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ON THE TRACTION PROBLEM FOR THE LAMÉ SYSTEM ON CURVILINEAR POLYGONS

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ABSTRACT. We give a complete description of the spectra of certain elastostatic and hydrostatic boundary layer potentials in L^p, $1 < p < \infty$, on bounded curvilinear polygons. In particular, our analysis shows that the spectral radii of these operators on L^p are less than one if p is large enough. This holds for the case of the boundary layer potential operator associated to the *traction* conormal derivative. Such results are important when dealing with the issue of constructively solving boundary value problems for the Lamé system of elasticity and for the Stokes system of hydrostatic in domains with isolated singularities. Our approach is based on Mellin transform techniques and Calderón-Zygmund theory.

1. Introduction. Quite often, solving a boundary value problem such as

$$\text{(BVP)} \quad \begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ \mathcal{B}u = f & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for an elliptic operator $\mathcal{L}$ with the boundary condition $\mathcal{B}$, reduces to inverting an operator of the form "identity+K," where K is a singular integral operator, on some appropriate L^p boundary function spaces. In turn, the question of expanding the aforementioned inverse in a strongly convergent Neumann series comes down to checking whether $\rho(K; L^p)$, the spectral radius of the operator K on the L^p function space under discussion, satisfies

$$\rho(K; L^p) < 1. \tag{2}$$

We recall here that $\rho(K; L^p) := \sup\{|w|; w \in \mathbb{C} \text{ and } wI - K \text{ is not invertible on } L^p\}$, where I stands for the identity operator. This

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abstract scenario applies, for instance, to the case when $\mathcal{L} = \triangle$, the Laplacian, with Dirichlet or Newmann boundary conditions, as well as when $\mathcal{L} = \mu\triangle + (\lambda+\mu)\nabla\,\mathrm{div}$ is the elastostatic operator with Dirichlet or conormal type boundary conditions.

The smoothness of the underlying domain Ω plays a crucial role since, in many important instances, it affects the singularity of the kernel $k(X,Y)$ of K. This point is most transparent when

$$(3)\qquad k(X,Y) \text{ contains the factor } \frac{\langle X-Y, N(Y)\rangle}{|X-Y|^n}, \quad n = \dim\Omega,$$

as is the case with, e.g., the so-called double layer potential operator for the Laplacian. If Ω is a bounded domain with $\mathcal{C}^\infty$ boundary then K is a pseudodifferential operator of order -1, see, e.g., Proposition 11.2 in [**35**, Chapter 7]. Also, if $\partial\Omega$ is of class $\mathcal{C}^2$ then K becomes a compact operator on $L^p(\partial\Omega)$, $1 < p < \infty$, whenever (3) holds. This, in turn, greatly facilitates the analysis of (2).

The question (2) has a fundamentally different nature when $\partial\Omega$ is merely Lipschitz, a context in which (2) is sometimes referred to as the spectral radius conjecture, cf., [**4, 18**]. This general issue remains open at the time being, as it has been solved only in some special cases. Some of the authors that have dealt with this or some closely related problems are Fabes and collaborators in [**13, 16**], Shelepov in [**33**], Elschner in [**10, 12**], Rathsfeld in [**32**], Lewis in [**24**], Diomeda and Lisena in [**5, 6, 26, 27**], Maz'ya and collaborators in [**19, 20, 28**] and Duduchava in [**7**–**9**].

In the case of a system of PDEs, $\mathcal{L}$, the writing $(\mathcal{L}u)^i = a^{ij}_{kl}\partial_k\partial_l u^j$ is not unique and the algebraic structure of K depends on the specific choice of the tensor $(a^{ij}_{kl})_{i,j,k,l}$. A case in point is the Lamé system for which both, physically relevant, choices

$$(4)\qquad a^{ij}_{kl} := \mu\delta_{ij}\delta_{kl} + \left(\mu + \lambda - \frac{\mu(\mu+\lambda)}{3\mu+\lambda}\right)\delta_{ik}\delta_{jl} + \frac{\mu(\mu+\lambda)}{3\mu+\lambda}\delta_{il}\delta_{jk},$$

and

$$(5)\qquad a^{ij}_{kl} := \mu\delta_{ij}\delta_{kl} + \lambda\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk},$$

will do. The two choices in (4)–(5) lead to, respectively, the so-called pseudostress, $K_{\text{pseudostress}}$, and traction, K_{traction} (elastic) double layer potential operators. See Section 3 for a more detailed discussion on this. In connection with (2), quite recently, in [**30**] it has been shown that

$$\rho(K_{\text{pseudostress}} : (L^p(\partial\Omega)/\mathbb{R})^2) < 1, \quad \text{if } 2 \le p < \infty, \tag{6}$$

for any curvilinear polygon $\Omega \subseteq \mathbb{R}^2$. It is therefore natural to make the following

Conjecture. *If $\Omega \subseteq \mathbb{R}^2$ is a curvilinear polygon, then*

$$\rho(K_{\text{traction}} : (L^p(\partial\Omega))^2/\Psi) < 1, \quad \text{for each } 2 \le p < \infty, \tag{7}$$

where by Ψ we denote the space of vector valued functions ψ on $\mathbb{R}^2$ satisfying the three equations $\partial_i\psi^j + \partial_j\psi^i = 0$, $i, j = 1, 2$, restricted to $\partial\Omega$.

We point out here that the natural spaces to look for spectral radius estimates (2) are $(L^p(\partial\Omega)/\mathbb{R})^2$ in the case of $K_{\text{pseudostress}}$ and $(L^p(\partial\Omega))^2/\Psi$ for K_{traction}. We refer the reader to e.g., [**3**] and Section 6 for more details.

An important technical distinction between (6) and (7) is as follows. In the case of a polygonal domain $\Omega \subset \mathbb{R}^2$, the integral kernel of $K_{\text{pseudostress}}$ satisfies (3) and, thus, vanishes whenever X, Y belong to the same side of $\partial\Omega$. This property no longer holds in the case of K_{traction} which makes the analysis of this latter operator considerably more difficult and subtle.

One possible line of attack is as follows: (i) prove (7) when p is sufficiently large, (ii) prove (7) when $p = 2$; then one can invoke an interpolation type argument in order to conclude that (7) holds for $2 \le p < \infty$. In this paper we provide a partial solution to the above conjecture. Here we are able to prove that (i) is valid in general, i.e., there exists $p_0 > 1$ such that (7) holds for $p_0 < p < \infty$, whereas (ii) holds for a special subclass of curvilinear polygons and elastic media. More specifically, it is assumed that all angles θ of Ω satisfy

$$\theta \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \quad \text{and} \quad \frac{\mu}{\lambda + 2\mu} \in \left(0, \frac{31}{100}\right]. \tag{8}$$

See also Theorem 7.1 which is one of the main results of this paper.

The organization of the paper is as follows. In Section 2 we collect some useful, known results about the Mellin transform and Hardy kernels. Elastostatic layer potentials are introduced in Section 3. Section 4 contains the Mellin analysis of their kernels in a plane sector. In this setting, spectral radius estimates are derived in Section 5. We also include a more detailed analysis of the spectrum in Section 6. In Section 7 we consider the case $p=2$. Finally, in Section 8 we extend the scope of our analysis in order to include spectral results for a certain family of hydrostatic layer potential operators.

2. The Mellin transform and Hardy kernels on $(L^p(\mathbb{R}_+))^2$.

This section contains some notation and preliminaries on the Mellin transform together with the rudiments on the spectral analysis for the algebra of Mellin convolution operators generated by Hardy kernels and the Hilbert transform on $(L^p(\mathbb{R}_+))^2$.

Let $\mathcal{C}_0^\infty(\mathbb{R}_+)$ be the space of infinitely many times differentiable functions, compactly supported on $[0,\infty)$. The Mellin transform of a function $f\in\mathcal{C}_0^\infty(\mathbb{R}_+)$ is defined as

$$\mathcal{M}f(z):=\int_0^\infty x^{z-1}f(x)\,dx,\quad z\in\mathbb{C}. \tag{9}$$

For any $f\in\mathcal{C}_0^\infty(\mathbb{R}_+)$, $\mathcal{M}f(z)$ is an entire function and the following inversion formula holds

$$f(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty}x^{-z}\mathcal{M}f(z)\,dz, \tag{10}$$

where the above path integral is taken over the contour $z=1/p+i\xi$, $-\infty<\xi<\infty$.

For any $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta$, define the strip $\Gamma_{\alpha,\beta}:=\{z\in\mathbb{C}\,;\alpha<\operatorname{Re}z<\beta\}$, and let $\Gamma_\alpha:=\{z\in\mathbb{C}\,;\operatorname{Re}z=\alpha\}$. If f is measurable on $\mathbb{R}_+$ and the integral in (9) converges absolutely for all z in some strip $\Gamma_{\alpha,\beta}$ we shall call the integral $\mathcal{M}f(z)$ the *Mellin transform* of the function f. Note that $\mathcal{M}f$ is a holomorphic function in the strip $\Gamma_{\alpha,\beta}$. Next we make the following definition.

Definition 2.1. Let $k(x,y)$ be a measurable function defined on $\mathbb{R}_+\times\mathbb{R}_+$. Then k is a Hardy kernel for $L^p(\mathbb{R}_+)$, $1<p<\infty$, provided that $k(x,y)$ is homogeneous of degree -1, i.e., for any $\lambda > 0$ we have $k(\lambda x, \lambda y) = \lambda^{-1} k(x,y)$, and there holds

$$\int_0^\infty |k(1,y)|\, y^{-1/p}\, dy \left(= \int_0^\infty |k(x,1)|\, x^{1/p-1}\, dx \right) < \infty.$$

Also, a matrix $k = (k_{ij})_{i,j=1,2}$ of measurable functions on $\mathbb{R}_+ \times \mathbb{R}_+$ is called a Hardy kernel for $(L^p(\mathbb{R}_+))^2$, provided that each individual entry k_{ij} is a Hardy kernel for $L^p(\mathbb{R}_+)$.

Going further, let $k = (k_{ij})_{i,j=1,2}$ be a Hardy kernel for $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$, and for each $\vec{f} \in (L^p(\mathbb{R}_+))^2$ set

$$K\vec{f}(x) := \int_0^\infty k(x,y)\vec{f}(y)\, dy, \quad x \in \mathbb{R}_+. \tag{11}$$

To state the characterization of the spectrum of operators K as in (11) on $(L^p(\mathbb{R}_+))^2$ we need one more piece of notation. Let $\mathcal{X}$ be a Banach space and $T : \mathcal{X} \to \mathcal{X}$ be a linear and continuous operator. We denote by $\sigma(T;\mathcal{X})$ the spectrum of the operator, i.e.,

$$\sigma(T;\mathcal{X}) := \{w \in \mathbb{C}\,;\, wI - T \text{ is not invertible on } \mathcal{X}\}. \tag{12}$$

Also the spectral radius of T is

$$\rho(T;\mathcal{X}) := \sup\{|w|\,;\, w \in \sigma(T;\mathcal{X})\}, \tag{13}$$

i.e., the radius of the smallest closed circular disk centered at the origin which contains $\sigma(T,\mathcal{X})$. We have

Theorem 2.2 *Let K be an element in the algebra of Mellin convolution operators generated by Hardy kernels and the Hilbert transform for $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$. Then K is a bounded operator on $(L^p(\mathbb{R}_+))^2$ and its spectrum is the closure of the range of the Mellin transform $\mathcal{M}k(1/p + i\xi)$, i.e., it is the closure in the plane of the set of all points $w \in \mathbb{C}$ such that*

$$\det(wI - \mathcal{M}k)(1/p + i\xi) = 0 \quad \text{for some} \quad \xi \in \mathbb{R}. \tag{14}$$

Above, k is the kernel of the operator K, I is the identity matrix operator, and $\mathcal{M}k := (\mathcal{M}k_{ij})_{i,j=1,2}$.

When k is a Hardy kernel for $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$, and K is as in (11), then the conclusion of Theorem 2.2 follows in this special case by adapting the argument in [**14**] or [**1**] to the matrix setting described above. For the definition of the algebra of Mellin convolution operators generated by Hardy kernels and the Hilbert transform for $(L^p(\mathbb{R}_+))^2$ and the proof of Theorem 2.2 we refer the reader to the exposition in Section 3 of [**25**].

We shall call the matrix $\mathcal{M}k$ the *matrix of the Mellin symbols* of the operator K on $(L^p(\mathbb{R}_+))^2$, $1 < p < \infty$. In the notation of [**23, 25, 11**], of the algebra of pseudodifferential operators of Mellin type, the condition (14) reads $\det \operatorname{Smbl}^{1/p}(wI - K)(0, z) = 0$ for some $z = 1/p + i\xi$, $\xi \in \mathbb{R}$.

