\nabla w_4, \nabla u_4)_0 - h^2\gamma(\nabla w_4, \partial_t u)_0 + h^2\gamma(\nabla u_4, w)_0\big]\, dt = (B_0\psi, \gamma_0 w)_0,$$

and we seek $U \in \mathbf{H}_{1,\kappa}(G)$ that satisfies (9) for all $W \in \mathrm{C}_0^\infty(\bar{G})$ and such that

$$\gamma_0 U = U_0(x) = (\varphi(x)^{\mathrm{T}}, \theta(x))^{\mathrm{T}}.$$

In what follows, we give an explicit analytic expression for the solution of (9) as a combination of two special potential-type integrals called "initial" potentials of the first and second kinds. In order to do that, we need to study the properties of a matrix of fundamental solutions $D(x,t)$ for the system (1).

$D(x,t)$ is a (distributional) solution of the problem

$$\begin{gathered} \mathcal{B}_0\partial_t^2 D(x,t) + \mathcal{B}_1\partial_t D(x,t) + \mathcal{A}D(x,t) = \delta(x,t)I, \quad (x,t) \in \mathbf{R}^3, \\ D(x,t) = 0, \quad t < 0, \end{gathered}$$

where I is the identity (4×4)-matrix and δ is the Dirac delta. Its Fourier transform $\widetilde{D}(\xi,t)$ satisfies

(10)
$$\begin{gathered} \mathcal{B}_0\partial_t^2 \widetilde{D}(\xi,t) + \mathcal{B}_1(\xi)\partial_t \widetilde{D}(\xi,t) + \mathcal{A}(\xi)\widetilde{D}(\xi,t) = \delta(t)I, \quad (\xi,t) \in \mathbf{R}^3, \\ \widetilde{D}(\xi,t) = 0, \quad t < 0, \end{gathered}$$

where

$$\mathcal{B}_1(\xi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\eta\xi_1 & -i\eta\xi_2 & 0 & \varkappa^{-1} \end{pmatrix}, \; \mathcal{A}(\xi) = \begin{pmatrix} & & & -ih^2\gamma\xi_1 \\ & A(\xi) & & -ih^2\gamma\xi_2 \\ & & & 0 \\ 0 & 0 & 0 & |\xi|^2 \end{pmatrix},$$

$$A(\xi) = \begin{pmatrix} h^2\mu|\xi|^2 + h^2(\lambda+\mu)\xi_1^2 + \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -i\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu|\xi|^2 + h^2(\lambda+\mu)\xi_2^2 + \mu & -i\mu\xi_2 \\ i\mu\xi_1 & i\mu\xi_2 & \mu|\xi|^2 \end{pmatrix}.$$

We represent $D(x,t)$ and $\widetilde{D}(\xi,t)$ in the form

$$D(x,t) = \chi(t)\Phi(x,t), \qquad \widetilde{D}(\xi,y) = \chi(t)\tilde{\Phi}(\xi,t), \tag{11}$$

where $\chi(t)$ is the characteristic function of the positive semi-axis. Substituting (11) in (10), we find that the matrix $\tilde{\Phi}(\xi,t)$ satisfies the Cauchy problem

$$\begin{gathered} \mathcal{B}_0\partial_t^2\tilde{\Phi}(\xi,t) + \mathcal{B}_1(\xi)\partial_t\tilde{\Phi}(\xi,t) + \mathcal{A}(\xi)\tilde{\Phi}(\xi,t) = 0, \quad \xi \in \mathbf{R}^2, \; t > 0, \\ \mathcal{B}_0\tilde{\Phi}(\xi,0) = 0, \quad \mathcal{B}_0\partial_t\tilde{\Phi}(\xi,0) + \mathcal{B}_1(\xi)\tilde{\Phi}(\xi,0) = I. \end{gathered}$$

In what follows, the columns of a (4×4)-matrix K are denoted by $K^{(j)}$, $j=1,2,3,4$. Clearly, each $\tilde{\Phi}^{(j)}(\xi,t)$ satisfies

$$
\begin{aligned}
&\mathcal{B}_0\partial_t^2\tilde{\Phi}^{(j)}(\xi,t)+\mathcal{B}_1(\xi)\partial_t\tilde{\Phi}^{(j)}(\xi,t)+\mathcal{A}(\xi)\tilde{\Phi}^{(j)}(\xi,t)=0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xi\in\mathbf{R}^2,\ t>0, \\
&\mathcal{B}_0\tilde{\Phi}^{(j)}(\xi,0)=0,\quad \mathcal{B}_0\partial_t\tilde{\Phi}^{(j)}(\xi,0)+\mathcal{B}_1(\xi)\tilde{\Phi}^{(j)}(\xi,0)=I^{(j)}.
\end{aligned}
\tag{12}
$$

If we write

$$
\begin{aligned}
\tilde{\Phi}^{(j)}(\xi,t)&=(\tilde{u}^{(j)}(\xi,t)^{\mathrm{T}},\tilde{u}_4^{(j)}(\xi,t))^{\mathrm{T}},\\
\tilde{u}^{(j)}(\xi,t)&=(\tilde{u}_1^{(j)}(\xi,t),\tilde{u}_2^{(j)}(\xi,t),\tilde{u}_3^{(j)}(\xi,t))^{\mathrm{T}},\\
\tilde{u}^{(j)}(\xi,0)&=\tilde{\varphi}^{(j)}(\xi)=(\tilde{\varphi}_1^{(j)}(\xi),\tilde{\varphi}_2^{(j)}(\xi),\tilde{\varphi}_3^{(j)}(\xi))^{\mathrm{T}},\\
\tilde{u}_4^{(j)}(\xi,0)&=\tilde{\theta}^{(j)}(\xi),\\
\partial_t\tilde{u}^{(j)}(\xi,0)&=\tilde{\psi}^{(j)}(\xi)=(\tilde{\psi}_1^{(j)}(\xi),\tilde{\psi}_2^{(j)}(\xi),\tilde{\psi}_3^{(j)}(\xi))^{\mathrm{T}},\quad j=1,2,3,4,
\end{aligned}
$$

then the initial conditions in (12) become

$$
\begin{aligned}
\tilde{\varphi}^{(j)}(\xi)&=0,\qquad \tilde{\theta}^{(j)}(\xi)=\varkappa\delta_{4j},\\
\tilde{\psi}^{(j)}(\xi)&=B_0^{-1}(\delta_{1j},\delta_{2j},\delta_{3j})^{\mathrm{T}},\quad j=1,2,3,4,
\end{aligned}
\tag{13}
$$

where δ_{ij} is the Kronecker delta.

