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NON-AUTONOMOUS INTEGRODIFFERENTIAL EQUATIONS WITH NON-LOCAL CONDITIONS

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ABSTRACT. Recent results concerning the existence and uniqueness of mild and classical solutions for non-local Cauchy problems are extended to the following non-autonomous semi-linear integrodifferential equation

$$u'(t) = A(t)\left[u(t) + \int_0^t F(t,s)u(s)\,ds\right] + f(t,u(t)),$$
$$0 \le t \le T,$$
$$u(0) + g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = u_0,$$

in a Banach space X, with $A(\cdot)$ the generators of strongly continuous semigroups. The non-local condition can be applied in physics with better effect than the classical Cauchy problem $u(0) = u_0$, since more measurements at t_is are allowed. The variation of constants formula for solutions via a resolvent operator is first derived in order to carry out the study.

1. Introduction. In this paper, we will study the existence and uniqueness of mild and classical solutions for the non-autonomous semi-linear integrodifferential equation with non-local Cauchy problems.

To begin, let us consider the following semi-linear problem

$$u'(t) = Au(t) + f(t,u(t)), \quad 0 \le t \le T, \tag{1.1}$$
$$u(0) = u_0, \tag{1.2}$$

in a Banach space X, with A the generator of a strongly continuous semigroup $T(\cdot)$. Here in (1.2), $u(0) = u_0$ is referred to as the initial value problem, or Cauchy problem.

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Motivated by questions from physics, the existence and uniqueness of mild and classical solutions for the following non-local Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \quad 0 \le t \le T, \tag{1.3}$$

$$u(0) + g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = u_0, \tag{1.4}$$

where $0 < t_1 < t_2 < \cdots t_p = T$, is investigated recently in a series of papers, e.g., Byszewski [**1**], Byszewski and Lakshmikantham [**2**], Jackson [**5**], Lin [**6**] and references therein. As remarked in those papers, non-local condition (1.4) can be applied to physics with better effect than (1.2).

Now, look at the following classical heat equation for material with memory [**4**]

$$\begin{cases} q(t,x) = -Eu_x(t,x) - \int_0^b (t-s) u_x(s,x)\, ds, \ t \ge 0, \quad x \in [0,1], \\ u_t(t,x) = -\partial q(t,x)/\partial x + f(t,x), \\ u(0,x) = u_0(x). \end{cases} \tag{1.5}$$

The first equation gives the heat flux and the second is the balance equation. Eq. (1.5) can be written as (assuming $E = 1$)

$$\begin{aligned} u_t(t,x) &= \frac{\partial^2}{\partial x^2}\left[u(t,x) + \int_0^t b(t-s) u(s,x)\, ds\right] + f(t,x), \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.6}$$

It is clear that if non-local condition (1.4) is introduced to Eq. (1.6), then it will also have better effect than the classical condition $u(0,x) = u_0(x)$, since the same comments as above apply here. Therefore we need to extend the study to the following semi-linear integrodifferential equation with non-local Cauchy problem

$$u'(t) = A\Big[u(t) + \int_0^t F(t-s)u(s)\, ds\Big] + f(t, u(t)), \quad 0 \le t \le T, \tag{1.7}$$

$$u(0) + g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = u_0, \tag{1.8}$$

and its corresponding non-autonomous version,

$$u'(t) = A(t)\Big[u(t) + \int_0^t F(t,s)u(s)\, ds\Big] + f(t, u(t)), \quad 0 \le t \le T, \tag{1.9}$$

$$u(0) + g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = u_0, \tag{1.10}$$

in a Banach space X, with $A(\cdot)$ the generators of strongly continuous semigroups, and F a bounded operator. For example, in Eq. (1.6), $A = \partial^2/\partial x^2$ on $H^2(0,1) \cap H_0^1(0,1)$ generates a strongly continuous semigroup in $L^2(0,1)$. Also, see [**8**]where a non-autonomous viscoelasticity is formulated as Eq. (1.9).

