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FAST BOOLEAN APPROXIMATION METHODS FOR SOLVING INTEGRAL EQUATIONS IN HIGH DIMENSIONS

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ABSTRACT. Solving integral equations in high dimensions requires a huge computational effort and hence fast methods are desirable. We develop and analyze Boolean approximation methods using the piecewise constant functions for solving integral equations of the second kind on a unit cube in R^d, including the Boolean Galerkin method and the Boolean collocation method. These schemes are based on an idea from Boolean sum approximation to obtain a linear combination of multiple coarse levels of approximations. We prove that these schemes provide fast computational methods. Specifically, they have convergence in order $\mathcal{O}(h\log^{d-1}(h^{-1}))$, with computational cost in order $\mathcal{O}(h^{-1}\log^{d-1}(h^{-1}))$, as $h \to 0$, where h is the mesh size used in the methods. For the special case when $d = 2$, we develop an iterated Boolean Galerkin method and prove the super-convergence property of this method.

1. Introduction. Integral equations of the second kind with smooth kernels in *high* dimensions have important applications in many areas such as physics, engineering and finance. Regularization of integral equations of the first kind also leads to integral equations of the second kind with smooth kernels (see, for example, [**12**] and the references cited therein). In particular, for applications of high-dimensional integral kernels in learning theory, see a recent paper [**6**]. In some areas of machine learning, a meaningful dimension is in the hundreds. Solving integral equations in high dimensions is a very challenging task

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due to *huge* computational cost, and thus, fast numerical schemes are highly desirable. In this paper we propose fast methods using a combination technique based on Boolean approximations for solving high-dimensional integral equations of the second kind with a smooth kernel. Due to the high-dimensional nature of these equations, we will use the space of the piecewise constant functions as our approximation space and consider both Boolean Galerkin and Boolean collocation methods. In these methods, we construct Boolean sums of approximate solutions obtained from the Galerkin method or the collocation method using different mesh sizes to reduce order of computational complexity while preserving order of convergence of the corresponding standard method. Specifically, the proposed methods have convergence of order $\mathcal{O}(h\log^{d-1}(h^{-1}))$, with computational cost in order $\mathcal{O}(h^{-1}\log^{d-1}(h^{-1}))$, where d is the dimension and h is the mesh size used in the methods. Let us now explain what we mean in this paper by the computational cost. The term *computational cost* used in this paper refers to the total number of grid points in the partitions with different mesh sizes used in the methods. These methods have almost the same order of convergence as that $\mathcal{O}(h)$ for the standard methods and reduce significantly the computational cost $\mathcal{O}(h^{-d})$ for the standard methods. Another important property of this method is that it can be parallelized in a very efficient manner.

We now use the two-dimensional case as an example to demonstrate the key idea of these methods. Let h_x and h_y denote the mesh-size in x-axis and y-axis, respectively, and let $R_{h_x,h_y}u$ be the Galerkin approximation of the solution u of an equation on a uniform rectangle grid of h_x and h_y. Suppose that $h := 2^{-n}$ is the finest mesh-size. The combination solution of level n is defined by

$$u_h^c := \sum_{i=1}^{n} R_{2^{-i},2^{i-n-1}}u - \sum_{i=1}^{n-1} R_{2^{-i},2^{i-n}}u.$$

We will call u_h^c the multi-level Boolean-Galerkin approximation of u. Usually the approximate solution u_h^c is nearly as accurate as the approximate solution $R_{h,h}u$ while the computational cost required for the calculation of solutions $R_{h_x,h_y}u$, which are used for the calculation of u_h^c, is much less than that for the calculation of $R_{h,h}u$. Note that the computational cost for $R_{h,h}u$ is $\mathcal{O}(2^{2n})$ while that for u_h^c is $\mathcal{O}(n2^n)$.

The Boolean sum approximation is also used in [**11**] in conjunction with the degenerate kernel scheme for solving integral equations to achieve a higher order of convergence. The d-dimensional Boolean sum approximation studied in this paper is closely related to the d-dimensional Boolean interpolation which is first constructed in [**7**]. This technique is originally used in [**9**] to reduce computational complexity in the numerical solutions of partial differential equations and it has been extensively studied in, e.g., [**3, 4, 16**–**18**]. This combination technique is analogous to the sparse grid method and the multi-parameter extrapolation method discussed in [**19, 21, 22**].

In this paper we will study the Boolean approximation methods for solving integral equations of the second kind, including the Boolean Galerkin and Boolean collocation methods, aiming at reducing the computational complexity with preserving the order of accuracy. In Section 2 we describe a combination technique based on the Boolean approximation and review results on the convergence order of the Boolean approximations. Sections 3 and 4 are devoted to the development of the Boolean Galerkin method and Boolean collocation method for solving integral equations of the second kind, respectively. In Section 5 we develop the Boolean iterated Galerkin approximation for solving two-dimensional integral equations of the second kind and prove the super-convergence property of the iterated method.

2. The Boolean approximation.

In this section we describe a combination technique based on the Boolean approximation. This technique will be used in the next three sections to develop Boolean approximation schemes for solving Fredholm integral equations of the second kind defined on the unit cube in R^d.

We begin with the definition of notations. Let $\square := [0,1]^d$ be the unit cube in R^d. We use $W^{s,p}(\square)$ to denote the standard Sobolev spaces of functions whose derivatives of order less than or equal to s are in $L^p(\square)$. We denote by N_0 the set of all nonnegative integers. For a function $v \in W^{s,p}(\square)$, a point $x := (x_0, x_1, \ldots, x_{d-1}) \in \square$ and an index $\alpha := (\alpha_0, \alpha_1, \ldots, \alpha_{d-1}) \in N_0^d$, we let

$$(D^\alpha v)(x) := \left(\frac{\partial^{\alpha_0}}{\partial x_0^{\alpha_0}} \cdots \frac{\partial^{\alpha_{d-1}} v}{\partial x_{d-1}^{\alpha_{d-1}}} \right)(x).$$

The norms and semi-norms for the space $W^{s,p}(\square)$ are defined by

$$\|v\|_{s,p} := \left(\sum_{|\alpha|\le s} \|D^\alpha v\|_p^p\right)^{1/p} \quad \text{and} \quad |v|_{s,p} := \left(\sum_{|\alpha|=s} \|D^\alpha v\|_p^p\right)^{1/p},$$

respectively, see, e.g., [**1, 5**]. When $p = 2$, we let $H^s(\square) = W^{s,p}(\square)$, $\|\cdot\|_s = \|\cdot\|_{s,p}$, $\|\cdot\| = \|\cdot\|_0$ and we use $(\cdot,\cdot)$ for the standard $L^2(\square)$ inner product. We will also use the negative norm $\|\cdot\|_{-1}$, which is defined for $w \in H^{-1}(\square) := (H^1(\square))^*$ by

$$\|w\|_{-1} = \sup_{\phi\in H^1(\square)} \frac{(w,\phi)}{\|\phi\|_1}.$$

One of our purposes in this section is to describe the multi-dimensional tensor product interpolation operator. To this end, we first define the interpolation operator in one dimension. For a positive integer n, we let $Z_n := \{0,1,\ldots,n-1\}$ and $h := 1/n$. Let $T^h[0,1]$ be a uniform mesh of the interval $[0,1]$ with the mesh size h, i.e.,

$$T^h[0,1] := \{[ih,(i+1)h) : i \in Z_n\}.$$

We use $\partial^2 T^h[0,1]$ to denote the set of the midpoints of the subintervals in the mesh $T^h[0,1]$, namely,

$$\partial^2 T^h[0,1] = \left\{\left(i+\frac{1}{2}\right)h : i \in Z_n\right\}.$$