3. The layer potentials. Consider the system of elastostatic $\mathcal{L}\vec{u} = 0$ in an open subset of $\mathbb{R}^2$, given by

$$\mathcal{L} = \mu\triangle + (\lambda + \mu)\nabla \operatorname{div}, \tag{15}$$

where μ and λ are the Lamé moduli such that $\mu > 0$ and $\lambda + \mu \geq 0$. We represent the operator $\mathcal{L}$ in the notation

$$\mathcal{L} = A(D) = \left(a_{ij}^{kl}\partial_i\partial_j\right)_{k,l}, \tag{16}$$

with the coefficient matrix A given by

$$a_{ij}^{kl} = a_{ij}^{kl}(r) := \mu\,\delta_{ij}\delta_{kl} + (\mu + \lambda - r)\delta_{ik}\delta_{jl} + r\,\delta_{il}\delta_{jk}, \tag{17}$$

with $i, j, k, l = 1, 2$, and $r \in \mathbb{R}$. The conormal derivative for the operator $\mathcal{L}$ associated to $A(r) := (a_{ij}^{kl}(r))_{i,j,k,l}$ is given by

$$\left(\frac{\partial\vec{u}}{\partial N_{A(r)}}\right)^j := N_i a_{ik}^{jl}(r)\partial_k u^l = \mu\frac{\partial u^j}{\partial N} + (\mu + \lambda - r)N_j \operatorname{div}\vec{u} + rN_i\partial_j u^i,$$
$$j = 1, 2.$$

The choice $r := \mu$ in (17) gives the *traction* (*stress*) conormal derivative which has the form

$$\frac{\partial\vec{u}}{\partial N_{A(\mu)}} := \mu(\nabla\vec{u} + (\nabla\vec{u})^t)\cdot N + \lambda(\operatorname{div}\vec{u})N, \tag{18}$$

where the superscript t indicates transposition of matrices and $\nabla\vec{u}\colon = \left(\partial_j u^i\right)_{1\le i,j\le 2}$. For $r \in \mathbb{R}$ arbitrary, the conormal derivative associated to the matrix (17) is

$$\left(\frac{\partial \vec{u}}{\partial N_{A(r)}}\right)^j = \mu\frac{\partial u^j}{\partial N} + (\mu+\lambda-r)N_j\,\mathrm{div}\vec{u} + rN_i\partial_j u^i. \tag{19}$$

Recall the Kelvin matrix-valued fundamental solution for the system of elastostatics

$$G_{ij}(X) = \frac{1}{2\mu(2\mu+\lambda)\pi}\left[\frac{3\mu+\lambda}{2}\delta_{ij}\log|X|^2 - (\mu+\lambda)\frac{X_iX_j}{|X|^2}\right], \tag{20}$$

where $X \in \mathbb{R}^2 \setminus \{0\}$ and $i,j = 1,2$, see, e.g., (9.2) in [**21**, Chapter 9]. Denote by G^j the jth column of $G = (G_{kl})_{k,l=1,2}$, with $j = 1,2$. A straightforward computation gives that the ith component of $\partial G^j/\partial N_{A(r)}$ is

$$\begin{aligned}
&\left(\frac{\partial G^j}{\partial N_{A(r)}}\,(X-\cdot)\right)^i(Q)\\
&\qquad = -\frac{\gamma_1(r)\delta_{ij}}{\pi}\cdot\frac{\langle X-Q, N(Q)\rangle}{|X-Q|^2}\\
&\qquad\quad - \frac{\gamma_2(r)}{\pi}\cdot\frac{(X_i-Q_i)N_j(Q) - (X_j-Q_j)N_i(Q)}{|X-Q|^2}\\
&\qquad\quad - \frac{2(1-\gamma_1(r))}{\pi}\cdot\frac{\langle X-Q, N(Q)\rangle(X_i-Q_i)(X_j-Q_j)}{|X-Q|^4}
\end{aligned} \tag{21}$$

where
(22)

$$\gamma_1(r) = \frac{\mu(3\mu+\lambda) - r(\mu+\lambda)}{2\mu(2\mu+\lambda)} \quad\text{and}\quad \gamma_2(r) = \frac{\mu(\mu+\lambda) - r(3\mu+\lambda)}{2\mu(2\mu+\lambda)}.$$

In the case of the stress conormal derivative, $r = \mu$, we have

$$\gamma_1(\mu) = \frac{\mu}{\lambda+2\mu} \quad\text{and}\quad \gamma_2(\mu) = -\frac{\mu}{\lambda+2\mu}. \tag{23}$$

Denote by K_r the double layer potential operator associated to the choice of the coefficient matrix $A(r) = \left(a_{ij}^{kl}(r)\right)_{i,j,k,l}$ as in (17). We have

$$\left(K_r\vec{f}\right)^i(P) := p.v.\int_{\partial\Omega} k_r^{ij}(P,Q)f^j(Q)\,d\sigma(Q), \quad P \in \partial\Omega, \tag{24}$$

where the integral kernel $k_r(P,Q) := \big(k_r^{ij}(P,Q)\big)_{i,j}$ is given by $k_r(P,Q) := \big[(\partial G/\partial N_{A(r)})(P-\cdot)\big]^t\,(Q)$. Using (21) we obtain

(25)

$$k_r(P,Q) = \frac{\langle Q-P, N(Q)\rangle}{\pi|P-Q|^2} \cdot \begin{bmatrix} 1+\Lambda_1(r)\frac{(P_1-Q_1)^2-(P_2-Q_2)^2}{|P-Q|^2} & 2\Lambda_1(r)\frac{(P_1-Q_1)(P_2-Q_2)}{|P-Q|^2} \\ 2\Lambda_1(r)\frac{(P_1-Q_1)(P_2-Q_2)}{|P-Q|^2} & 1-\Lambda_1(r)\frac{(P_1-Q_1)^2-(P_2-Q_2)^2}{|P-Q|^2} \end{bmatrix} + \frac{\gamma_2(r)}{\pi}\cdot\frac{(P_1-Q_1)N_2(Q)-(P_2-Q_2)N_1(Q)}{|P-Q|^2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $P=(P_1,P_2)$ and $Q=(Q_1,Q_2)$ and

$$\Lambda_1(r) := 1-\gamma_1(r), \tag{26}$$

with $\gamma_1(r)$ as in (22).

4. The Mellin symbol in a plane sector. In this section we assume that Ω is the domain consisting of the interior of an angle of measure $\theta\in(0,2\pi)$. Our main goal is to find an explicit description of the spectrum of the operator K_r as in (24) on $(L^p(\partial\Omega))^2$, $1<p<\infty$. By rotation and translation invariance we assume without loss of generality that the domain Ω is the region above the graph of

$$h(x) := \cot(\theta/2)|x|, \quad x\in\mathbb{R}.$$

We set $s:=|P|$ and $t:=|Q|$. Denoting by $(\partial\Omega)_1$ the right ray of the angle and by $(\partial\Omega)_2$ the left ray, we distinguish four cases.

Case I. $P,Q\in(\partial\Omega)_1$. In this situation we have $\langle P-Q, N(Q)\rangle = 0$, $P=(s\sin(\theta/2), s\cos(\theta/2))$, and $Q=\Big(t\sin(\theta/2), t\cos(\theta/2)\Big)$, $N(Q) = (\cos(\theta/2), -\sin(\theta/2))$, $|P-Q|^2 = (s-t)^2$ and $(P_1-Q_1)N_2(Q)-(P_2-Q_2)N_1(Q) = -(s-t)$. Thus $k_r(P,Q)$ takes the form

$$k_r(s,t) = -\frac{\gamma_2(r)}{\pi}\cdot\frac{1}{s-t}\cdot\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{27}$$

and is a matrix multiple of the kernel of the Hilbert transform. The Mellin symbol of k_r is

$$\mathcal{M}k_r(z) := \mathcal{M}(k_r(\cdot,1))(z) = \gamma_2(r)\cdot\frac{\cos(\pi z)}{\sin(\pi z)}\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{28}$$

Introduce

$$\begin{gathered} A := (1-\gamma_1(r))z\sin\theta, \quad B := \sin((\pi-\theta)z),\\ C := \cos((\pi-\theta)z), \quad D := \sin(\pi z), \quad E := \cos(\pi z). \end{gathered} \tag{29}$$

In the new notation (28) gives

$$\mathcal{M}k_r(z) = \gamma_2(r)\cdot\frac{E}{D}\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{30}$$

Case II. $P, Q \in (\partial\Omega)_2$. Straightforward calculations similar to the ones in the previous case show that now $k_r(P,Q)$ takes the form

$$k_r(s,t) = \frac{\gamma_2(r)}{\pi}\cdot\frac{1}{s-t}\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{31}$$

and again is a matrix multiple of the kernel of the Hilbert transform. The Mellin symbol of k_r in the notation (29) is given by

$$\begin{aligned} \mathcal{M}k_r(z) &:= \mathcal{M}(k_r(\cdot,1))(z) = \gamma_2(r)\cdot\frac{\cos(\pi z)}{\sin(\pi z)}\cdot\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\ &= \gamma_2(r)\frac{E}{D}\cdot\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \end{aligned} \tag{32}$$

Case III. $P\in(\partial\Omega)_1$, $Q\in(\partial\Omega)_2$. We have $P=(s\sin(\theta/2), s\cos(\theta/2))$, $Q = (-t\sin(\theta/2), t\cos(\theta/2))$, $N(Q) = (-\cos(\theta/2), -\sin(\theta/2))$ and $\langle P-Q, N(Q)\rangle = -s\sin\theta$. Also $|P-Q|^2 = s^2 - 2st\cos\theta + t^2$, $(P_1-Q_1)N_2(Q)-(P_2-Q_2)N_1(Q) = s\cos\theta - t$, $(P_1-Q_1)^2-(P_2-Q_2)^2 = -s^2\cos\theta + 2st - t^2\cos\theta$ and $(P_1-Q_1)(P_2-Q_2) = ((s^2-t^2)/2)\sin\theta$.

Straightforward manipulations show that in this case (25) can be written in the form

$$k_r(s,t) = \begin{bmatrix} k^1(s,t)+k^2(s,t) & k^3(s,t) \\ k^3(s,t) & k^1(s,t)-k^2(s,t) \end{bmatrix} + k^4(s,t)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{33}$$

where

$$\begin{aligned} &k^1(s,t) := \frac{1}{\pi}\cdot\frac{s\sin\theta}{s^2-2st\cos\theta+t^2}, \\ &k^2(s,t) := \frac{\Lambda_1(r)}{\pi}\cdot\frac{s\sin\theta(-s^2\cos\theta+2st-t^2\cos\theta)}{(s^2-2st\cos\theta+t^2)^2}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &k^3(s,t) := \frac{\Lambda_1(r)}{\pi}\cdot\frac{s(s^2-t^2)\sin^2\theta}{(s^2-2st\cos\theta+t^2)^2}, \\ &k^4(s,t) := e\frac{\gamma_2(r)}{\pi}\cdot\frac{s\cos\theta-t}{s^2-2st\cos\theta+t^2}. \end{aligned} \tag{35}$$

In particular, k_r from (33) is a Hardy kernel. The Mellin transforms of the kernels introduced in (34)–(35) are computed below. We have $\mathcal{M}k^j(z) := \mathcal{M}(k^j(\cdot,1))(z)$ for $j=1,\dots,4$, and

$$\begin{aligned} \mathcal{M}k^1(z) &= \frac{\sin((\pi-\theta)z)}{\sin(\pi z)}, \\ \mathcal{M}k^2(z) &= \Lambda_1(r)\frac{z\sin\theta\cos((\pi-\theta)z)}{\sin(\pi z)}, \\ \mathcal{M}k^3(z) &= \Lambda_1(r)\frac{z\sin\theta\sin((\pi-\theta)z)}{\sin(\pi z)}, \\ \mathcal{M}k^4(z) &= -\gamma_2(r)\frac{\cos((\pi-\theta)z))}{\sin(\pi z)}. \end{aligned} \tag{36}$$

Now, using (29) and (33)–(36) we conclude

$$\mathcal{M}k_r(z) := \mathcal{M}(k_r(\cdot,1))(z) = \frac{1}{D}\begin{bmatrix} B+AC & AB+\gamma_2(r)C \\ AB-\gamma_2(r)C & B-AC \end{bmatrix}. \tag{37}$$

Case IV. $P \in (\partial\Omega)_2$, $Q \in (\partial\Omega)_1$. In this setting, via similar manipulations as the ones we used before, we have

(38)
$$k_r(s,t) = \begin{bmatrix} k^1(s,t)+k^2(s,t) & -k^3(s,t) \\ -k^3(s,t) & k^1(s,t)-k^2(s,t) \end{bmatrix} + k^4(s,t)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where k^j, $j = 1, \dots, 4$, are as in (34) and (35). Again, k_r in (38) is a Hardy kernel. Using (36), we obtain

(39)
$$\mathcal{M}k_r(z) := \mathcal{M}(k_r(\cdot,1))(z) = \frac{1}{D}\begin{bmatrix} B+AC & -AB+\gamma_2(r)C \\ -AB-\gamma_2(r)C & B-AC \end{bmatrix}.$$

Now, based on (30), (32), (37) and (39) we write the Mellin transform of the kernel of the operator $wI - K_r$, $w \in \mathbb{C}$, as the 4×4 matrix

$$(40)\quad \mathcal{M}(wI - k_r)(z) = \frac{1}{D}\begin{bmatrix} wD & \gamma_2(r)E & -B-AC & -AB-\gamma_2(r)C \\ -\gamma_2(r)E & wD & -AB+\gamma_2(r)C & -B+AC \\ -B-AC & AB-\gamma_2(r)C & wD & -\gamma_2(r)E \\ AB+\gamma_2(r)C & -B+AC & \gamma_2(r)E & wD \end{bmatrix}.$$

A straightforward calculation gives

$$\begin{aligned} &\det(\mathcal{M}(wI - k_r)(z)) \\ (41)\qquad &= \frac{1}{D^4}[(wD-AC)^2 - B^2 - \gamma_2(r)^2C^2 + (\gamma_2(r)E + AB)^2] \\ &\quad\cdot [(wD+AC)^2 - B^2 - \gamma_2(r)^2C^2 + (\gamma_2(r)E - AB)^2]. \end{aligned}$$

We point out that for $\operatorname{Re} z \neq 0$, $z = x + iy, x, y \in \mathbb{R}$, the denominator D does not vanish for any $y \in \mathbb{R}$.

Recall the definition of the spectrum from (12). Appealing to (41) and Theorem 2.2 and noticing that $\pm\gamma_2(r)$ are the roots of the righthand side of (41) in the limit case $z = 1/p \pm i\infty$ we conclude the following.