For Eqs. (1.7)–(1.8), the existence and uniqueness of mild and classical solutions are obtained in Lin and Liu [**7**]. Thus, our purpose here is to extend the study to Eqs. (1.9)–(1.10). Observe that the technique used in the study of Eqs. (1.3)–(1.4) is to first obtain mild solutions using a fixed point argument when $f(t,u)$ is Lipschitz in u, where the mild solutions are defined to be the functions satisfying the variation of constants formula

$$\begin{aligned} u(t) &= T(t)[u0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))] \\ &\quad + \int_0^t T(t-s) f(s, u(s))\, ds, \quad 0 \le t \le T, \end{aligned} \tag{1.11}$$

with $T(\cdot)$ the semigroup generated by A. Then mild solutions are shown to be classical solutions if $f \in C^1([0,T] \times X, X)$.

In the study of Eqs. (1.7)–(1.8) in [**7**], the same technique is used. That is, mild solutions are first obtained via the variation of constants formula

$$\begin{aligned} u(t) &= R(t)[u_0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))] \\ &\quad + \int_0^t R(t-s) f(s, u(s))\, ds, \quad 0 \le t \le T, \end{aligned} \tag{1.12}$$

where the semigroup $T(\cdot)$ in (1.11) is now replaced by the resolvent operator $R(\cdot)$, the counterpart of semigroup $T(\cdot)$ for integrodifferential equations. Then mild solutions are shown to be classical solutions if $f \in C^1([0,T] \times X, X)$.

This indicates that for non-autonomous integrodifferential equation (1.9), we need a family of operators which will play the same role as $R(\cdot)$ does for Eq. (1.7), and this creates certain difficulties. However, we will show that we can rewrite Eq. (1.9) as an equation studied in Grimmer [**3**], and hence we can obtain a resolvent operator $R(t,s)$, which yields mild solutions defined by

$$\begin{aligned} u(t) &= R(t,0)[u_0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))] \\ &\quad + \int_0^t R(t,s) f(s, u(s))\, ds, \quad 0 \le t \le T. \end{aligned} \tag{1.13}$$

After this, with some trial and error, we are able to find an appropriate way to estimate mild solutions, which enables us to prove that mild solutions are classical solutions if $f \in C^1([0,T] \times X, X)$.

The organization of this paper is as follows. In Section 2, we provide some results about resolvent operator $R(t,s)$, including the representation of solutions via the variation of constants formula. Then in Section 3, these results are used to obtain the existence and uniqueness of mild and classical solutions for the non-local Cauchy problem Eqs. (1.9)–(1.10). Finally, in Section 4, we study a special case when $\|R(t,s)\|_{B(X)} \leq Me^{-\alpha(t-s)}$, $0 \leq s \leq t \leq T$, for some constant $\alpha > 0$, and when the function g in non-local condition (1.10) is given by $g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = \sum_{i=1}^{p} c_i u(t_i)$, where c_i's are given constants. We will see that, in this case, conditions in Assumption (H5) in Section 3 can be improved.

2. Resolvent operators. We first list the notations and assumptions from Grimmer [**3**]; they are needed to obtain a resolvent operator for Eq. (1.9). Let $B(E,Z)$ be the Banach space of linear bounded operators from E to Z and $B(E,Z) = B(E)$ if $E = Z$. We assume that in Eq. (1.9), $A(\cdot)$ has the common domain $D \subset X$ for $0 \leq t \leq T$, where X is a Banach space. We denote Y the Banach space formed from D with the graph norm. Furthermore, we denote $BU(X)$ the Banach space of bounded uniformly continuous functions on $[0,\infty)$ into X and S_X a subspace of $BU(X)$ but with a stronger norm than the sup norm on $BU(X)$. It is also assumed that the translation $f(s) \to f(t+s)$ defines a strongly continuous semigroup on S_X with generator D_s on domain $D(D_s)$. The following conditions are also assumed.