Define the space of piecewise constant functions in $L^\infty[0,1]$ by setting

$$S^h[0,1] := \{v \in L^\infty[0,1] : v|_\tau \text{ is constant}, \ \tau \in T^h[0,1],\ v(0) = v(1)\}.$$

We consider the space of continuous periodic functions on $[0,1]$

$$C_p[0,1] := \{v \in C[0,1] : v(0) = v(1)\}$$

and note that the derivative of a differentiable function in $C_p[0,1]$ is also periodic. Let $I_h : C_p[0,1] \to S^h[0,1]$ be the Lagrange interpolation operator defined for $v \in C_p[0,1]$ by the equations

$$(I_h v)(t) = v(t), \quad t \in \partial^2 T^h[0,1], \quad \text{and} \quad (I_h v)(1) = v\left(\frac{h}{2}\right).$$

We next describe the multi-dimensional notation. For $\mathbf{h} = (h_0, \dots, h_{d-1})$, where h_j is either 0 or 2^{-i_j} with $i_j \in N_0$, define a mesh of the unit cube $\square$ in R^d by

$$T^{\mathbf{h}} = T^{h_0}[0,1] \times \cdots \times T^{h_{d-1}}[0,1]$$

and note that $T^{\mathbf{h}}$ provides a partition for the unit cube $\square$. The corresponding space of piecewise constant functions on $\square$ is then defined by

$$S^{\mathbf{h}}(\square) = S^{h_0}[0,1] \otimes \cdots \otimes S^{h_{d-1}}[0,1].$$

We remark that $S^{\mathbf{h}}(\square)$ is the tensor product space of the space of piecewise constant functions on the interval $[0,1]$. The interpolation operator from $C_p(\square) := C_p[0,1] \times \cdots \times C_p[0,1]$, d times, into $S^{\mathbf{h}}(\square)$ is defined by

$$I_{\mathbf{h}} = I_{h_0} \circ \cdots \circ I_{h_{d-1}}.$$

For $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{d-1})$ we set

$$\mathbf{h}^\alpha := h_0^{\alpha_0} \cdots h_{d-1}^{\alpha_{d-1}},$$

and

$$\mathbf{h}\alpha := (h_0\alpha_0, \dots, h_{d-1}\alpha_{d-1}).$$

We define the order $\alpha \le \beta$ for the elements $\alpha, \beta \in \{0,1\}^d$ by $\alpha_i \le \beta_i$ for all $i \in Z_d$. Furthermore, we denote $\mathbf{0} := (0, \dots, 0) \in R^d$, $\mathbf{e} := (1, \dots, 1) \in R^d$ and for $i \in Z_d$, $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots, 0) \in R^d$ whose ith component is one and zero otherwise. We observe that, if $\alpha + \beta \le \mathbf{e}$, then

$$D^\alpha I_{\mathbf{h}(\mathbf{e}-\beta)} = I_{\mathbf{h}(\mathbf{e}-\beta)} D^\alpha. \tag{2.1}$$

We also need the notion of the mixed Sobolev space (cf., [**18**]), defined for $\alpha \in \{0,1\}^d$ and $1 \le p \le \infty$ by

$$W^{\alpha,p}_{\text{mix}}(\square) := \{w \in L^p(\square) : D^\beta w \in L^p(\square),\ \text{for all } \beta \text{ with } \mathbf{0} \le \beta \le \alpha\}$$

with the associated norm given for $v \in W^{\alpha,p}_{\text{mix}}(\square)$ by

$$\|v\|_{W^{\alpha,p}_{\text{mix}}} := \left(\sum_{0 \le \beta \le \alpha} \|D^\beta v\|_p^2 \right)^{1/2}.$$

In particular, we denote

$$H^{\alpha}(\square) := W^{\alpha,2}_{\mathrm{mix}}(\square).$$

Noticing the basic interpolation estimate, that is, there exists a positive constant c such that, for $i \in Z_d$,

$$\|v - I_{\mathbf{h}\mathbf{e}_i} v\| \le c h_i \|D^{\mathbf{e}_i} v\|, \quad \text{for all } v \in H^{\mathbf{e}_i}(\square), \tag{2.2}$$

we conclude that there exists a positive constant c for all $v \in H^{\mathbf{e}}(\square)$ such that

$$\|v - I_{\mathbf{h}} v\| \le c \max\{h_0, \dots, h_{d-1}\} \|v\|_{H^{\mathbf{e}}}. \tag{2.3}$$

For $\mathbf{0} \le \alpha \le \mathbf{e}$, we define the *error* operator $\Pi^{\alpha}_{\mathbf{h}}$ by

$$\Pi^{\alpha}_{\mathbf{h}} := \prod_{\substack{\mathbf{0} \le \beta \le \alpha \\ |\beta| = 1}} (I - I_{\mathbf{h}\beta})$$

and record in the next lemma a result regarding this operator which was established in [**18**].

Lemma 2.1. *There hold the following statements.*

(i) *For* $\mathbf{0} \le \beta \le \mathbf{e}$,

$$I - I_{\mathbf{h}\beta} = - \sum_{\substack{\mathbf{0} \le \alpha \le \beta \\ |\alpha| \ge 1}} (-1)^{|\alpha|} \Pi^{\alpha}_{\mathbf{h}}.$$

(ii) *For* $\alpha, \beta, \alpha + \beta \in \{0,1\}^d$, *if* $v \in H^{\alpha+\beta}(\square)$, *then there exists a positive constant* c *such that*

$$\|\Pi^{\alpha}_{\mathbf{h}} v\|_{H^{\beta}} \le c \mathbf{h}^{\alpha} \|v\|_{H^{\alpha+\beta}}.$$

We next define the orthogonal projection from $L^2(\square)$ onto $S^{\mathbf{h}}(\square)$. Let $P_{\mathbf{h}} : L^2(\square) \to S^{\mathbf{h}}(\square)$ be the L^2 projection defined as follows. For every $v \in L^2(\square)$

$$(v - P_{\mathbf{h}} v, s) = 0 \quad \text{for all } s \in S^{\mathbf{h}}.$$

It can be verified that, for any $\mathbf{0} \leq \alpha,\ \beta \leq \mathbf{e}$ with $\alpha + \beta \leq \mathbf{e}$, there holds the identity that, for every $v \in H^{\alpha}(\square) \cap C_p(\square)$,

$$D^{\alpha} P_{\mathbf{h}\beta} v = P_{\mathbf{h}\beta} D^{\alpha} v. \tag{2.4}$$

We also have the estimates of the order of convergence for the projection $P_{\mathbf{h}}$ which we describe below. If $v \in H^{\mathbf{e}}(\square)$, then there exists a positive constant c for all $i \in Z_d$,

$$\|v - P_{\mathbf{h}\mathbf{e}_i} v\| \leq c h_i \|D^{\mathbf{e}_i} v\| \tag{2.5}$$

and

$$\|v - P_{\mathbf{h}} v\| \leq c \max\{h_0, \ldots, h_{d-1}\} \|v\|_{H^{\mathbf{e}}}. \tag{2.6}$$

Next we describe the Boolean approximation to a given function. For this purpose we define the index set $J_n^d := \{1, 2, \ldots, n\}^d$ for a given integer $n \in N$. For a vector $\mathbf{i} = (i_0, \ldots, i_{d-1}) \in J_n^d$, we choose

$$\mathbf{h} = \mathbf{h}_i = (h_0, \ldots, h_{d-1}) := (2^{-i_0}, \ldots, 2^{-i_{d-1}}).$$