Theorem 4.1. *Let Ω be the domain consisting of the interior of an angle of measure $\theta \in (0, 2\pi)$ and recall A, B, C, D, E from (29). The* *spectrum of the operator* K_r *on* $(L^p(\partial\Omega))^2$, $1 < p < \infty$, *is given by*

$$\begin{aligned}(42)\quad & \sigma\left(K_r; (L^p(\partial\Omega))^2\right) \\ & \quad = \left\{ w \in \mathbb{C}; (wD \pm AC)^2 = Q_{\mp} \text{ for some } z \in \frac{1}{p} + i\mathbb{R} \right\} \\ & \quad \cup \{\gamma_2(r), -\gamma_2(r)\},\end{aligned}$$

where

$$Q_{\pm} = B^2 + \gamma_2(r)^2 C^2 - (\gamma_2(r)E \pm AB)^2. \tag{43}$$

An immediate consequence of Theorem 4.1 is the fact that

$$\sigma(K_r; (L^p(\partial\Omega))^2) = \Sigma_\theta^r(p) := \bigcup_{i=1}^{4} \Sigma_i(r,\theta,p), \tag{44}$$

with

(45)

$$\begin{aligned}\Sigma_1(r,\theta,p)(y) &:= \frac{\sqrt{Q_-} - AC}{D}, \quad \Sigma_2(r,\theta,p)(y) := \frac{-\sqrt{Q_-} - AC}{D}, \\ \Sigma_3(r,\theta,p)(y) &:= \frac{\sqrt{Q_+} + AC}{D}, \quad \Sigma_4(r,\theta,p)(y) := \frac{-\sqrt{Q_+} + AC}{D},\end{aligned}$$

where A, B, C, D, E are evaluated at $z = 1/p + iy$, $y \in \mathbb{R}$, see (29). A simple application of L'Hopital's rule gives

$$\lim_{y\to\pm\infty} \Sigma_j(r,\theta,p)(y) = \gamma_2(r), \quad j = 1, 3,$$

and

$$\lim_{y\to\pm\infty} \Sigma_j(r,\theta,p)(y) = -\gamma_2(r), \quad j = 2, 4. \tag{46}$$

Examples of the curves $\Sigma_i(r,\theta,p)$ for $\lambda = 0$, $\theta = \pi/2$, $p = 10$ and $r = \mu$ are included in Figures 1–4. In this case (corresponding to the stress conormal derivative) we have $\gamma_1(\mu) = -\gamma_2(\mu) = 0.5$. Consequently, the spectrum of the operator K_{traction} on $\left(L^{10}(\partial\Omega)\right)^2$ in the case Ω

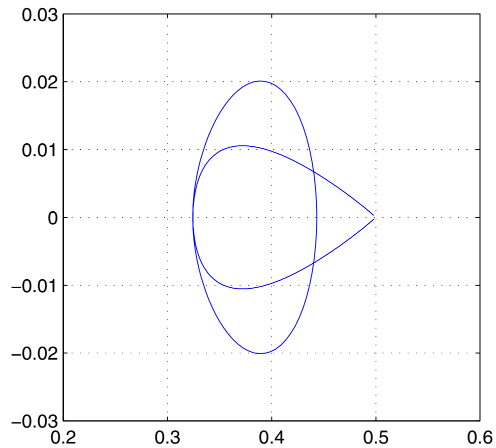

FIGURE 1. The curve $\Sigma_1(\mu, (\pi/2), 10)$.

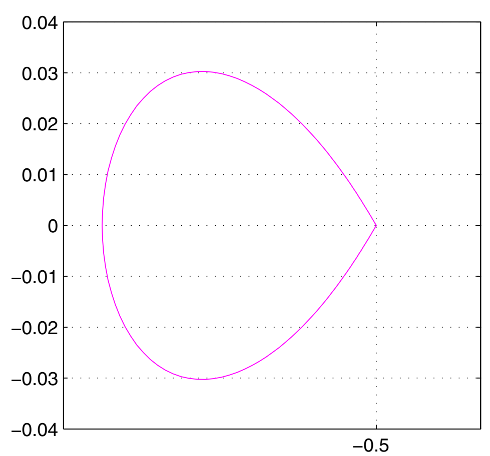

FIGURE 2. The curve $\Sigma_2(\mu, (\pi/2), 10)$.

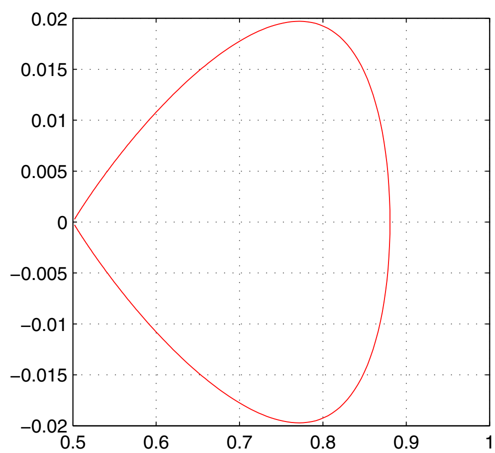

FIGURE 3. The curve $\Sigma_3(\mu, (\pi/2), 10)$.

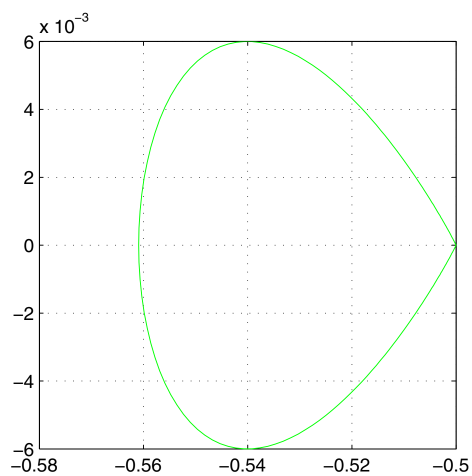

FIGURE 4. The curve $\Sigma_4(\mu, (\pi/2), 10)$.

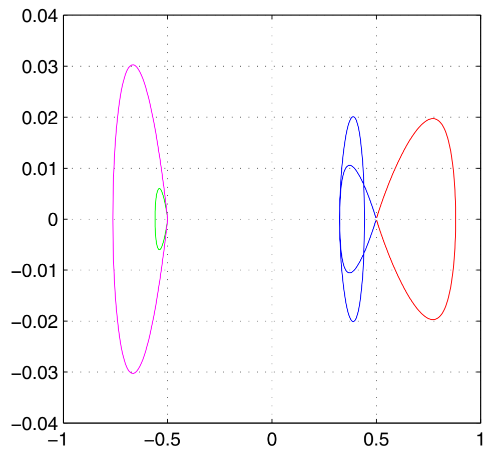

FIGURE 5. The L^{10} spectrum of the traction elastostatic double layer potential operator on angle of measure $\pi/2$ for $\lambda = 0$.

is the domain consisting of the interior of an angle of measure $\pi/2$ is presented in Figure 5.

5. Spectral results on L^p spaces; the sector case. The main result of this section is the following. To state it, recall the elastostatic double layer potential operator K_r from (24), associated to the choice of the coefficients matrix $A(r)$ given in (17).

Theorem 5.1. *Let Ω be the domain consisting of the interior of an angle of aperture $\theta \in (0, 2\pi)$. Then for any $r \in (-\mu, \mu]$ there exists $p(\theta, r)$, depending only on θ and r, such that*

$$\rho(K_r; (L^p(\partial\Omega))^2) < 1, \quad \forall\, p \geq p(\theta, r). \tag{47}$$

In particular

$$\rho\left(K_{\text{traction}}; (L^p(\partial\Omega))^2\right) < 1 \quad \text{for } p \quad \text{sufficiently large.} \tag{48}$$

Proof. For $i, j = 1, 2$ and $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ let

(49)
$$k^{ij}_{\gamma_1,\gamma_2}(P,Q) := \frac{\langle Q-P, N(Q)\rangle}{\pi|P-Q|^2}\left[\gamma_1\delta_{ij} + 2(1-\gamma_1)\frac{(P_i-Q_i)(P_j-Q_j)}{|P-Q|^2}\right] - \frac{\gamma_2}{\pi}\cdot\frac{(P_i-Q_i)N_j(Q)-(P_j-Q_j)N_i(Q)}{|P-Q|^2},$$

and

$$\left(K_{\gamma_1,\gamma_2}\vec{f}\right)^i(P) := p.v.\int_{\partial\Omega} k^{ij}_{\gamma_1,\gamma_2}(P,Q)f^j(Q)\,d\sigma(Q), \quad P \in \partial\Omega. \tag{50}$$

In (50) we have $\vec{f} := (f^1, f^2)$. We will show that

$$\sigma\left(K_{\gamma_1,\gamma_2}; (L^p(\partial\Omega))^2\right) < 1, \quad (\gamma_1,\gamma_2) \in \mathcal{T}\setminus\{(1,1),(1,-1)\} \tag{51}$$

for p large enough, where $\mathcal{T}$ is the triangle with vertices $(0,0), (1,1)$ and respectively $(1,-1)$. Since $r \in (-\mu, \mu]$ forces $(\gamma_1(r), \gamma_2(r)) \in \mathcal{T}\setminus\{(1,1),(1,-1)\}$ and $K_r = K_{\gamma_1(r),\gamma_2(r)}$ we have that (51) implies (47). The case of the *traction* (*stress*) conormal derivative corresponds to $(\gamma_1,\gamma_2) = (\mu/(2\mu+\lambda), -\mu/(2\mu+\lambda))$, that is, (γ_1,γ_2) belongs to the side of $\mathcal{T}$ joining $(0,0)$ with $(1,-1)$. Therefore (48) also follows from (51).

Fix now $(\gamma_1,\gamma_2) \in \mathcal{T}\setminus\{(1,1),(1,-1)\}$. In the light of Theorem 4.1 in order to obtain (51) it suffices to show that for p large enough

$$(wD \pm AC)^2 - B^2 - \gamma_2^2C^2 + (\gamma_2E \mp AB)^2 = 0 \Longrightarrow |w| < 1, \tag{52}$$

where A, B, C, D, E are as in (29) evaluated at $z = x + iy$ for some $y \in \mathbb{R}$ and $x = 1/p$, $p \in (1,\infty)$. Note that $x \in [0,1]$. To stress the dependence of w on x and y we will write $w = w(x,y)$. Consider the function

$$F(x,s) := |w(x,1/s)|. \tag{53}$$

In this new notation we have to show that there exists $\varepsilon > 0$ such that for any $x \in (0,\varepsilon)$ and $s \in \mathbb{R}$ we have

$$F(x,s) < 1. \tag{54}$$

We proceed in two steps. First we prove that there exist constants $\delta > 0$ and $M > 0$ such that

$$|(\partial_2 F)(x,s)| < M < \infty, \quad \forall\, x \in [0,1] \quad \text{and} \quad \forall\, s \in [-\delta, \delta]. \tag{55}$$

In the second step we show that

$$F(0,s) < 1, \quad \forall\, s \in \mathbb{R}. \tag{56}$$

This implies $|w(0,y)| < 1$ where w is a root of (52) for A, B, C, D, E evaluated at $z = iy$, $y \in \mathbb{R}$ which corresponds to $p = \infty$.

Let us indicate how (55) and (56) give (54) and, consequently, the conclusion of Theorem 5.1. We have $F(x,s) = F(x,0) + \int_0^1 (d/dt)[F(x,ts)]\,dt = F(x,0) + s\int_0^1 (\partial_2 F)(x,ts)\,dt$. This and (55) give $|F(x,s) - F(x,0)| \le |s|M$, for any $x \in [0,1]$ and $s \in [-\delta, \delta]$. Since $F(x,0) = |\gamma_2|$, a consequence of (46), and $|\gamma_2| < 1$, by choosing δ small enough we conclude that $F(x,s) < 1$ for any $x \in [0,1]$ and $s \in [-\delta, \delta]$. This means

$$|w(x,y)| < 1, \quad \forall\, x \in [0,1] \quad \text{and} \quad \forall\, y \in \left(-\infty, -\frac{1}{\delta}\right] \cup \left[\frac{1}{\delta}, \infty\right). \tag{57}$$

Finally, let us assume by contradiction that (54) is not satisfied. This implies that there exist two sequences $\{p_k\}_k, \{y_k\}_k \subset \mathbb{R}$ with $\lim_{k\to\infty} p_k = \infty$ such that $F(x_k, 1/y_k) = |w(x_k, y_k)| \ge 1$, where $x_k := 1/p_k \to 0$ as $k \to \infty$. Using (57) we can conclude that $y_k \in [-1/\delta, 1/\delta]$. By compactness, and eventually passing to a subsequence, we have that $y_k \to y_0 \in \mathbb{R}$ as $k \to \infty$. Taking the limit as $k \to \infty$ in (57) we obtain $F(0, 1/y_0) = |w(0, y_0)| \ge 1$ and this contradicts (56). Consequently (54) is satisfied and the conclusion of Theorem 5.1 holds. The proof of Theorem 5.1 is therefore completed provided we show (55) and (56). $\square$

Next we present the proofs of the two steps alluded to above in the proof of Theorem 5.1.

Proof of Step I. Recall that the main goal of this step is to show that (55) holds. This follows from a tedious but elementary calculation that we omit based on the fact that

$$F(x,s) = \left| \frac{\pm\left[B^2 + \gamma_2(r)^2 C^2 - (\gamma_2(r)E \pm AB)^2\right]^{1/2} \pm AC}{D} \right|, \tag{58}$$

where A, B, C, D, E are evaluated at $z = x + (i/s)$. The main idea is the fact that $(\partial_2 F)(x, s)$ can be written as a sum of factors that can be estimated by $s^{-m} \cdot e^{(-C\theta/s)}$ for some positive constants m and C independently of $x \in [0, 1]$. Then (55) follows. $\square$

We continue with

Proof of Step II. The main goal of this step is to show that (56) holds. Equivalently, this is (52) for A, B, C, D, E evaluated at $z = iy$, for any $y \in \mathbb{R}$. We proceed for now with the analysis under the assumption $y \in \mathbb{R} \setminus \{0\}$ and treat the case $y = 0$ at the end. In (52) make $z = iy$ with $y \in \mathbb{R} \setminus \{0\}$, and divide both sides by -1. We are left with proving that

$$\begin{aligned} \mathcal{W}(\theta) &:= \big\{ w \in \mathbb{C}; (wd \pm ac)^2 = b^2 - \gamma_2^2 c^2 + (\gamma_2 e \pm ab)^2 \big\} \\ &\subseteq D_1(0) := \{ w \in \mathbb{C}\,;\, |w| < 1 \}\,, \end{aligned} \tag{59}$$

where

$$\begin{gathered} a := (1 - \gamma_1) y \sin\theta, \quad b := \sinh((\pi - \theta) y), \\ c := \cosh((\pi - \theta) y), \quad d := \sinh(\pi y), \quad e := \cosh(\pi y). \end{gathered} \tag{60}$$

We point out that since $\mathcal{W}(\theta) = -\mathcal{W}(2\pi - \theta)$ we can assume without loss of generality for the remaining part of the section that $\theta \in (0, \pi]$. Finally (59) is a consequence of the sequence of lemmas that we present next. $\square$

Lemma 5.2. *Let $\theta \in (0, \pi]$ and a, c, e be as in* (60)*. Then for any $\gamma_1 \in [0, 1]$ and $y \in \mathbb{R} \setminus \{0\}$ we have*

$$e^2 \geq c^2 (a^2 + 1). \tag{61}$$

Proof. We write (61) as $\cosh^2(\pi y) \geq ((1 - \gamma_1)^2 y^2 \sin^2\theta + 1) \times \cosh^2((\pi - \theta) y)$. Since $\gamma_1 \in [0, 1]$, it suffices to show that the inequality holds for $\gamma_1 = 0$, i.e., $\cosh^2(\pi y) \geq (y^2 \sin^2\theta + 1) \cosh^2((\pi - \theta) y)$, or equivalently,

$$\cosh(2\pi y) \geq (y^2 \sin^2\theta + 1) \cosh(2(\pi - \theta) y) + 2y^2 \sin^2\theta. \tag{62}$$

If $\theta = \pi$ the inequality (62) becomes $\cosh(2\pi y) + 2 \geq 3$ and it is obviously satisfied as $\cosh(2\pi y) \geq 1$ for any $y \in \mathbb{R}$. We assume from now on that $\theta \in (0, \pi)$.