Comparing the initial conditions in (12), (13), and (9), we see that $\Phi^{(j)}(x,t)$ (or $D^{(j)}(x,t)$) is the solution in G of the variational equation

$$
\begin{aligned}
&\int_0^\infty \big[a(u^{(j)},w)-(B_0^{1/2}\partial_t u^{(j)},B_0^{1/2}\partial_t w)_0+h^2\gamma\eta^{-1}\varkappa^{-1}(w_4,\partial_t u_4^{(j)})_0\\
&\qquad +h^2\gamma\eta^{-1}(\nabla w_4,\nabla u_4^{(j)})_0-h^2\gamma(\nabla w_4,\partial_t u^{(j)})_0+h^2\gamma(\nabla u_4^{(j)},w)_0\big]\,dt\\
&\qquad\quad =(B_0\psi^{(j)},\gamma_0 w)_0\quad \forall\, W\in \mathrm{C}_0^\infty(\bar{G}),\ \gamma_0 U=(0,0,0,\varkappa\delta_{4j}\delta(x)),
\end{aligned}
\tag{14}
$$

where $u^{(j)}$, $u_4^{(j)}$, and $\psi^{(j)}$, $j=1,2,3,4$, are the inverse Fourier transforms with respect to ξ of $\tilde{u}^{(j)}(\xi,t)$, $\tilde{u}_4^{(j)}(\xi,t)$, and $\tilde{\psi}^{(j)}(\xi)$, respectively.

For simplicity, in what follows all positive constants in estimates, which do not depend on the functions and variables occurring in those estimates, are denoted by the same symbol c.

Lemma 1. *There are positive constants c such that*

$$|\tilde{u}^{(j)}(\xi,t)| \le c(1+t)(1+|\xi|)^{-1}, \qquad |\partial_t \tilde{u}^{(j)}(\xi,t)| \le c, \tag{15}$$
$$|\tilde{u}_4^{(j)}(\xi,t)| \le c, \quad j=1,2,3,4,$$

and, for any $l \ge 1$,

$$|\partial_t^{l+1} \tilde{u}^{(j)}(\xi,t)| \le c(1+|\xi|)^{2l-1}, \quad j=1,2,3,4,$$
$$|\partial_t^l \tilde{u}_4^{(j)}(\xi,t)| \le c \begin{cases} (1+t)(1+|\xi|)^{2l-1}, & j=1,2,3, \\ (1+|\xi|)^{2l}, & j=4. \end{cases} \tag{16}$$

Proof. First, we establish an energy estimate for the matrix of fundamental solutions. Multiplying the ith component, $i=1,2,3$, of the vector differential equation in (12) by the complex conjugate of $\partial_t \tilde{u}_i^{(j)}(\xi,t)$ and the complex conjugate of the fourth equation in (12) by $h^2\gamma\varkappa^{-1}\tilde{u}_4^{(j)}(\xi,t)$, and adding the new equalities, we arrive at

$$\begin{aligned}\big(B_0\partial_t^2\tilde{u}^{(j)}(\xi,t), \partial_t\tilde{u}^{(j)}(\xi,t)\big) + \big(A(\xi)\tilde{u}^{(j)}(\xi,t), \partial_t\tilde{u}^{(j)}(\xi,t)\big) \\ + h^2\gamma\eta^{-1}\varkappa^{-1}\big(\tilde{u}_4^{(j)}(\xi,t), \partial_t\tilde{u}_4^{(j)}(\xi,t)\big) = -h^2\gamma\eta^{-1}|\xi|^2|\tilde{u}_4^{(j)}(\xi,t)|^2,\end{aligned}$$

or

$$\begin{aligned}\frac{d}{dt}\big\{|B_0^{1/2}\partial_t\tilde{u}^{(j)}(\xi,t)|^2 + \big(A(\xi)\tilde{u}^{(j)}(\xi,t), \tilde{u}^{(j)}(\xi,t)\big) \\ + h^2\gamma\eta^{-1}\varkappa^{-1}|\tilde{u}_4^{(j)}(\xi,t)|^2\big\} = -2h^2\gamma\eta^{-1}|\xi|^2|\tilde{u}_4^{(j)}(\xi,t)|^2.\end{aligned} \tag{17}$$

From (17) and (13) it follows that

$$\begin{aligned}|B_0^{1/2}\partial_t\tilde{u}^{(j)}(\xi,t)|^2 + (A(\xi)\tilde{u}^{(j)}(\xi,t), \tilde{u}^{(j)}(\xi,t)) + h^2\gamma\eta^{-1}\varkappa^{-1}|\tilde{u}_4^{(j)}(\xi,t)|^2 \\ \le c;\end{aligned}$$

therefore,

$$(A(\xi)\tilde{u}^{(j)}(\xi,t), \tilde{u}^{(j)}(\xi,t)) \le c, \qquad |\partial_t\tilde{u}^{(j)}(\xi,t)| \le c, \tag{18}$$
$$|\tilde{u}_4^{(j)}(\xi,t)| \le c.$$

In [**3**] it is shown that there are positive constants k_1 and k_2 such that

$$\big[k_1(1+|\xi|)^2 - k_2\big]|\psi|^2 \le \big(A(\xi)\psi,\psi\big) \quad \forall\,\psi\in\mathbf{C}^3; \tag{19}$$

hence,

$$(1+|\xi|)^2|\psi|^2 \le c\big[(A(\xi)\psi,\psi)+|\psi|^2\big],$$

and from (18) it follows that

$$(1+|\xi|)^2|\tilde u^{(j)}(\xi,t)|^2 \le c\big[1+|\tilde u^{(j)}(\xi,t)|^2\big]. \tag{20}$$

On the other hand,

$$\tilde u^{(j)}(\xi,t) = \int_0^t \partial_\tau \tilde u^{(j)}(\xi,\tau)\,d\tau,$$

and again from (18) it follows that

$$|\tilde u^{(j)}(\xi,t)| \le ct. \tag{21}$$

Combining (20) and (21), we obtain the estimate

$$|\tilde u^{(j)}(\xi,t)| \le c(1+t)(1+|\xi|)^{-1}, \quad j=1,2,3,4. \tag{22}$$

Estimates (18) and (22) prove (15).