Let $W_n := \{w_{\mathbf{h}_\mathbf{i}} : \mathbf{i} \in J_n^d\}$ be a sequence of approximations to a function $w \in L^2(\square)$. We construct from this sequence of approximations the Boolean approximation w_h^c, $h := 2^{-n}$ to w by the formula

$$w_h^c := \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} \sum_{\substack{|\mathbf{i}| = n+d-k-1 \\ \mathbf{i} \in J_n^d}} w_{\mathbf{h}_\mathbf{i}}. \tag{2.7}$$

In particular, for interpolation projection $I_{\mathbf{h}_\mathbf{i}}$, we have the Boolean interpolation $I_h^c u$ for $u \in C(\square)$ defined by

$$I_h^c u := \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} \sum_{\substack{|\mathbf{i}| = n+d-k-1 \\ \mathbf{i} \in J_n^d}} I_{\mathbf{h}_\mathbf{i}} u.$$

Likewise, for the orthogonal projection $P_{\mathbf{h}_\mathbf{i}}$, we have the Boolean orthogonal projection $P_h^c u$ for $u \in L^2(\square)$ defined by

$$P_h^c u := \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} \sum_{\substack{|\mathbf{i}| = n+d-k-1 \\ \mathbf{i} \in J_n^d}} P_{\mathbf{h}_\mathbf{i}} u.$$

When $d = 2$ formula (2.7) reduces to

$$w_h^c := \sum_{k=1}^{n} w_{(2^{-k}, 2^{-(n+1-k)})} - \sum_{k=1}^{n-1} w_{(2^{-k}, 2^{-(n-k)})}. \tag{2.8}$$

We also have similar formulas for $I_h^c u$ and $P_h^c u$ in the two-dimensional case. Formulas of this type use a combination of approximations from W_n to construct a better approximation for w. This technique was used in [**7**] to develop a Boolean interpolation for a function of d variables, in [**11**] to obtain an approximation solution of integral equations of higher convergence order with a fixed computational complexity and also in [**16, 17**] and [**18**] to construct fast numerical solutions of partial differential equations with preserving order of convergence.

We now remark on the computational complexity of this Boolean combination technique. To do this, we define the sparse grid and sparse space associated with the partition $T^{h\mathbf{e}}(\square)$, respectively, by

$$G_d^n := \bigcup_{\substack{|\mathbf{i}|=n+d-1 \\ \mathbf{i} \in J_n^d}} \partial^2 T^{\mathbf{h_i}}(\square),$$

and

$$S_d^n := \bigcup_{\substack{|\mathbf{i}|=n+d-1 \\ \mathbf{i} \in J_n^d}} S^{\mathbf{h_i}}(\square).$$

Clearly, $w_h^c \in S_d^h$ is defined on G_d^h. It follows from [**16**] and [**8**] that both the cardinality of G_d^n and dimension of S_d^h are of order $\mathcal{O}(2^n n^{d-1})$. Thus the computational complexity for w_h^c is $\mathcal{O}(2^n n^{d-1})$, which is significantly less than $\mathcal{O}(2^{nd})$.

We next demonstrate that the Boolean approximation preserves more or less the order of convergence. The convergence of the Boolean approximations depends on the properties of the hierarchical surplus which will be defined below. For $\alpha \in J_n^d$, we define the hierarchical surplus operator for δ^α by

$$\delta^\alpha w_{\mathbf{h}} := \sum_{\mathbf{0} \le \beta \le \alpha} (-1)^{|\beta|} w_{\mathbf{h}(\mathbf{e}+\beta)}$$

and call $\delta^\alpha w_{\mathbf{h}}$ an $|\alpha|$-dimensional hierarchical surplus of $w_{\mathbf{h}}$. Let us illustrate this operator by two simple examples in the case $d = 2$. When $\alpha = (1, 0)$, we have that

$$\delta^{(1,0)} w_{h_0,h_1} = w_{h_0,h_1} - w_{2h_0,h_1}$$

and when $\alpha = (1, 1)$ we have that

$$\delta^{(1,1)} w_{h_0,h_1} = w_{h_0,h_1} - w_{2h_0,h_1} - w_{2h_0,h_1} + w_{2h_0,h_1}.$$

We now return to the discussion of the hierarchical surplus in a general case. Note that

$$\begin{aligned} w_{h\mathbf{e}} &= \sum_{\substack{\mathbf{0}\le\alpha\le\mathbf{e}\\ \mathbf{0}\le\mathbf{i}\le\mathbf{e}}} \delta^\alpha w_{\mathbf{h}} = \sum_{1\le|\mathbf{h}\alpha|\le nd} \delta^\alpha w_{\mathbf{h}} \\ &= \sum_{|\mathbf{h}\alpha|\ge n+d} \delta^\alpha w_{\mathbf{h}} + \sum_{|\mathbf{h}\alpha|\le n+d-1} \delta^\alpha w_{\mathbf{h}}. \end{aligned}$$

By an induction argument we obtain the equation

$$\sum_{|\mathbf{h}\alpha|\le n+d-1} \delta^\alpha w_{\mathbf{h}} = w_h^c,$$

which implies that

$$\sum_{|\mathbf{h}\alpha|\ge n+d} \delta^\alpha w_{\mathbf{h}} = w_{h\mathbf{e}} - w_h^c. \tag{2.9}$$

Moreover, if there exists a positive constant $c(w)$ depending only on w such that

$$\|\delta^\alpha w_{\mathbf{h}}\| \le c(w)\mathbf{h}^\alpha \tag{2.10}$$

then there holds the estimate

$$\left\|\sum_{|\mathbf{h}\alpha|\ge n+d} \delta^\alpha w_{\mathbf{h}}\right\| \le c(w) \sum_{|\mathbf{h}\alpha|\ge n+d} 2^{-|\mathbf{h}\alpha|} \le c(w)h(\log h^{-1})^{d-1}.$$

Using this estimate, we then obtain an estimate of convergence order of the combination solution, cf. [**16**] and [**18**], which we state in the next theorem.

Theorem 2.2. *Suppose that there exists a positive constant $c(w)$ depending only on w such that inequality* (2.10) *holds for every $\mathbf{0} \leq \alpha \leq \mathbf{e}$ and*

$$\|w - w_{\mathbf{h}}\| \leq c(w) \max\{h_0, \dots, h_{d-1}\}.$$

Then

$$\|w - w_h^c\| \leq c(w) h \log^{d-1}(h^{-1}).$$

Let $w_{\mathbf{h}}$ denote either the interpolation projection or the orthogonal projection of w. From (2.1) and (2.4), it is not difficult to show that estimate (2.10) holds for these cases.

Lemma 2.3. *Let $w \in H^{\alpha}(\square)$. If $w_{\mathbf{h}} := I_{\mathbf{h}} w$ or $w_{\mathbf{h}} := P_{\mathbf{h}} w$, then*

$$\|\delta^{\alpha} w_{\mathbf{h}}\| \leq c\mathbf{h}^{\alpha} \|w\|_{H^{\alpha}}.$$

By Lemma 2.3 and Theorem 2.2, we obtain the next result.

Theorem 2.4. *Let $w \in H^{\mathbf{e}}(\square)$ and $w_{\mathbf{h}} := I_{\mathbf{h}} w$ or $w_{\mathbf{h}} := P_{\mathbf{h}} w$. Then*

$$\|w - w_h^c\| \leq ch \log^{d-1}(h^{-1}) \|w\|_{H^{\mathbf{e}}}.$$

3. The Boolean Galerkin method. This section is devoted to the development of the Boolean Galerkin method for solving integral equations. The main purpose of this section is to estimate the order of convergence and computational complexity of the method.