Using the Taylor series expansion of the function $\cosh x$ given by

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

we rewrite (62) as

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(2\pi)^k}{(2k)!} y^{2k} \geq 1 + &\left(\frac{[2(\pi-\theta)]^2}{2!} + 2\sin^2\theta \right) y^2 \\ &+ \sum_{k=2}^{\infty} \left(\frac{[2(\pi-\theta)]^{2k}}{(2k)!} + \sin^2\theta \frac{[2(\pi-\theta)]^{2k-2}}{(2k-2)!} \right) y^{2k}.\end{aligned} \tag{63}$$

Therefore, in order to prove (62) it suffices to show

$$\frac{(2\pi)^2}{2!} \geq \frac{[2(\pi-\theta)]^2}{2!} + 2\sin^2\theta$$

and

$$\frac{(2\pi)^{2k}}{(2k)!} \geq \frac{[2(\pi-\theta)]^{2k}}{(2k)!} + \sin^2\theta \frac{[2(\pi-\theta)]^{2k-2}}{(2k-2)!}, \tag{64}$$

for all integers $k \geq 2$. The first inequality in (64) follows easily from straightforward algebraic manipulations that we omit. As for the second inequality in (64) we rewrite it in the equivalent form

$$(\pi-\theta)^2 \left[\left(\frac{\pi}{\pi-\theta} \right)^{2k} - 1 \right] \geq k\left(k - \frac{1}{2}\right) \sin^2\theta. \tag{65}$$

We have

$$\left(\frac{\pi}{\pi-\theta} \right)^{2k} = \left[\left(1 + \frac{\theta}{\pi-\theta} \right)^k \right]^2 \geq \left(k\frac{\theta}{\pi-\theta} + 1 \right)^2 \geq k^2 \frac{\theta^2}{(\pi-\theta)^2} + 1, \tag{66}$$

where the first inequality above follows from $(1+x)^n > nx + 1$ for any $n \geq 0$ and $x \geq 0$. This finally gives (65) and finishes the proof of Lemma 5.2. $\square$

Notice that the expression $b^2-\gamma_2^2c^2+(\gamma_2 e\pm ab)^2=\gamma_2^2(e^2-c^2)\pm 2\gamma_2 abe+b^2+a^2b^2$, with a,b,c,d,e as in (60), has the discriminant $4b^2[e^2-c^2(a^2+1)]$ when regarded as quadratic in γ_2. Therefore an immediate corollary of Lemma 5.2 is the following.

Corollary 5.3. *Let $\theta\in(0,\pi]$, $\gamma_1\in[0,1]$, $\gamma_2\in\mathbb{R}$ and $y\in\mathbb{R}\setminus\{0\}$. Then*

$$b^2-\gamma_2^2c^2+(\gamma_2 e\pm ab)^2\geq 0. \tag{67}$$

In particular this implies that the roots w of the equation $(wd\pm ac)^2=b^2-\gamma_2^2c^2+(\gamma_2 e\pm ab)^2$ are real, i.e.,

$$\mathcal{W}(\theta)\subseteq\mathbb{R}. \tag{68}$$

We include next two technical results useful in the sequel.

Lemma 5.4. *For any $\theta\in(0,\pi]$ and $y\in\mathbb{R}\setminus\{0\}$ the following holds*:

(69)

$$\sinh^2(\pi y)-\sinh^2((\pi-\theta)y)-2y\sin\theta\sinh(\pi y)\cosh((\pi-\theta)y)+y^2\sin^2\theta\geq 0.$$

Proof. We have

$$\sinh x=\sum_{k=1}^{\infty}\frac{x^{2k-1}}{(2k-1)!},\quad \cosh x=\sum_{k=1}^{\infty}\frac{x^{2k-2}}{(2k-2)!},$$

and

$$\sinh^2 x=\frac{\cosh(2x)-1}{2}=\sum_{k=1}^{\infty}2^{2k-1}\frac{x^{2k}}{(2k)!}.$$

Therefore

$$\begin{aligned}\sinh^2(\pi y)&=\sum_{k=1}^{\infty}2^{2k-1}\frac{(\pi y)^{2k}}{(2k)!},\\ \sinh^2((\pi-\theta)y)&=\sum_{k=1}^{\infty}2^{2k-1}\frac{((\pi-\theta)y)^{2k}}{(2k)!},\\ \sinh(\pi y)\cosh((\pi-\theta)y)&=\sum_{k=0}^{\infty}c_{2k+1}y^{2k+1},\end{aligned} \tag{70}$$

where

$$c_1 = \pi, \quad c_3 = \frac{\pi^3}{3!} + \frac{\pi(\pi-\theta)^2}{2!}$$

and

$$\begin{aligned} (71) \qquad c_{2k+1} &= \frac{\pi^{2k+1}}{(2k+1)!} + \frac{\pi^{2k-1}(\pi-\theta)^2}{(2k-1)!2!} + \cdots + \frac{\pi(\pi-\theta)^{2k}}{(2k)!} \\ &= \frac{\left[(2\pi-\theta)^{2k+1} + \theta^{2k+1}\right]}{2(2k+1)!}, \end{aligned}$$

for $k \geq 2$. The last equality in (71) follows from the binomial formulas for $[\pi + (\pi-\theta)]^{2k+1}$ and $[\pi - (\pi-\theta)]^{2k+1}$. Consequently (69) becomes

$$\begin{aligned} (72) \qquad &\sum_{k=1}^{\infty} 2^{2k-1}\frac{(\pi y)^{2k}}{(2k)!} - \sum_{k=1}^{\infty} 2^{2k-1}\frac{((\pi-\theta)y)^{2k}}{(2k)!} \\ &\qquad - 2y\sin\theta\left(\sum_{k=0}^{\infty} c_{2k+1}y^{2k+1}\right) + y^2\sin^2\theta \geq 0, \end{aligned}$$

with c_{2k+1} as in (71). In (72) we have

$$\begin{aligned} (73) \qquad &\text{the coefficient of} \quad y^2 = \pi^2 - (\pi-\theta)^2 - 2\pi\sin\theta + \sin^2\theta, \\ &\text{the coefficient of} \quad y^{2k} = \frac{2^{2k-1}(\pi^{2k} - (\pi-\theta)^{2k})}{(2k)!} - 2c_{2k-1}\sin\theta, \end{aligned}$$

for $k \geq 2$. It is easy to see that the coefficient of y^2 can be written as $(\theta - \sin\theta)(2\pi - \theta + \sin\theta)$ which is strictly positive for $\theta \in (0, \pi]$. Next, we prove that for any $k \geq 2$ the coefficient of y^{2k} is positive. A moment's reflection shows that this comes down to showing

$$(74) \qquad \frac{2^{2k-1}(\pi^{2k} - (\pi-\theta)^{2k})}{2k} \geq (\sin\theta)\left[(2\pi-\theta)^{2k-1} + \theta^{2k-1}\right].$$

Rewriting the lefthand side as

$$(75) \quad \frac{2^{2k-1}(\pi^{2k} - (\pi-\theta)^{2k})}{2k} = \frac{1}{4k}\left[((2\pi-\theta)+\theta)^{2k} - ((2\pi-\theta)-\theta)^{2k}\right],$$

and using the binomial formula for the expansion of $((2\pi-\theta)+\theta)^{2k}$ and respectively $((2\pi-\theta)-\theta)^{2k}$ in terms of $2\pi-\theta$ and θ we obtain

$$\begin{aligned}((2\pi-\theta)+\theta)^{2k} &- ((2\pi-\theta)-\theta)^{2k}\\ &\geq 2(2k)(2\pi-\theta)^{2k-1}\theta + 2(2k)(2\pi-\theta)\theta^{2k-1}\\ &\geq 4k\theta\left[(2\pi-\theta)^{2k-1}+\theta^{2k-1}\right],\end{aligned} \tag{76}$$

where the last inequality in (76) follows from the fact that $2\pi-\theta\geq\theta$. Since $\theta>\sin\theta$ for $\theta\in(0,\pi]$ and appealing to (75)–(76) we conclude that (74) holds. This finishes the proof of Lemma 5.4. □

Lemma 5.5. *For any $\theta\in(0,\pi]$ and $y\in\mathbb{R}\setminus\{0\}$ the following inequality holds*

$$\begin{aligned}\sinh^2(\pi y)-\sinh^2((\pi-\theta)y)\pm 2y\sin\theta\cosh(\pi y)\sinh((\pi-\theta)y)&\\ -y^2\sin^2\theta>0.&\end{aligned} \tag{77}$$

Proof. First note that the lefthand side of (77) is an even function in the variable y. Therefore, we can assume without loss of generality that $y>0$. Next, since $2y\sin\theta\cosh(\pi y)\sinh((\pi-\theta)y)\geq 0$ it is enough to prove (77) with the choice minus in the lefthand side. That is,

$$\begin{aligned}\sinh^2(\pi y)-\sinh^2((\pi-\theta)y)-2y\sin\theta\cosh(\pi y)\sinh((\pi-\theta)y)&\\ -y^2\sin^2\theta>0,&\end{aligned} \tag{78}$$

for $y>0$ and $\theta\in(0,\pi]$. We have

$$\sinh((\pi-\theta)y)\cosh(\pi y)=\sum_{k=0}^{\infty}d_{2k+1}y^{2k+1},$$

where

$$d_1=\pi-\theta,\quad d_3=\frac{(\pi-\theta)^3}{3!}+\frac{(\pi-\theta)\pi^2}{2!}$$

and

$$d_{2k+1}=\frac{(\pi-\theta)^{2k+1}}{(2k+1)!}+\frac{(\pi-\theta)^{2k-1}\pi^2}{(2k-1)!2!}+\cdots+\frac{(\pi-\theta)\pi^{2k}}{(2k)!},\quad k\geq 2. \tag{79}$$

Therefore, based on (70) and (79), an equivalent form of (78) is

$$\sum_{k=1}^{\infty} 2^{2k-1} \frac{(\pi y)^{2k} - ((\pi-\theta)y)^{2k}}{(2k)!} - 2y\sin\theta\Big(\sum_{k=0}^{\infty} d_{2k+1} y^{2k+1}\Big) - y^2 \sin^2\theta \geq 0, \tag{80}$$

where d_{2k+1} are defined above. The coefficients of the powers of y in the lefthand side of (80) are

$$\begin{aligned} &\text{the coefficient of} \quad y^2 = \pi^2 - (\pi-\theta)^2 - 2\sin\theta(\pi-\theta) - \sin^2\theta, \\ &\text{the coefficient of} \quad y^{2k} = \frac{2^{2k-1}\left[\pi^{2k} - (\pi-\theta)^{2k}\right]}{(2k)!} - 2d_{2k-1}\sin\theta, \end{aligned} \tag{81}$$

for $k \geq 2$. We rewrite the coefficient of y^2 in the form $\pi^2 - (\pi-\theta+\sin\theta)^2$. Since $\theta > \sin\theta$ for $\theta \in (0,\pi]$ it follows that $\pi^2 - (\pi-\theta+\sin\theta)^2 > 0$. Next, since $\theta \in (0,\pi]$ we have $d_{2k+1} \leq c_{2k+1}$ for any natural number k. Here c_{2k+1} are as in (71) and d_{2k+1} are as in (79). Therefore

$$\begin{aligned} &\frac{2^{2k-1}\left[\pi^{2k} - (\pi-\theta)^{2k}\right]}{(2k)!} - 2d_{2k-1}\sin\theta \\ &\qquad \geq \frac{2^{2k-1}\left[\pi^{2k} - (\pi-\theta)^{2k}\right]}{(2k)!} - 2c_{2k-1}\sin\theta \\ &\qquad \geq 0, \end{aligned} \tag{82}$$

where the last inequality is an equivalent form of (74) and it has been shown in the proof of Lemma 5.4. This finishes the proof of Lemma 5.5. □

We return now to the proof of Step II.

End of proof of Step II. Recall that we are left with showing (59). That is, for any $(\gamma_1,\gamma_2) \in \mathcal{T} \setminus \{(1,1),(1,-1)\}$ and $\theta \in (0,\pi]$ the roots w of the equation $(wd \pm ac)^2 = b^2 - \gamma_2^2 c^2 + (\gamma_2 e \pm ab)^2$ satisfy $|w| < 1$.

