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CONDITION NUMBERS IN NUMERICAL METHODS FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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Dedicated to Professor Bernd Silbermann for his 60th birthday

ABSTRACT. The authors study the conditioning of linear systems arising from the numerical solution of Fredholm integral equations of the second kind. Convergent and stable numerical procedures to compute polynomial approximate solutions are proposed.

1. Introduction. The paper deals with the condition numbers of linear systems arising from the numerical solution of Fredholm integral equations of the second kind. The problem is well known. Let

$$(I-A)f=g \tag{1.1}$$

with A a linear and compact operator on a weighted space L^p_u. If we look for a polynomial sequence $\{f_n\}$ converging in L^p_u to the solution of (1.1), then we are led to consider a finite dimensional equation of the type

$$(I-A_n)\,f_n=g_n, \tag{1.2}$$

where $A_n : L^p_u \longrightarrow \mathbf{P}_{n-1}$ is some approximation of A, and f_n, $g_n \in \mathbf{P}_{n-1}$. This procedure includes projection methods and their discretized versions. If suitable consistency conditions are fulfilled (for instance, (2.3), (2.4), then $I-A_n$ is invertible if $(I-A)^{-1}$ exists, the condition number of $I-A_n$ is a good approximation of the condition number of $I-A$ and f_n converges to f in L^p_u.

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The numerical problem consists in the accurate computation of the solution of (1.2). If we represent f_n and g_n by a pair of arbitrary bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbf{P}_{n-1}$ we obtain a linear system $M_n\mathbf{a_n} = \mathbf{b_n}$ equivalent to (1.2). Yet the numerical solution of that system can give an unsatisfactory solution if the condition number of the matrix M_n is very large.

In this paper we prove that, for special bases $\mathcal{B}$ and $\mathcal{B}'$, the condition number of M_n is independent of the dimension of the system. Then, if the weight u of the space L^p_u is a generalized Jacobi weight, we characterize the previous bases. These results, established in Section 2, are used in the Section 3 to study some integral equations with special kernels. In Theorem 3.1 the mapping properties of a particular Fredholm operator are established. In Section 3 some numerical tests are given and Section 4 contains the proofs of some theorems.

2. Main results. For $X \subset [-1,1]$ and $1 \le p < \infty$, let $L^p(X)$ be the space of all measurable functions f such that

$$\|f\|^p_{L^p(X)} = \int_X |f(x)|^p \, dx < \infty.$$

If $X = [-1,1]$ then we use the notation $\|f\|_{L^p([-1,1])} \equiv \|f\|_p$, $L^p([-1,1]) \equiv L^p$. If u is a weight function on $[-1,1]$, then L^p_u is the set of all functions f for which $fu \in L^p$. The set L^p_u with the norm $\|f\|_{L^p_u} = \|fu\|_p$ is a Banach space. In the sequel we shall assume u is a generalized Jacobi weight, $u \in GJ$, i.e., $u(x) = v^{\alpha,\beta}(x) \prod_{k=1}^r |t_k - x|^{\gamma_k}$ where $v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $-1 < t_1 < \ldots < t_r < 1$, $\alpha, \beta, \gamma_k > -1$. The symbol C will denote a positive constant which may take different values in different formulae. Sometimes we shall write $C \neq C(a, b, \ldots)$ if C is a constant independent of the parameters $a, b, \ldots$, and $A \sim B$ if there exists a positive constant M, such that $\left(\frac{A}{B}\right)^{\pm 1} \le M$.

Let us consider the following operator equation

$$(I - A)f = g \tag{2.1}$$

where A denotes a compact operator on the space L^p_u and I the identity operator. By the Fredholm alternative theorem the equation (2.1) has a unique solution $f \in L^p_u$ for any $g \in L^p_u$ if and only if the homogeneous equation $(I-A)h=0$ has only the trivial solution. We shall assume the existence and uniqueness of the solution of (2.1).

In order to construct a polynomial approximation of the solution f of (2.1), we consider the finite dimensional problem

$$(I - A_n)\, f_n = g_n \tag{2.2}$$

where f_n and g_n belong to the subspace $\mathbf{P}_{n-1}$ of all polynomials of degree at most $n-1$ and $A_n : L^p_u \longrightarrow \mathbf{P}_{n-1}$ is a linear operator.

Let

$$\|T\| = \|T\|_{L^p_u \to L^p_u} = \sup_{\|f\|_{L^p_u=1}} \|(Tf)u\|_p$$

denote the norm of a linear operator $T : L^p_u \to L^p_u$. If we assume that

$$\|A - A_n\| \longrightarrow 0, \quad n \to \infty \tag{2.3}$$

and

$$\|g - g_n\|_{L^p_u} \longrightarrow 0, \quad n \to \infty, \tag{2.4}$$

then, by a standard argument, we derive the following result.

Theorem 2.1. *Assume that* $\ker(I-A) = \{0\}$ *in* L^p_u *and conditions* (2.3) *and* (2.4) *are fulfilled. Then, for all sufficiently large* n*, the equation* (2.2) *has a unique solution* $f_n \in \mathbf{P}_{n-1}$*. Moreover*

$$\|f - f_n\|_{L^p_u} \le C \left(\|g - g_n\|_{L^p_u} + \|A - A_n\| \cdot \|g\|_{L^p_u} \right) \tag{2.5}$$

with C *independent of* f *and* n*, and*

$$|\mathrm{cond}\,(I-A) - \mathrm{cond}\,(I-A_n)| = O(\|A - A_n\|) \tag{2.6}$$

where $\mathrm{cond}(T) = \|T\| \cdot \left\|T^{-1}\right\|$ *denotes the condition number of an invertible operator* T*.*

For completeness we shall give the proof of the previous theorem in Section 4. For the time being we observe that, if $g \in L^p_u$ and $A : L^p_u \longrightarrow L^p_u$, $1 < p < \infty$, is a compact operator, then there exist an operator $A_n : L^p_u \longrightarrow \mathbf{P}_{m-1}$ and a polynomial g_n that satisfy (2.3) and (2.4). In fact, for $u \in GJ$, we can find a sequence $\{S_m(w)\}$ of Fourier projectors uniformly bounded in L^p_u, see the proof of Theorem 2.3. Therefore, with $g_n = S_n(w,g)$, (2.4) is satisfied. Setting $A_n f = S_n(w, Af)$, we also have:

$$\|[Af - S_n(w, Af)]u\|_p \le C \inf_{P \in \mathbf{P}_{n-1}} \|[Af - P]u\|_p =: E_{n-1}(Af)_{u,p}.$$

Then (2.3) follows, since the compactness of A is equivalent to

$$\lim_{n\to\infty} \sup_{f \in L^p_u} \frac{E_{n-1}(Af)_{u,p}}{\|fu\|_p} = 0$$

(see [**25**]).

At this point, from a theoretical point of view, we can obtain the solution f_n of equation (2.2), by solving a system of linear equations equivalent to (2.2). The procedure is well known. Consider the restriction $(I - A_n)\,|_{\mathbf{P}_{n-1}}$ of the operator $I - A_n$ to the subspace $\mathbf{P}_{n-1}$ of L^p_u. If $\mathcal{B} = \{\varphi_i : i = 1, \dots, n\}$ and $\mathcal{B}' = \{\varphi'_i : i = 1, \dots, n\}$ are two arbitrary bases of $\mathbf{P}_{n-1}$, then we can represent the functions f_n and g_n as

$$f_n = \sum_{i=1}^{n} a_{ni}\varphi_i, \qquad g_n = \sum_{i=1}^{n} b_{ni}\varphi'_i. \tag{2.7}$$

Thus we can write the matrix $M_n = (m_{ij})_{i,j=1,\dots,n}$ of the isomorphism $(I - A_n)\ |_{\mathbf{P}_{n-1}}$ with respect to the pair of bases $(\mathcal{B}, \mathcal{B}')$ and the following system

$$M_n \mathbf{a_n} = \mathbf{b_n} \tag{2.8}$$

where

$$\mathbf{a_n} = (a_{n1}, a_{n2}, \dots, a_{nn})^T \text{ and } \mathbf{b_n} = (b_{n1}, b_{n2}, \dots, b_{nn})^T.$$

It is well known that the polynomial $f_n \in \mathbf{P}_{n-1}$ given in (2.7) is the solution of the equation (2.2) if and only if the vector $\mathbf{a_n}$ is the solution of (2.8).