Suppose that kernel $k \in C_p(\square \times \square)$ is periodic, i.e.,

$$k(x + \mathbf{e}_i, y) = k(x, y) = k(x, y + \mathbf{e}_i), \quad x, y \in \square, \quad i \in Z_d$$

and is nonnegative, i.e.,

$$k(x, y) \geq 0 \quad \text{and} \quad k(x, y) \not\equiv 0, \quad x, y \in \square.$$

As a result, the operator $K : L^2(\square) \to L^2(\square)$ defined by

$$(Ku)(x) = \int_{\square} k(x, y) u(y)\, dy, \quad x \in \square, \quad u \in L^2(\square)$$

is a compact, positive integral operator. With the integral operators of this type, we consider the Fredholm integral equations of the second kind

$$u + Ku = f \tag{3.1}$$

where $u \in L^2(\square)$ is the unknown to be determined and $f \in L^2(\square)$ is a given function. Obviously, the inverse operator $(I+K)^{-1}$ exists as a bounded operator on $L^2(\square)$. In other words, equation (3.1) has a unique solution u in $L^2(\square)$ and there exists a positive constant c such that

$$\|u\| \le c\|f\|.$$

The Galerkin projection $R_{\mathbf{h}} : L^2(\square) \to S^{\mathbf{h}}(\square)$ is defined by

$$((I+K)(R_{\mathbf{h}}u - u), v) = 0, \quad \text{for all } v \in S^{\mathbf{h}}(\square).$$

In the next theorem we present an estimate for the derivatives of the Galerkin projection $R_{\mathbf{h}}u$ for any $u \in H^\alpha(\square) \cap C_p(\square)$. For this purpose, in addition, we require that kernel $k \in H^\alpha(\square \times \square) \cap C_p(\square \times \square)$.

Theorem 3.1. *Suppose kernel $k \in H^\alpha(\square \times \square) \cap C_p(\square \times \square)$ is nonnegative. Let $\alpha, \beta \in \{0,1\}^d$ with $\alpha + \beta \le \mathbf{e}$. If $w \in H^\alpha(\square) \cap C_p(\square)$, then*

$$\|D^\alpha R_{\mathbf{h}\beta} w\| \le c(\|D^\alpha w\| + \|w\|).$$

Proof. From (2.4) and the identity

$$(I + P_{\mathbf{h}\beta}K)R_{\mathbf{h}\beta}w = P_{\mathbf{h}\beta}(I+K)w,$$

we obtain

$$D^\alpha R_{\mathbf{h}\beta}w = P_{\mathbf{h}\beta}D^\alpha(I+K)w - P_{\mathbf{h}\beta}D^\alpha K R_{\mathbf{h}\beta}w. \tag{3.2}$$

Since K is positive, we have that the operators

$$(I + P_{\mathbf{h}\beta}K)^{-1} : L^2(\square) \to L^2(\square)$$

exist and are uniformly bounded. As a result, we get

$$\|R_{\mathbf{h}\beta} w\| \le c\|w\|. \tag{3.3}$$

Combining (3.2) and (3.3) we obtain

$$\begin{aligned}\|D^\alpha R_{\mathbf{h}\beta} w\| &\le \|P_{\mathbf{h}\beta} D^\alpha (I+K) w\| + \|P_{\mathbf{h}\beta} D^\alpha K R_{\mathbf{h}\beta} w\| \\ &\le c(\|D^\alpha w\| + \|w\|) + c\|R_{\mathbf{h}\beta} w\| \\ &\le c(\|D^\alpha w\| + \|w\|).\end{aligned}$$

This completes the proof. $\square$

We will assume that the kernel k is nonnegative without further mention.

Corollary 3.2. (i) *If $k \in H^{\mathbf{e}}(\square \times \square) \cap C_p(\square \times \square)$ and $u \in H^{\mathbf{e}}(\square)$, then there exists a positive constant c such that*

$$\|u - R_{\mathbf{h}} u\| \le c \inf_{v \in S^{\mathbf{h}}} \|u - v\| \le c \max\{h_0, \dots, h_{d-1}\} \|u\|_{H^{\mathbf{e}}}. \tag{3.4}$$

(ii) *If $k \in H^{\mathbf{e}_i}(\square \times \square) \cap C_p(\square \times \square)$ and $u \in H^{\mathbf{e}_i}(\square) \cap C_p(\square)$ for $i \in Z_d$, then there exists a positive constant c such that*

$$\|\delta^{\mathbf{e}_i} R_{\mathbf{h}} u\| \le c h_i (\|D_{x_i} u\| + \|u\|), \quad i \in Z_d. \tag{3.5}$$

Proof. Part (i) follows immediately from estimate (2.5). It remains to prove part (ii). To this end, for $i \in Z_d$, we define $\hat{\mathbf{e}}_i = \mathbf{e} - \mathbf{e}_i$, noticing that the ith component of the vector $\hat{\mathbf{e}}_i$ is zero and one otherwise. Note that

$$R_{\mathbf{h}} u - R_{\mathbf{h}\hat{\mathbf{e}}_i} u = (R_{\mathbf{h}} - I) R_{\mathbf{h}\hat{\mathbf{e}}_i} u.$$

By part (i) of this corollary and Theorem 3.1, we have that there exists a positive constant c such that

$$\|R_{\mathbf{h}} u - R_{\mathbf{h}\hat{\mathbf{e}}_i} u\| \le c h_i \|D^{\mathbf{e}_i} R_{\mathbf{h}\hat{\mathbf{e}}_i} u\| \le c h_i (\|D^{\mathbf{e}_i} u\| + \|u\|).$$

The triangle inequality then completes the proof. $\square$

We now turn our attention to the convergence analysis of the combination solution based on the Galerkin method for integral equation (3.1).

Theorem 3.3. *Suppose that* $k \in H^{\mathbf{e}}(\square \times \square) \cap C_p(\square \times \square)$ *and* $u \in H^{\mathbf{e}}(\square) \cap C_p(\square)$. *If* $\mathbf{0} \le \alpha \le \mathbf{e}$, *then*

$$\|\delta^\alpha R_{\mathbf{h}} u\| \le c\mathbf{h}^\alpha \|u\|_{H^\alpha}. \tag{3.6}$$

Proof. Since for $\mathbf{0} \le \alpha \le \mathbf{e}$ and $u \in H^\alpha(\square) \cap C_p(\square)$ we have that

$$\delta^\alpha R_{\mathbf{h}} u = \delta^\alpha R_{\mathbf{h}} R_{\mathbf{h}(\mathbf{e}-\alpha)} u,$$

and there exists a positive constant c such that

$$\|R_{\mathbf{h}(\mathbf{e}-\alpha)} u\|_{H^\alpha} \le c\|u\|_{H^\alpha},$$

we only need to prove (3.6) for $u \in S^{\mathbf{h}(\mathbf{e}-\alpha)} \cap H^\alpha(\square) \cap C_p(\square)$.