Fix $y \in \mathbb{R} \setminus \{0\}$ for the moment (note that in this case we have $d \neq 0$) and consider the functions

$$F_\pm(w) := (wd \pm ac)^2 - b^2 + \gamma_2^2 c^2 - (\gamma_2 e \pm ab)^2, \quad w \in \mathbb{R} \setminus \{0\}. \tag{83}$$

We have $F'_\pm(w) = 2(wd \pm ac)$ and, therefore, the minimum of $F_\pm(w)$ is attained when $w = \mp ac/d$. Next, recall a, c, d from (60) and make the observation that

$$|ac| < |d|, \quad \text{for any } y \in \mathbb{R}. \tag{84}$$

Using the fact that a and d are odd in the variable y and c is even, in order to prove (84) we can assume without loss of generality that $y > 0$. By the mean value theorem there exists $\xi \in [\pi - \theta, \pi]$ such that $\sinh(\pi y) - \sinh(\pi - \theta)y = \theta y \cosh(\xi y)$. Since cosh is an increasing function on $(0, \infty)$ we have $\cosh(\xi y) \geq \cosh((\pi - \theta)y)$. This implies

$$\begin{aligned} \sinh(\pi y) \geq \sinh(\pi y) - \sinh(\pi - \theta)y &\geq \theta y \cosh((\pi - \theta)y) \\ &> y \sin\theta \cos((\pi - \theta)y), \end{aligned} \tag{85}$$

as $\theta > \sin\theta$ for $\theta \in (0, \pi]$ and $y > 0$. This gives (84). Going further, as consequence of Corollary 5.3 we have

$$F_\pm\left(\mp \frac{ac}{d}\right) = -\left[b^2 - \gamma_2^2 c^2 + (\gamma_2 e \pm ab)^2\right] \leq 0, \tag{86}$$

and the roots w of $F_\pm(w) = 0$ are real. Recall $\mathcal{T}$ is the triangle with vertices $(0,0)$, $(1,1)$ and $(1,-1)$. In the new notation our goal is to show that for $(\gamma_1, \gamma_2) \in \mathcal{T} \setminus \{(1,1), (1,-1)\}$ all the roots of $F_\pm(w) = 0$ belong to the interval $(-1, 1)$. Using (84) and (86) it suffices to show

$$F_\pm(1), F_\pm(-1) > 0. \tag{87}$$

This is $(d \pm ac)^2 - \left[b^2 - \gamma_2^2 c^2 + (\gamma_2 e \pm ab)^2\right] > 0$, and $(-d \pm ac)^2 - \left[b^2 - \gamma_2^2 c^2 + (\gamma_2 e \pm ab)^2\right] > 0$, for a, b, c, d, e as in (60), $y \in \mathbb{R} \setminus \{0\}$, $\theta \in (0, \pi]$ and $(\gamma_1, \gamma_2) \in \mathcal{T} \setminus \{(1,1), (1,-1)\}$. Note that the lefthand sides of the above inequalities are even functions in y and therefore we can assume without loss of generality that $y > 0$. Since when $y > 0$ we have that $(|\gamma_2|\, e + ab)^2 \geq (\gamma_2 e - ab)^2$, it suffices to prove

$$(d \pm ac)^2 - \left[b^2 - \gamma_2^2 c^2 + (\gamma_2 e + ab)^2\right] > 0, \tag{88}$$

for $(\gamma_1, \gamma_2) \in \mathcal{T} \setminus \{(1,1), (1,-1)\}$, $\gamma_1 \geq \gamma_2 \geq 0$ and $y > 0$. We will divide our analysis into two cases.

Case I. We first treat the situation $(\gamma_1, \gamma_2) \in \mathcal{T} \setminus \{\gamma_1 = 1\}$. To this end, fix $\theta \in (0, \pi]$, $y > 0$ and $\gamma_1 \in [0, 1)$ and consider

$$G_\pm(\gamma_2) := (d \pm ac)^2 - \left[b^2 - \gamma_2^2 c^2 + (\gamma_2 e + ab)^2\right], \quad \text{for} \quad \gamma_2 \in [0, \gamma_1] \quad \text{and} \quad y \in (0, \infty). \tag{89}$$

We have $G'_\pm(\gamma_2) = 2\gamma_2(c^2 - e^2) - 2abe$ and therefore $G'_\pm(\gamma_2) \leq 0$ for $\gamma_2 \in \big(abe/(c^2 - e^2), \infty\big)$, as $c^2 < e^2$ for $y \in \mathbb{R}$ and $\theta \in (0, \pi]$. Also, $[0, \gamma_1] \subset \big(abe/(c^2 - e^2), \infty\big)$ which, in turn, implies $G_\pm(\gamma_2) \geq G_\pm(\gamma_1)$ for any $y > 0$, $\theta \in (0, \pi]$, $\gamma_1 \in [0, 1)$ and $\gamma_2 \in [0, \gamma_1]$. Hence, to have (88) it suffices

$$G_\pm(\gamma_1) \geq 0, \quad \theta \in (0, \pi], \quad \gamma_1 \in [0, 1), \tag{90}$$

where $G_\pm$ is as in (89).

It is straightforward to see that $e^2 - d^2 = 1$ and $c^2 - b^2 = 1$. This allows us to rewrite $G_\pm(\gamma_1)$ as $G_\pm(\gamma_1) = (1 - \gamma_1^2)(d^2 - b^2) \pm 2dac + a^2 - 2\gamma_1 eab$. Recall that $a := (1 - \gamma_1) y \sin\theta = (1 - \gamma_1)\tilde{a}$, where $\tilde{a} := y \sin\theta$. In this notation $G_\pm(\gamma_1) = (1 - \gamma_1)\left[(1 + \gamma_1)(d^2 - b^2) \pm 2d\tilde{a}c + (1 - \gamma_1)\tilde{a}^2 \mp 2\gamma_1 \tilde{a}b\right]$. Since $\gamma_1 \in [0, 1)$, the inequality (90) is therefore equivalent to

$$H_\pm(\gamma_1) := (1 + \gamma_1)(d^2 - b^2) \pm 2d\tilde{a}c + (1 - \gamma_1)\tilde{a}^2 - 2\gamma_1 \tilde{a}b \geq 0, \tag{91}$$

for $y \in \mathbb{R} \setminus \{0\}$, $\theta \in (0, \pi]$ and $\gamma_1 \in [0, 1)$. Differentiating with respect to γ_1 gives $H'_\pm(\gamma_1) = d^2 - b^2 - 2e\tilde{a}b - \tilde{a}^2$. Using (60) and $\tilde{a} := y \sin\theta$ we get

$$\begin{aligned} H'_\pm(\gamma_1) &= \sinh^2(\pi y) - \sinh^2((\pi - \theta) y) \\ &\quad \mp 2y \sin\theta \sinh((\pi - \theta) y) \cosh(\pi y) - y^2 \sin^2\theta \geq 0, \end{aligned} \tag{92}$$

where the inequality in (92) is a consequence of Lemma 5.5. Next, $H_\pm(0) = d^2 - b^2 \pm 2d\tilde{a}c + \tilde{a}^2$, and using (60) and $\tilde{a} := y \sin\theta$, that is,

$$\begin{aligned} H_\pm(0) &= \sinh^2(\pi y) - \sinh^2((\pi - \theta) y) \\ &\quad \pm 2y \sin\theta \sinh(\pi y) \cosh((\pi - \theta) y) + y^2 \sin^2\theta. \end{aligned} \tag{93}$$

Appealing to Lemma 5.4 we obtain $H_\pm(0) \geq 0$ for any $y \in \mathbb{R} \setminus \{0\}$, $\theta \in (0, \pi]$. Finally, since $H_\pm$ is increasing in γ_1, this implies $H_\pm(\gamma_1) \geq 0$

for $\gamma_1 \in [0,1)$. This gives (91) and, in turn, (90). The analysis of Case I is now complete and the conclusion of Theorem 5.1 holds for $\theta \in (0,\pi]$, $y \in \mathbb{R} \setminus \{0\}$ and $(\gamma_1,\gamma_2) \in \mathcal{T} \setminus \{\gamma_1 = 1\}$.

Case II. Consider now $\theta \in (0,\pi]$, $y \in \mathbb{R} \setminus \{0\}$, $\gamma_1 = 1$ and $\gamma_2 \in (-1,1)$. In this setting we have that $a = 0$ and (86) becomes $F_\pm(\pm 1) = d^2 - b^2 - \gamma_2^2 c^2 + \gamma_2^2 e^2 > 0$. This is clearly true as $e^2 > c^2$ and $d^2 > b^2$. The analysis of this situation is therefore finished.

To complete the proof of Theorem 5.1 we consider now $y = 0$, $\theta \in (0,\pi]$ and $(\gamma_1,\gamma_2) \in \mathcal{T} \setminus \{(1,1),(1,-1)\}$. In this case we write (52) as

$$\left(w \pm \frac{AC}{D}\right)^2 - \left(\frac{B}{D}\right)^2 + \gamma_2^2 \frac{E^2 - C^2}{D^2} \mp 2\gamma_2 \frac{EAB}{D^2} + \left(\frac{AB}{D}\right)^2. \tag{94}$$

Taking into account that $\lim_{y\to 0} AC/D = ((1-\gamma_1)/\pi)\sin\theta$,$\lim_{y\to 0} AB/D = 0$, $\lim_{y\to 0} B/D = (\pi-\theta)/\pi$, $\lim_{y\to 0}(E^2 - C^2)/(D^2) = ((\pi-\theta)^2 - \pi^2)/(\pi^2)$, and $\lim_{y\to 0} EAB/(D^2) = ((1-\gamma_1)/(\pi^2))(\pi-\theta)\sin\theta$, in this case matters reduce to show that the roots w of the equations

$$F_\pm(w) = 0, \tag{95}$$

satisfy $|w| < 1$, where

$$\begin{aligned} F_\pm(w) := \left(w \pm (1-\gamma_1)\frac{\sin\theta}{\pi}\right)^2 - \frac{(\pi-\theta)^2}{\pi^2} - \gamma_2^2\,\frac{\pi^2 - (\pi-\theta)^2}{\pi^2} \\ \pm 2\gamma_2(1-\gamma_1)\frac{(\pi-\theta)\sin\theta}{\pi^2}. \end{aligned} \tag{96}$$

We first claim that the roots w of the equations $F_\pm(w) = 0$ are real. To see this it is enough to show that $(\pi-\theta)^2/(\pi^2) + \gamma_2^2(\pi^2 - (\pi-\theta)^2)/(\pi^2) \mp 2\gamma_2((1-\gamma_1)/(\pi^2))(\pi-\theta)\sin\theta \geq 0$. Since $\gamma_1 \in [0,1]$, $\pi^2 - (\pi-\theta)^2 > \theta^2$, and $\theta \geq \sin\theta$, for $\theta \in (0,\pi]$, this is a consequence of the fact that

$$\frac{(\pi-\theta)^2}{\pi^2} + \gamma_2^2 \frac{\pi^2 - (\pi-\theta)^2}{\pi^2} - 2|\gamma_2|\frac{(\pi-\theta)\sin\theta}{\pi^2} > \left(\frac{\pi-\theta}{\pi} - \frac{\gamma_2\theta}{\pi}\right)^2 \geq 0. \tag{97}$$

The vertices of the parabolas $F_\pm(w)$ are at $w^0_\pm = \mp((1-\gamma_1)/\pi)\sin\theta \in (-1,1)$ and based on (97) we have $F_\pm(w^0_\pm) < 0$. To conclude that the roots w of $F_\pm(w) = 0$ belong to the interval $(-1,1)$, it suffices to show

$$F_\pm(1) > 0 \quad \text{and} \quad F_\pm(-1) > 0. \tag{98}$$

When $\gamma_1 = 1$ and $\gamma_2 \in (-1,1)$ we have that $F_\pm(1) = ((\pi^2 - (\pi-\theta)^2)/(\pi^2))(1-\gamma_2^2) > 0$, which gives (98). Next assume $(\gamma_1,\gamma_2) \in \mathcal{T}\backslash\{\gamma_1 = 1\}$. Then (98) is $1 \pm 2((1-\gamma_1)/\pi)\ \sin\ \theta + ((1-\gamma_1)^2/(\pi^2))\ \sin^2\ \theta - (\pi-\theta)^2/(\pi^2) - \gamma_2^2(\pi^2 - (\pi-\theta)^2)/(\pi^2) \mp 2\gamma_2((1-\gamma_1)/(\pi^2))(\pi-\theta)\sin\theta > 0$. Clearly it suffices to assume $\gamma_2 \ge 0$ and prove the above inequality for the choice of minus signs in its lefthand side. That is,

(99)

$$1 - 2(1-\gamma_1)\frac{\sin\theta}{\pi} + (1-\gamma_1)^2\frac{\sin^2\theta}{\pi^2} - \frac{(\pi-\theta)^2}{\pi^2} - \gamma_2^2\frac{\pi^2 - (\pi-\theta)^2}{\pi^2} - 2\gamma_2(1-\gamma_1)\frac{(\pi-\theta)\sin\theta}{\pi^2} > 0.$$

Differentiating the lefthand side of (99) with respect to γ_2 we obtain $-2\gamma_2(\pi^2 - (\pi-\theta)^2)/(\pi^2) - 2((1-\gamma_1)/(\pi^2))(\pi-\theta)\sin\theta$ which is ≤ 0, for $\gamma_2 \in [0,\infty)$, $\gamma_1 \in [0,1]$, $\theta \in (0,\pi]$. Since we are interested to prove (99) for pairs $(\gamma_1,\gamma_2) \in \mathcal{T}\setminus\{\gamma_1 = 1\}$ it therefore suffices to assume $\gamma_2 = \gamma_1$. Making $\gamma_2 = \gamma_1$ in (99) and dividing both sides by $1-\gamma_1 > 0$ we are left with proving

(100)

$$T(\gamma_1) := 1 + \gamma_1 + \frac{(1-\gamma_1)\sin^2\theta}{\pi^2} - 2\frac{\sin\theta}{\pi} - (1+\gamma_1)\frac{(\pi-\theta)^2}{\pi^2} - 2\gamma_1\frac{(\pi-\theta)\sin\theta}{\pi^2} > 0.$$

We have $T(0) = ((\theta - \sin\theta)/\pi)\cdot((2\pi - \theta - \sin\theta)/\pi) > 0$, for $\theta \in (0,\pi]$ as $2\pi - \theta \ge \theta > \sin\theta$ in this case. Also a direct calculation gives $T'(\gamma_1) = ((\theta - \sin\theta)/\pi)\cdot((2\pi - \theta + \sin\theta)/\pi) > 0$, for any $\theta \in (0,\pi]$ as before. Finally this shows that $T(\gamma_1) > 0$ for any $\gamma_1 \in [0,1)$ which is (100). As pointed out before this implies (98) and the analysis in the case $y = 0$ is therefore complete. This finishes the proof of Theorem 5.1. □

We conclude this section with the following.