We now claim that the inequalities

$$\begin{aligned} &|\partial_t^{l+1}\tilde u^{(j)}(\xi,t)| + |\partial_t^l \tilde u_4^{(j)}(\xi,t)| \le c(1+|\xi|)^{2l-1}, \quad j=1,2,3,\\ &|\partial_t^{l+1}\tilde u^{(4)}(\xi,t)| + |\partial_t^l \tilde u_4^{(4)}(\xi,t)| \le c(1+|\xi|)^{2l} \end{aligned} \tag{23}$$

hold for any $l\ge 1$. Differentiating the equation in (12), we find that

$$\mathcal{B}_0\partial_t^2\partial_t^l\tilde\Phi^{(j)}(\xi,t) + \mathcal{B}_1(\xi)\partial_t\partial_t^l\tilde\Phi^{(j)}(\xi,t) + \mathcal{A}(\xi)\partial_t^l\tilde\Phi^{(j)}(\xi,t) = 0, \qquad \xi\in\mathbf{R}^2,\ t>0; \tag{24}$$

therefore,

$$\frac{d}{dt}\big\{|B_0^{1/2}\partial_t\partial_t^l\tilde u^{(j)}(\xi,t)|^2 + \big(A(\xi)\partial_t^l\tilde u^{(j)}(\xi,t),\partial_t^l\tilde u^{(j)}(\xi,t)\big) + h^2\gamma\eta^{-1}\varkappa^{-1}|\partial_t^l\tilde u_4^{(j)}(\xi,t)|^2\big\} \le 0$$

and

$$\begin{aligned}|B_0^{1/2}\partial_t\partial_t^l\tilde{u}^{(j)}(\xi,t)|^2 + \big(A(\xi)\partial_t^l\tilde{u}^{(j)}(\xi,t),\partial_t^l\tilde{u}^{(j)}(\xi,t)\big)\\ + h^2\gamma\eta^{-1}\varkappa^{-1}|\partial_t^l\tilde{u}_4^{(j)}(\xi,t)|^2\\ \leq c\big\{|B_0^{1/2}\partial_t^{l+1}\tilde{u}^{(j)}(\xi,0)|^2 + \big(A(\xi)\partial_t^l\tilde{u}^{(j)}(\xi,0),\partial_t^l\tilde{u}^{(j)}(\xi,0)\big)\\ + h^2\gamma\eta^{-1}\varkappa^{-1}|\partial_t^l\tilde{u}_4^{(j)}(\xi,0)|^2\big\}.\end{aligned}$$

For $l=1$,

$$\begin{aligned}&|\partial_t^2\tilde{u}^{(j)}(\xi,t)|^2 + |\partial_t\tilde{u}_4^{(j)}(\xi,t)|^2\\ &\quad\leq c\big\{|\partial_t^2\tilde{u}^{(j)}(\xi,0)|^2 + |\partial_t\tilde{u}_4^{(j)}(\xi,0)|^2 + (1+|\xi|)^2|\partial_t\tilde{u}^{(j)}(\xi,0)|^2\big\}.\end{aligned}\tag{25}$$

Since $\tilde{\Phi}^{(j)}(\xi,t)$ is smooth for $t\in[0,\infty)$, from (12) we deduce that

$$\begin{aligned}|\partial_t^2\tilde{u}^{(j)}(\xi,0)| &\leq c\big\{(1+|\xi|)^2|\tilde{\Phi}^{(j)}(\xi)| + |\xi||\,\tilde{\theta}^{(j)}(\xi)|\big\},\\ |\partial_t\tilde{u}_4^{(j)}(\xi,0)| &\leq c\big\{|\xi||\tilde{\psi}^{(j)}(\xi)| + |\xi|^2|\tilde{\theta}^{(j)}(\xi)|\big\};\end{aligned}$$

hence,

$$\begin{aligned}\partial_t^2\tilde{u}^{(j)}(\xi,0) = 0, \qquad |\partial_t\tilde{u}_4^{(j)}(\xi,0)| \leq c|\xi|, \quad j=1,2,3,\\ |\partial_t^2\tilde{u}^{(4)}(\xi,0)| \leq c|\xi|, \qquad |\partial_t\tilde{u}_4^{(4)}(\xi,0)| \leq c|\xi|^2.\end{aligned}\tag{26}$$

Substituting (26) in (25), we obtain

$$\begin{aligned}|\partial_t^2\tilde{u}^{(j)}(\xi,t)| + |\partial_t\tilde{u}_4^{(j)}(\xi,t)| &\leq c(1+|\xi|), \quad j=1,2,3,\\ |\partial_t^2\tilde{u}^{(4)}(\xi,t)| + |\partial_t\tilde{u}_4^{(4)}(\xi,t)| &\leq c(1+|\xi|)^2;\end{aligned}$$

therefore, (23) is proved for $l=1$. Suppose now that (23) holds for all $k\leq l$, $l>1$. Then

$$\begin{aligned}\mathcal{B}_0\partial_t^2\partial_t^{l+1}\tilde{\Phi}^{(j)}(\xi,t) + \mathcal{B}_1(\xi)\partial_t\partial_t^{l+1}\tilde{\Phi}^{(j)}(\xi,t) + \mathcal{A}(\xi)\partial_t^{l+1}\tilde{\Phi}^{(j)}(\xi,t) = 0,\\ \xi\in\mathbf{R}^2,\ t>0;\end{aligned}$$

hence,

$$\begin{aligned}&|\partial_t^{l+2}\tilde{u}^{(j)}(\xi,t)| + |\partial_t^{l+1}\tilde{u}_4^{(j)}(\xi,t)|\\ &\quad\leq c\big\{|\partial_t^{l+2}\tilde{u}^{(j)}(\xi,0)| + |\partial_t^{l+1}\tilde{u}_4^{(j)}(\xi,0)| + (1+|\xi|)|\partial_t^{l+1}\tilde{u}^{(j)}(\xi,0)|\big\}.\end{aligned}\tag{27}$$

On the other hand, from (24) it follows that

$$|\partial_t^{l+2}\tilde{u}^{(j)}(\xi,0)| \le c\big\{(1+|\xi|)^2|\partial_t^l\tilde{u}^{(j)}(\xi,0)| + |\xi||\partial_t^l\tilde{u}_4^{(j)}(\xi,0)|\big\},$$
$$|\partial_t^{l+1}\tilde{u}_4^{(j)}(\xi,0)| \le c\big\{|\xi||\partial_t^{l+1}\tilde{u}^{(j)}(\xi,0)| + |\xi|^2|\partial_t^l\tilde{u}_4^{(j)}(\xi,0)|\big\}.$$