Now define by

$$\|\mathbf{v_n}\|_{l_p} = \left(\sum_{k=1}^{n} |v_{nk}|^p \right)^{1/p}, \quad 1 < p < \infty,$$

the l_p-norm of a vector $\mathbf{v_n} = (v_{n1}, \dots, v_{nn}) \in \mathring{R}^n$ and by

$$\|L_n\| = \sup_{\|\mathbf{v_n}\|_{l_p}=1} \|L\mathbf{v_n}\|_{l^p}$$

the related l_p-norm of a matrix L_n. Let δM_n and $\delta \mathbf{b_n}$ be the perturbations of the matrix M_n and $\mathbf{b_n}$ in the system (2.8), generated by the finite accuracy of the computer and denote by $\mathbf{a_n} + \delta \mathbf{a_n}$ the solution of the system

$$(2.9) \qquad (M_n + \delta M_n)(\mathbf{a_n} + \delta \mathbf{a_n}) = \mathbf{b_n} + \delta \mathbf{b_n}.$$

Then, for $\|\delta M_n\| \le \|M_n\|/2$, we have

$$(2.10) \qquad \frac{\|\delta \mathbf{a_n}\|_{l_p}}{\|\mathbf{a_n}\|_{l_p}} \le 2 \,\mathrm{cond}\, (M_n) \left(\frac{\|\delta M_n\|}{\|M_n\|} + \frac{\|\delta \mathbf{b_n}\|_{l_p}}{\|\mathbf{b_n}\|_{l_p}} \right).$$

Since the equivalence between (2.2) and (2.8) does not imply

$$(2.11) \qquad \sup_n \mathrm{cond}\,(M_n) < \infty,$$

$\mathrm{cond}\,(M_n)$ can be "very large" and the computation of f_n is unsatisfactory, see Example 1 in Section 3. Consequently it is crucial to choose the bases $\mathcal{B}$ and $\mathcal{B}'$ such that (2.11) is satisfied. To this end we give the following definition.

We shall say that $\mathcal{B} = \{\varphi_1, \dots, \varphi_n\}$ is a *Marcinkiewicz basis*, *M-basis*, in L^p_u if, for any polynomial $q = \sum_{i=1}^{n} v_{ni}\varphi_i$, we have:

$$(2.12) \qquad \frac{1}{C} \|\mathbf{v_n}\|_{l_p} \le \|q\|_{L^p_u} \le C \|\mathbf{v_n}\|_{l_p}, \quad 1 < p < \infty,$$

with $\mathbf{v_n} = (v_{n1}, \dots, v_{nn})$ and $C \ne C(n, q)$. Marcinkiewicz first proved in [**13**] (see also [**27**, p. 28]) inequalities of type (2.12), with $p \ne 2$, for trigonometric polynomials. In the algebraic case the reader can consult

[**14**], [**15**] and the references therein. In $L^2_{\sqrt{w}}$, $w \in GJ$, if $\{\varphi_n\}_n$ is an orthonormal set of polynomials, then $\mathcal{B} = \{\varphi_1, \dots, \varphi_n\}$ is an M-basis since $\|\mathbf{v_n}\|_{l_2} = \|q\|_{L^2_{\sqrt{w}}}$, but in L^p_u for $p \neq 2$ we cannot expect a similar situation.

By using M-bases, we can give a simple estimate of the perturbation of the polynomial f_n arising from the numerical solution of the system (2.8). In fact if $\mathcal{B}$ is an M-basis, then, denoting by $\mathbf{a_n}^* = \mathbf{a_n} + \delta\mathbf{a_n}$ the numerical solution of (2.9) and by $f_n^* = \sum_{j=1,n} a^*_{nj}\varphi_j$ the corresponding polynomial, the estimate (2.5) has to be replaced by the following one

$$\|f - f_n^*\|_{L^p_u} \leq \|f - f_n\|_{L^p_u} + \|f_n - f_n^*\|_{L^p_u}\,, \tag{2.13}$$

where the second addendum, recalling that $\mathcal{B}$ is an M-basis, can be estimated as follows

$$\begin{aligned}\frac{\|f_n - f_n^*\|_{L^p_u}}{\|f_n\|_{L^p_u}} &= \frac{\left\|\left(\sum_{j=1}^n \delta a_{nj}\varphi_j\right)u\right\|_p}{\|f_n\|_{L^u_p}} \sim \frac{\|\delta\mathbf{a_n}\|_{l_p}}{\|\mathbf{a_n}\|_{l_p}} \\ &\leq C\text{cond }(M_n)\left(\frac{\|\delta M_n\|}{\|M_n\|} + \frac{\|\delta\mathbf{b_n}\|_{l_p}}{\|\mathbf{b_n}\|_{l_p}}\right).\end{aligned}$$

Here C and the constants in "$\sim$" are independent of n and the functions.

Then estimate (2.13) is comparable with (2.5) if $\sup_n \text{cond}\,(M_n) < \infty$ and the quantity in the brackets is very small, e.g., as the machine precision. Moreover, the following proposition holds.

Proposition 2.2. *If the polynomials f_n and g_n in* (2.2) *are represented by means of M-bases, then the matrix of the system* (2.8) *satisfies*

$$\frac{1}{C}\,\text{cond}\left((I - A_n)\,|_{\mathbf{P}_{n-1}}\right) \leq \text{cond}(M_n) \leq C\,\text{cond}\left((I - A_n)\,|_{\mathbf{P}_{n-1}}\right), \tag{2.14}$$

with a positive constant C independent of n. In particular, if B and B' are orthonormal bases in L^2_u, then we have

$$\text{cond}\,(M_n) = \text{cond}\,\left((I - A_n)\,|_{\mathbf{P}_{n-1}}\right). \tag{2.15}$$

Proof. Let $\mathbf{a_n} = (a_{n1}, \ldots, a_{nn})^T \in \overset{\circ}{R}^n$, $f_n = \sum_{i=1}^n a_{ni}\varphi_i$. Let us recall that if $\mathbf{b_n} = M_n \mathbf{a_n}$, $\mathbf{b_n} = (b_{n1}, \ldots, b_{nn})^T$ and $g_n = (I - A_n) f_n$ then the second formula in (2.7) holds. Thus, by (2.12), we can write

$$\|f_n\|_{L_u^p} \sim \|\mathbf{a_n}\|_{l_p} \tag{2.16}$$

and, also,

$$\|M_n \mathbf{a_n}\|_{l^p} \sim \|(I - A_n) f_n\|_{L_u^p}. \tag{2.17}$$

Here and in the sequel the constants in "∼" are independent of n and the functions. According to the definition of the operator norm and taking into account conditions (2.16) and (2.17), one has

$$\begin{aligned}\|M_n\| &= \sup_{\substack{\mathbf{a_n} \in \overset{\circ}{R}^n \\ \|\mathbf{a_n}\|_{l_p} \neq 0}} \frac{\|M_n \mathbf{a_n}\|_{l_p}}{\|\mathbf{a_n}\|_{l_p}} \\ &\sim \sup_{\substack{f_n \in \mathbf{P}_{n-1} \\ \|f_n\|_{L_u^p} \neq 0}} \frac{\|(I - A_n) f_n\|_{L_u^p}}{\|f_n\|_{L_u^p}} = \left\|(I - A_n)\,|_{\mathbf{P}_{n-1}}\right\|.\end{aligned}$$

In the same way, for the inverse matrix, we can write

$$\begin{aligned}\left\|M_n^{-1}\right\| &= \sup_{\substack{\mathbf{b_n} \in \overset{\circ}{R}^n \\ \|\mathbf{b_n}\|_{l_p} \neq 0}} \frac{\left\|M_n^{-1} \mathbf{b_n}\right\|_{l_p}}{\|\mathbf{b_n}\|_{l_p}} \\ &\sim \sup_{\substack{g_n \in \mathbf{P}_{n-1} \\ \|g_n\|_{L_u^p} \neq 0}} \frac{\left\|(I - A_n)^{-1} g_n\right\|_{L_u^p}}{\|g_n\|_{L_u^p}} = \left\|\left[(I - A_n)\,|_{\mathbf{P}_{n-1}}\right]^{-1}\right\|.\end{aligned}$$

Then, by the definition of the condition number, (2.14) follows. Finally (2.15) follows by replacing in the previous relations the symbol "∼" by "=". □

Now, we are able to state the following result.