(H1). $A(t)$, $0 \leq t \leq T$, is a family of generators of strongly continuous semigroups. And $A(\cdot)$ is stable, that is, there exist $\omega, M \in \Re$, the reals, such that $(\omega, \infty) \subset \rho(A(t))$, $t \in [0,T]$, and, for each $n \in N$, the integers,

$$(\lambda - \omega)^n \|R(\lambda, t_1, \dots, t_n)\|_{B(X)} \leq M,$$

when $0 \leq t_1 \leq \cdots \leq t_n \leq T$ and $\lambda > \omega$. Here $\rho(A(t))$ is the resolvent set of $A(t)$ and

$$R(\lambda, t_1, \dots, t_n) = (\lambda - A(t_n))^{-1} \cdots (\lambda - A(t_1))^{-1}.$$

(H2). $F(t,s) \in B(X)$, $0 \leq s \leq t \leq T$. For $x \in X$, $F_1(t+\cdot, t)x \in S_X$, where F_1 denotes the partial derivative to the first variable and

$[F_1(t+\cdot,t)x](s) = F_1(t+s,t)x$ for $s \geq 0$. This then defines for $t \geq 0$ an operator $F_0(t) \equiv F_1(t+\cdot,t) : X \to S_X$. For $x \in X$, $F_0(\cdot)x \in C^1([0,\infty),S_X)$. $F_0(\cdot) \in C([0,\infty),B(X,S_X))$. $F_0(t) : X \to D(D_s)$ for $t \geq 0$. $D_sF_0(\cdot) \in C([0,\infty),B(X,S_X))$.

Next, we give the following definitions.

Definition 2.1. A resolvent operator of Eq. (1.9), with $f \equiv 0$, is an operator-valued function $R(t,s) \in B(X)$ for $0 \leq s \leq t \leq T$, satisfying

1. $R(t,s)$ is strongly continuous in s and t. $R(t,t) = I$, the identity operator on X, $0 \leq t \leq T$. $\|R(t,s)\| \leq Me^{\beta(t-s)}$ for some constants M and β.
2. $R(t,s)Y \subset Y$. $R(t,s)$ is strongly continuous in s and t on Y.
3. For $y \in Y$, $R(t,s)y$ is continuously differentiable in s and t, and

$$\frac{\partial}{\partial t} R(t,s)y = A(t)\Big[R(t,s)y + \int_s^t F(t,r)R(r,s)y\,dr\Big], \quad 0 \leq s \leq t \leq T. \tag{2.1}$$

Definition 2.2. $u(\cdot) \in C([0,T],X)$ is a mild solution of Eqs. (1.9)–(1.10) if it satisfies

$$\begin{aligned} u(t) = R(t,0)\Big[u_0 - g(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\Big] \\ + \int_0^t R(t,s)f(s,u(s))\,ds, \quad 0 \leq t \leq T. \end{aligned} \tag{2.2}$$

Definition 2.3. A classical solution of Eqs. (1.9)–(1.10) is a function $u(\cdot) \in C([0,T],Y) \cap C^1([0,T],X)$, which satisfies Eqs. (1.9)–(1.10) on $[0,T]$.

The existence of a resolvent operator for Eq. (1.9), with $f \equiv 0$, is given by the following result.

Theorem 2.4. *Let Assumptions* (H1) *and* (H2) *be satisfied. Then Eq.* (1.9)*, with* $f \equiv 0$*, has a resolvent operator* R*.*

Proof. First, let

$$w(t) = u(t) + \int_0^t F(t,s)u(s)\,ds, \quad t \geq 0,$$

so that we can rewrite Eq. (1.9), with $f \equiv 0$, into a form studied in Grimmer [**3**] as

(2.3)
$$\begin{bmatrix} u(t) \\ w(t) \end{bmatrix}' = \begin{bmatrix} 0 & A(t) \\ F(t,t) & A(t) \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 0 \\ F_1(t,s) & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \end{bmatrix} ds$$

on $\tilde{X} = X \times X$. As shown in [**8**], the leading operator in Eq. (2.3) is stable, see (H1). Next, it can be checked that other conditions in [**3**, Theorem 3.7] are satisfied. Thus by [**3**, Theorem 3.7], Eq. (2.3) has a resolvent operator $\tilde{R}$ such that, for $y \in Y$,