Making use of (27), we obtain

$$\begin{aligned}|\partial_t^{l+2}\tilde{u}^{(j)}(\xi,t)| &+ |\partial_t^{l+1}\tilde{u}_4^{(j)}(\xi,t)| \\ &\le c\big\{(1+|\xi|)^2|\partial_t^l\tilde{u}^{(j)}(\xi,0)| + (1+|\xi|)^2|\partial_t^l\tilde{u}_4^{(j)}(\xi,0)| \\ &\qquad + (1+|\xi|)|\partial_t^{l+1}\tilde{u}^{(j)}(\xi,0)|\big\} \\ &\le c\begin{cases}(1+|\xi|)^{2l+1}, & j=1,2,3,\\ (1+|\xi|)^{2l+2}, & j=4.\end{cases}\end{aligned}$$

This proves (23) for all $l \ge 1$ and $j = 1,2,3,4$, which, in turn, proves statement (16) of the lemma for $l \ge 1$ and $j = 1,2,3$. To complete the proof of the lemma, we need to improve estimates (23) for $\partial_t^{l+1}\tilde{u}_4^{(j)}(\xi,t)$, $l \ge 1$, namely, we need to show that

$$|\partial_t^{l+1}\tilde{u}_4^{(j)}(\xi,t)| \le c(1+t)(1+|\xi|)^{2l-1}, \quad l \ge 1. \tag{28}$$

First, we establish (28) for $l = 1$. Since

$$\mathcal{B}_0\partial_t^2\tilde{\Phi}^{(4)}(\xi,t) = -\mathcal{B}_1(\xi)\partial_t\tilde{\Phi}^{(4)}(\xi,t) - \mathcal{A}(\xi)\tilde{\Phi}^{(4)}(\xi,t), \quad \xi \in \mathbf{R}^2,\ t>0,$$

we have

$$\mathcal{B}_0\partial_t^2\tilde{u}^{(4)}(\xi,t) = -A(\xi)\tilde{u}^{(4)}(\xi,t) - ih^2\gamma\xi\tilde{u}_4^{(4)}(\xi,t). \tag{29}$$

From (15) and (29) it follows that

$$|\partial_t^2\tilde{u}^{(4)}(\xi,t)| \le c(1+t)(1+|\xi|),$$

which proves (16) for $l = 1$. Suppose now that (16) holds for all $k \le l$, $l > 1$. By (24),

$$\mathcal{B}_0\partial_t^{l+2}\tilde{\Phi}^{(4)}(\xi,t) = -\mathcal{B}_1(\xi)\partial_t^{l+1}\tilde{\Phi}^{(4)}(\xi,t) - \mathcal{A}(\xi)\partial_t^l\tilde{\Phi}^{(4)}(\xi,t),$$
$$\xi \in \mathbf{R}^2,\ t>0$$

and

$$\mathcal{B}_0 \partial_t^{l+2} \tilde{u}^{(4)}(\xi,t) = -A(\xi)\partial_t^l \tilde{u}^{(4)}(\xi,t) - ih^2\gamma\xi\partial_t^l \tilde{u}_4^{(4)}(\xi,t).$$

Using the inductive assumption and (23), we obtain

$$|\partial_t^{l+2} \tilde{u}^{(4)}(\xi,t)| \le c(1+t)(1+|\xi|)^{2l+1},$$

which completes the proof of the lemma. □

We now introduce the initial potentials that generate the solution of the Cauchy problem for the homogeneous system (1). The initial potential of the first kind of density

$$F(x) = (f(x)^{\mathrm{T}}, f_4(x))^{\mathrm{T}}, \qquad f(x) = (f_1(x), f_2(x), f_3(x))^{\mathrm{T}},$$

is defined by

$$\begin{aligned}\mathcal{J}(x,t) = (\mathcal{J}F)(x,t) &= \int_{\mathbf{R}^2} D(x-y,t)F(y)\,dy \\ &= \int_{\mathbf{R}^2} \Phi(x-y,t)F(y)\,dy, \quad t>0.\end{aligned}$$

The initial potential of the second kind of density

$$R(x) = (r(x)^{\mathrm{T}}, r_4(x))^{\mathrm{T}}, \qquad r(x) = (r_1(x), r_2(x), r_3(x))^{\mathrm{T}},$$

is defined by

$$\begin{aligned}\mathcal{E}(x,t) = (\mathcal{E}R)(x,t) &= \int_{\mathbf{R}^2} \partial_t D(x-y,t)R(y)\,dy \\ &= \int_{\mathbf{R}^2} \partial_t \Phi(x-y,t)R(y)\,dy = \partial_t(\mathcal{J}R)(x,t), \quad t>0.\end{aligned}$$

Their Fourier transforms, either in the classical or in the distributional sense, are

$$\left.\begin{aligned}\tilde{\mathcal{J}}(\xi,t) &= (\tilde{\mathcal{J}}\tilde{F})(\xi,t) = \widetilde{D}(\xi,t)\tilde{F}(\xi) = \tilde{\Phi}(\xi,t)\tilde{F}(\xi), \\ \tilde{\mathcal{E}}(\xi,t) &= (\tilde{\mathcal{E}}\tilde{R})(\xi,t) = \partial_t\widetilde{D}(\xi,t)\tilde{R}(\xi) = \partial_t\tilde{\Phi}(\xi,t)\tilde{R}(\xi),\end{aligned}\right. \quad t>0,$$

where $\tilde{F}(\xi)$ and $\tilde{R}(\xi)$ are the Fourier transforms of $F(x)$ and $R(x)$, respectively.

Lemma 2. (i) *If* $f \in H_1(\mathbf{R}^2)$ *and* $f_4 \in H_2(\mathbf{R}^2)$, *then* $\mathcal{J} \in \mathbf{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$.

(ii) *If* $r \in H_3(\mathbf{R}^2)$ *and* $r_4 \in H_4(\mathbf{R}^2)$, *then* $\mathcal{E} \in \mathbf{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$.