Theorem 2.3. *Assume that the system* (2.8) *is obtained from* (2.2) *by representing f_n and g_n by means of the M-bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Moreover, under the assumptions of Theorem* 2.1, *the following results*

$$\begin{aligned}\frac{1}{C}\operatorname{cond}(I-A) &\le \liminf_n \operatorname{cond}(M_n)\\ &\le \limsup_n \operatorname{cond}(M_n) \le C \operatorname{cond}(I-A),\end{aligned} \tag{2.18}$$

with a positive constant C independent of n. In particular, if both $\mathcal{B}$ and $\mathcal{B}'$ are orthonormal bases, in L^2_u, then

$$\lim_n \operatorname{cond}(M_n) = \operatorname{cond}(I-A). \tag{2.19}$$

Obviously Theorem 2.3 can be applied if the Marcinkiewicz bases are known. Yet the existence of such bases in arbitrary Banach spaces seems to be an open problem. To confirm this fact, until now, (polynomial) M-bases are not known in L^p_u with u arbitrary and $p \neq 2$. In L^p_u with $p \in (1,\infty)$ and u a generalized Jacobi weight, such bases do exist. To show this fact we need some notations. Let $w \in GJ$ be a weight function (equal or different from u). Denote by $\{p_n(w)\}_n$ the system of the orthonormal polynomials with respect to the weight w having positive leading coefficients and let $x_{nj} = x_{nj}(w)$, $j = 1,\dots,n$, be the zeros of $p_n(w)$. Furthermore, let $L_n(w,F)$ be the Lagrange polynomial interpolating a function F at the zeros of $p_n(w)$, i.e., $L_n(w,F,x) = \sum_{j=1}^n l_{nj}(w,x)F(x_{nj})$ with $l_{nj}(w,x) = (p_n(w,x))/(p_n'(w,x_{nj})(x-x_{nj}))$. Finally, for a given weight σ, let

$$\lambda_n(\sigma,x) = \left[\sum_{i=0}^{n-1} p_i^2(\sigma,x)\right]^{-1}, \quad n = 1,2,\dots$$

be the n-th Christofel functions. Obviously

$$\mathcal{B}^p_n(u,w) = \left\{\lambda_n^{-1/p}\left(u^p, x_{nj}(w)\right) l_{nj}(w)\right\}_{j=1,\dots,n} \tag{2.20}$$

is a basis of $\mathbf{P}_{n-1}$, but, in general, it is not an M-basis. The following proposition characterizes the M-bases of type (2.20).

Proposition 2.4. *$\mathcal{B}_n^p(u,w)$ is an M-basis if and only if the weights u and w satisfy the conditions*

$$\frac{u}{\sqrt{w\varphi}} \in L^p, \quad \frac{\sqrt{w\varphi}}{u} \in L^q, \quad \frac{1}{p}+\frac{1}{q}=1, \quad \varphi(x)=\sqrt{1-x^2}\,. \tag{2.21}$$

Proof. Every polynomial $q\in\mathbf{P}_{n-1}$, can be written in a unique way as

$$q=\sum_{j=1}^{n} a_{nj}\lambda_n^{-1/p}\left(u^p,x_{nj}\right) l_{nj}(w), \quad a_{nj}=\lambda_n^{1/p}\left(u^p,x_{nj}\right) q\left(x_{nj}\right).$$

By Theorem 2.6 and Theorem 2.7 in [**14**], if u and w satisfy (2.21), and only in this case, Marcinkiewicz inequalities hold and, then, the equivalence

$$\frac{1}{C}\|qu\|_p \le \left(\sum_{j=1}^{n}|a_{nj}|^p\right)^{1/p} \le C\|qu\|_p \tag{2.22}$$

is fulfilled for some C independent of n and q. □

Remark 1. Since the conditions (2.21) are equivalent ([**14**]) to the uniform boundedness of the operators $L_n(w)$ in the Sobolev type spaces

$$W_r^p(u)=\left\{f\in L_u^p : \|fu\|_p+\left\|f^{(r)}\varphi^r u\right\|_p<\infty\right\}, \quad r\ge 1,$$

we can rewrite Proposition 2.4 as follows.

$B_n^p(u,w)$ is an M-basis if and only if $\sup_n \|L_n(w)\|_{W_r^p(u)\to W_r^p(u)}<\infty$.

In this section we assumed $1<p<\infty$; the case $p\in\{1,\infty\}$ is an open problem.

3. Special cases and numerical tests. In this section, in order to apply the results in Section 2, we shall consider some special cases. To this end let $(I-A)f=g$ in L_u^p where $1<p<\infty$ and $u=v^{\alpha,\beta}$ is a Jacobi weight ($u\in J$). We also assume that $(I-A)f=g$ has a unique solution for any g.

Moreover we recall some basic facts about the polynomial approximation in L^p_u (see [**5**], [**14**]). The error of the best approximation is defined as

$$E_m(f)_{u,p} = \inf_{p\in\mathbf{P}_m} \|(f-P)u\|_p, \quad 1<p<\infty,$$
$$u = v^{\alpha,\beta}, \quad \alpha,\beta > -\frac{1}{p},$$

and, in order to estimate $E_m(f)_{u,p}$, we can use the following modulus of smoothness

$$\Omega^k_\varphi(f,t)_{u,p} = \sup_{0<h\le t} \|(\Delta^k_{h\varphi} f)u\|_{L^p(I_{rh})}$$

where $\varphi(x) = \sqrt{1-x^2}$, $\Delta^k_{h\varphi} f(x) = \sum_{i=0}^k \binom{k}{i}(-1)^i f(x+((k/2)-i)h\varphi(x))$ and $I_{rh} = [-1+(2rh)^2, 1-(2rh)^2]$. Then we can write the Jackson theorem and the Stechkin inequality by

$$\begin{aligned} &E_m(f)_{u,p} \le C\int_0^{1/m} \Omega^k_\varphi(f,t)\frac{dt}{t} \\ &\Omega^k_\varphi\left(f,\frac{1}{m}\right)_{u,p} \le \frac{C}{m^k}\sum_{i=0}^m (1+i)^{k-1} E_i(f)_{u,p}, \end{aligned} \tag{3.1}$$

for some $C>0$ independent of m and f. By (3.1) the equivalence

$$\sup_{t>0} \frac{\Omega^k_\varphi(f,t)_{u,p}}{t^r} \sim \sup_{k\ge 1} k^r E_k(f)_{u,p} \tag{3.2}$$

follows, with $k>r>0$, $r\in \mathring{R}$. Z^p_r denotes the Zygmund space with index r, defined as

$$Z^p_r = \left\{ f\in L^p_u : \sup_{t>0}\frac{\Omega^k_\varphi(f,t)_{u,p}}{t^r} < \infty, \quad k>r\right\},$$

where $u=v^{\alpha,\beta}$, $1<p<\infty$, $0<r\in\mathring{R}$, equipped with the norm

$$\|f\|_{Z^p_r} = \|f\|_{L^p_u} + \sup_{t>0}\frac{\Omega^k_\varphi(f,t)_{u,p}}{t^r}.$$

Note that, in virtue of (3.2), the semi-norm in Z_r^p can be expressed by $E_k(f)_{u,p}$. Moreover for $r > (1/p)$, Z_r^p admits a function continuous in $(-1,1)$, as a representative. Finally we recall a result about the polynomial interpolation.

Let $L_m(w,f)$, $w \in J$, be the Lagrange polynomial interpolating a function f at the zeros of Jacobi polynomial $p_m(w)$. Then, for any $g \in C^0(-1,1)$, there exists a constant $C > 0$, independent of m and f, such that

$$(3.3)\quad \|[g - L_m(w,g)]\|_{L_u^p} \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\Omega_\varphi^k(g,t)_{u,p}}{t^{1+\frac{1}{p}}}\, dt, \quad 1 < p < \infty,$$

if and only if the weights u and w satisfy (2.21) [**14**].

Case 1. For the equation $(I - A)f = g$ we assume

(i) For all $f \in L_u^p$, $\Omega_\varphi^k(Af,t)_{u,p} \le Ct^r\|fu\|_p$, $C \ne C(f,t)$ and $k > r > \frac{1}{p}$

(ii) $g \in Z_r^p$, $r > (1/p)$.