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{R}(t,s)\begin{bmatrix} y \\ y \end{bmatrix} &= \begin{bmatrix} 0 & A(t) \\ F(t,t) & A(t) \end{bmatrix} \tilde{R}(t,s)\begin{bmatrix} y \\ y \end{bmatrix} \\ &\quad + \int_s^t \begin{bmatrix} 0 & 0 \\ F_1(t,r) & 0 \end{bmatrix} \tilde{R}(r,s)\begin{bmatrix} y \\ y \end{bmatrix} dr. \end{aligned} \tag{2.4}$$

Now we write $\tilde{R}$ as (R_{ij}), $i,j = 1,2$, and let $R \equiv R_{11} + R_{12}$ and $R_0 \equiv R_{21} + R_{22}$. Then

(2.5)
$$\frac{\partial}{\partial t} R(t,s)y = A(t)R_0(t,s)y,$$

$$\frac{\partial}{\partial t} R_0(t,s)y = A(t)R_0(t,s)y + F(t,t)R(t,s)y + \int_s^t F_1(t,r)R(r,s)y\,dr$$

$$= \frac{\partial}{\partial t}\Big[R(t,s)y + \int_s^t F(t,r)R(r,s)y\,dr\Big]. \tag{2.6}$$

Hence, as $R(s,s) = I$ and $R_0(s,s) = I$, we have

$$R_0(t,s)y = R(t,s)y + \int_s^t F(t,r)R(r,s)y\,dr. \tag{2.7}$$

Therefore, from (2.5),

$$\frac{\partial}{\partial t} R(t,s)y = A(t)\Big[R(t,s)y + \int_s^t F(t,r)R(r,s)y\,dr\Big]. \tag{2.8}$$

That is, $R(t,s)$ is a resolvent operator for Eq. (1.9), with $f \equiv 0$. □

We also need the following result concerning the classical solutions and the variation of constants formula for linear Cauchy problem when $f(t,u) \equiv f(t)$ and $g \equiv 0$ in Eqs. (1.9)–(1.10).

Theorem 2.5. *Let Assumptions* (H1) *and* (H2) *be satisfied, and assume that* $f(t,u) \equiv f(t)$, $g \equiv 0$, $u_0 \in Y$ *and* $f(\cdot) \in C^1([0,T],X)$. *Then Eqs.* (1.9)–(1.10) *have a unique classical solution given by*

$$u(t) = R(t,0)u_0 + \int_0^t R(t,s)f(s)\,ds, \quad 0 \le t \le T, \tag{2.9}$$

where R is the resolvent operator obtained in Theorem 2.4.

Proof. From the results of Corollary 3.8 in [**3**], we see that

$$\begin{cases} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix}' = \begin{bmatrix} 0 & A(t) \\ F(t,t) & A(t) \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix} \\ \qquad\qquad + \displaystyle\int_0^t \begin{bmatrix} 0 & 0 \\ F_1(t,s) & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \end{bmatrix} ds + \begin{bmatrix} f(t) \\ f(t) \end{bmatrix}, \\ (u(0),w(0)) = (u_0,u_0), \end{cases} \tag{2.10}$$

has a unique classical solution given by

$$\begin{bmatrix} u(t) \\ w(t) \end{bmatrix} = \tilde{R}(t,0)\begin{bmatrix} u_0 \\ u_0 \end{bmatrix} + \int_0^t \tilde{R}(t,s)\begin{bmatrix} f(s) \\ f(s) \end{bmatrix} ds, \quad 0 \le t \le T, \tag{2.11}$$

where $\tilde{R}(t,s)$ is the resolvent operator obtained in Theorem 2.4.