For this purpose, following [**18**], we employ induction on $|\alpha| = 1, 2, \ldots, d$. When $|\alpha| = 1$, this is the case in part (ii) of Corollary 3.2. We now assume that (3.6) holds for the case $|\alpha| = l - 1$ and proceed to the case when $|\alpha| = l$. Because $u \in S^{\mathbf{h}(\mathbf{e}-\alpha)} \cap C_p(\square)$, we obtain that $I_{\mathbf{h}\alpha} u \in S^{\mathbf{h}} \cap C_p(\square)$. Thus, by Lemma 2.1, we conclude that

$$\begin{aligned}\delta^\alpha R_{\mathbf{h}} u &= \delta^\alpha R_{\mathbf{h}}(I - I_{\mathbf{h}\alpha})u + \delta^\alpha R_{\mathbf{h}} I_{\mathbf{h}\alpha} u \\ &= -\delta^\alpha R_{\mathbf{h}} \Bigg(\sum_{\substack{\beta \le \alpha \\ |\beta| \ge 1}} (-1)^{|\beta|} \Pi_{\mathbf{h}}^\beta u \Bigg) + \delta^\alpha I_{\mathbf{h}\alpha} u.\end{aligned}$$

Lemma 2.3 ensures that there exists a positive constant c such that

$$\|\delta^\alpha I_{\mathbf{h}\alpha} u\| \le c\mathbf{h}^\alpha \|u\|_{H^\alpha}.$$

Therefore, it suffices to show that there exists a positive constant c such that for $\beta \le \alpha$ and $|\beta| \ge 1$,

$$\|\delta^\alpha R_{\mathbf{h}} \Pi_{\mathbf{h}}^\beta u\| \le c\mathbf{h}^\alpha \|u\|_{H^\alpha}.$$

Suppose that $|\beta| \geq 1$ and $\beta \leq \alpha$. This implies $|\alpha - \beta| < |\alpha| \leq l$ and thus, by the induction hypothesis, we conclude that there exists a positive constant c such that

$$\begin{aligned}\|\delta^\alpha R_{\mathbf{h}} \Pi_{\mathbf{h}}^\beta u\| &= \|\delta^\beta \delta^{\alpha-\beta} R_{\mathbf{h}} \Pi_{\mathbf{h}}^\beta u\| \\ &\leq c \max_\beta (\|\delta^{\alpha-\beta} R_{\mathbf{h}} \Pi_h^\beta u\|) \\ &\leq c \max_\beta (\|\delta^{\alpha-\beta} R_{\mathbf{h}} R_{\mathbf{h}(\mathbf{e}-(\alpha-\beta))} \Pi_{\mathbf{h}}^\beta u\|) \\ &\leq c \max_\beta (\mathbf{h}^{\alpha-\beta} \|R_{\mathbf{h}(\mathbf{e}-(\alpha-\beta))} \Pi_{\mathbf{h}}^\beta u\|_{H^{\alpha-\beta}}),\end{aligned}$$

where

$$\max_\beta (w_{\mathbf{h}}) = \max_{\mathbf{0} \leq \gamma \leq \beta} |w_{\mathbf{h}(\mathbf{e}+\gamma)}|.$$

By employing Theorem 3.1 and Lemma 2.1, we obtain that there exists a positive constant c such that

$$\begin{aligned}\|\delta^\alpha R_{\mathbf{h}} \Pi_{\mathbf{h}}^\beta u\| &\leq c \max_\beta (\mathbf{h}^{\alpha-\beta} \|\Pi_{\mathbf{h}}^\beta u\|_{H^{\alpha-\beta}}) \\ &\leq c \max_\beta (\mathbf{h}^{\alpha-\beta} \mathbf{h}^\beta \|u\|_{H^\alpha}) \\ &\leq c \mathbf{h}^\alpha \|u\|_{H^\alpha},\end{aligned}$$

which advances the induction process and proves the result. □

We now present the main result of this section. We will use $\#(u_h^c)$ for the number of multiplications required to construct u_h^c. Throughout this paper, we assume that solving $u_{\mathbf{h}}$ requires the number of multiplications proportional to the dimension of the vector representing $u_{\mathbf{h}}$. This can be done by using matrix compression techniques.

Theorem 3.4. *Suppose that $k \in H^{\mathbf{e}}(\square \times \square) \cap C_p(\square \times \square)$ and $u \in H^{\mathbf{e}}(\square) \cap C_p(\square)$. If $u_{\mathbf{h}} = R_{\mathbf{h}} u$, then there exists a positive constant c such that*

$$\|u - u_h^c\| \leq c 2^{-n} n^{d-1} \|u\|_{H^{\mathbf{e}}}$$

and

$$\#(u_h^c) = \mathcal{O}(2^n n^{d-1}).$$

Proof. The first result follows directly from Theorems 3.3 and 2.2 and the convergence of the finite element solution $R_{\mathbf{h}}u$.

To obtain the estimate for $\#(u_h^c)$, for each $l \in Z_d$, we let

$$J_l := \{\mathbf{i} \in J_n^d : |\mathbf{i}| = n + d - l - 1\}$$

and note that

$$\#(J_l) \leq (n + d - l - 1)^{d-1}.$$

Therefore, there exists a positive constant c such that

$$\#(u_h^c) \leq \sum_{l=0}^{d-1} \binom{d-1}{l} \#(J_l) 2^{n+d-l-1} \leq c 2^n n^{d-1},$$

which completes the proof. $\square$

We remark that the standard Galerkin method using the piecewise constant functions has order of convergence given by $\mathcal{O}(2^{-n})$ and requires computational cost at order $\mathcal{O}(2^{dn})$. Theorem 3.4 shows that, with a little price at order of convergence, our method reduces the computational complexity from $\mathcal{O}(2^{dn})$ to $\mathcal{O}(2^n n^{d-1})$. This reduction is significant when d or n is large.

4. The Boolean collocation method. In this section we discuss the Boolean collocation method for solving integral equation (3.1) when $f \in L^\infty(\square)$. Let $K : L^\infty(\square) \to L^\infty(\square)$ be a bounded integral operator with a kernel k satisfying the conditions described in Section 3. Thus the inverse operator $(I + K)^{-1}$ exists as a bounded operator on $L^\infty(\square)$. Namely, equation (3.1) has a unique solution $u \in L^\infty(\square)$ and there exists a positive constant c such that

$$\|u\|_\infty \leq c\|f\|_\infty.$$

The collocation projection $R_{\mathbf{h}} : L^\infty(\square) \to S^{\mathbf{h}}$ is defined by

$$(I + I_{\mathbf{h}}K)R_{\mathbf{h}}u = I_{\mathbf{h}}f, \tag{4.1}$$

recalling that $I_{\mathbf{h}}$ is the interpolation projection defined in Section 2. We have the following result similar to Theorem 3.1.

Theorem 4.1. *Suppose that the kernel* $k \in W^{\alpha,\infty}_{\mathrm{mix}}(\square \times \square)$. *Let* $\alpha, \beta \in \{0,1\}^d$ *with* $\alpha + \beta \leq \mathbf{e}$. *If* $w \in W^{\alpha,\infty}_{\mathrm{mix}}(\square) \cap C_p(\square)$, *then there exists a positive constant* c *such that*

$$\|D^\alpha R_{\mathbf{h}\beta} w\|_\infty \leq c(\|D^\alpha w\|_\infty + \|w\|_\infty). \tag{4.2}$$

Proof. In the proof of Theorem 3.1, replacing the orthogonal projection $P_{\mathbf{h}}$ by the interpolation projection $I_{\mathbf{h}}$, we prove this result. $\square$

Corollary 4.2. *If* $k \in W^{\mathbf{e},\infty}_{\mathrm{mix}}(\square \times \square)$ *and* $u \in W^{\mathbf{e},\infty}_{\mathrm{mix}}(\square)$, *then there exists a positive constant* c *such that*

$$\|u - R_{\mathbf{h}} u\|_\infty \leq c \max\{h_0, \ldots, h_{d-1}\} \|u\|_{W^{\mathbf{e},\infty}_{\mathrm{mix}}}. \tag{4.3}$$