This case appears frequently in applications. Condition (i) can be verified by estimating directly $\Omega_\varphi^r(Af,t)_{u,p}$. For instance in the case $(Af)(t) = \lambda \int_{-1}^1 k(x,t)f(x)\, dx$, $\lambda \in \mathring{R}$, it is not hard to prove that, if $\sup_{|x|\le 1} \|k(x,\cdot)\|_{Z_r^p} < \infty$ and $sup_{|t|\le 1}\|k(\cdot,t)\|_{Z_r^p} < \infty$, then Af satisfies (i). Now, if A and g satisfy (i) and (ii) respectively, we choose a Jacobi weight w verifying (2.21) and we set $A_n f = L_n(w, Af)$ and $g_n = L_n(w,g)$ where $L_n(w,F,x) = \sum_{k=1}^n l_k(w)F(x_k)$, $p_n(w,x_k) = 0$. Hence we solve the finite-dimensional equation

$$(3.4)\qquad (I - A_n)f_n = g_n.$$

By representing f_n and g_n by the M-basis $\mathcal{B}_n^p(u,w) = \{l_k(w)/(\lambda_n^{1/p}(u^p, x_k))\}_{k=1,\dots,n}$, (3.4) is equivalent to the system

$$(3.5)\quad \lambda^{1/p}(u^p,x_i)\sum_{j=1}^n a_{n,j}\,\lambda_n^{-1/p}(u^p,x_j)[(I-A)l_j(w)](x_i) = \lambda_n^{1/p}(u^p,x_i)g(x_i) \quad i = 1,\dots,n,$$

where $a_{n,j} = \lambda_n^{1/p}(u^p, x_j) f_n(x_j)$, that is well conditioned. Moreover by (ii) and (3.3) it follows

$$\|(A - A_n)f\|_{L^p_u} \le \frac{c}{n^r}\|f\|_{L^p_u}.$$

Therefore by Theorem 2.1, the system (3.5) has a unique solution, say $(\bar{a}_{n1}, \dots, \bar{a}_{nn})$, and we set

$$\overline{f}_n = \sum_{i=1}^{n} \bar{a}_{ni} \frac{l_i(w)}{\lambda_n^{1/p}(u^p, x_i)}$$

as the approximate solution of $(I - A)f = g$. Indeed by Theorem 2.1 it follows

$$\|f - \bar{f}_n\|_{L^p_u} \le \frac{C}{n^r}\|g\|_{Z^r_r}. \tag{3.6}$$

Consequently the solution f belongs to Z^p_r, in view of the Steckhin inequality (3.1).

Remark 2. Collocating the equation (3.4) on the interpolation knots $x_1, \dots, x_n$, we obtain the following system

$$\sum_{j=1}^{n} f_n(x_j)[(I - A)l_j(w)](x_i) = g(x_i), \quad i = 1, \dots, n. \tag{3.7}$$

Now if we multiply (3.7) to the left by the diagonal matrix, whose entries are $\lambda_n^{1/p}(u^p, x_i)$, $i = 1, \dots, n$ we obtain (3.5). In this context condition (2.21), characterizing the interpolation nodes (and, in this example, also the collocation nodes), seems to be crucial. Indeed, if (2.21) is not satisfied, then (3.6) is not true and $\mathcal{B}^p_n(u, w)$ is not an M-basis.

If we denote by B_n the matrix of the system (3.7), then $\sup_n \operatorname{cond}(B_n)$ may be infinite. In fact, for each polynomial $q = \sum_{i=1}^n l_i(w)q(x_i)$, by using the Marcinkiewicz inequality, we have

$$\frac{C_1}{\Lambda}\|qu\|_p \le \left(\sum_{i=1}^{n} |q(x_i)|^p\right)^{1/p} \le C_2\frac{\|qu\|_p}{\lambda}$$

where $\lambda = \min_i \lambda_n^{1/p}(u^p, x_i)$ and $\Lambda = \max_i \lambda_n^{1/p}(u^p, x_i)$. Then working as in the proof of Proposition 2.2, we get

$$\operatorname{cond}(B_n) \le C \operatorname{cond}[(I - A_n)_{|\mathbf{P}_{n-1}}] \left(\frac{\Lambda}{\lambda}\right)^2.$$

Since $\lambda_n^{1/p}(u^p, x_i) \sim u(x_i)(\sqrt{1-x_i^2}/n)^{1/p}$, for $u(x) = (1-x^2)^\alpha$, $\alpha > 0$, we have $(\Lambda/\lambda)^2 \sim n^{4\alpha+\frac{2}{p}}$.

Then the procedure in Case 1 can be seen as a preconditioned collocation method. Nevertheless the results of Section 2 can be used in different contexts. In this example we deduce the space of the solution by (3.6) while other methods are based on the a priori knowledge of the smoothness of the solution (see [**18**] and enclosed references). Finally we observe that the system (3.5) requires the computation of $c_{ij} = [(I - A)l_j(w)](x_i)$. Sometimes this can be avoided if g and Af satisfy (i) and (ii) with r sufficiently large. To this end we show the following

Case 2. With $(Af)(x) = \lambda(Kf)(x) = \lambda \int_{-1}^{1} k(x,y)f(y)dy$, $\lambda \in \mathring{R}$, we consider the equation $(I - \lambda K)f = g$ in L^2, i.e., $p = 2$ and $u = 1$, and assume $g \in Z_r^2$, $\sup_{|x|<1} \|k(x,\cdot)\|_{Z_r^2} < \infty$ and $\sup_{|y|<1} \|k(\cdot,y)\|_{Z_r^2} < \infty$ with r sufficiently large. Then we can use the following procedure. Define the integral operator

$$K_n f(x) = \int_{-1}^{1} L_{n,y}[k(x,y)]f(y)\,dy. \tag{3.8}$$

Throughout L_n denotes the Lagrange interpolation operator based on the Legendre nodes and the subscript y in $L_{n,y}$ means that the interpolation is done with respect to the variable y. Then, consider the following finite dimensional equation

$$(I - \lambda L_n K_n) f_n = L_n g \tag{3.9}$$

where f_n is an unknown polynomial of degree at most $n-1$ (see also [**10**], [**11**]). Represent (3.9) in the M-basis $\{\lambda_{nj}^{-1/2} l_{nj}\}_{j=1,\dots,n}$, i.e., (2.20) with $u = w = 1$ and obtain the system

$$\lambda_{ni}^{1/2} \sum_{j=1}^{n} a_{nj}[\lambda_{nj}^{-1/2}\,\delta_{ij} - \lambda\lambda_{nj}^{1/2}\,k(x_{ni},x_{nj})] = \lambda_{ni}^{1/2}\,g(x_{ni}), \quad i = 1,\dots,n, \tag{3.10}$$

that is well conditioned. By solving (3.10), we construct the polynomial

$$f_n = \sum_{j=1}^{n} a_{nj}\lambda_{nj}^{-\frac{1}{2}} l_{nj} \in \mathbf{P}_{n-1}, \tag{3.11}$$

that satisfies (3.6) with $p = 2$ and $u = 1$.

Remark 3. The described numerical method can be considered as a preconditioned Nyström method (see, for instance, [**1**]) using the Gauss-Legendre quadrature rule approximating the integral $Kf(x)$, that is,

$$Kf(x) \cong \sum_{j=1}^{n} \lambda_{nj} k(x,x_{nj}) f(x_{nj}).$$

Recall that the Nyström method consists in solving the following system

$$\sum_{j=1}^{n} f_{nj}\,[\delta_{ij} - \lambda\lambda_{nj}k(x_{ni},x_{nj})] = g(x_{ni}), \quad i = 1,\dots,n, \tag{3.12}$$

and in constructing the approximating solution by the Nyström interpolation formula

$$\tilde{f}_n(x) = g(x) + \lambda \sum_{j=1}^{n} \lambda_{nj} k(x,x_{nj}) f_{nj}. \tag{3.13}$$

Let us observe that the unknown coefficients f_{nj}, $j = 1,\dots,n$ can be also obtained by solving the system (3.10) and computing $f_{nj} = \lambda_{nj}^{-1/2} a_{nj}$, $j = 1,\dots,n$, with the advantage of solving a system which is well conditioned.

Case 3. Finally we consider the equation

$$(I - \lambda K^{\mu}) f = g \tag{3.14}$$

where $g \in Z_r^2$, $(K^\mu f)(x) = \int_{-1}^1 k^\mu(x,y) f(y)\,dy$, $k^\mu(x,y) = |x-y|^\mu$, $-1 < \mu < 0$ and $k^0(x,y) = \log|x-y|$. $k^\mu(x,y)$ is a classical kernel and the equation $(I - \lambda K^\mu) f = g$ was considered by several authors (see, e.g., [**17**], [**18**], [**21**] and references therein). The mapping properties of the operator K^μ in some spaces are well known [**9**]. Here we state a more accurate result.