Next we verify that the first component of a solution of Eq. (2.10) is a solution of Eqs. (1.9)–(1.10) when $u(0) = u_0$ and $f(t,u) \equiv f(t)$. Because, if $\begin{bmatrix} u(t) \\ w(t) \end{bmatrix}$ is a solution of Eqs. (2.10), then

$$u'(t) = A(t)w(t) + f(t), \tag{2.12}$$

and

$$
\begin{aligned}
w'(t) &= F(t,t)u(t) + A(t)w(t) + \int_0^t F_1(t,s)u(s)\,ds + f(t) \\
&= \frac{d}{dt}\int_0^t F(t,s)u(s)\,ds + [u'(t) - f(t)] + f(t) \\
&= \frac{d}{dt}\int_0^t F(t,s)u(s)\,ds + u'(t), \quad w(0) = u(0) = u_0;
\end{aligned} \tag{2.13}
$$

therefore,

$$
w(t) = \int_0^t F(t,s)u(s)\,ds + u(t), \tag{2.14}
$$

hence, (2.12) becomes

$$
u'(t) = A(t)\Big[\int_0^t F(t,s)u(s)\,ds + u(t)\Big] + f(t). \tag{2.15}
$$

Accordingly, $u(t)$ is a solution of Eqs. (1.9)–(1.10) when $u(0) = u_0$ and $f(t,u) \equiv f(t)$.

Therefore, the first component of (2.11), which is given by (2.9), is a solution of Eqs. (1.9)–(1.10) when $u(0) = u_0$ and $f(t,u) \equiv f(t)$. The uniqueness is obtained since it is clear that a solution of Eqs. (1.9)–(1.10) in this case leads to a solution of Eq. (2.10). □

3. Non-local Cauchy problems. In this section we will use the techniques developed in Pazy [**9**], Byszewski [**1**] and Lin and Liu [**7**] to study Eqs. (1.9)–(1.10). First we will use a fixed point argument to study the existence and uniqueness of mild solutions, under the following assumptions:

(H3). $f : [0,T] \times X \to X$ is continuous in $t \in [0,T]$, and there exists a constant $L > 0$ such that

$$
\|f(t,u) - f(t,v)\|_X \le L\|u - v\|_X, \quad t \in [0,T], \quad u,v \in X. \tag{3.1}
$$

(H4). $g : [0,T]^p \times X^p \to X$, and there exists a constant $K > 0$ such that

$$
\|g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) - g(t_1,\dots,t_p,v(t_1),\dots,v(t_p))\|_X \le K\|u - v\|_{C([0,T],X)}. \tag{3.2}
$$

(H5). Denote

$$M \equiv \max_{0 \le s \le t \le T} \|R(t,s)\|_{B(X)}, \tag{3.3}$$

then

$$M(K + TL) < 1. \tag{3.4}$$

Under these assumptions, we can obtain the following result concerning the existence and uniqueness of mild solutions. The proof is similar to the one in [**7**], with $R(t-s)$ replaced by $R(t,s)$, hence it is omitted.

Theorem 3.1. *Let Assumptions* (H1)–(H5) *be satisfied. Then, for every* $u_0 \in X$*, Eqs.* (1.9)–(1.10) *have a unique mild solution.*

Next we prove that mild solutions are classical solutions if $f \in C^1([0,T] \times X, X)$.

Theorem 3.2. *Let Assumptions* (H1)–(H5) *be satisfied, and let* $u(\cdot)$ *be the unique mild solution of Eqs.* (1.9)–(1.10) *guaranteed by Theorem* 3.1*. Assume further that* $u_0 \in Y$*,* $g : [0,T]^p \times X^p \to Y$ *and that* $f \in C^1([0,T] \times X, X)$*. Then* $u(\cdot)$ *gives rise to a unique classical solution of Eqs.* (1.9)–(1.10).