If $k \in W^{\mathbf{e}_i,\infty}_{\mathrm{mix}}(\square \times \square)$ *and* $u \in W^{\mathbf{e}_i,\infty}(\square) \cap C_p(\square)$ *for* $i = 0, 1, \ldots, d-1$, *then there exists a positive constant* c *such that*

$$\|\delta^{\mathbf{e}_i} R_{\mathbf{h}} u\|_\infty \leq c h_i (\|D^{\mathbf{e}_i} u\|_\infty + \|u\|_\infty). \tag{4.4}$$

Proof. Estimate (4.3) follows directly from the fact that there exists a positive constant c such that

$$\|u - R_{\mathbf{h}} u\|_\infty \leq c \inf_{v \in S^{\mathbf{h}}} \|u - v\|_\infty \leq c \max\{h_0, \ldots, h_{d-1}\} \|u\|_{W^{\mathbf{e},\infty}_{\mathrm{mix}}}.$$

To show estimate (4.4), it suffices to prove the identity

$$R_{\mathbf{h}} R_{\mathbf{h}\hat{\mathbf{e}}_i} u = R_{\mathbf{h}} u, \quad i = 0, 1, \ldots, d-1. \tag{4.5}$$

Note that

$$I_{\mathbf{h}} I_{\mathbf{h}\hat{\mathbf{e}}_i} = I_{\mathbf{h}}.$$

This identity ensures that

$$\begin{aligned} I_{\mathbf{h}}(I + K) R_{\mathbf{h}\hat{\mathbf{e}}_i} u &= I_{\mathbf{h}}((I + I_{\mathbf{h}\hat{\mathbf{e}}_i} K) R_{\mathbf{h}\hat{\mathbf{e}}_i} u) \\ &= I_{\mathbf{h}} I_{\mathbf{h}\hat{\mathbf{e}}_i} (I + K) u) \\ &= I_{\mathbf{h}}(I + K) u. \end{aligned}$$

Hence, from definition (4.1), we have that

$$(I + I_{\mathbf{h}}K)R_{\mathbf{h}}R_{\mathbf{h}\hat{\mathbf{e}}_i}u = (I + I_{\mathbf{h}}K)R_{\mathbf{h}}u.$$

This yields equation (4.5). $\square$

Using Corollary 4.2 and replacing the orthogonal projections in Theorem 3.3 by the corresponding interpolation projections, we obtain the next result regarding the surplus of the collocation projection of u.

Theorem 4.3. *Suppose that* $k \in W_{\text{mix}}^{\mathbf{e},\infty}(\square \times \square)$ *and* $u \in W_{\text{mix}}^{\mathbf{e},\infty}(\square) \cap C_p(\square)$. *If* $\mathbf{0} \le \alpha \le \mathbf{e}$, *then there exists a positive constant* c *such that*

$$\|\delta^\alpha R_{\mathbf{e}} u\|_\infty \le c\mathbf{h}^\alpha \|u\|_{W_{\text{mix}}^{\alpha,\infty}}.$$

The main result of this section follows directly from Theorem 4.3.

Theorem 4.4. *Suppose that* $k \in W_{\text{mix}}^{\mathbf{e},\infty}(\square \times \square)$ *and* $u \in W_{\text{mix}}^{\mathbf{e},\infty}(\square) \cap C_p(\square)$. *If* $u_{\mathbf{h}} := R_{\mathbf{h}}u$, *then there exists a positive constant* c *such that*

$$\|u - u_h^c\|_\infty \le c2^{-n}n^{d-1}\|u\|_{W_{\text{mix}}^{\mathbf{e},\infty}}$$

and

$$\#(u_h^c) = \mathcal{O}(2^n n^{d-1}).$$

5. The Boolean iterated Galerkin approximation. In this section we derive the Boolean iterated Galerkin approximation for integral equations on a unit cube $\square := [0,1] \times [0,1]$ in R^2. Specifically, we consider Fredholm integral equations of the second kind

$$u(x,y) + \int_0^1 \int_0^1 k(x,y;s,t)u(s,t)\,ds\,dt = f(x,y), \quad (x,y) \in \square, \tag{5.1}$$

where $k \in H^1(\square \times \square)$. We develop a Boolean combination solution based on the iterated Galerkin approximation for the solution of equation

(5.1) and show such a solution possesses super-convergence. Throughout this section we will write h_x and h_y for h_0 and h_1, D_x and D_y for $D^{(1,0)}$ and $D^{(0,1)}$. In order to obtain a super-convergence result, we assume that the kernel k and the solution u of equation (5.1) has a higher order of smoothness. Precisely, we consider the space

$$H^{1,2}(\square) := \{w \in H^1(\square) : D_x D_y w \in L^2(\square)\}, \tag{5.2}$$

with the norm defined for $w \in H^{1,2}(\square)$ by

$$\|w\|_{H^{1,2}} := \|w\|_{H^1} + \|D_x D_y w\|,$$

and assume that $k \in C^2(\square \times \square)$. For the Galerkin approximation $u_h := u_{\mathbf{h}} = R_{h,h}u$ to the solution of equation (5.1) with $h := 2^{-n}$, we define the *iterated Galerkin approximation* by the equation

$$\tilde{u}_h := f - Ku_h \tag{5.3}$$

and the *Boolean iterated Galerkin approximation* $\tilde{u}_h^c$ by formula (2.7) with $w_{\mathbf{h}}$ replaced by

$$\tilde{u}_{\mathbf{h}} := f - Ku_{\mathbf{h}},$$

where $u_{\mathbf{h}} := R_{\mathbf{h}}u$. The main purpose of this section is to prove the super-convergence property of the Boolean iterated Galerkin approximation $\tilde{u}_h^c$. To this end, we first develop a sequence of estimates.

Proposition 5.1. *There exists a positive constant c such that*

$$\|u - u_h\|_{-1} + \|u - \tilde{u}_h\| \le c \max\{h_0, \dots, h_{d-1}\}\|u - u_h\|.$$

Lemma 5.2. *For $w \in H^1(\square)$, set*

$$g := P_{h_x,0}(I + K)(I - P_{h_x,0})w. \tag{5.4}$$

Then there exists a positive constant c such that

$$\|g\| \le ch_x^2\|D_x w\|. \tag{5.5}$$

Moreover, if $w \in H^{1,2}(\square)$, then there exists a positive constant c such that

$$\|D_y g\| \le c h_x^2 \|D_x D_y w\|. \tag{5.6}$$

Proof. Let $\mathbf{h}_1 := (h_x, 0) \in R^2$. For $v \in S^{\mathbf{h}_1}(\square)$, we set

$$\phi := (I + K)v.$$

Since the kernel of the operator K is symmetric, we have that

$$(I + K)^* = I + K. \tag{5.7}$$

Therefore, using the definition of g and equation (5.7), we conclude that

$$(g, v) = ((I + K)(I - P_{h_x,0})w, v) = ((I - P_{h_x,0})w, \phi).$$

Noting that there holds the identity $(I - P_{h_x,0})^2 = I - P_{h_x,0}$ and the operator $I - P_{h_x,0}$ is self-adjoint, it follows that

$$(g, v) = ((I - P_{h_x,0})w, (I - P_{h_x,0})\phi).$$

The fact that, for any $v \in S^{\mathbf{h}_1}(\square)$,

$$((I - P_{h_x,0})w, v) = 0$$

implies that

$$(g, v) = ((I - P_{h_x,0})w, (I - P_{h_x,0})K_v).$$

Consequently, there exists a positive constant c such that

$$|(g, v)| \le c h_x^2 \|D_x w\| \|v\|.$$

Choosing $v := g$ in the last inequality proves (5.5). Finally, from the identity