Theorem 3.1. *Let $k(x,y)$ be as in* (3.14). *Then the operator K^μ satisfies the following estimate*

$$E_n\,(K^\mu f)_2 \le \frac{C}{n^{1+\mu}}\,\|f\|_2, \tag{3.15}$$

with a constant C independent of n and f. Consequently $K^\mu : L^2 \longrightarrow Z_s^2$ is a compact operator for all $s < 1 + \mu$.

In virtue of (3.15) and by assuming $g \in Z_r^2$, $r > 1/2$, we construct a polynomial approximation of the solution of (3.14), as in Case 1. We set $(K_n^\mu f)(x) = L_m(K^\mu f, x)$ and $g_m = L_m g$, where L_m is the Lagrange operator based on Legendre zeros and we consider the equation

$$(I - \lambda K_n^\mu) f_n = g_n. \tag{3.16}$$

Representing f_n by means of orthonormal Legendre polynomials $\{p_i\}$, i.e., $f_n = \sum_{i=1}^{n-1} c_{ni} p_i$ and g_n by $\{l_j/\sqrt{\lambda_j}\}_{j=1,\dots,n}$ (i.e. (2.20) with $u = w = 1$), we have the system

$$\lambda_{ni}^{1/2} \sum_{j=0}^{n-1} c_{nj}\left[p_j(x_{ni}) - \lambda M_j^\mu(x_{ni})\right] = \lambda_{ni}^{1/2}\, g(x_{ni}), \qquad i = 1, \dots, n, \tag{3.17}$$

where $M_j^\mu(x_i) = K^\mu p_j(x_{n,i})$ can be computed by a recurrence relation (see for instance [**4**], [**18**]). The system (3.17) is well conditioned and has a unique solution (for n large) since by (3.15)and (3.3) we get

$$\|K - K_n\| = O(n^{-(1+\mu)})$$

and

$$\|f - f_n\|_{L^2} = O(n^{-s}), \quad s = \min(1+\mu, r).$$

Notice that, if g is sufficiently smooth, e.g., $g \in C^{(k)}(-1,1)$ k large, then, by [**26**], f and Af belong to $Z_{2\mu+3}$ and $\|f - f_n\|_2 = O(m^{-2\mu-3})$. Here we considered the case $-1 < \mu \leq 0$. The case $\mu > 0$ is similar. In particular for $\mu > 0$ "large" the system (3.10) is more suitable.

In order to show some numerical tests, we consider in L^2 equations of the type $f(x) - \lambda \int_{-1}^{1} k(x,t) f(t)\, dt = g(x)$.

Example 1. Let $\lambda = 1$, $k(x,y) = x + y$, $g(x) = x^2 + 2$.

Following Case 2, from the finite dimensional equation (3.9) we have, by using an M-basis, the system (3.10). The numerical results are in the following tables

TABLE 1.1

n	$f_n(-1)$	$f_n(-0.5)$	$f_n(0.5)$
8	7.666666666666	-8.33333333333e-2	-1.4083333333333e+1
16	7.666666666666	-8.333333333332e-2	-1.4083333333333e+1

TABLE 1.2

n	cond (M_n)
8	1.3928203230275e+1
16	1.3928203230275e+1
32	1.3928203230275e+1

If we represent f_n and $g_n = L_n g$ in (3.4) by using the fundamental Lagrange polynomials $\tilde{l}_1, \dots, \tilde{l}_n$ based on given zeros $\tilde{x}_1, \dots, \tilde{x}_n$, we have the linear system

$$\sum_{j=1}^{n} \left[\delta_{i,j} - \lambda \sum_{k=1}^{n} l_k(\tilde{x}_i) \sum_{l=1}^{n} \lambda_l k(x_l, x_k) \tilde{l}_j(x_l) \right] f_n(\tilde{x}_j) = g_n(\tilde{x}_i), \quad i = 1, \dots, n. \tag{3.18}$$

Denote by C_n the matrix associated with the system (3.18). Then, if $x_i = \{-1 + (2/n-1)i\}$, $i = 0. \dots, n-1$, the condition number of C_n has the following behavior

TABLE 1.3

n	cond (C_n)
8	3.552202729523286e+001
16	4.340167849100752e+004
32	8.615969917640776e+012
64	3.079632038081276e+017

If $\tilde{x}_i = x_{m,i}(v^{7,7})$ are the zeros of Jacobi polynomial $p_n^{\alpha,\beta}$ with $\alpha = \beta = 7$, then we have

TABLE 1.4

n	cond (C_n)
8	1.599545062878915e+004
16	2.343021438144193e+006
32	9.165456903104157e+008
64	7.229867516189905e+011

Example 2. Let $\lambda = 0.2$, $k(x, y) = |x - y|^{5/2}$, $g(x) = |x|^{3/2}$.

The exact solution f belongs to $Z^2_{3/2}$. Tables (2.1)–(2.2) show the behavior of the solution and of the condition number of the system.

TABLE 2.1

n	$f_n(\pm 1)$	$f_n(\pm 0.75)$	$f_n(\pm 0.25)$	$f_n(\pm 0.5)$
8	1.51	9.9e-1	2.e-1	5.7e-1
16	1.51	9.91e-1	2.71e-1	5.73e-1
32	1.512	9.913e-1	2.719e-1	5.7392e-1
64	1.5127	9.9139e-1	2.719e-1	5.739e-1
128	1.5127	9.91395e-1	2.7199e-1	5.73932e-1
256	1.512704	9.91395e-1	2.719916e-1	5.739322722e-1

TABLE 2.2

n	cond (M_n)
8	2.0897
16	2.0897
32	2.089735
64	2.0897353
128	2.089735
256	2.089735296
512	2.0897352963

Example 3. Let $\lambda = 0.2$, $k(x,y) = |x-y|^{-0.4}$, $g(x) = \cos(x^2+1)$.

From the theoretical results, we have $f \in Z_r^2$ with $r = 2.2$, and the numerical tests are

TABLE 3.1

n	$f_n(0)$	$f_n(\pm 0.25)$	$f_n(\pm 0.75)$	$f_n(\pm 1)$
8	1.059	0.987	0.366	-0.14
16	1.0597	0.9870	0.3667	-0.14
32	1.05972	0.987089	0.366791	-0.141
64	1.059726	0.98708924	0.36679105	-0.141
128	1.0597264	0.98708924	0.36679105	-0.141
256	1.05972645	0.98708924707	0.36679105	-0.1410
512	1.0597264512	0.987089247072	0.3667910549	-0.14101

TABLE 3.2

n	cond (M_n)
8	2.
16	2.6
32	2.6
64	2.7
128	2.7
256	2.73
512	2.733

4. Proofs.

Proof of Theorem 2.1. At first, let us observe that, by applying the Fredholm alternative theorem to the equation (2.1), we can deduce that it has a unique solution $f \in L^p_u$ for any $g \in L^p_u$.

Moreover, since the condition (2.3) holds, it is a well known result (see for instance [**1**]) that, for sufficiently large n, the inverse operators $(I - A_n)^{-1}$ exist and are uniformly bounded with respect to n and satisfy the relations:

$$(I - A_n)^{-1} = \left[I - (I - A)^{-1}(A - A_n)\right]^{-1}(I - A)^{-1}, \tag{4.1}$$

and taking the operator norm (i.e., $\|\cdot\| = \|\cdot\|_{L^p_u \to L^p_u}$), we have

$$\left\|(I - A_n)^{-1}\right\| \le \frac{\left\|(I - A)^{-1}\right\|}{1 - \|(I - A)^{-1}\| \cdot \|A - A_n\|}. \tag{4.2}$$

Then the finite dimensional equation (2.2) has a unique solution f_n which belongs to $\mathbf{P}_{n-1}$, since $f_n = g_n + A_n f_n$. Now, to prove the error estimate (2.5), let us observe that

$$f - f_n = (I - A_n)^{-1}\left[g - g_n + (A - A_n)f\right]$$

and, using (4.2), one has

$$\begin{aligned}\|f-f_n\|_{L^p_u} &\le \left\|(I-A_n)^{-1}\right\| \|g-g_n+(A-A_n)f\|_{L^p_u}\\ &\le C\left(\|g-g_n\|_{L^p_u}+\|A-A_n\|\cdot\|f\|_{L^p_u}\right)\\ &\le C\left(\|g-g_n\|_{L^p_u}+\|A-A_n\|\cdot\|g\|_{L^p_u}\right),\end{aligned}$$

with $C \neq C(n)$.