Proof. We will first show that $u(\cdot) \in C^1([0,T], X)$. To this end, we set

$$B(s) \equiv \frac{\partial}{\partial u} f(s,u), \quad s \in [0,T], \tag{3.5}$$

and set, for $t \in [0,T]$,

(3.6)

$$\begin{aligned} k(t) \equiv {} & R(t,0)f(0,u(0)) + A(t)\Big[R(t,0)u(0) + \int_0^t F(t,r)R(r,0)u(0)\,dr\Big] \\ & + \int_0^t R(t,s)\frac{\partial}{\partial s} f(s,u(s))\,ds + \int_0^t \Big[\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big]R(t,s)f(s,u(s))\,ds, \end{aligned}$$

where $u(0) = u_0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))$.

Note that, from Definition 2.1 and our assumptions, $k(\cdot) \in C([0,T], X)$. Thus, the fixed point argument used in Pazy [**9**, pp. 184–187] can be applied here to show that

$$w(t) = k(t) + \int_0^t R(t,s)B(s)w(s)\,ds, \quad t \in [0,T], \tag{3.7}$$

has a unique solution $w(\cdot) \in C([0,T], X)$. Moreover, from our assumptions we have

$$f(s, u(s+h)) - f(s, u(s)) = B(s)[u(s+h) - u(s)] + \omega_1(s,h), \tag{3.8}$$

and

$$f(s+h, u(s+h)) - f(s, u(s+h)) = \frac{\partial}{\partial s} f(s, u(s+h))h + \omega_2(s,h), \tag{3.9}$$

where

$$h^{-1}\|\omega_i(s,h)\| \longrightarrow 0, \quad h \to 0, \tag{3.10}$$

uniformly on $s \in [0,T]$ for $i = 1, 2$. Now define

$$w_h(t) \equiv \frac{u(t+h) - u(t)}{h} - w(t), \quad t \in [0,T]. \tag{3.11}$$

Then, from (3.6), (3.7), (3.11) and the fact that $u(\cdot)$ is a mild solution, we obtain

(3.12)

$$\begin{aligned} w_h(t) = h^{-1}\Big\{&R(t+h,0)u(0) + \int_0^{t+h} R(t+h,s)f(s,u(s))\,ds \\ &- R(t,0)u(0) - \int_0^t R(t,s)f(s,u(s))\,ds\Big\} \\ &- R(t,0)f(0,u(0)) - A(t)\Big[R(t,0)u(0) + \int_0^t F(t,r)R(r,0)u(0)\,dr\Big] \\ &- \int_0^t R(t,s)\frac{\partial}{\partial s}f(s,u(s))\,ds - \int_0^t \Big[\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big]R(t,s)f(s,u(s))\,ds \\ &- \int_0^t R(t,s)B(s)w(s)\,ds. \end{aligned}$$

Now, using Definition 2.1 for resolvent operator R, we have, as $h \to 0$,

$$h^{-1}\{R(t+h,0)u(0)-R(t,0)u(0)\} - A(t)\Big[R(t,0)u(0)+\int_0^t F(t,r)R(r,0)u(0)\,dr\Big] \to 0. \tag{3.13}$$

Furthermore,

(3.14)

$$\begin{aligned} &h^{-1}\Big\{\int_0^{t+h} R(t+h,s)f(s,u(s))\,ds - \int_0^t R(t,s)f(s,u(s))\,ds\Big\} \\ &\quad = h^{-1}\int_0^h R(t+h,s)f(s,u(s))\,ds \\ &\qquad + h^{-1}\Big\{\int_h^{t+h} R(t+h,s)f(s,u(s))\,ds-\int_0^t R(t,s)f(s,u(s))\,ds\Big\}, \end{aligned}$$

and

(3.15)

