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DECONVOLUTION USING MEYER WAVELETS

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ABSTRACT. In this paper, we have a procedure based on band limited orthogonal wavelets, the Meyer wavelets, to solve convolution equations of the first kind, which is usually an ill-posed problem. The problem will be converted into a well-posed problem in the scaling subspaces, provided that the kernel $k \in L^1(R)$ and $\hat{k}(\omega) \neq 0$. In the case $\hat{k}(\omega)$ has a single zero, we search for a solution in the wavelet subspaces, which can be used to solve the problem numerically. Results related to the convergence rate and error bounds are obtained. However, the stability of the discrete system depends on the resolution level m.

1. Introduction. The problem of deconvolution is pervasive in many applications. It consists of solving the convolution equation:

$$\int_{-\infty}^{\infty} k(t-s)f(s)\,ds = g(t), \quad t \in R, \tag{1}$$

with associated linear operator $\mathbf{K}$ defined by

$$\mathbf{K} : f(t) \longrightarrow \int_{-\infty}^{\infty} k(t-s)f(s)\,ds, \quad t \in R, \tag{2}$$

where $k(t)$ is a known fixed function or distribution, i.e., finding f in terms of g. Such problems arise in mixture problems in statistics; in this case k is a probability measure and $g(t)$ is the density function of the sum of two random variables [**10**]. In signal processing k is the impulse response of a filter and f and g are respectively the input and output [**3**]. In biomathematics, g might be the weight distribution and f the age distribution of a fish population.

The difficulty is that (1) is a first kind integral equation which usually leads to ill-posed problems. The subject has an extensive literature.

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Many procedures have been proposed to solve such problems [**8**], and to improve the approximation, but none is completely satisfactory.

The regularization method, e.g., Tikhonov's method [**8**], or Towney-Phillips's method [**6, 5**], is the most popular one, but it may suffer from the fact that the approximate solution totally depends on the chosen regularization parameter α. The probabilistic methods are based on replacing the function f and g by stationary random processes; however, they need to have certain prior information about f itself to make any progress, see [**1, 2**] and [**8**].

In fact, the problem may not have a solution. This occurs when g is not in the range of the operator. Even if g is in the closure of the range, the solution may not exist since the inverse operator (deconvolution) is usually not continuous.

In this paper we have a procedure based on band limited orthogonal wavelets, the Meyer wavelets. The problem (1) will be converted into a well posed problem in the scaling subspaces, provided that $k \in L^1(R)$ and $\hat{k}(\omega) \neq 0$. In the case $\hat{k}(\omega)$ has a single zero, we look for an approximate solution in both scaling subspaces and wavelet subspaces, which can be used to solve the problem numerically.

Results related to the convergence rate and error bounds are obtained for functions g in a Sobolev space. However, the stability of the resulting discrete system is related to the resolution level m. For larger m it has a larger condition number.

2. Background in Meyer wavelets. There are many wavelet bases created and employed for different purposes. In this work the scaling function of the Meyer wavelet is used to construct a wavelet basis.

Recall the construction of Meyer wavelets. Let h be a probability density function with support in $[-(\pi/3), (\pi/3)]$ and define $\phi(t)$ as the function whose Fourier transform is the nonnegative square root of the integral $\hat{\phi}(\omega) = [\int_{\omega-\pi}^{\omega+\pi} h(x)dx]^{1/2}$. Then $\hat{\phi}$ has support in $[-(4\pi/3), (4\pi/3)]$, and $\hat{\phi} = 1$, for $\omega \in [-(2\pi/3), (2\pi/3)]$. It is easy to check that the orthogonality condition is satisfied [**9**, p. 37]. The mth scaling space V_m is composed of $2^{m+2}\pi/3$ band limited functions. It has good frequency localization but relatively poor time localization. In addition, the Fourier transform of the mother wavelet vanishes in a neighborhood of the origin.

This approach to constructing a wavelet basis is different from the one that is based on finite impulse response filters in the time domain. The associated filters have an infinite number of nonzero coefficients (IIR). For this reason, it is more difficult to apply the Mallat algorithm to this type of wavelet directly. However, we can carry out the decomposition and reconstruction in the frequency domain for Meyer wavelets.