It remains to prove (2.6). It is clear that

$$|\|I-A_n\|-\|I-A\||\le\|A-A_n\|\,. \tag{4.3}$$

For the inverse operators, we can write (see, for instance, [**22**])

$$\begin{aligned}(I-A_n)^{-1}-(I-A)^{-1} &= (I-A)^{-1}\left[(I-A)(I-A_n)^{-1}-I\right]\\ &= (I-A)^{-1}[(I-A)-(I-A_n)](I-A_n)^{-1}\\ &= (I-A)^{-1}(A_n-A)(I-A_n)^{-1}.\end{aligned}$$

Then, taking the norms, we get

$$\begin{aligned}&\left\|(I-A_n)^{-1}-(I-A)^{-1}\right\|\\ &\qquad\le\left\|(I-A)^{-1}\right\|\|A-A_n\|\left\|(I-A_n)^{-1}\right\|\end{aligned} \tag{4.4}$$

and, consequently, taking also into account (4.2),

$$\left|\left\|(I-A_n)^{-1}\right\|-\left\|(I-A)^{-1}\right\|\right|\le C\,\|A-A_n\|\,. \tag{4.5}$$

Combining (4.3) and (4.5), the relation (2.6) follows. In fact we can write

$$\begin{aligned}&|\mathrm{cond}\,(I-A_n)-\mathrm{cond}(I-A)|\\ &\quad=\left|\|I-A_n\|\left\|(I-A_n)^{-1}\right\|-\|I-A\|\left\|(I-A)^{-1}\right\|\right|\\ &\quad\le\left|\|I-A_n\|\left\|(I-A_n)^{-1}\right\|-\left\|(I-A_n)^{-1}\right\|\|I-A\|\right|\\ &\qquad+\left|\left\|(I-A_n)^{-1}\right\|\|I-A\|-\|I-A\|\left\|(I-A)^{-1}\right\|\right|\\ &\quad=\left\|(I-A_n)^{-1}\right\|\,|\|I-A_n\|-\|I-A\||\\ &\qquad+\|I-A\|\left|\left\|(I-A_n)^{-1}\right\|-\left\|(I-A)^{-1}\right\|\right|\\ &\quad\le C\,\|A-A_n\|\,,\end{aligned}$$

with $C \neq C(n)$. □

Proof of Theorem 2.3. Set $B = I - A$ and $B_n = I - A_n$. Let $\varepsilon > 0$ be arbitrarily chosen, but fixed. By the definition of the operator norm, a function $f_\varepsilon \in L^p_u$ such that

$$\|Bf_\varepsilon\|_{L^p_u} > \|B\| - \frac{\varepsilon}{2}, \quad \|f_\varepsilon\|_{L^p_u} = 1.$$

Now, let P_n be a projection of L^p_u onto $\mathbf{P}_{n-1}$ such that

$$\sup_n \|P_n\| < \infty. \tag{4.6}$$

Let us observe that a projector satisfying (4.6) exists in L^p_u. It is sufficient to consider the Fourier operator

$$S_n(w,f) = \sum_{j=0}^{n-1} c_j p_j(w), \quad c_j = \int_{-1}^{1} f(x)p_j(w,x)w(x)\,dx$$

under the assumptions

$$\frac{u}{\sqrt{w\varphi}} \in L^p, \quad \text{and} \quad \frac{w}{u}, \sqrt{\frac{w}{\varphi}}\,\frac{1}{u} \in L^q,$$
$$\varphi(x) = \sqrt{1-x^2}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

(see [**2**], [**3**], [**22**]).

Then, as is well known, one has

$$\|P_n f - f\|_{L^p_u} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty \tag{4.7}$$

for any $f \in L^p_u$. By applying (4.7) to the function f_ε, we can deduce that there exists an index $n_0 \in \mathbf{N}$ such that

$$\|P_n f_\varepsilon - f_\varepsilon\|_{L^p_u} < \frac{\varepsilon}{2\|B\|}, \quad \forall\, n \geq n_0.$$

Then, for any $n \geq n_0$, one has

$$\begin{aligned}\|BP_n f_\varepsilon\|_{L^p_u} &\geq \|Bf_\varepsilon\|_{L^p_u} - \|B\left(P_n f_\varepsilon - f_\varepsilon\right)\|_{L^p_u} \\ &> \|B\| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \|B\| - \varepsilon,\end{aligned}$$

from which it follows that

$$\|B\| - \varepsilon < \|BP_n f_\varepsilon\|_{L^p_u} = \left\|B\,|_{\mathbf{P}_{n-1}}\, P_n f_\varepsilon\right\|_{L^p_u} \le \left\|B\,|_{\mathbf{P}_{n-1}}\right\| \cdot \|P_n f_\varepsilon\|_{L^p_u}.$$

Therefore, taking into account (4.7), we obtain

$$\|B\| - \varepsilon \le \liminf \left\|B\,|_{\mathbf{P}_{n-1}}\right\| \cdot \|f_\varepsilon\|_{L^p_u} = \liminf \left\|B\,|_{\mathbf{P}_{n-1}}\right\|$$

and then

$$\|B\| - \varepsilon \le \liminf \left\|B\,|_{\mathbf{P}_{n-1}}\right\| \le \limsup \left\|B\,|_{\mathbf{P}_{n-1}}\right\| \le \|B\|.$$

We can deduce that there exists $\lim_n \left\|B\,|_{\mathbf{P}_{n-1}}\right\|$ and one has

$$\lim_n \left\|B\,|_{\mathbf{P}_{n-1}}\right\| = \|B\|. \tag{4.8}$$

Now, let us observe that, by (2.3), we have $\|B - B_n\| \longrightarrow 0$, as $n \to \infty$ from which it follows that $\left\|B_n\,|_{\mathbf{P}_{n-1}} - B\,|_{\mathbf{P}_{n-1}}\right\| \longrightarrow 0$ as $n \to \infty$ and then

$$\left|\left\|B_n\,|_{\mathbf{P}_{n-1}}\right\| - \left\|B\,|_{\mathbf{P}_{n-1}}\right\|\right| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{4.9}$$

Since

$$\begin{aligned}\left|\left\|B_n\,|_{\mathbf{P}_{n-1}}\right\| - \|B\|\right| &\le \left|\left\|B_n\,|_{\mathbf{P}_{n-1}}\right\| - \left\|B\,|_{\mathbf{P}_{n-1}}\right\|\right| \\ &\quad + \left|\left\|B\,|_{\mathbf{P}_{n-1}}\right\| - \|B\|\right|,\end{aligned}$$

by (4.8), (4.9) and taking into account the relation $\|M_n\| \sim \left\|B_n\,|_{\mathbf{P}_{n-1}}\right\|$, with the constant in "$\sim$" independent of n, we conclude that

$$\frac{1}{C}\,\|B\| \le \liminf_n \|M_n\| \le \limsup_n \|M_n\| \le C\|B\| \tag{4.10}$$

with C constant and $C \ne C(n)$.

A similar result can be obtained for the inverse operators M_n^{-1} and B^{-1}, taking into account the relations (4.2), (4.4) and $\left\|M_n^{-1}\right\| \sim \left\|\left(B_n\,|_{\mathbf{P}_{n-1}}\right)^{-1}\right\|$, i.e.,

$$\frac{1}{C}\,\left\|B^{-1}\right\| \le \liminf_n \left\|M_n^{-1}\right\| \le \limsup_n \left\|M_n^{-1}\right\| \le C\|B^{-1}\|. \tag{4.11}$$

Combining (4.10) and (4.11) we get (2.18). Equation (2.19) follows by replacing the symbol "$\sim$" by "$=$". □

Lemma 4.1. *Let* $K^\mu f(x) = \int_{-1}^{1} |x-y|^\mu f(y)\,dy$, *with* $-1 < \mu < 0$. *Then for every* $f \in L^2$ *one has*

$$\sup_{t>0} \frac{\Omega_\varphi\left(K^\mu f, t\right)_2}{t^{1+\mu}} \le C\|f\|_2, \tag{4.12}$$

where the constant C *depends only on* μ.