Theorem 5.6. *Let Ω be the domain consisting of the interior of an angle of measure $\theta \in (0, 2\pi)$ and let $r \in (-\mu, \mu]$. Then there exists $p(\theta, r) > 1$, depending only on θ and r, such that*

$$\left\{p \in (1, \infty);\ \rho\left(K_r; (L^p(\partial\Omega))^2\right) < 1\right\} = [p(\theta, r), \infty). \tag{101}$$

Before proceeding with the proof of Theorem 5.6, let us record a useful result in the sequel.

Proposition 5.7. *Let $\mathcal{X}_0, \mathcal{X}_1$ be a compatible couple of Banach spaces and set $\mathcal{X}_s := [\mathcal{X}_0, \mathcal{X}_1]_s$, $s \in (0, 1)$, via complex interpolation. Consider T a linear and continuous operator on $\mathcal{X}_0$ and $\mathcal{X}_1$. Then, for any $s \in (0, 1)$ we have*

$$\rho\left(T; \mathcal{X}_s\right) < \rho\left(T; \mathcal{X}_0\right)^{1-s} \rho\left(T; \mathcal{X}_1\right)^s. \tag{102}$$

In particular, if $\rho\left(T; \mathcal{X}_0\right) < M$ and $\rho\left(T; \mathcal{X}_1\right) < M$, then $\rho\left(T; \mathcal{X}_s\right) < M$ for all $s \in [0, 1]$.

Proof. As a consequence of the Riesz-Thorin interpolation theorem we have that $T^n : \mathcal{X}_s \to \mathcal{X}_s$ is a well-defined linear and continuous operator, for any $s \in [0, 1]$ and any natural number n. Moreover, denoting by $\|T^n\|_{\mathcal{X}_s}$ the norm of the operator T^n on the space $\mathcal{X}_s$, the following holds

$$\|T^n\|_{\mathcal{X}_s} \le \|T^n\|_{\mathcal{X}_0}^{1-s} \cdot \|T^n\|_{\mathcal{X}_1}^{s}, \tag{103}$$

for any $n \in \mathbb{N}$. Taking nth roots in both sides of (103) we obtain $\|T^n\|_{\mathcal{X}_s}^{1/n} \le \|T^n\|_{\mathcal{X}_0}^{(1-t)/n} \cdot \|T^n\|_{\mathcal{X}_1}^{t/n}$. Now, a standard functional analysis gives $\rho(T, \mathcal{X}_s) = \lim_{n\to\infty} \|T^n\|_{\mathcal{X}_s}^{1/n}$, for any $s \in [0, 1]$. Passing to the limit as $n \to \infty$ in the previous inequality we therefore get (102) and the proof of Proposition 5.7 is completed. $\square$

We return now to the proof of Theorem 5.6.

Proof of Theorem 5.6. Fix $\theta \in (0, 2\pi)$ and $r \in (-\mu, \mu]$. As a consequence of Theorem 5.1 and Proposition 5.7 applied for the operator K_r, the $L^p(\partial\Omega)$ interpolation scale, and $M = 1$, it follows that for any $p_0 \in (1, \infty)$ such that $\rho(K_r; (L^{p_0}(\partial\Omega))^2) < 1$, we have

$$[p_0, \infty) \subseteq \Big\{ p \in (1, \infty);\ \ \rho(K_r; (L^p(\partial\Omega))^2) < 1 \Big\}. \tag{104}$$

This gives (101) and finishes the proof of Theorem 5.6. □

6. The spectrum of K_r on curvilinear polygons. In this section we provide an explicit description for the spectrum of the elastostatic layer potential operator K_r, $r \in \mathbb{R}$ on curvilinear polygons. Recall that K_r defined in (24) is the elastostatic double layer potential operator associated to choice of coefficient tensor $A(r) = \big(a_{ij}^{kl}(r)\big)_{i,j,k,l}$ as in (17). To state our main results in this section let us introduce the following notation. For $1 \le p < \infty$ consider

$$L_0^p(\partial\Omega) := \Big\{ f \in L^p(\partial\Omega);\ \int_{\partial\Omega} f \, d\sigma = 0 \Big\}. \tag{105}$$

Also, let Ψ denote the space of vector valued functions ψ on $\mathbb{R}^2$ satisfying the three equations $\partial_i \psi^j + \partial_j \psi^i = 0$, $i, j = 1, 2$, restricted to $\partial\Omega$. Define

$$L_\Psi^p(\partial\Omega) := \Big\{ f \in (L^p(\partial\Omega))^2;\ \int_{\partial\Omega} f \cdot \psi \, d\sigma = 0, \text{ for all } \psi \in \Psi \Big\}. \tag{106}$$

Note that $L_\Psi^p(\partial\Omega)$ is a subspace of $(L^p(\partial\Omega))^2$ of codimension three. Using the Hahn-Banach and Riesz representation theorems we obtain that for any $1 < p < \infty$ the dual space of $L_\Psi^p(\partial\Omega)$ is $(L^q(\partial\Omega))^2 / \Psi$, where $1/p + 1/q = 1$.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, simply connected curvilinear polygon with angles θ_i, $i = 1, \ldots, n$, and let $p \in (1, \infty)$. For each $1 \le i \le n$ consider the curve $\Sigma_{\theta_i}^r(p)$ as in (44) associated to r, the angle θ_i and the integrability exponent p. Set $\widehat{\Sigma_{\theta_i}^r(p)}$ for the closure of its interior. Then*

$$\sigma\Big(K_r; (L^p(\partial\Omega))^2\Big) = \Big(\bigcup_{1 \le i \le n} \widehat{\Sigma_{\theta_i}^r(p)} \Big) \cup \{\lambda_j\}_j, \tag{107}$$

where $\{\lambda_j\}_j$ *consists of eigenvalues of the operator* K_r *on* $(L^p(\partial\Omega))^2$. *Whenever* $r \in (-\mu, \mu]$ *the set* $\{\lambda_j\}_j \subset (-1, 1]$ *has finitely many points.*

For $w \in \mathbb{C}$, $w \in \cup_{1\le i\le n}\Sigma^r_{\theta_i}(p)$ *the operator* $wI - K_r$ *is not Fredholm on* $(L^p(\partial\Omega))^2$.

For $w \in \mathbb{C}$, $w \notin \cup_{1\le i\le n}\Sigma^r_{\theta_i}(p)$ *the operator* $wI - K_r$ *is Fredholm on* $(L^p(\partial\Omega))^2$ *and its index is given by*

$$\text{index}\left(wI - K_r; (L^p(\partial\Omega))^2\right) = \sum_{i=1}^{n} W\left(w, \Sigma^r_{\theta_i}(p)\right), \tag{108}$$

where $W\left(w, \Sigma^r_{\theta_i}(p)\right)$ *stands for the sum of the winding numbers of the point* $w \notin \Sigma^r_{\theta_i}(p)$ *with respect to each one of the four closed curves* $\Sigma_j(r, \theta_i, p)$, $j = 1, \ldots, 4$, *constituting* $\Sigma^r_{\theta_i}(p)$.

Proof. The proof of Theorem 6.1 follows in a similar manner as the proof of the spectral structure Theorem 7.1 from [**30**]. Here we just sketch the main steps. Using the symbolic calculus of pseudodifferential operators of Mellin type developed by [**25, 23, 11**] one gets

$$\sigma_e\left(K_r; (L^p(\partial\Omega))^2\right) = \bigcup_{1\le i\le n} \Sigma^r_{\theta_i}(p), \tag{109}$$

where $\Sigma^r_{\theta_i}(p)$ are as in (44) and

$$\begin{aligned} \sigma_e(K_r; (L^p(\partial\Omega))^2) \\ := \{w \in \mathbb{C}\,;\, wI - K_r \text{ is not Fredholm on } (L^p(\partial\Omega))^2\} \end{aligned} \tag{110}$$

is the essential spectrum of K_r on $(L^p(\partial\Omega))^2$. Also, as a consequence of Theorem 1 in [**23**] we have

$$\bigcup_{1\le i\le n} \widehat{\Sigma^r_{\theta_i}(p)} \subseteq \sigma\left(K_r; (L^p(\partial\Omega))^2\right), \tag{111}$$

and the index formula (108) holds (see the proof of Theorem 7.1 in [**30**] for more details). Moreover $\sigma\left(K_r; (L^p(\partial\Omega))^2\right) \setminus \left(\bigcup_{1\le i\le n} \widehat{\Sigma^r_{\theta_i}(p)}\right)$ consists of $w \in \mathbb{C}$ such that $wI - K_r$ is Fredholm with index zero on

$(L^p(\partial\Omega))^2$. In particular, any such w has to be an L^p eigenvalue of K_r. This gives (107).

Recall now the coefficient matrix $A(r) := \left(a_{ij}^{kl}(r)\right)_{i,j,k,l}$, $i,j,k,l = 1,2$, from (17). When $r \in (-\mu,\mu)$ the matrix $A(r)$ is *strictly positive definite*, i.e., there exists $c > 0$ such that

$$a_{ij}^{kl}(r)\xi_i^k\xi_j^l \geq c|\xi|^2, \tag{112}$$

for any $\xi \in \mathbb{R}^4$. In this case, the $(L^p(\partial\Omega))^2$ eigenvalues of the operator K_r belong to $(-1,1]$. See, e.g., Theorem 3 in [**29**] for a proof in the case $p = 2$. Since $L^p(\partial\Omega) \subset L^2(\partial\Omega)$ for $p \geq 2$ the case $p \in [2,\infty)$ is completed. We refer the reader to [**31**] for a proof in the case $1 < p < 2$. In the traction case, when $r = \mu$, the matrix $A(\mu)$ is only *semi-positive definite*, i.e.,

$$a_{ij}^{kl}(\mu)\xi_i^k\xi_j^l \geq 0, \tag{113}$$

for any $\xi \in \mathbb{R}^4$. Nonetheless, Theorem 3 in [**29**] applies and gives that the $(L^p(\partial\Omega))^2$ eigenvalues of the operator K_{traction} lie in $(-1,1]$ for $p \geq 2$. For a proof in the case $1 < p < 2$ see e.g., [**31**]. This implies $\{\lambda_j\}_j \subset (-1,1]$ when $r \in (-\mu,\mu]$. Also

$$\{\lambda_j\}_j \subseteq \partial\sigma\left(K_r; (L^p(\partial\Omega))^2\right) \setminus \sigma_e\left(K_r; (L^p(\partial\Omega))^2\right). \tag{114}$$

Generally, for a linear and continuous operator $T : \mathcal{X} \longrightarrow \mathcal{X}$, where $\mathcal{X}$ is a Banach space, one has that $\partial\sigma(T;\mathcal{X}) \setminus \sigma_e(T;\mathcal{X})$ contains only isolated points, see, e.g., [**17**, p. 102]. In the case $r \in (-\mu,\mu]$ this gives that $\{\lambda_j\}_j \subset (-1,1]$ consists of isolated points and therefore has finite cardinality. This completes the proof of Theorem 6.1. $\square$

Next, based on the Mellin analysis of the operator K_r for $r \in (-\mu,\mu]$ in the sector case, we present the corresponding result for Theorem 5.6 when Ω is a curvilinear polygon. Recall the space Ψ introduced at the beginning of this section.

Theorem 6.2. *Let Ω be a curvilinear polygon with angles $\theta_1,\ldots,\theta_n$. Then, for any $r \in (-\mu,\mu)$ there exists $p(\theta_1,\ldots,\theta_n,r) > 1$ depending only on the angles of Ω and r, such that*

$$\left\{p \in (1,\infty)\,;\, \rho(K_r;(L^p(\partial\Omega)/\mathbb{R})^2) < 1\right\} = [p(\theta_1,\ldots,\theta_n,r),\infty). \tag{115}$$

Also, in the traction case, i.e., $r = \mu$, there exists $p(\theta_1, \dots, \theta_n, \mu) > 1$ depending only on the angles of Ω and μ, such that

$$\left\{p \in (1, \infty)\,;\, \rho\left(K_{\text{traction}}; (L^p(\partial\Omega))^2 / \Psi\right) < 1\right\} = [p(\theta_1, \dots, \theta_n, \mu), \infty). \tag{116}$$

Proof. By applying Proposition 5.7 to the operator K_r and the interpolation scale $(L^p(\partial\Omega)/\mathbb{R})^2$, and also to K_{traction} and $(L^p(\partial\Omega))^2/\Psi$, it suffices to show that for any $r \in (-\mu, \mu)$ we have

$$\rho(K_r; (L^p(\partial\Omega)/\mathbb{R})^2\, t) < 1 \quad \text{and} \quad \rho(K_{\text{traction}}; (L^p(\partial\Omega))^2 / \Psi) < 1, \tag{117}$$

for p large enough.