Let $a_{m,n}[f]$ denote the scaling function coefficients of f and $b_{m,n}[f]$ the corresponding wavelet coefficients. Then denote

$$a_m^f(\omega) = 2^{-m/2} \sum_{n=-\infty}^{\infty} a_{m,n}[f] e^{-i(n/2^m)\omega}, \quad m \in Z \tag{3}$$

and

$$b_m^f(\omega) = 2^{-m/2} \sum_{n=-\infty}^{\infty} b_{m,n}[f] e^{-i(n/2^m)\omega}, \quad m \in Z. \tag{4}$$

To simplify the notation, in the case of only one function involved in the discussion, we will drop the superscript f in (3) and(4).

For $f \in L^2(R)$, we have the projections onto the subspaces V_0 and W_0 respectively given by

$$f_0(t) = P_0 f(t) = \sum_{n=-\infty}^{\infty} a_{0,n} \phi(t-n),$$
$$f^0(t) = P^0 f(t) = \sum_{n=-\infty}^{\infty} b_{0,n} \psi(t-n).$$

In the frequency domain, the above equations are expressed by

$$\hat{f}_0(\omega) = a_0(\omega)\hat{\phi}(\omega),$$
$$\hat{f}^0(\omega) = b_0(\omega)\hat{\psi}(\omega).$$

At the resolution level $m = 1$,

$$f_1(t) = \sum_{n=-\infty}^{\infty} a_{1,n} \sqrt{2} \phi(2t-n) = f_0(t) + f^0(t).$$

Correspondingly, in the frequency domain,

$$\hat{f}_1(\omega) = \sum_{n=-\infty}^{\infty} a_{1,n} e^{-i(n/2)\omega} 2^{-1/2} \hat{\phi}\left(\frac{\omega}{2}\right) = a_1(\omega)\hat{\phi}\left(\frac{\omega}{2}\right). \tag{5}$$

We can express a_1 in terms of b_0 and a_0 by using the dilation equations $\varphi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty} c_n \phi(2t-n)$, and $\psi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty} d_n \phi(2t-n)$, where $d_n = c_{1-n}(-1)^n$. Then we may write

$$f_0(t) = \sum_{n=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} a_{0,j} c_{n-2j}\right) \sqrt{2}\phi(2t-n),$$

and

$$f^0(t) = \sum_{n=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} b_{0,j} d_{n-2j}\right) \sqrt{2}\phi(2t-n).$$

By (5), we have

$$a_{1,n} = \sum_{k=-\infty}^{\infty} a_{0,k} c_{n-2k} + \sum_{k=-\infty}^{\infty} b_{0,k} d_{n-2k}.$$

We then have,

$$a_1(\omega) = m_0\left(\frac{\omega}{2}\right) a_0(\omega) + e^{-i(\omega/2)} \overline{m_0\left(\frac{\omega}{2}+\pi\right)} b_0(\omega),$$

where $m_0(\omega) = \sum_{k=-\infty}^{\infty} (c_k/\sqrt{2}) e^{-ik\omega}$. These are the same functions that appear in the frequency domain version of the dilation equations.

This is the reconstruction formula, given $a_0(\omega)$ and $b_0(\omega)$, we can find $a_1(\omega)$. This works at each scale to give us the tree algorithm for the reconstruction in the frequency domain. In general, we have the reconstruction algorithm:

(6)

$$a_{m+1}(\omega) = m_0\left(\frac{\omega}{2^{m+1}}\right) a_m(\omega) + e^{-i(\omega/2^{m+1})} \overline{m_0\left(\frac{\omega}{2^{m+1}}+\pi\right)} b_m(\omega).$$

Similarly, we can show that the decomposition algorithm in the frequency domain takes the form:

$$\begin{aligned} a_m(\omega) &= m_0\left(\frac{\omega}{2^{m+1}}\right) a_{m+1}(\omega), \\ b_m(\omega) &= e^{-i(\omega/2^{m+1})}\overline{m_0\left(\frac{\omega}{2^{m+1}}+\pi\right)}\, a_{m+1}(\omega). \end{aligned} \tag{7}$$

Next we state two lemmas related to the properties of Meyer wavelets for later use.