$$\begin{aligned} &h^{-1}\Big(\int_h^{t+h} R(t+h,s)f(s,u(s))\,ds-\int_0^t R(t,s)f(s,u(s))\,ds\Big) \\ &= h^{-1}\Big(\int_0^t R(t+h,s+h)f(s+h,u(s+h))\,ds-\int_0^t R(t,s)f(s,u(s))\,ds\Big) \\ &= h^{-1}\Big\{\int_0^t R(t+h,s+h)f(s+h,u(s+h))\,ds \\ &\qquad - \int_0^t R(t,s+h)f(s+h,u(s+h))\,ds \\ &\qquad + \int_0^t R(t,s+h)f(s+h,u(s+h))\,ds-\int_0^t R(t,s)f(s+h,u(s+h))\,ds\Big\} \\ &\quad + h^{-1}\Big[\int_0^t R(t,s)f(s+h,u(s+h))\,ds - \int_0^t R(t,s)f(s,u(s+h))\,ds\Big] \\ &\quad + h^{-1}\Big(\int_0^t R(t,s)f(s,u(s+h))\,ds-\int_0^t R(t,s)f(s,u(s))\,ds\Big). \end{aligned}$$

In (3.15), for $h \approx 0$, one has

(3.16)

$$h^{-1}\Big\{ \int_0^t R(t{+}h, s{+}h) f(s{+}h, u(s{+}h))\, ds - \int_0^t R(t, s{+}h) f(s{+}h, u(s{+}h))\, ds + \int_0^t R(t, s{+}h) f(s{+}h, u(s{+}h))\, ds - \int_0^t R(t, s) f(s{+}h, u(s{+}h))\, ds \Big\} \approx \int_0^t \Big[\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big] R(t,s) f(s, u(s))\, ds,$$

and

(3.17)

$$h^{-1}\Big[\int_0^t R(t,s) f(s+h, u(s+h))\, ds - \int_0^t R(t,s) f(s, u(s+h))\, ds \Big] \approx \int_0^t R(t,s) \frac{\partial}{\partial s} f(s, u(s))\, ds,$$

and

$$h^{-1}\Big(\int_0^t R(t,s) f(s, u(s+h))\, ds - \int_0^t R(t,s) f(s, u(s))\, ds \Big) \approx \int_0^t R(t,s) B(s) [w_h(s) + w(s)]\, ds. \tag{3.18}$$

Using the definition of the resolvent operator and our assumptions, it is clear from (3.13)–(3.18) that (3.12) becomes

$$\|w_h(t)\|_X \le \varepsilon(h) + M_* \int_0^t \|w_h(s)\|_X\, ds, \tag{3.19}$$

where

$$M_* = \max_{0 \le s \le t \le T} \|R(t,s)B(s)\|_{B(X)}, \tag{3.20}$$

and

$$\varepsilon(h) \to 0, \quad h \to 0. \tag{3.21}$$

From (3.19), it follows by Gronwall's inequality that

$$\|w_h(t)\|_X \le \varepsilon(h)e^{TM_*}, \quad t \in [0,T], \tag{3.22}$$

and therefore

$$\|w_h(t)\|_X \to 0, \quad h \to 0, \quad t \in [0,T]. \tag{3.23}$$

This implies that $u(t)$ is differentiable on $[0,T]$ and that its derivative is $w(t)$. Since $w(\cdot) \in C([0,T],X)$, we obtain $u(\cdot) \in C^1([0,T],X)$.

Finally we show that $u(\cdot)$ is a classical solution of Eqs. (1.9)–(1.10). Note that, since $u(\cdot) \in C^1([0,T],X)$ and $f \in C^1([0,T] \times X, X)$, we see that, for the known function $u(t)$, $f(t,u(t))$ is a function in t, and $t \to f(t,u(t))$ is in $C^1([0,T],X)$. Therefore, Theorem 2.5 implies that the linear Cauchy problem

$$v'(t) = A(t)\Big[v(t) + \int_0^t F(t,s)v(s)\,ds\Big] + f(t,u(t)), \quad 0 \le t \le T, \tag{3.24}$$

$$v(0) = u_0 - g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)), \tag{3.25}$$

has a unique classical solution $v(\cdot)$, given by

$$\begin{aligned} v(t) &= R(t,0)[u_0 - g(t_1,\dots,t_p,u(t_1),\dots,u(t_p))] \\ &\quad + \int_0^t R(t,s)f(s,u(s))\,ds, \quad 0 \le t \le T. \end{aligned} \tag{3.26}$$