Proof. We write $\tilde{\mathcal{J}}(\xi, t)$ in the form

$$\tag{30} \tilde{\mathcal{J}}(\xi,t) = \begin{pmatrix} \tilde{u}_1^{(1)}(\xi,t))\tilde{f}_1(\xi) + \tilde{u}_1^{(2)}(\xi,t))\tilde{f}_2(\xi) + \tilde{u}_1^{(3)}(\xi,t))\tilde{f}_3(\xi) + \tilde{u}_1^{(4)}(\xi,t))\tilde{f}_4(\xi) \\ \tilde{u}_2^{(1)}(\xi,t))\tilde{f}_1(\xi) + \tilde{u}_2^{(2)}(\xi,t))\tilde{f}_2(\xi) + \tilde{u}_2^{(3)}(\xi,t))\tilde{f}_3(\xi) + \tilde{u}_2^{(4)}(\xi,t))\tilde{f}_4(\xi) \\ \tilde{u}_3^{(1)}(\xi,t))\tilde{f}_1(\xi) + \tilde{u}_3^{(2)}(\xi,t))\tilde{f}_2(\xi) + \tilde{u}_3^{(3)}(\xi,t))\tilde{f}_3(\xi) + \tilde{u}_3^{(4)}(\xi,t))\tilde{f}_4(\xi) \\ \tilde{u}_4^{(1)}(\xi,t))\tilde{f}_1(\xi) + \tilde{u}_4^{(2)}(\xi,t))\tilde{f}_2(\xi) + \tilde{u}_4^{(3)}(\xi,t))\tilde{f}_3(\xi) + \tilde{u}_4^{(4)}(\xi,t))\tilde{f}_4(\xi) \end{pmatrix}.$$

From (15) it follows that

$$|\tilde{u}_i^{(k)}(\xi,t))\tilde{f}_k(\xi)| \le c(1+t)(1+|\xi|)^{-1}|\tilde{f}_k(\xi)|, \quad k=1,2,3,4, \quad i=1,2,3,$$
$$|\tilde{u}_4^{(k)}(\xi,t))\tilde{f}_k(\xi)| \le c|\tilde{f}_k(\xi)|, \quad k=1,2,3,4;$$

hence,

$$\int_G e^{-2\kappa t}(1+|\xi|)^2|\tilde{\mathcal{J}}(\xi,t)|^2\,d\xi\,dt < \infty.$$

Also by (15) and (16),

$$\begin{aligned} |\partial_t \tilde{u}^{(k)}(\xi,t)\tilde{f}_k(\xi)| &\le c|\tilde{f}_k(\xi)|, \quad k=1,2,3,4, \\ |\partial_t \tilde{u}_4^{(k)}(\xi,t)\tilde{f}_k(\xi)| &\le c(1+t)(1+|\xi|)|\tilde{f}_k(\xi)|, \quad k=1,2,3, \\ |\tilde{u}_4^{(4)}(\xi,t)\tilde{f}_4(\xi)| &\le c(1+|\xi|)^2|\tilde{f}_4(\xi)|, \end{aligned}$$

and

$$\int_G e^{-2\kappa t}|\partial_t\tilde{\mathcal{J}}(\xi,t)|^2\,d\xi\,dt < \infty.$$

This proves (i). Assertion (ii) is proved similarly. It is obvious that

$$\tag{31} \|\tilde{\mathcal{J}}\|_{1,\kappa;G} \le c(\|f\|_1 + \|f_4\|_2), \qquad \|\tilde{\mathcal{E}}\|_{1,\kappa;G} \le c(\|r\|_3 + \|r_4\|_4),$$

where $\|\cdot\|_m$ is the norm in $H_m(\mathbf{R}^2)$. □

Remark. If instead of $\mathbf{H}_{1,\kappa}(G)$ we use $\mathbf{H}'_{1,\kappa}(G)$ with norm

$$\|U\|'_{1,\kappa;G} = \left\{ \int_G e^{-2\kappa t}\big[(1+|\xi|)^2|\tilde{U}(\xi,t)|^2 + |\partial_t \tilde{u}(\xi,t)|^2\big]\, d\xi\, dt \right\}^{1/2},$$

then formulas (31) are replaced by

$$\|\tilde{\mathcal{J}}\|'_{1,\kappa;G} \le c(\|f\|_1 + \|f_4\|_1), \qquad \|\tilde{\mathcal{E}}\|'_{1,\kappa;G} \le c(\|r\|_2 + \|r_4\|_3).$$

Hence, if $F \in H_1(\mathbf{R}^2)$, $r \in H_2(\mathbf{R}^2)$, and $r_4 \in H_3(\mathbf{R}^2)$, then $\tilde{\mathcal{J}}(\xi,t)$ and $\tilde{\mathcal{E}}(\xi,t)$ belong to $\mathbf{H}'_{1,\kappa}(G)$. □

We are now interested to know what problems are satisfied by the initial potentials. Suppose that their densities $F(x)$ and $R(x)$ have the properties in Lemma 2. Then both $\mathcal{J}(x,t)$ and $\mathcal{E}(x,t)$ belong to $\mathbf{H}_{1,\kappa}(G)$. We start with $\mathcal{J}(x,t)$.

Obviously,

$$\mathcal{J}(x,t) = \sum_{i=1}^{4} \int_{\mathbf{R}^2} \Phi^{(i)}(x-y,t) f_i(y)\, dy.$$

On the other hand, from (13) and (14) it follows that $\Phi^{(i)}(x,t) = (u^{(i)}(x,t)^{\mathrm{T}}, u_4^{(i)}(x,t))^{\mathrm{T}}$ satisfies the variational equation

$$\begin{aligned}\int_0^\infty \big[a(u^{(i)},w) - (B_0^{1/2}\partial_t u^{(i)}, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t u_4^{(i)})_0 \\ + h^2\gamma\eta^{-1}(\nabla w_4, \nabla u_4^{(i)})_0 - h^2\gamma(\nabla w_4, \partial_t u^{(i)})_0 + h^2\gamma(\nabla u_4^{(i)}, w)_0\big]\, dt \\ = \begin{cases} (\gamma_0 \bar{w}_i)(0), & i=1,2,3, \\ 0, & i=4, \end{cases} \qquad \forall\, W \in \mathrm{C}_0^\infty(\bar{G});\end{aligned}$$

therefore, $\mathcal{J}(x,t) = (j(x,t)^{\mathrm{T}}, j_4(x,t))^{\mathrm{T}}$, $j(x,t) = (j_1(x,t), j_2(x,t), j_3(x,t))^{\mathrm{T}}$, satisfies

$$\begin{aligned}(32)\quad \int_0^\infty \big[a(j,w) - (B_0^{1/2}\partial_t j, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t j_4)_0 \\ + h^2\gamma\eta^{-1}(\nabla w_4, \nabla j_4)_0 - h^2\gamma(\nabla w_4, \partial_t j)_0 + h^2\gamma(\nabla j_4, w)_0\big]\, dt \\ = (f, \gamma_0 w)_0 \quad \forall\, W \in \mathrm{C}_0^\infty(\bar{G}).\end{aligned}$$

Finally, from (13) we see that $j(x,t) \to 0$ and $\tilde{j}(\xi,t) \to 0$ as $t \to 0+$ for almost all x and ξ in $\mathbf{R}^2$. But by (15) and (30),

$$\|j(x,t)\|_2^2 = \int_{\mathbf{R}^2} (1+|\xi|)^4 |\tilde{j}(\xi,t)|^2 \, d\xi \\ \leq c(1+t) \int_{\mathbf{R}^2} (1+|\xi|)^2 |\tilde{F}(\xi)|^2 \, d\xi < \infty, \quad t \in [0,T].$$

Applying Lebesgue's dominated convergence theorem, we conclude that $j(x,t) \to 0$, as $t \to 0$, in $H_2(\mathbf{R}^2)$.