$$D_y g = P_{h_x,0}(I - K)(I - P_{h_x,0})D_y w$$

and the same argument used in proving (5.5), we obtain (5.6). $\square$

Proposition 5.3. *Suppose that* $u \in H^{1,2}(\square) \cap C_p(\square)$. *Then there exists a function* $\psi \in L^2(\square)$ *satisfying*

$$R_{h_x,h_y}((I - P_{h_x,0})u) = R_{h_x,h_y}\psi \tag{5.8}$$

and there exists a positive constant c *such that*

$$\|D_y\psi\| \le ch_x^2\|u\|_{H^{1,2}}. \tag{5.9}$$

Proof. As in (5.4), we let

$$g := P_{h_x,0}(I + K)(I - P_{h_x,0})u.$$

Then we have that

$$g = P_{h_x,0}g$$

and, by Lemma 5.2, there exists a positive constant c such that

$$\|D_y g\| \le ch_x^2\|u\|_{H^{1,2}}. \tag{5.10}$$

Now let

$$\psi := (I + K)^{-1}g.$$

Then ψ satisfies equation (5.8) and has the estimate

$$\|D_y\psi\| \le c\|D_y g\|. \tag{5.11}$$

Combining (5.10) and (5.11) yields (5.9). $\square$

Now we study the hierarchical surplus, the difference between Boolean Galerkin and Galerkin approximations. First we prove two consequences of Sections 2 and 3.

Proposition 5.4. *If* $w \in H^1(\square)$, *then*

$$\|\delta^{(0,1)}P_{h_x,h_y}w\| \le ch_y\|D_y w\|, \tag{5.12}$$

and

$$\|\delta^{(0,1)}P_{h_x,h_y}w\|_{-1} \le ch_y^2\|D_y w\|. \tag{5.13}$$

If $w \in H^{1,2}(\square)$, *then*

$$\|\delta^{(0,1)} \circ \delta^{(1,0)} P_{h_x,h_y} w\|_{-1} \le c h_x h_y \min\{h_x, h_y\} \|w\|_{H^{1,2}}. \tag{5.14}$$

Proof. Estimate (5.12) follows directly from the definition of $\delta^{(0,1)}$. We now prove estimate (5.13). For $\phi \in H^1(\square)$, since

$$((P_{h_x,h_y} - P_{h_x,2h_y})w, P_{0,2h_y} P_{h_x,0}\phi) = 0,$$

and there exists a positive constant c such that

$$\|(I - P_{0,2h_y}) P_{h_y,0}\phi\| \le c h_y \|D_y P_{h_x,0}\phi\| \le c h_y \|P_{h_x,0} D_y \phi\| \le c h_y \|\phi\|_1$$

we conclude that there exists a positive constant c such that

$$\begin{aligned}(\delta^{(0,1)} P_{h_x,h_y} w, \phi) &= (\delta^{(0,1)} P_{h_x,h_y} w, P_{h_x,0}\phi) \\ &= (\delta^{(0,1)} P_{h_x,h_y} w, (I - P_{0,2h_y})\phi) \\ &\le c h_y \|\delta^{(0,1)} P_{h_x,h_y} w\| \|\phi\|_1.\end{aligned}$$

This ensures that there exists a positive constant c such that

$$\|\delta^{(0,1)} P_{h_x,h_y} w\|_{-1} \le c h_y \|\delta^{(0,1)} P_{h_x,h_y} w\|.$$

Combining this estimate with (5.12) gives estimate (5.13). Similarly, we obtain that there exists a positive constant c such that

$$\|\delta^{(0,1)} \delta^{(1,0)} P_{h_x,h_y} w\|_{-1} \le c \min\{h_x, h_y\} \|\delta^{(1,0)} \delta^{(0,1)} P_{h_x,h_y} w\|.$$

Moreover, there exists a positive constant c such that

$$\|\delta^{(1,0)} \delta^{(0,1)} P_{h_x,h_y} w\| \le c h_x h_y \|w\|_{H^{1,2}}.$$

Combining the last two estimates yields estimate (5.14). $\square$

Proposition 5.5. *If* $w \in H^1(\square) \cap C_p(\square)$, *then there exists a positive constant* c *such that*

$$\|\delta^{(0,1)} R_{h_x,h_y} w\| \le c h_y \|D_y w\|, \tag{5.15}$$

and

$$\|\delta^{(0,1)} R_{h_x,h_y} w\|_{-1} \le c h_y^2 \|D_y w\|. \tag{5.16}$$

Proof. Estimate (5.15) is a special case of Corollary 3.2(ii) when $d = 2$. It remains to prove estimate (5.16). To this end, for any $\phi \in H^1(\square)$, we let $\psi := R_{h_x,0}(I+K)^{-1}\phi$. Then there exists a positive constant c such that $\|D_y\psi\| \le c\|\phi\|_1$ and $P_{0,h_y}\psi \in S^{(h_x,h_y)}(\square)$. It follows that

$$\begin{aligned}((R_{h_x,h_y} - R_{h_x,0})w, \phi) &= ((R_{h_x,h_y} - R_{h_x,0})w, (I+K)\psi) \\ &= ((I+K)(R_{h_x,h_y} - R_{h_x,0})w, \psi) \\ &= ((I+K)(R_{h_x,h_y} - R_{h_x,0})w, (I - P_{0,h_y})\psi).\end{aligned}$$

Using estimate (5.15) and the approximation order of the space $S^{(0,h_y)}(\square)$, we conclude that there exists a positive constant c such that

$$((R_{h_x,h_y} - R_{h_x,0})w, \phi) \le c h_y^2 \|D_y w\| \|D_y\psi\| \le c h_y^2 \|D_y w\| \|\phi\|_1,$$

which leads to estimate (5.16). $\square$

Proposition 5.6. *If $u \in H^{1,2}(\square)$, then there exists a positive constant c such that*

$$\|\delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u\| \le c h_x h_y \|u\|_{H^{1,2}}, \tag{5.17}$$

and

$$\|\delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u\|_{-1} \le c h_x^2 h_y^2 \|u\|_{H^{1,2}}. \tag{5.18}$$

Proof. Let $\psi \in L^2(\square)$ satisfy equation (5.8). By Propositions 5.3 and 5.5, we have that there exists a positive constant c such that

$$\begin{aligned}\|\delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u\| &= \|\delta^{(0,1)} R_{h_x,h_y}\psi\| \\ &\le c h_y \|D_y\psi\| \le c h_x h_y \|u\|_{H^{1,2}},\end{aligned}$$

and

$$\begin{aligned}\|\delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u\|_{-1} &= \|\delta^{(0,1)} R_{h_x,h_y}\psi\|_{-1} \\ &\le ch_y^2\|D_y\psi\| \le ch_x^2 h_y^2 \|u\|_{H^{1,2}}. \qquad \square\end{aligned}$$

Proposition 5.7. *If $u \in H^{1,2}(\square)$, then there exists a positive constant c such that*

$$\|R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u\| \le ch_x h_y \min\{h_x, h_y\}\|u\|_{H^{1,2}}. \tag{5.19}$$

Proof. For any $v \in S^{\mathbf{h}}(\square)$ with $\mathbf{h} := (h_x, h_y)$, we let $\phi := (I + K)^{-1}v$. We now estimate the quantity

$$\gamma := (R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u, v).$$

Since the operator $I + K$ is self-adjoint, we have that

$$\gamma = ((I + K)R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u, \phi).$$