Proof. We set $\varphi_1(x) = \sqrt{1-|x|}$. Since $(\varphi(x)/\sqrt{2}) \le \varphi_1(x) \le \varphi(x)$, $\varphi(x) = \sqrt{1-x^2}$, the φ-modulus Ω_φ is equivalent to Ω_{φ_1} [**5**]. Thus we prove (4.12) with Ω_φ replaced by Ω_{φ_1}.

We can write

$$\begin{aligned}
\|\Delta_{h\varphi_1}(K^\mu f)\|_{L^2(I_h)} &= \left(\int_{-1+4h^2}^{1-4h^2} \left|\Delta_{h\varphi_1}(K^\mu f)(x)\right|^2\,dx\right)^{1/2} \\
&\le \left(\int_{-1+4h^2}^{0} \left|\Delta_{h\varphi_1}(K^\mu f)(x)\right|^2\,dx\right)^{1/2} \\
&\quad + \left(\int_{0}^{1-4h^2} \left|\Delta_{h\varphi_1}(K^\mu f)(x)\right|^2\,dx\right)^{1/2} \\
&:= I_1 + I_2.
\end{aligned}$$

Let us estimate I_1. We have

$$\begin{aligned}
\left|\Delta_{h\varphi_1}(K^\mu f)(x)\right| &= \left|\int_{-1}^{1}\left[\left|x+\frac{h}{2}\varphi_1(x)-y\right|^\mu - \left|x-\frac{h}{2}\varphi_1(x)-y\right|^\mu\right] f(y)\,dy\right| \\
&\le \int_{-1}^{1}\left|\left|x+\frac{h}{2}\varphi_1(x)-y\right|^\mu - \left|x-\frac{h}{2}\varphi_1(x)-y\right|^\mu\right||f(y)|\,dy \\
&= \left\{\int_{-1}^{-1+(1+x/2)} + \int_{-1+(1+x/2)}^{x-h\varphi_1(x)} + \int_{x-h\varphi_1(x)}^{x+h\varphi_1(x)} + \int_{x+h\varphi_1(x)}^{x+(1+x/2)}\right. \\
&\quad \left. + \int_{x+(1+x/2)}^{2x+1} + \int_{2x+1}^{1}\right\}\left|\left|x+\frac{h}{2}\varphi_1(x)-y\right|^\mu \right. \\
&\quad \left. - \left|x-\frac{h}{2}\varphi_1(x)-y\right|^\mu\right||f(y)|\,dy = \sum_{i=1}^{6} G_i(x).
\end{aligned}$$

It follows that

$$\begin{aligned} I_1 &\le \left(\int_{-1+4h^2}^{0} \left[\sum_{i=1}^{6} G_i(x) \right]^2 dx \right)^{1/2} \\ &\le \sum_{i=1}^{6} \left(\int_{-1+4h^2}^{0} [G_i(x)]^2 \, dx \right)^{1/2} \\ &:= \sum_{i=1}^{6} A_i. \end{aligned}$$

To estimate A_1 and A_2 we shall use the following bound:

$$\left| \left| x + \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} - \left| x - \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} \right| \le h\varphi_1(x) \left(x - \frac{h}{2}\,\varphi_1(x) - y \right)^{\mu-1}, \tag{4.13}$$

since $|\mu| < 1$. Let us estimate A_1. We have

$$\begin{aligned} G_1(x) &= \left| \int_{-1}^{-1+(x+1/2)} \left| \left| x + \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} \right.\right. \\ &\qquad \left.\left. - \left| x - \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} \right| |f(y)| \, dy \right| \\ &\le h\sqrt{1+x} \int_{-1}^{-1+(x+1/2)} \left(x - \frac{h}{2}\,\varphi_1(x) - y \right)^{\mu-1} |f(y)| \, dy \\ &\le Ch(1+x)^{\mu-(1/2)} \int_{-1}^{-1+(x+1/2)} |f(y)| \, dy \\ &\le Ch(1+x)^{\mu-(1/2)} \|f\|_2 \left(\int_{-1}^{-1+(x+1/2)} dy \right)^{1/2} \\ &\le Ch(1+x)^{\mu} \|f\|_2, \end{aligned}$$

where we have taken into account the following facts

$$\begin{aligned} &x \ge -1 + 4h^2 \Longleftrightarrow h \le \frac{\sqrt{1+x}}{2} \Longleftrightarrow \frac{h}{2}\,\varphi_1(x) \le \frac{1+x}{4} \\ &y \le -1 + \frac{1+x}{2} \Longrightarrow x - \frac{h}{2}\,\varphi_1(x) - y \ge \frac{1+x}{4}. \end{aligned}$$

It follows, by using

$$h \leq \frac{\sqrt{1+x}}{2} \Longrightarrow h^{\mu} \geq C(1+x)^{(\mu/2)},$$

that

$$\begin{aligned} A_1 &= \left(\int_{-1+4h^2}^{0} |G_1(x)|^2 \, dx \right)^{1/2} \\ &\leq C h^{1+\mu} \|f\|_2 \left(\int_{-1+4h^2}^{0} (1+x)^{\mu} \, dx \right)^{1/2} \\ &\leq C h^{1+\mu} \|f\|_2. \end{aligned}$$

To estimate $G_2(x)$ (and consequently A_2) we use (4.13) and the change of variable $x - y = \tau\sqrt{1+x}$. One has

$$\begin{aligned} G_2(x) &\leq h\sqrt{1+x} \int_{-1+(x+1/2)}^{x-h\varphi_1(x)} \left(x - \frac{h}{2}\,\varphi_1(x) - y \right)^{\mu-1} |f(y)| \, dy \\ &= h(1+x)^{(1/2)+(\mu/2)} \int_{h}^{(\sqrt{1+x}/2)} \left| f\left(x - \tau\sqrt{1+x} \right) \right| \left(\tau - \frac{h}{2} \right)^{\mu-1} d\tau. \end{aligned}$$

Setting

$$x_+^0 = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

since $\sqrt{1+x}/2 \leq (1/2)$, we have

$$\begin{aligned} |G_2(y)| &\leq h(1+x)^{(1/2)+(\mu/2)} \\ &\quad \times \int_{h}^{1/2} \left| f\left(x - \tau\sqrt{1+x} \right) \right| \left(\tau - \frac{h}{2} \right)^{\mu-1} \left(\frac{\sqrt{1+x}}{2} - \tau \right)_+^0 d\tau \end{aligned}$$

from which, by the generalized Minkowski inequality, we deduce

$$\begin{aligned}
A_2 &= \left(\int_{-1+4h^2}^{0} |G_2(x)|^2 \, dx \right)^{1/2} \\
&\le h \biggl(\int_{-1+4h^2}^{0} (1+x)^{\mu+1} \biggl| \int_h^{1/2} \left| f\left(x - \tau\sqrt{1+x}\right) \right| \left(\tau - \frac{h}{2}\right)^{\mu-1} \\
&\qquad \times \left(\frac{\sqrt{1+x}}{2} - \tau \right)_+^0 d\tau \biggr|^2 dx \biggr)^{1/2} \\
&\le h \int_h^{1/2} \left(\tau - \frac{h}{2}\right)^{\mu-1} \biggl(\int_{-1+4h^2}^{0} \left| f\left(x - \tau\sqrt{1+x}\right) \right|^2 \\
&\qquad \times \left[\left(\frac{\sqrt{1+x}}{2} - \tau \right) +^0 \right]^2 (1+x)^{\mu+1} \, dx \biggr)^{1/2} d\tau \\
&\le Ch \int_h^{1/2} \left(\tau - \frac{h}{2}\right)^{\mu-1} \left(\int_{-1+4\tau^2}^{0} \left| f\left(x - \tau\sqrt{1+x}\right) \right|^2 dx \right)^{1/2} d\tau.
\end{aligned}$$