Also, $\tilde{j}_4(\xi,t) \to \varkappa \tilde{f}_4(\xi)$, as $t \to 0$, for almost all $\xi \in \mathbf{R}^2$. Again from (15) and (30) it follows that

$$\|j_4(x,t) - \varkappa f_4(x,t)\|_1^2 = \int_{\mathbf{R}^2} (1+|\xi|)^2 |\tilde{j}_4(\xi,t) - \varkappa \tilde{f}_4(\xi)|^2 \, d\xi \\ \leq c \int_{\mathbf{R}^2} (1+|\xi|)^2 |\tilde{F}(\xi)|^2 \, d\xi < \infty;$$

therefore, $\tilde{j}_4(x,t) \to \varkappa \tilde{\Phi}_4(x)$, as $t \to 0$, in $H_1(\mathbf{R}^2)$.

We now turn our attention to the properties of the initial potential of the second kind $\mathcal{E}(x,t) = (e(x,t)^{\mathrm{T}}, e_4(x,t))^{\mathrm{T}}$, where $e(x,t) = (e_1(x,t), e_2(x,t), e_3(x,t))^{\mathrm{T}}$. Its Fourier transform is $\tilde{\mathcal{E}}(\xi,t) = (\tilde{e}(\xi,t)^{\mathrm{T}}, \tilde{e}_4(\xi,t))^{\mathrm{T}}$, $\tilde{e}(\xi,t) = (\tilde{e}_1(\xi,t), \tilde{e}_2(\xi,t), \tilde{e}_3(\xi,t))^{\mathrm{T}}$. First, we study the matrix $K(x,t) = \partial_t \Phi(x,t)$ and its Fourier transform $\tilde{K}(\xi,t)$. Let $K^{(i)}(x,t) = (k^{(i)}(x,t)^{\mathrm{T}}, k_4^{(i)}(x,t))^{\mathrm{T}}$, where $k^{(i)}(x,0) = \alpha^{(i)}(x)$, $k_4^{(i)}(x,0) = \gamma^{(i)}(x)$, and $\partial_t k^{(i)}(x,0) = \beta^{(i)}(x)$. The corresponding Fourier transforms are denoted by $\tilde{K}^{(i)}(\xi,t) = (\tilde{k}^{(i)}(\xi,t)^{\mathrm{T}}, \tilde{k}_4^{(i)}(\xi,t))^{\mathrm{T}}$, $\tilde{k}^{(i)}(\xi,0) = \tilde{\alpha}^{(i)}(\xi)$, $\tilde{k}_4^{(i)}(\xi,0) = \tilde{\gamma}^{(i)}(\xi)$, and $\partial_t \tilde{k}^{(i)}(\xi,0) = \tilde{\beta}^{(i)}(\xi)$, $i = 1,2,3,4$. From (12) it follows that each $\tilde{K}^{(i)}(\xi,t)$ satisfies both

$$\mathcal{B}_0 \partial_t^2 \tilde{K}^{(i)}(\xi,t) + \mathcal{B}_1(\xi) \partial_t \tilde{K}^{(i)}(\xi,t) + \mathcal{A}(\xi) \tilde{K}^{(i)}(\xi,t) = 0 \tag{33}$$

and

$$\mathcal{B}_0 \partial_t \tilde{K}^{(i)}(\xi,t) + \mathcal{B}_1(\xi) \tilde{K}^{(i)}(\xi,t) + \mathcal{A}(\xi) \tilde{\Phi}^{(i)}(\xi,t) = 0. \tag{34}$$

To find the initial data for $\tilde{K}^{(i)}(\xi,t)$, $i = 1,2,3,4$, we note that

$$\tilde{\alpha}^{(i)}(\xi) = \partial \tilde{u}^{(i)}(\xi,0) = \tilde{\psi}^{(i)}(\xi), \quad i = 1,2,3,4,$$

so

$$\tilde{\alpha}^{(i)}(\xi) = B_0^{-1}(\delta_{1i}, \delta_{2i}, \delta_{3i})^{\mathrm{T}}, \quad i = 1, 2, 3, 4. \tag{35}$$

For $t = 0$, the forth component of (12) yields

$$\tilde{\gamma}^{(i)}(\xi) = i\varkappa\eta(\xi_1\tilde{\alpha}_1^{(i)}(\xi) + \xi_2\tilde{\alpha}_2^{(i)}(\xi)) - |\xi|^2\tilde{\theta}^{(i)}(\xi);$$

therefore,

$$\tilde{\gamma}^{(i)}(\xi) = \begin{cases} i(\eta\varkappa/(\rho h^2))\xi_i & i = 1, 2, \\ 0 & i = 3, \\ -\varkappa^2|\xi|^2 & i = 4. \end{cases} \tag{36}$$

To find $\tilde{\beta}^{(i)}(\xi)$, we consider the first three components of (34), from which

$$\tilde{\beta}^{(i)}(\xi) = i\varkappa h^2\gamma B_0^{-1}(\xi_1, \xi_2, 0)^{\mathrm{T}}\delta_{i4}, \quad i = 1, 2, 3, 4. \tag{37}$$