We write the righthand side of the equation above as

$$\begin{aligned}\gamma &= ((I + K)R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u, (I - R_{h_x,h_y})\phi) \\ &\quad + ((I + K)R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u, R_{h_x,h_y}\phi).\end{aligned}$$

By the definition of R_{h_x,h_y}, we find that

$$((I + K)R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u, (I - R_{h_x,h_y})\phi) = 0.$$

It follows from these equations that

$$\begin{aligned}\gamma &= ((I + K)R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u, R_{h_x,h_y}\phi) \\ &= ((I - P_{h_x,0})(I - P_{0,h_y})u, (I + K)R_{h_x,h_y}\phi) \\ &= ((I - P_{h_x,0})(I - P_{0,h_y})u, (I - P_{h_x,0})(I - P_{0,h_y})KR_{h_x,h_y}\phi).\end{aligned}$$

This ensures that there exists a positive constant c such that

$$|\gamma| \le ch_x h_y \min\{h_x, h_y\}\|u\|_{H^{1,2}}\|KR_{h_x,h_y}\phi\|_1.$$

Noticing that there exists a positive constant c such that

$$\|KR_{h_x,h_y}\phi\|_1 \le c\|R_{h_x,h_y}\phi\| \le c\|\phi\| \le c\|v\|,$$

we conclude that

$$|\gamma| \le ch_xh_y \min\{h_x,h_y\}\|u\|_{H^{1,2}}\|\phi\| \le ch_xh_y \min\{h_x,h_y\}\|u\|_{H^{1,2}}\|v\|,$$

which proves (5.19) by choosing $v := R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u$. $\square$

Next we estimate the error of the Boolean Galerkin approximation. For this purpose we define

$$B_{h_x,h_y}u := u_{h_x,2h_y} + u_{2h_x,h_y} - u_{2h_x,2h_y}.$$

Theorem 5.8. *If $u \in H^{1,2}(\square)$, then there exists a positive constant c such that*

$$\|B_{h_x,h_y}u - R_{h_x,h_y}u\| \le ch_xh_y\|u\|_{H^{1,2}}$$

and

$$\|B_{h_x,h_y}u - R_{h_x,h_y}u\|_{-1} \le ch_xh_y \min\{h_x,h_y\}\|u\|_{H^{1,2}}.$$

Proof. Since

$$I - P_{h_x,h_y} = (I - P_{h_x,0}) + (I - P_{0,h_y}) - (I - P_{h_x,0})(I - P_{0,h_y}),$$

and

$$R_{h_x,h_y}P_{h_x,h_y} = P_{h_x,h_y},$$

we have that

$$\begin{aligned} R_{h_x,h_y} &= R_{h_x,h_y}(I - P_{h_x,h_y}) + P_{h_x,h_y} \\ &= R_{h_x,h_y}(I - P_{h_x,0}) + R_{h_x,h_y}(I - P_{0,h_y}) \\ &\quad + R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y}) + P_{h_x,h_y}. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
B_{h_x,h_y}u - R_{h_x,h_y}u &= \delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y} u \\
&= \delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u \\
&\quad + \delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y}(I - P_{0,h_y})u \\
&\quad + \delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u \\
&\quad + \delta^{(1,0)} \circ \delta^{(0,1)} P_{h_x,h_y} u.
\end{aligned}$$

Note that

$$\begin{aligned}
&\|\delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u\| \\
&\qquad \le \|\delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})u\| + \|\delta^{(0,1)} R_{2h_x,h_y}(I - P_{2h_x,0})u\| \\
&\qquad \le 2 \max_{\tilde{h} \in \{h_x, 2h_x\}} \|\delta^{(0,1)} R_{\tilde{h},h_y}(I - P_{\tilde{h},0})u\|.
\end{aligned}$$

Likewise, we have that

$$\|\delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y}(I - P_{0,h_y})u\| \le 2 \max_{\tilde{h} \in \{h_y, 2h_y\}} \|\delta^{(0,1)} R_{h_x,\tilde{h}}(I - P_{0,\tilde{h}})u\|$$

and

$$\begin{aligned}
&\|\delta^{(1,0)} \circ \delta^{(0,1)} R_{h_x,h_y}(I - P_{h_x,0})(I - P_{0,h_y})u\| \\
&\qquad \le 2 \max_{\substack{\tilde{h}_x \in \{h_x, 2h_x\} \\ \tilde{h}_y \in \{h_y, 2h_y\}}} \|\delta^{(0,1)} R_{\tilde{h}_x,\tilde{h}_y}(I - P_{\tilde{h}_x,0})(I - P_{0,\tilde{h}_y})u\|.
\end{aligned}$$

It follows that there exists a positive constant c such that

$$\begin{aligned}
\|B_{h_x,h_y}u - R_{h_x,h_y}u\| &\le c \max_{\tilde{h} \in \{h_x, 2h_x\}} \|\delta^{(0,1)} R_{\tilde{h},h_y}(I - P_{\tilde{h},0})u\| \\
&\quad + c \max_{\tilde{h} \in \{h_y, 2h_y\}} \|\delta^{(1,0)} R_{h_x,\tilde{h}}(I - P_{0,\tilde{h}})u\| \\
&\quad + c \max_{\substack{\tilde{h}_x \in \{h_x, 2h_x\} \\ \tilde{h}_y \in \{h_y, 2h_y\}}} \|R_{\tilde{h}_x,\tilde{h}_y}(I - P_{\tilde{h}_x,0})(I - P_{0,\tilde{h}_y})u\| \\
&\quad + c\|\delta^{(1,0)} \circ \delta^{(0,1)} P_{h_x,h_y} u\|.
\end{aligned}$$

This estimate with Propositions 5.4, 5.6 and 5.7 proves the first estimate of this theorem. The second estimate is proved similarly. □

As we explained in the introduction, the combination solution u_h^c can be viewed as a multi-level Boolean-Galerkin approximation. By the last theorem, we now obtain the following result. We define

$$\tilde{u}_h^c = f - Ku_h^c.$$

Theorem 5.9. *If $u \in H^{1,2}(\square)$, then there exists a positive constant c such that*

$$\|u - \tilde{u}_h^c\| + \|u - u_h^c\|_{-1} \le ch^{3/2}\|u\|_{H^{1,2}}$$

and

$$\#(u_h^c) + \#(\tilde{u}_h^c) = \mathcal{O}(2^n n).$$

Proof. Using the identity

$$u - \tilde{u}_h^c = K(u - u_h^c),$$

we have that there exists a positive constant c such that

$$\|u - \tilde{u}_h^c\|_{-1} \le c\|u - u_h^c\|_{-1}.$$

It suffices to prove that there exists a positive constant c such that

$$\|u - u_h^c\|_{-1} \le ch^{3/2}\|u\|_{H^{1/2}}.$$

To this end, we note from a direct computation that

$$R_{h,h}u - u_h^c = \sum_{i=2}^{n} \sum_{j=n-i+2}^{n} (B_{2^{-i},2^{-j}}u - R_{2^{-i},2^{-j}}u).$$

Using Theorem 5.5, we conclude that there exists a positive constant c such that

$$\begin{aligned}\left\| \sum_{i=2}^{n} \sum_{j=n-i+2}^{n} (B_{2^{-i},2^{-j}}\, u - R_{2^{-i},2^{-j}}u) \right\|_{-1} \\ &\le c \sum_{i=2}^{n} \sum_{j=n-i+2}^{n} 2^{-(i+j)} 2^{-\max\{i,j\}} \\ &\le c2^{-n-n/2} = ch^{3/2}.\end{aligned}$$

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