Lemma 1. *Let ϕ be a symmetric scaling function of a Meyer wavelet. Then*

$$\hat{\phi}^*(\omega) := \sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2k\pi) \geqslant \frac{\sqrt{2}}{2}, \quad \textit{for all } \omega \in R. \tag{8}$$

Proof. We observe that the minimum value of $\hat{\phi}(\omega)$ on the interval $[-\pi, \pi]$ occurs at the end points since $\hat{\phi}(\omega)$ is nonincreasing on the interval $[0, (4\pi/3)]$ and is even. But

$$\hat{\phi}(\pi) = \left(\int_0^{2\pi} h(u)\, du\right)^{1/2} = \left(\int_0^{\pi/3} h(u)\, du\right)^{1/2} = \frac{\sqrt{2}}{2} = \hat{\phi}(-\pi).$$

Hence on $[-\pi, \pi]$, $\hat{\phi}(\omega) \geqslant \sqrt{2}/2$. Since $\hat{\phi}^*(\omega)$ is periodic, (8) holds for all $\omega \in R$. $\square$

Lemma 2. *Let ϕ be the same as in Lemma* 1*. Then*

$$\hat{\phi}\left(\frac{\omega}{2}\right) = \frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} + e^{-i(\omega/2)}\, \overline{\frac{\hat{\psi}(\omega)}{\hat{\phi}^*(\omega)}}. \tag{9}$$

Proof. Since $\phi(t)$ is real and nonnegative, we have by the dilation equations in the frequency domain for ϕ and ψ,

$$\hat{\phi}(\omega) + e^{-i(\omega/2)}\hat{\psi}(\omega) = m_0\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) + m_0\left(\frac{\omega}{2}+\pi\right)\hat{\phi}\left(\frac{\omega}{2}\right).$$

On the other hand, we have

$$m_0\left(\frac{\omega}{2}\right) = \sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 4k\pi), \quad \text{and}$$
$$m_0\left(\frac{\omega}{2} + \pi\right) = \sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 4k\pi + 2\pi),$$

and therefore

$$m_0\left(\frac{\omega}{2}\right) + m_0\left(\frac{\omega}{2} + \pi\right) = \sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2k\pi) = \hat{\phi}^*(\omega).$$

By Lemma 1, $\hat{\phi}^*(\omega) \geqslant \sqrt{2}/2$, so its reciprocal exists and is continuous (at least), and the conclusion follows. □

Kernels with nonvanishing Fourier transform. In this section we deal with the case when $\hat{k}(\omega) \neq 0$. Thus a solution to (1) is given formally by taking the inverse Fourier transform of (1), if it exists,

$$\hat{f}(\omega) = \hat{g}(\omega)/\hat{k}(\omega). \tag{10}$$

Unfortunately, the inverse Fourier transform may not exist (even in the sense of distributions). To overcome this shortcoming we use the Meyer wavelets introduced in the last section. We approximate g by its projection

$$g_m = \mathbf{P}_m g \in V_m,$$

for which (10) has an inverse Fourier transform. Consequently, this gives a function in V_{m+1} which is an approximation, in some sense, to f.

We start with an important lemma.

Lemma 3. *Let $g \in V_m$, let $k \in L^1(R)$, and let $\hat{k}(\omega) \neq 0$, for $\omega \in [-2^{(m+1)}(4\pi/3), 2^{(m+1)}(4\pi/3)]$. Then the convolution equation* (1) *has a unique solution in the subspace V_{m+1}.*

Proof. Since $k * f = g$, by taking the Fourier transform on both sides, we have

$$\hat{k}(\omega)\hat{f}(\omega) = \hat{g}(\omega),$$

and hence, since $\hat{g}(\omega)$ has support in the same interval,

$$\hat{f}(\omega) = \begin{cases} (\hat{g}(\omega)/\hat{k}(\omega)) & |\omega| \leq 2^m(4\pi/3), \\ 0 & |\omega| \geq 2^m(4\pi/3), \end{cases}$$

is a formal solution of (1).