Combining (33), (35), (36), and (37), we see that the $\tilde{K}^{(i)}(\xi, t)$, $i = 1, 2, 3, 4$, satisfy

$$\begin{gathered} \mathcal{B}_0\partial_t^2\tilde{K}^{(i)}(\xi,t) + \mathcal{B}_1(\xi)\partial_t\tilde{K}^{(i)}(\xi,t) + \mathcal{A}(\xi)\tilde{K}^{(i)}(\xi,t) = 0, \\ \tilde{k}^{(i)}(\xi,0) = \tilde{\alpha}^{(i)}(\xi) = B_0^{-1}(\delta_{1i}, \delta_{2i}, \delta_{3i})^{\mathrm{T}}, \\ \tilde{k}_4^{(i)}(\xi,0) = \tilde{\gamma}^{(i)}(\xi) = \begin{cases} i(\eta\varkappa/(\rho h^2))\xi_i & i = 1, 2, \\ 0 & i = 3, \\ -\varkappa^2|\xi|^2 & i = 4, \end{cases} \\ \partial_t\tilde{k}^{(i)}(\xi,0) = \tilde{\beta}^{(i)}(\xi) = i\varkappa h^2\gamma B_0^{-1}(\xi_1, \xi_2, 0)^{\mathrm{T}}\delta_{i4}. \end{gathered} \tag{38}$$

Consequently, the $K^{(i)}(x, t)$, $i = 1, 2, 3, 4$, are solutions of the variational equations

$$\begin{aligned} \int_0^\infty \big[a(k^{(i)}, w) &- (B_0^{1/2}\partial_t k^{(i)}, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t k_4^{(i)})_0 \\ &+ h^2\gamma\eta^{-1}(\nabla w_4, \nabla k_4^{(i)})_0 - h^2\gamma(\nabla w_4, \partial_t k^{(i)})_0 + h^2\gamma(\nabla k_4^{(i)}, w)_0 \big]\, dt \\ &= \begin{cases} 0 & i = 1, 2, 3, \\ \varkappa h^2\gamma \operatorname{div}(\gamma_0\bar{w})(0) & i = 4, \end{cases} \qquad \forall\, W \in \mathrm{C}_0^\infty(\bar{G}). \end{aligned}$$

Since

$$\mathcal{E}(x,t)=\sum_{i=1}^{4}\int_{\mathbf{R}^2}K^{(i)}(x-y,t)r_i(y)\,dy,$$

the initial potential of the second kind satisfies

$$\begin{aligned}(39)\quad \int_0^\infty\big[a(e,w)-(B_0^{1/2}\partial_t e,B_0^{1/2}\partial_t w)_0+h^2\gamma\eta^{-1}\varkappa^{-1}(w_4,\partial_t e_4)_0\\ +h^2\gamma\eta^{-1}(\nabla w_4,\nabla e_4)_0-h^2\gamma(\nabla w_4,\partial_t e)_0+h^2\gamma(\nabla e_4,w)_0\big]\,dt\\ =\varkappa h^2\gamma(r_4,\operatorname{div}(\gamma_0 w))_0\quad \forall\, W\in \mathrm{C}_0^\infty(\bar G).\end{aligned}$$

To determine the initial conditions for this potential, we see that

$$\tilde{\mathcal{E}}(\xi,t)=\sum_{i=1}^{4}\tilde K^{(i)}(\xi,t)\tilde r_i(\xi),$$

which, along with the initial data in (38), yields

$$\begin{aligned}\lim_{t\to0+}\tilde{\mathcal{E}}(\xi,t)=(\rho h^2)^{-1}\big(\tilde r_1(\xi),\tilde r_2(\xi),h^2\tilde r_3(\xi),i\varkappa\eta\,[\xi_1\tilde r_1(\xi)+\xi_2\tilde r_2(\xi)]\\ -\varkappa^2\rho h^2|\xi|^2\tilde r_4(\xi)\big)^{\mathrm{T}},\end{aligned}$$

or

$$\begin{aligned}\lim_{t\to0+}\mathcal{E}(x,t)=(\rho h^2)^{-1}\big(r_1(x),r_2(x),h^2r_3(x),-\varkappa\eta\operatorname{div}r(x)\\ +\varkappa^2\rho h^2\Delta r_4(x)\big)^{\mathrm{T}},\end{aligned}$$

for almost all ξ or x, respectively. On the other hand, from (30) and the conditions on $R(x)$ it follows that

$$\int_{\mathbf{R}^2}(1+|\xi|)^6|\tilde e(\xi,t)-B_0^{-1}\tilde r(\xi)|^2\,d\xi\le c\int_{\mathbf{R}^2}(1+|\xi|)^6|\tilde R(\xi)|^2\,d\xi<\infty.$$

Therefore, by Lebesgue's theorem,

$$\lim_{t\to0+}e(x,t)=B_0^{-1}r(x)\quad\text{in}\quad H_3(\mathbf{R}^2).$$

In addition, it is easily verified that

$$\lim_{t\to0+}e_4(x,t)=-\frac{\varkappa\eta}{\rho h^2}\operatorname{div}r(x)+\varkappa^2\Delta r_4(x)\quad\text{in}\quad H_2(\mathbf{R}^2).$$

The above results are now combined in the following assertion.

Lemma 3. (i) *Let* $F = (f^{\rm T}, f_4)^{\rm T}$, *with* $f \in H_1(\mathbf{R}^2)$ *and* $f_4 \in H_2(\mathbf{R}^2)$. *Then* $\mathcal{J}(x,t) = (\mathcal{J}F)(x,t)$ *belongs to* $\mathbf{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$, $j(x,t) \to 0$, *as* $t \to 0$, *in* $H_2(\mathbf{R}^2)$, $j_4(x,t) \to \varkappa f_4(x)$, *as* $t \to 0$, *in* $H_1(\mathbf{R}^2)$, *and* $\mathcal{J}(x,t)$ *satisfies* (32).

(ii) *Let* $R = (r^{\rm T}, r_4)^{\rm T}$, *with* $r \in H_3(\mathbf{R}^2)$ *and* $r_4 \in H_4(\mathbf{R}^2)$. *Then* $\mathcal{E}(x,t) = (\mathcal{E}R)(x,t)$ *belongs to* $\mathbf{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$, $e(x,t) \to B_0^{-1}r(x)$, *as* $t \to 0$, *in* $H_3(\mathbf{R}^2)$, $e_4(x,t) \to -(\rho h^2)^{-1}\varkappa\eta \operatorname{div} r(x) + \varkappa^2 \Delta r_4(x)$, *as* $t \to 0$, *in* $H_2(\mathbf{R}^2)$, *and* $\mathcal{E}(x,t)$ *satisfies* (39).

The main results of this section are expressed in the next two assertions.