But the right-hand side of (3.26) is exactly $u(t)$ since $u(\cdot)$ is a mild solution. So we have $v(t) = u(t)$, $t \in [0,T]$ and hence $u(\cdot)$ is a classical solution of Eqs. (1.9)–(1.10). This proves the result. □

4. When $\|R(t,s)\|_{B(X)} \le Me^{-\alpha(t-s)}$, $0 \le s \le t \le T$, $\alpha > 0$. In this section we study a special case when $\|R(t,s)\|_{B(X)} \le Me^{-\alpha(t-s)}$, $0 \le s \le t \le T$, for some constant $\alpha > 0$, and when the function g in non-local condition (1.10) is given by $g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) = \sum_{i=1}^p c_i u(t_i)$, where c_is are given constants. We will see that, in this case, conditions in Assumption (H5) in Section 3 can be improved. Because now, using a fixed point argument, we can first prove the existence and uniqueness of a mild solution $u(\cdot,v)$, for any $v \in X$, of the Cauchy problem

$$\begin{cases} u'(t) = A[u(t) + \int_0^t F(t-s)u(s)\,ds] + f(t,u(t)) \quad 0 \le t \le T, \\ u(0) = v, \end{cases} \tag{4.1}$$

that is, a solution of

$$u(t) = R(t,0)v + \int_0^t R(t,s)f(s,u(s))\,ds, \quad 0 \le t \le T. \tag{4.2}$$

And then we are able to define, for any $u_0 \in X$, an operator along the trajectory of the mild solution $u(\cdot) = u(\cdot, v)$ of Eq. (4.1),

$$Q_1 v = u_0 - \sum_{i=1}^{p} c_i u(t_i), \tag{4.3}$$

and show that the operator is a contraction. Thus, by plugging in the fixed point $v = Q_1 v = u_0 - \sum_{i=1}^{p} c_i u(t_i)$ into (4.2) we obtain a unique mild solution of Eqs. (1.9)–(1.10). Finally, similar to Theorem 3.2, we can show that mild solutions are classical solutions if $f \in C^1([0,T]\times X, X)$.

We now list the following assumptions in order to carry out the ideas we just mentioned.

(H6). For some constant $\alpha > 0$, the resolvent operator of Eq. (1.9), with $f \equiv 0$, satisfies

$$\|R(t,s)\|_{B(X)} \le Me^{-\alpha(t-s)}, \quad 0 \le s \le t \le T. \tag{4.4}$$

(H7). Function g in non-local condition (1.10) is given by

$$g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = \sum_{i=1}^{p} c_i u(t_i), \tag{4.5}$$

where c_i's are given constants. And

$$\lambda \equiv \alpha - ML > 0, \quad M \sum_{i=1}^{p} |c_i| e^{-\lambda t_i} < 1 \quad (L \text{ is from (3.1)}). \tag{4.6}$$

Remark 4.1. Note that condition (4.6) is better than (3.4) in some situations.

Now we state the following results concerning mild and classical solutions of Eqs. (1.9)–(1.10). Since the proofs are similar to those in [**7**], they are omitted.

Theorem 4.3. *Let Assumptions* (H1)–(H3), (H6) *and* (H7) *be satisfied. Then for every* $u_0 \in X$, *Eqs.* (1.9)–(1.10) *has a unique mild solution.*

Theorem 4.3. *Let Assumptions* (H1)–(H3), (H6) *and* (H7) *be satisfied, and let* $u(\cdot)$ *be the unique mild solution of Eqs.* (1.9)–(1.10) *guaranteed by Theorem* 4.2. *Assume further that* $u_0 \in Y$, $\sum_{i=1}^{p} c_i u(t_i) \in Y$ *and that* $f \in C^1([0,T] \times X, X)$. *Then* $u(\cdot)$ *gives rise to a unique classical solution of Eqs.* (1.9)–(1.10).

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