Fix $r \in (-\mu, \mu]$. Based on (107) and the fact that $\{\lambda_j\}_j \subset (-1, 1]$ we conclude that for p large enough we have

$$\sigma\left(K_r; (L^p(\partial\Omega))^2\right) \subset D_R(0) \cup \{1\}, \quad \text{for some} \quad 0 < R < 1, \tag{118}$$

where $D_R(0)$ is the disk of radius R centered at the origin. This is because, using Theorem 5.6, for p large enough we have $\cup_{1 \le i \le n} \widehat{\Sigma^r_{\theta_i}(p)} \subseteq D_R(0)$ for some $R < 1$ and $i = 1, \dots, n$. For the moment fix $p \in (1, \infty)$ such that (118) holds. Then, for any $w \in \mathbb{C}$, $|w| \ge 1$ and $w \neq 1$ we have $wI - K_r : (L^p(\partial\Omega))^2 \to (L^p(\partial\Omega))^2$ is an invertible operator. Also $I - K_r : (L^p(\partial\Omega))^2 \to (L^p(\partial\Omega))^2$ is a Fredholm operator with index zero. Passing to the dual we obtain that $wI - K_r^* : (L^q(\partial\Omega))^2 \to (L^q(\partial\Omega))^2$ is invertible, where $1/p + 1/q = 1$ and $|w| \ge 1$ with $w \neq 1$. Also $I - K_r^* : (L^q(\partial\Omega))^2 \to (L^q(\partial\Omega))^2$ is a Fredholm operator with index zero. Reasoning as in the proof of Theorem 1 in [**29**] where the case $p = q = 2$ has been considered, for any $r \in (-\mu, \mu)$ we have that $I - K_r^* : (L^q(\partial\Omega))^2 \to (L^q_0(\partial\Omega))^2$ is well defined. Also, in the traction case, i.e., $r = \mu$, the operator $I - K^*_{\text{traction}} : (L^q(\partial\Omega))^2 \to L^q_\Psi(\partial\Omega)$ is well defined, where $L^2_\Psi(\partial\Omega)$ is as in (106). Based on this, as in the proof of Theorem 1 in [**29**], we conclude

$$\begin{aligned} wI - K_r^* &: (L^q_0(\partial\Omega))^2 \longrightarrow (L^q_0(\partial\Omega))^2, \\ wI - K^*_{\text{traction}} &: L^q_\Psi(\partial\Omega) \longrightarrow L^q_\Psi(\partial\Omega), \end{aligned} \tag{119}$$

are isomorphisms for all $w \in \mathbb{C}$, $|w| \geq 1$. In particular, by passing to the dual, this shows that (117) is satisfied. This completes the proof of Theorem 6.2. $\square$

We conclude this section by presenting some examples for the L^{10} spectrum of the operator K_{stress} on a curvilinear polygon of angles $\pi/5$ and $9\pi/10$. First we consider a glass elastic medium, i.e., $\lambda = \mu = 2.2 \times 10^5 \text{kg/cm}^2$. In this case $\gamma_1(\mu) = 1/3$ and $\gamma_2(\mu) = -1/3$. We have

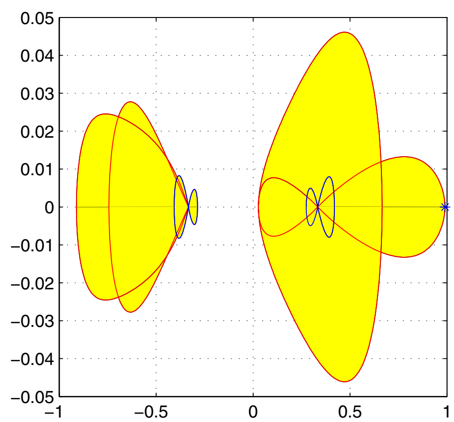

FIGURE 6. The L^{10} spectrum of the traction elastostatic double layer potential operator on a curvilinear polygon with angles $\pi/5$ and $9\pi/10$ on glass.

where the $*$ stands for the generic location of the set of eigenvalues $\{\lambda_j\}_j$ as in the statement of Theorem 6.1. Next we consider the case of a lead material when $\mu/(2\mu + \lambda) = 0.1075$, which in turn implies $\gamma_1(\mu) = -\gamma_2(\mu) = 0.1075$. We present now the L^{10} spectrum of the operator K_{traction} on a curvilinear polygon of angles $\pi/5$ and $9\pi/10$ in this case.

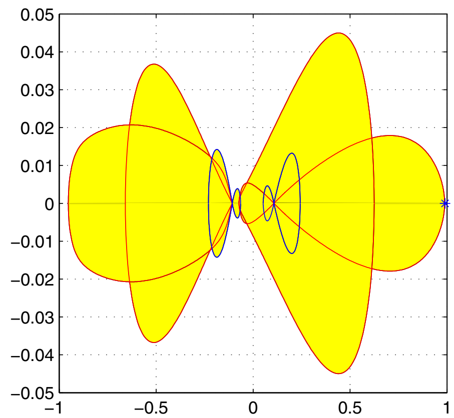

FIGURE 7. The L^{10} spectrum of the traction elastostatic double layer potential operator on a curvilinear polygon with angles $\pi/5$ and $9\pi/10$ on lead.

7. The L^2 case for the traction layer potential. The goal of this section is to provide some partial results for the Conjecture (7). Recall the operator $K_{\text{traction}} = K_{\gamma_1,-\gamma_1}$ with $\gamma_1 := \mu/(\lambda+2\mu)$ from (50) and the space Ψ introduced in Section 6. Our main result in this section is the following.

Theorem 7.1. *Let Ω be a curvilinear polygon in $\mathbb{R}^2$ with angles in the interval $[3\pi/4, 5\pi/4]$ and assume that the Lamé moduli μ, λ satisfy $\mu/(\lambda+2\mu) \in (0, 31/100]$. Then*

$$\rho\left(K_{\text{traction}}; \left(L^2(\partial\Omega)\right)^2/\Psi\right) < 1. \tag{120}$$

Moreover,

$$\rho\left(K_{\text{traction}}; \left(L^p(\partial\Omega)\right)^2/\Psi\right) < 1 \quad \text{for all} \quad 2 \leq p < \infty. \tag{121}$$

Before proceeding with the proof of Theorem 7.1 let us point out that the condition

$$\gamma_1 := \frac{\mu}{\lambda+2\mu} \in \left(0, \frac{31}{100}\right]$$

is satisfied for the following elastic materials, see [**2**, p. 129]; here the Lamé moduli λ and μ are given in $10^5\,\mathrm{kg/cm}^2$.

Elastic material	λ	μ	γ_1	Elastic material	λ	μ	γ_1
Iron	9.9	7.8	0.3059	Copper	8.7	4.1	0.2426
Bronze	6.2	3.8	0.2754	Aluminum	5.6	2.6	0.2407
Nickel	1.3	0.85	0.2833	Rubber	0.40	0.012	0.0283

We now present the

Proof of Theorem 7.1. First, let us point out that (120) implies (121). Assume (120) holds. Then

$$2 \in \left\{p \in (1,\infty);\, \rho\left(K_{\text{traction}}; \left(L^p(\partial\Omega)\right)^2/\Psi\right) < 1\right\}, \tag{122}$$

and (121) follows from Theorem 6.2. Matters can be reduced to the case Ω is the domain consisting of the interior of an angle of aperture $\theta \in [3\pi/4, \pi]$. This is because, once the sector case is settled, we have

$$\Sigma^\mu_\theta(2) = \Sigma^\mu_{2\pi-\theta}(2) \subseteq D_1(0) := \{w \in \mathbb{C};\, |w| < 1\}, \tag{123}$$

and then (122) follows from Theorem 6.1 and the proof of Theorem 6.2. In (123), the curves $\Sigma^\mu_\theta(2)$ and $\Sigma^\mu_{2\pi-\theta}(2)$ are as in (44) associated to $r := \mu$, the angles $\theta, 2\pi - \theta \in [3\pi/4, 5\pi/4]$, and the integrability exponent 2.

In the light of (44) and (45) it suffices to show that for $r = \mu$, i.e. the traction case, and $\mu/(\lambda + 2\mu) \in (0, 31/100]$ we have

$$\left|\frac{\pm\sqrt{Q_\pm} \pm AC}{D}\right| < 1, \tag{124}$$

where $Q_\pm$ are as in (43) and A, B, C, D, E are as in (29) evaluated at $z := (1/2) + iy$, for any $y \in \mathbb{R}$, $\theta \in [3\pi/4, \pi]$. In this case, to prove (124) it suffices to show that the following inequalities hold

$$|Q_\pm| < \frac{|D|^2}{4} \quad \text{and} \quad |AC|^2 < \frac{|D|^2}{4}. \tag{125}$$

We prove (125) in the next two lemmas, and this finishes the proof of Theorem 7.1. □

Before going further let us notice that in the case under discussion (traction, $p = 2$) the correspondent of (29) is

$$A = (1-\gamma_1)\Big(\frac{1}{2}+iy\Big)\sin\theta, \qquad B = \sin\Big((\pi-\theta)\Big(\frac{1}{2}+iy\Big)\Big),$$
$$C = \cos\Big((\pi-\theta)\Big(\frac{1}{2}+iy\Big)\Big), \quad D = \sin\Big(\pi\Big(\frac{1}{2}+iy\Big)\Big),$$
$$E = \cos\Big(\pi\Big(\frac{1}{2}+iy\Big)\Big), \qquad \text{with } \gamma_1 := \frac{\mu}{\lambda+2\mu} \in \Big(0, \frac{31}{100}\Big]. \tag{126}$$

Straightforward calculations give

$$\begin{gathered} |A|^2 = (1-\gamma_1)^2(\sin^2\theta)\Big(\frac{1}{4}+y^2\Big), \quad |D|^2 = \frac{1}{2}\cosh(2\pi y) + \frac{1}{2}, \\ |E|^2 = \frac{1}{2}\cosh(2\pi y) - \frac{1}{2}, \quad |B|^2 = \frac{1}{2}\cosh(2(\pi-\theta)y) + \frac{1}{2}\cos\theta, \\ |C|^2 = \frac{1}{2}\cosh(2(\pi-\theta)y) - \frac{1}{2}\cos\theta. \end{gathered} \tag{127}$$

Also, based on (43), we have

$$\begin{aligned} |Q_\pm| &= \big|B^2 + \gamma_1^2 C^2 - (-\gamma_1 E \pm AB)^2\big| \\ &= \big|\gamma_1^2 + (1-\gamma_1^2)B^2 - (\gamma_1 E \mp AB)^2\big| \\ &\leq \gamma_1^2 + (1-\gamma_1^2)|B|^2 + 2\gamma_1^2|E|^2 + 2|A|^2|B|^2, \end{aligned} \tag{128}$$

where γ_1 is as in (126). This is because in the traction case $r = \mu$ and $\gamma_1(\mu) = -\gamma_2(\mu) = \gamma_1$. In (128) we used $B^2 + C^2 = 1$ and $|\gamma_1 E \pm AB|^2 \leq 2\gamma_1^2|E|^2 + 2|A|^2|B|^2$. Now we are ready to present

Lemma 7.2. *Let A, C, D be as in* (126) *with $\theta \in [3\pi/4, \pi]$ and $\gamma_1 := \mu/(\lambda+2\mu) \in (0,1]$. Then*

$$|A|^2|C|^2 < \frac{|D|^2}{4}. \tag{129}$$

Proof. Based on (127) and straightforward algebraic manipulations, we rewrite (129) in the sufficient form $\cosh(2\pi y) + 1 > 4 \times (1-\gamma_1)^2 \left((1/4) + y^2\right) (\cosh(2(\pi-\theta)y) + 1) \sin^2\theta$. Using

$$\cosh(my) = \sum_{k=0}^{\infty} \frac{m^{2k}}{(2k)!} y^{2k} \quad \text{for} \quad m = 2\pi \quad \text{and} \quad m = 2(\pi - \theta)$$

in the above inequality and rearranging terms in increasing powers of y we have to show

$$\begin{aligned} &2 + \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}}{(2k)!} y^{2k} \\ &\quad > 2(1-\gamma_1)^2(\sin^2\theta) + (1-\gamma_1)^2 \cdot (\sin^2\theta) \left(8 + \frac{(2(\pi-\theta))^2}{2!}\right) y^2 \\ &\quad\quad + \sum_{k=2}^{\infty} (1-\gamma_1)^2 \cdot (\sin^2\theta) \left(\frac{(2(\pi-\theta))^{2k}}{(2k)!} + 4 \frac{(2(\pi-\theta))^{2k-2}}{(2k-2)!}\right) y^{2k}. \end{aligned} \tag{130}$$

In (130) the constant terms satisfy $2 > 2(\sin^2\theta)(1-\gamma_1)^2$ as $\gamma_1 \in (0,1]$. For the terms in y^2 we need

$$\pi^2 > (1-\gamma_1)^2 \left(\sin^2\theta\right) \left((\pi-\theta)^2 + 4\right). \tag{131}$$

It is easy to see that actually $\pi^2 > (\sin^2\theta)\left((\pi-\theta)^2 + 4\right)$ for $\theta \in [(3\pi/4), \pi]$ and this gives (131). Finally, for the terms in y^{2k}, $k \geq 2$, we require

$$\frac{(2\pi)^{2k}}{(2k)!} > (1-\gamma_1)^2 (\sin^2\theta) \left(\frac{(2(\pi-\theta))^{2k}}{(2k)!} + 4\, \frac{(2(\pi-\theta))^{2k-2}}{(2k-2)!}\right), \tag{132}$$

for θ and γ_1 as in the hypothesis. Note that for (132) it suffices to show

$$\frac{(2\pi)^{2k}}{(2k)!} > \frac{(2(\pi-\theta))^{2k}}{(2k)!} + 4(1-\gamma_1)^2 (\sin^2\theta) \frac{(2(\pi-\theta))^{2k-2}}{(2k-2)!}. \tag{133}$$

Finally this is true following from (66) in the proof of Lemma 5.2 as $\theta^2 > 5 > 4(1-\gamma_1)^2 \sin^2\theta$ for $\theta \geq 3\pi/4$. This finishes the proof of Lemma 7.2. $\square$

Lemma 7.3. *Let A, B, C, D, E be as in* (126) *with $\theta \in [3\pi/4, \pi]$ and $\gamma_1 \in (0, 31/100]$. Then*

$$\frac{|D|^2}{4} > \gamma_1^2 + (1-\gamma_1^2)|B|^2 + 2\gamma_1^2|E|^2 + 2|A|^2|B|^2. \tag{134}$$

In particular, for $r := \mu$ (the traction case) and $Q_\pm$ as in (43) *we have*

$$\frac{|D|^2}{4} > |Q_\pm|. \tag{135}$$

Proof. First, let us note that (135) follows from (134) and (128). Next, multiplying both sides by 4 and using that $|E|^2 = |D|^2 - 1$, the inequality (134) becomes

$$(1-8\gamma_1^2)|D|^2 + 8\gamma_1^2 > 4\gamma_1^2 + 4(1-\gamma_1^2)|B|^2 + 8|A|^2|B|^2. \tag{136}$$

Using (127) and multiplying both sides by 2 this reads

$$\begin{aligned}(1-8\gamma_1^2)\cosh(2\pi y) + 1 > &\left[4(1-\gamma_1^2) + 8(1-\gamma_1)^2\left(\frac{1}{4}+y^2\right)\sin^2\theta\right] \\ &\cdot \left(\cosh(2(\pi-\theta)y) + \cos\theta\right).\end{aligned} \tag{137}$$