Theorem 2. *Let* $\varphi \in H_3(\mathbf{R}^2)$, $\theta \in H_2(\mathbf{R}^2)$, *and* $\psi \in H_1(\mathbf{R}^2)$, *and let*

$$f = B_0\psi, \quad f_4 = \varkappa^{-1}\theta + \eta \operatorname{div}\varphi, \quad r = B_0\psi, \quad r_4 = 0.$$

Then

$$\mathcal{L}(x,t) = (\mathcal{J}F)(x,t) + (\mathcal{E}R)(x,t) \tag{40}$$

is the solution of (9) *in* $\mathbf{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$ *and satisfies the initial condition*

$$\gamma_0\mathcal{L} = (\varphi^{\rm T}, \theta)^{\rm T}$$

and the estimate

$$\|\mathcal{L}\|_{1,\kappa;G} \le c\big\{\|\varphi\|_3 + \|\theta\|_2 + \|\psi\|_1\big\}. \tag{41}$$

Proof. By Lemma 3, $\mathcal{L} = (l^{\rm T}, l_4)^{\rm T} \in \mathbf{H}_{1,\kappa}(G)$ satisfies

$$\begin{aligned}\int_0^\infty \big[a(l,w) &- (B_0^{1/2}\partial_t l, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t l_4)_0 \\ &+ h^2\gamma\eta^{-1}(\nabla w_4, \nabla l_4)_0 - h^2\gamma(\nabla w_4, \partial_t l)_0 + h^2\gamma(\nabla l_4, w)_0\big]\, dt \\ &\qquad\qquad = (B_0\psi, \gamma_0 w)_0 \quad \forall\, W \in {\rm C}_0^\infty(\bar{G}).\end{aligned}$$

Again by Lemma 3,

$$\begin{aligned}
\gamma_0\mathcal{L} &= ((\gamma_0 l)^{\mathrm{T}}, \gamma_0 l_4)^{\mathrm{T}},\\
\gamma_0 l &= \gamma_0 j + \gamma_0 e = B_0^{-1} r = \varphi,\\
\gamma_0 l_4 &= \gamma_0 j_4 + \gamma_0 e_4 = \varkappa f_4 - \frac{\eta\varkappa}{\rho h^2}\operatorname{div} r + \varkappa^2 \Delta r_4 = \theta.
\end{aligned}$$

Finally, (41) follows from (31), and the lemma is proved. □

We conclude this section by looking at the relationship between the smoothness of the solution of the Cauchy problem and that of the initial data. This can be done by using various norms on the spaces of functions defined on G. Let us choose, for example, $\mathbf{H}_{m,\kappa}(G)$, consisting of elements $U = (u^{\mathrm{T}}, u_4)^{\mathrm{T}}$ with norm

$$\begin{aligned}
&\|U\|'_{m,\kappa;G}\\
&\quad = \biggl\{ \int_G e^{-2\kappa t}\biggl[(1+|\xi|)^{2m}|\tilde{U}(\xi,t)|^2 + \sum_{k=1}^{m}(1+|\xi|)^{2(m-k)}\\
&\qquad\qquad \times \bigl(|\partial_t^k \tilde{u}(\xi,t)|^2 + |\partial_t^{k-1}\tilde{u}_4(\xi,t)|^2\bigr)\biggr]\, d\xi\, dt\biggr\}^{1/2}, \quad m \in \mathbf{N}
\end{aligned}$$

(see the Remark).

Theorem 3. *Let*

$$\begin{aligned}
&\varphi \in H_{m+1}(\mathbf{R}^2), \quad \theta \in H_m(\mathbf{R}^2), \quad \psi \in H_m(\mathbf{R}^2), \quad m = 1, 2,\\
&\varphi \in H_{2m-1}(\mathbf{R}^2), \quad \theta \in H_{2m-2}(\mathbf{R}^2), \quad \psi \in H_{2m-3}(\mathbf{R}^2), \quad m \geq 3,
\end{aligned}$$

and let $f = B_0\psi$, $f_4 = \varkappa^{-1}\theta + \eta \operatorname{div}\varphi$, $r = B_0\varphi$, *and* $r_4 = 0$. *Then*

$$\mathcal{L}(x,t) = (\mathcal{J}F)(x,t) + (\mathcal{E}R)(x,t)$$

is the solution of (9) *in* $\mathbf{H}'_{m,\kappa}(G)$ *for any* $\varkappa > 0$ *and*

$$\begin{aligned}
&\|\mathcal{L}\|'_{m,\kappa;G} \leq c(\|\varphi\|_{m+1} + \|\theta\|_m + \|\psi\|_m), \quad m = 1, 2,\\
&\|\mathcal{L}\|'_{m,\kappa;G} \leq c(\|\varphi\|_{2m-1} + \|\theta\|_{2m-2} + \|\psi\|_{2m-3}), \quad m \geq 3.
\end{aligned}$$

Proof. First, from the conditions of the theorem it follows that

$$
\begin{aligned}
&f \in H_m(\mathbf{R}^2), &&f_4 \in H_m(\mathbf{R}^2), &&r \in H_{m+1}(\mathbf{R}^2), &&m = 1, 2,\\
&f \in H_{2m-3}(\mathbf{R}^2), &&f_4 \in H_{2m-2}(\mathbf{R}^2), &&r \in H_{2m-1}(\mathbf{R}^2), &&m \ge 3.
\end{aligned}
$$

Next, by Lemma 1 and (40),

$$
\begin{aligned}
|\tilde{l}(\xi,t)| &\le c\big[(1+t)(1+|\xi|)^{-1}|\tilde{\Phi}(\xi,t)| + |\tilde{r}(\xi,t)|\big],\\
|\tilde{l}_4(\xi,t)| &\le c\big[|\tilde{\Phi}(\xi,t)| + (1+t)(1+|\xi|)|\tilde{r}(\xi,t)|\big],\\
|\partial_t\tilde{l}(\xi,t)| &\le c\big[|\tilde{\Phi}(\xi,t)| + (1+|\xi|)|\tilde{r}(\xi,t)|\big],\\
|\partial_t^k\tilde{l}(\xi,t)| &\le c(1+|\xi|)^{2k-3}\big[|\tilde{\Phi}(\xi,t)| + (1+|\xi|)^2|\tilde{r}(\xi,t)|\big], \quad k \ge 2,\\
|\partial_t^{k-1}\tilde{l}_4(\xi,t)| &\le c(1+|\xi|)^{2k-3}\big[(1+t)|\tilde{\Phi}(\xi,t)| + (1+|\xi|)|\tilde{\Phi}_4(\xi,t)|\\
&\qquad + (1+t)(1+|\xi|)^2|\tilde{r}(\xi,t)|\big], \quad k \ge 2.
\end{aligned}
\tag{42}
$$

Using (42), we can easily verify the statements of the theorem. □

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