In order to evaluate the second integral at the righthand side we use the change of variable $s = x - \tau\sqrt{1+x}$, i.e., $x = s + (\tau^2/2) + \tau\sqrt{(\tau^2/4) + s + 1}$, $dx = \left(1 + (\tau/2\sqrt{(\tau^2/4) + s + 1})\right) ds$. We get

$$\begin{aligned}
\int_{-1+4\tau^2}^{0} & \left| f\left(x - \tau\sqrt{1+x}\right) \right|^2 dx \\
&= \int_{-1+2\tau^2}^{-\tau} |f(s)|^2 \left(1 + \frac{\tau}{2\sqrt{(\tau^2/4) + s + 1}} \right) ds \\
&\le C\|f\|_2^2,
\end{aligned}$$

since from $\tau \in [h, 1/2]$ and $s > -1 + 2\tau^2$ it follows that $s + 1 + (\tau^2/4) > (9/4)\tau^2$ and hence $1 < 1 + (\tau/2\sqrt{s + 1 + (\tau^2/4)}) < (4/3)$. Consequently,

$$\begin{aligned}
A_2 &\le Ch\|f\|_2 \int_h^{1/2} \left(\tau - \frac{h}{2}\right)^{\mu-1} d\tau \\
&= Ch\|f\|_2 \left[\left(\frac{h}{2}\right)^{\mu} - \left(\frac{1}{2} - \frac{h}{2}\right)^{\mu} \right] \\
&\le Ch^{1+\mu}\|f\|_2.
\end{aligned}$$

To estimate $G_3(y)$, and then A_3, we use the following decomposition

$$G_3(y) \le \left\{ \int_{x-h\varphi_1(x)}^{x} + \int_{x}^{x+h\varphi_1(x)} \right\} \left\| x + \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} - \left| x - \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} \Bigg| \, |f(y)|\,dy := F_1(x) + F_2(x).$$

Observe that

$$F_1(x) \le \int_{x-h\varphi_1(x)}^{x} \left| x - \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} |f(y)|\,dy \tag{4.14}$$

and

$$F_2(x) \le \int_{x}^{x+h\varphi_1(x)} \left| x + \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} |f(y)|\,dy. \tag{4.15}$$

By using the change of variable $x - (h/2)\varphi_1(x) - y = h\tau\varphi_1(x)$ in the first integral and $x + (h/2)\varphi_1(x) - y = h\tau\varphi_1(x)$ in the second one, we may estimate quantities such as

$$h^{1+\mu}[\varphi_1(x)]^{1+\mu} \int_0^{1/2} \left| f(x \pm h\left(\frac{1}{2} \pm \tau\right)\varphi_1(x) \right| \tau^{\mu}\,d\tau.$$

Consequently, to obtain an estimate for A_3 we can proceed as in the estimate of A_2. If we repeat this procedure by replacing τ by $(1/2 \pm \tau)\,h$ and assuming, as it is possible, $h < (1/2)$ in the estimate of the integrals

$$\int_{-1+4h^2}^{0} \left| f(x \pm h\left(\frac{1}{2} \pm \tau\right)\varphi_1(x) \right|^2 dx,$$

we get

$$A_3 \le C h^{1+\mu} \|f\|_2.$$

To estimate A_4 A_5 and A_6 we shall use the following bound:

$$\left\| x + \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} - \left| x - \frac{h}{2}\,\varphi_1(x) - y \right|^{\mu} \Bigg| \le h\varphi_1(x)\left(y - \frac{h}{2}\,\varphi_1(x) - x \right)^{\mu-1}. \tag{4.16}$$

The estimate of A_4 can be obtained by proceeding as for the estimate of A_2. One has

$$A_4 \le Ch^{1+\mu}\|f\|_2.$$

To bound A_5, observe that

$$\begin{aligned} G_5(x) &\le h\sqrt{1+x}\int_{x+(1+x/2)}^{2x+1}\left(y-\frac{h}{2}\,\varphi_1(x)-x\right)^{\mu-1}|f(y)|\,dy \\ &\le Ch^{1+\mu}(1+x)^{(\mu/2)-(1/2)}\int_{x+\frac{1+x}{2}}^{2x+1}|f(y)|dy \\ &\le Ch^{1+\mu}(1+x)^{\mu/2}\|f\|_2, \end{aligned}$$

from which we can deduce

$$A_5 \le Ch^{1+\mu}\|f\|_2\left(\int_{-1+4h^2}^{0}(1+x)^{\mu}\,dx\right)^{1/2} \le Ch^{1+\mu}\|f\|_2.$$

Let us estimate $G_6(x)$ and consequently A_6. We have

$$\begin{aligned} G_6(x) &\le h\sqrt{1+x}\int_{2x+1}^{1}\left(y-\frac{h}{2}\,\varphi_1(x)-x\right)^{\mu-1}|f(y)|\,dy \\ &\le h\sqrt{1+x}\int_{2x+1}^{1}(y+1)^{\mu-1}|f(y)|\,dy \\ &\le Ch\int_{2x+1}^{1}(y+1)^{\mu-\frac{1}{2}}|f(y)|\,dy \\ &\le Ch^{1+\mu}\int_{2x+1}^{1}(y+1)^{(\mu/2)-(1/2)}|f(y)|\,dy \\ &\le Ch^{1+\mu}\|f\|_2\left(\int_{2x+1}^{1}(y+1)^{\mu-1}\,dy\right)^{1/2} \\ &\le Ch^{1+\mu}\|f\|_2(1+x)^{\mu/2}, \end{aligned}$$

since $y \ge 2x+1 \Leftrightarrow y-x \ge (y+1)/2$ from which it follows that $y-(h/2)\varphi_1(x)-x \le (1+y)/4$. Therefore we get

$$A_6 \le Ch^{1+\mu}\|f\|_2\left(\int_{-1+4h^2}^{0}(1+x)^{\mu}\,dx\right)^{1/2} \le Ch^{1+\mu}\|f\|_2.$$

Finally we conclude

$$I_1 \le Ch^{1+\mu}\|f\|_2. \tag{4.17}$$

A similar estimate can be obtained also for I_2, recalling that in this case $\varphi_1(x) = \sqrt{1-x}$ and dividing the interval $[-1,1]$ as follows

$$\begin{aligned}[-1,1] = [-1,-1+2x] &\cup \left[-1+2x, x-\frac{1-x}{2}\right] \\ &\cup \left[x-\frac{1-x}{2}, x-h\varphi_1(x)\right] \\ &\cup [x-h\varphi_1(x), x+h\varphi_1(x)] \\ &\cup \left[x+h\varphi_1(x), 1-\frac{1-x}{2}\right] \cup \left[1-\frac{1-x}{2}, 1\right]. \qquad \square\end{aligned}$$

Proof of Theorem 3.1. For $-1 < \mu < 0$, (3.15) follows from (4.12) and (3.1). Let us prove the theorem in the case $\mu > 0$.

The following inequality (Favard's Theorem) holds (see [**5**]):

$$E_n(h)_2 \le \frac{C}{n} \inf_{P\in\mathbf{P}_{n-1}} \|(h'-P)\sqrt{\varphi}\,\|_2\,, \tag{4.18}$$

with $\varphi(x) = \sqrt{1-x^2}\,$.

By applying (4.18) to $K^\mu f$ in $[\mu]+1$ iterations, one has

$$\begin{aligned} E_n\left(K^\mu f\right)_2 &\le \frac{C}{n^{[\mu]+1}} \inf_{P\in\mathbf{P}_{n-([\mu]+1)}} \left\|(K^\mu f)^{([\mu]+1)} - P\right\|_2 \\ &\le \frac{C}{n^{[\mu]+1}} E_n\left((K^\mu f)^{([\mu]+1)}\right)_2 \\ &\le \frac{C}{n^{[\mu]+1+1-1+\mu-[\mu]}}\|f\|_2 \\ &= \frac{C}{n^{1+\mu}}\|f\|_2, \end{aligned}$$

where we used (3.15) by replacing μ by $-1 < \mu-[\mu]-1 < 0$.

Let us now consider $\mu = 0$. By applying (4.18) to $K^0 f$ we have

$$E_n(K^0 f)_2 \le \frac{C}{n} \inf_{P\in\mathbf{P}_n} \left\|\left[\left(K^0 f\right)' - P\right]\sqrt{\varphi}\right\|_2 \le \frac{C}{n}\left\|\left(K^0 f\right)'\sqrt{\varphi}\right\|_2.$$

Since

$$(K^0 f)'(x) = \int_{-1}^{1} \frac{f(y)}{x-y}\, dy =: Hf(x),$$

with Hf the Hilbert transform of f, using the boundedness of H in $L^2_{\sqrt{\varphi}}$ ([**17**]) one has

$$E_n(K^0 f)_2 \le \frac{C}{n} \|(Hf)\sqrt{\varphi}\|_2 \le \frac{C}{n} \|f\sqrt{\varphi}\|_2 \le \frac{C}{n}\|f\|_2$$

and the theorem is completely proved. □

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