We proceed as before by writing the Taylor expansions of $\cosh(2\pi y)$ and $\cosh(2(\pi-\theta)y)$ about $y=0$ and compare the coefficients of y^{2k} for $k \geq 0$ of the right- and lefthand side of (137). For the constant terms we need

$$2 - 8\gamma_1^2 > \left[4(1-\gamma_1^2) + 2(1-\gamma_1)^2\sin^2\theta\right](1+\cos\theta). \tag{138}$$

For the terms in y^2 we have to show

$$\begin{aligned}(1-8\gamma_1^2)\frac{(2\pi)^2}{2!} > &\left[4(1-\gamma_1^2) + 2(1-\gamma_1)^2\sin^2\theta\right]\frac{(2(\pi-\theta))^2}{2!} \\ &+ 8(1-\gamma_1)^2(\sin^2\theta)(1+\cos\theta).\end{aligned} \tag{139}$$

For the terms in y^{2k}, $k \geq 2$, we will prove

$$\begin{aligned}(1-8\gamma_1^2)\frac{(2\pi)^{2k}}{(2k)!} > &\left[4(1-\gamma_1^2) + 2(1-\gamma_1)^2\sin^2\theta\right]\frac{(2(\pi-\theta))^{2k}}{(2k)!} \\ &+ 8(1-\gamma_1)^2(\sin^2\theta)\frac{(2(\pi-\theta))^{2k-2}}{(2k-2)!}.\end{aligned} \tag{140}$$

Since $\theta \in [3\pi/4, \pi]$ we have that

$$1 + \cos\theta \le \frac{2-\sqrt{2}}{2} \le \frac{3}{10}$$

and $\sin^2\theta \le 1/2$. Therefore (138) is implied by $1 - 4\gamma_1^2 > (6/10) \times (1-\gamma_1^2) + (3/20)(1-\gamma_1)^2$ which can be seen to hold for $\gamma_1 \in [0, 31/100]$ via straightforward manipulations. Next we claim that for $\theta \in [3\pi/4, \pi]$ and $\gamma_1 \in [0, 31/100]$ we have

$$(141)\qquad \begin{gathered} \frac{3(1-8\gamma_1^2)}{4} \cdot \frac{(2\pi)^{2k}}{(2k)!} > \left[4(1-\gamma_1^2) + (1-\gamma_1)^2\right] \frac{(2(\pi-\theta))^{2k}}{(2k)!}, \\ \text{for } k \ge 1, \\ \frac{1-8\gamma_1^2}{4} \cdot \frac{(2\pi)^{2k}}{(2k)!} \ge 4(1-\gamma_1^2) \frac{(2(\pi-\theta))^{2k-2}}{(2k-2)!}, \quad \text{for } k \ge 2, \\ \frac{1-8\gamma_1^2}{4} \cdot \frac{(2\pi)^2}{2!} > 4(1-\gamma_1^2) \cdot \frac{3}{10}. \end{gathered}$$

Note that since $0 \le \sin^2\theta \le 1/2$ and $3/10 > 1 + \cos\theta \ge 0$ the first inequality for $k = 1$ and the last inequality in (141) give (140). Using similar arguments the first and the second inequalities in (141) give (140). As for (141) straightforward algebraic manipulations show that the third inequality holds for $\gamma_1 \in [0, 31/100]$. For the first inequality in (141) notice that it suffices to prove the case $k = 1$. In that case, using that for $\theta \in [3\pi/4, \pi]$ we have $\pi/(\pi - \theta) \ge 4$, the inequality can be easily verified to hold for $\gamma_1 \in [0, 31/100]$. Finally, the second inequality in (14) can be reduced to verifying the case $k = 2$ via a simple induction argument. Then, using again $\pi/(\pi-\theta) \ge 4$, the corresponding inequality for $k = 2$ can be easily checked for $\gamma_1 \in [0, 31/100]$. The proof of Lemma 7.3 is completed. □

We finish this section with the remark that the proof of Theorem 7.1 also gives

Theorem 7.4. *Let Ω be a curvilinear polygon in $\mathbb{R}^2$ with angles in the interval $[3\pi/4, 5\pi/4]$ and assume that the Lamé moduli μ, λ satisfy $\mu/(\lambda + 2\mu) \in (0, 31/100]$. Then*

$$(142)\qquad \rho(K_{\gamma_1,\gamma_1}; \left(L^2(\partial\Omega)/\mathbb{R}\right)^2) < 1.$$

Moreover,

$$\rho(K_{\gamma_1,\gamma_1}; (L^p(\partial\Omega)/\mathbb{R})^2) < 1 \quad \text{for all } 2 \leq p < \infty. \tag{143}$$

This is because throughout the proof of Theorem 7.1 we only made use of the fact $K_{\text{traction}} = K_{\gamma_1,\gamma_2}$ with $|\gamma_2| = \gamma_1$. □

8. The Stokes system. In this section we prove spectral radius estimates (2) for a certain family of hydrostatic layer potentials. We start by considering the linearized, homogeneous, time independent Navier-Stokes equations, i.e., the Stokes system

$$\begin{cases} \triangle \vec{u} = \nabla p, \\ \operatorname{div} \vec{u} = 0, \end{cases} \tag{144}$$

in an open set in $\mathbb{R}^2$, where $\vec{u}$ is the velocity field and p is the pressure function. If we define the matrix $A(r) := (a_{ij}^{kl}(r))_{i,j,k,l}$ by

$$a_{ij}^{kl}(r) := \delta_{ij}\delta_{kl} + r\,\delta_{il}\delta_{jk}, \quad \text{for } r \in \mathbb{R}, \tag{145}$$

then $a_{ij}^{kl}(r)\partial_i\partial_j u^l = \triangle u^k + r\partial_k(\operatorname{div}\vec{u})$. Hence, any solution $\vec{u}$, p of the Stokes system (144) satisfies $a_{ij}^{kl}(r)\partial_i\partial_j u^l = \partial_k p$. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and denote by N the outward unit normal vector almost everywhere on $\partial\Omega$. The conormal derivative that corresponds to the choice of coefficients $A(r) := (a_{ij}^{kl}(r))_{i,j,k,l}$ in (145) is

$$\left(\frac{\partial \vec{u}}{\partial N_{A(r)}}\right)^j := N_i a_{ik}^{jl}(r)\partial_k u^l - N_j p, \quad \text{where} \quad j = 1, 2. \tag{146}$$

Going further, denote by $G = (G_{ij})_{i,j}$ the Kelvin matrix valued fundamental solution for the system of hydrostatics, see, e.g., [**22**],

$$G_{ij}(X) := \frac{1}{2\pi}\left(\delta_{ij}\log|X|^2 - 2\frac{X_i X_j}{|X|^2}\right), \quad X \in \mathbb{R}^2 \setminus \{0\}.$$

Let $K_{S,r}$ be the double layer hydrostatic operator corresponding to the conormal derivative $\partial/\partial N_{A(r)}$ in (146) on the boundary of Ω. Also, set G^j for the jth column in the fundamental matrix. Then

(147)

$$(K_{S,r}(\vec{f}))^i(P) := p.v.\int_{\partial\Omega}\left(\frac{\partial G^j}{\partial N_{A(r)}}(P - \cdot)\right)^i(Q) f^j(Q)\,d\sigma(Q), \quad P \in \partial\Omega,$$

where $i = 1, 2$. The ith component of $(\partial G^j/\partial N_{A(r)})(P - \cdot)$ evaluated at Q, denoted by $k_{S,r}^{ij}(P,Q)$ is

$$
\begin{aligned}
k_{S,r}^{ij}(P,Q) = &-\frac{(1-r)\delta_{ij}}{2\pi} \cdot \frac{\langle X-Q, N(Q)\rangle}{|X-Q|^2} \\
&- \frac{1-r}{2\pi} \cdot \frac{(X_i - Q_i)N_j(Q) - (X_j - Q_j)N_i(Q)}{|X-Q|^2} \\
&- \frac{1+r}{\pi} \cdot \frac{\langle X-Q, N(Q)\rangle (X_i - Q_i)(X_j - Q_j)}{|X-Q|^4}
\end{aligned}
\tag{148}
$$

Taking into consideration (49) we have that

$$
K_{S,r} = K_{((1-r)/2,(1-r)/2)}. \tag{149}
$$

Our first result about the operator $K_{S,r}$ is the following.

Theorem 8.1. *Let Ω be the domain consisting of the interior of an angle of aperture θ. Then for any $r \in (-1, 1)$ there exists $p(\theta, r) \in (1, \infty)$ depending only on θ and r, such that*

$$
\rho(K_{S,r}; (L^p(\partial\Omega))^2) < 1 \quad \text{for } p \in [p(\theta, r), \infty)\,. \tag{150}
$$

Proof. This follows from (149) and Theorem 5.1 (more precisely (51)) since for $r \in (-1, 1)$ the point $((1-r)/2, (1-r)/2)$ belongs to the side of the triangle $\mathcal{T}$ joining $(0,0)$ with $(1,1)$. $\square$

Next, we consider the analogue of Theorem 7.1 in this context.

Theorem 8.2. *Let Ω be a curvilinear polygon in $\mathbb{R}^2$ with angles in the interval $[3\pi/4, 5\pi/4]$. Then for any $r \in [38/100, 1)$ we have*

$$
\rho\left(K_{S,r}; \left(L^2(\partial\Omega)/\,\mathbb{R}\right)^2\right) < 1. \tag{151}
$$

Moreover,

$$
\rho\left(K_{S,r}; (L^p(\partial\Omega)/\mathbb{R})^2\right) < 1 \quad \text{for all } 2 \le p < \infty. \tag{152}
$$

Proof. This follows from (149) and (51) in the proof of Theorem 7.4 since $(1-r)/2 \in [0, 31/100]$ for $r \in [38/100, 1)$. It is relevant here that for any $r \in (-1, 1)$ the coefficient matrix $A(r)$ given in (145) is *strictly positive definite* in the sense of (112). □

We point out that the choice $r := 1$ in (145) gives rise to the *stress* conormal derivative $\partial/\partial N_{A(1)}$, see (146), occasionally known as the *slip condition* when imposed at the boundary. Explicit spectral radius formulas for the associated boundary double layer potential operator $K_{\text{stress}} := K_{S,1}$ in the sector case have been established in [**29**]. In particular our analysis there gives

$$\rho\left(K_{\text{stress}}; (L^p(\partial\Omega))^2 / \Psi\right) < 1, \tag{153}$$

for any bounded curvilinear polygon in $\mathbb{R}^2$, where Ψ is defined at the beginning of Section 6.

We continue with an explicit description of the spectrum of the hydrostatic layer potential operator $K_{S,r}$, $r \in \mathbb{R}$ on curvilinear polygons.

Theorem 8.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, simply connected curvilinear polygon with angles θ_i, $i = 1, \dots, n$ and let $p \in (1, \infty)$. For each $1 \le i \le n$ consider the curve $\Sigma^r_{\theta_i}(p)$ as in (44) associated to r, the angle θ_i and the integrability exponent p. Set $\widehat{\Sigma^r_{\theta_i}(p)}$ for the closure of its interior. Then*

$$\sigma(K_{S,r}; (L^p(\partial\Omega))^2) = \Big(\bigcup_{1\le i\le n} \widehat{\Sigma^r_{\theta_i}(p)}\Big) \cup \{\lambda_j\}_j, \tag{154}$$

where $\{\lambda_j\}_j$ consists of eigenvalues of the operator $K_{S,r}$ on $(L^p(\partial\Omega))^2$. Whenever $r \in (-1, 1]$ the set $\{\lambda_j\}_j \subset (-1, 1]$ consists of finitely many points.

For $w \in \mathbb{C}$, $w \in \cup_{1\le i\le n}\Sigma^r_{\theta_i}(p)$ the operator $wI - K_{S,r}$ is not Fredholm on $(L^p(\partial\Omega))^2$.

For $w \in \mathbb{C}$, $w \notin \cup_{1\le i\le n}\Sigma^r_{\theta_i}(p)$ the operator $wI - K_{S,r}$ is Fredholm on $(L^p(\partial\Omega))^2$ and its index is given by

$$\text{index}\left(wI - K_{S,r}; (L^p(\partial\Omega))^2\right) = \sum_{i=1}^{n} W\left(w, \Sigma^r_{\theta_i}(p)\right), \tag{155}$$

where $W\left(w, \Sigma_{\theta_i}^r(p)\right)$ *stands for the sum of the winding numbers of the point* $w \notin \Sigma_{\theta_i}^r(p)$ *with respect to each one of the four closed curves* $\Sigma_j(r, \theta_i, p)$, $j = 1, \ldots, 4$, *constituting* $\Sigma_{\theta_i}^r(p)$.

Proof. The proof of Theorem 8.3 follows in a similar manner as the proof of Theorem 6.1. $\square$

We conclude this section by presenting an example for the L^{10} spectrum of the operator $K_{S,1/2}$ on a curvilinear polygon of angles $\pi/4$ and $\pi/7$. The value of the essential spectral radius in this case is 0.9967. Below, the $*$ stands for the generic location of the set of eigenvalues $\{\lambda_j\}_j$ as in the statement of Theorem 8.3. We have:

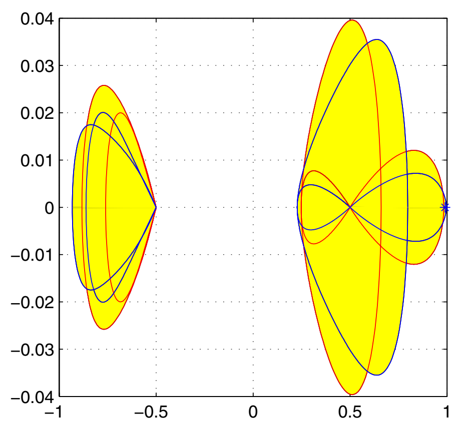

FIGURE 8. The L^{10} spectrum of the hydrostatic double layer potential operator $K_{S,1/2}$ on a curvilinear polygon with angles $\pi/4$ and $\pi/7$.

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