If $g \in V_m$, then we may write

$$g(t) = \mathbf{P}_m g(t) = g_m(t) = \sum_{n=-\infty}^{\infty} a_{m,n}[g] 2^{m/2} \phi(2^m t - n),$$

where

$$a_{m,n}[g] := \int_{-\infty}^{\infty} g(t) 2^{m/2} \phi(2^m t - n)\, dt.$$

We then have

$$\hat{g}_m(\omega) = \sum_{n=-\infty}^{\infty} g_{m,n} 2^{-m/2} \hat{\phi}\left(\frac{\omega}{2^m}\right) e^{-i(n/2^m)\omega} := a_m^g(\omega) \hat{\phi}\left(\frac{\omega}{2^m}\right),$$

and therefore have

$$\hat{f}(\omega) = \frac{\hat{g}_m(\omega)}{\hat{k}(\omega)} = a_m^g(\omega) \frac{\hat{\phi}(\omega/2^m)}{\hat{k}(\omega)}. \tag{11}$$

Since $\hat{\phi}(\omega/2^{m+1}) = 1$ on the support of $\hat{\phi}(\omega/2^m) \subseteq [-2^m(4\pi/3), 2^m(4\pi/3)]$, we may rewrite (11) as

$$\hat{f}(\omega) = a_m^g(\omega) \frac{\hat{\phi}(\omega/2^m)}{\hat{k}(\omega)} \hat{\phi}\left(\frac{\omega}{2^{m+1}}\right). \tag{12}$$

Notice that $a_m^g(\omega)$ is a periodic function with period $2^{m+1}\pi$. We extend $(\hat{\phi}(\omega/2^m)/\hat{k}(\omega)$ periodically $2^{m+2}\pi$, to get

$$y_{m+1}(\omega) = \sum_{k=-\infty}^{\infty} \frac{\hat{\phi}((\omega + 2^{m+2}\pi k)/2^m)}{\hat{k}(\omega + 2^{m+2}\pi k)}.$$

Since $y_{m+1}(\omega) = \hat{\phi}(\omega/2^m)/\hat{k}(\omega)$ on the support of $\hat{\phi}(\omega/2^m)$, we may write

$$\hat{f}(\omega) = a_m^g(\omega) y_{m+1}(\omega)\hat{\phi}\left(\frac{\omega}{2^{m+1}}\right) = a_{m+1}(\omega)\hat{\phi}\left(\frac{\omega}{2^{m+1}}\right), \tag{13}$$

where $a_{m+1}(\omega) = a_m^g(\omega)y_{m+1}(\omega)$ is a periodic function of period $2^{m+2}\pi$. Therefore $f \in V_{m+1}$. On the other hand, since the Fourier transform of the kernel is continuous and never vanishes in $[-2^{(m+1)}(4\pi/3), 2^{(m+1)}(4\pi/3)]$, the solution is unique. This completes the proof of the lemma. $\square$

In light of Lemma 3, if the righthand side function $g \in V_m$, we then have the expansion

$$f(t) = \mathbf{P}_{m+1} f(t) = \sum_{n=-\infty}^{\infty} a_{m+1,n}[f] 2^{(m+1/2)} \phi(2^{m+1}t - n). \tag{14}$$

Upon taking the Fourier transform, we have

$$a_{m+1}^f(\omega) = a_m^g(\omega) y_{m+1}(\omega),$$

which, on the interval $[-2^{(m+1)}(4\pi/3), 2^{(m+1)}(4\pi/3)]$, may be expressed as

$$a_{m+1}^f(\omega) = a_m^g(\omega)\hat{\phi}\left(\frac{\omega}{2^m}\right)/\hat{k}(\omega). \tag{15}$$

In order to get bounds on the Fourier coefficients of $a_{m+1}^f(\omega)$ in both n and m, we have to make some assumptions on the regularity. In what follows, we assume that $\hat{k}$, $\hat{g}$, and $\hat{\phi} \in C^p(R)$. We also now need the condition that $\hat{k}(\omega) \neq 0$ for all $\omega \in R$, since we allow m to vary.

Under above assumptions, we then have $1/\hat{k} \in C^p(R)$. As a result, $\hat{\phi}(\omega/2^m)/\hat{k}(\omega)$ and $y_{m+1}(\omega) \in C^p(R)$. So it follows easily that the Fourier coefficients in the expansion of $y_{m+1}(\omega)$ have order $O(|n|)^{-p})$ for fixed m. In fact, the Fourier coefficients of a_{m+1}^f are given by

$$\begin{aligned} a_{m,n}[f] &= \frac{1}{2^{m+1}\pi}\int_{-2^{m+1}\pi}^{2^{m+1}\pi} a_{m+1}^f(\omega) e^{-i(n/2^{m+1})\omega}\, d\omega \\ &= \frac{1}{2^{m+1}\pi}\int_{-2^{m+1}\pi}^{2^{m+1}\pi} a_m^g(\omega)\hat{\phi}\left(\frac{\omega}{2^m}\right)/\hat{k}(\omega) e^{-i(n/2^{m+1})\omega}\, d\omega. \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned}
&a_{m+1,n}[f] \\
&= \frac{1}{2^{m+1}\pi} \cdot \left(\frac{-in}{2^{m+1}}\right)^{-p} \int_{-2^{m+1}\pi}^{2^{m+1}\pi} \frac{d^p}{d\omega^p}\left[\frac{a_m^g(\omega)\hat{\phi}(\omega/2^m)}{\hat{k}(\omega)}\right] e^{-i(n/2^{m+1})\omega}\, d\omega \\
&= \frac{1}{2^{m+1}\pi} \cdot \left(\frac{2^{m+1}}{-in}\right)^{p} \int_{-2^{m+1}\pi}^{2^{m+1}\pi} \frac{d^p}{d\omega^p}\left[\frac{a_m^g(\omega)}{\widehat{k}(\omega)}\right] e^{-i(n/2^{m+1})\omega}\, d\omega.
\end{aligned}$$

Consequently, we have half of the proof of

Lemma 4. *The inverse operator of* $\mathbf{K}^{-1}$ *of* $\mathbf{K}$ *exists and is continuous from* V_m *to* V_{m+1} *and has norm satisfying*

$$||\mathbf{K}^{-1}|| \leq k_m, \tag{16}$$

where $k_m = \sup_{|\omega| \leq 2^m (4/3)\pi} \sup|\hat{k}(\omega)|^{-1}$.

Proof. By Lemma 3, for any $g \in V_m$, (1), has a unique solution $f \in V_{m+1}$. This enables us to define

$$\mathbf{K}^{-1} : V_m \longrightarrow V_{m+1}$$

by

$$g \longrightarrow f,$$

where $\mathbf{K}f = g$. Clearly, $\mathbf{K}$ is an injection. Then

$$||\mathbf{K}^{-1}g||_{L^2}^2 = \frac{1}{2\pi} \int_{-2^m(4\pi/3)}^{2^m(4\pi/3)} |g(\omega)/\hat{k}(\omega)|^2 d\omega \leq k_m^2 ||g||_{L^2}^2. \tag{17}$$

The following lemma is parallel to Lemma 3.

Lemma 5. *Let the kernel* $k \in L^1(R)$, *let the function* $f \in V_m$. *Then* $k * f \in V_{m+1}$.

The proof, which is similar to the proof of Lemma 3 for a general kernel k, is omitted.

If $g \notin V_m$, replace g by $g_m = \mathbf{P}_m g$, then (1) becomes

$$k * f = g_m, \tag{18}$$

which has a unique solution $\bar{f}_{m+1} \in V_{m+1}$ by Lemma 3, such that

$$k * \bar{f}_{m+1} = g_m.$$

We should like to know to what extent $\bar{f}_{m+1}$ approximates the "true" solution f. We need another lemma to reach the result that we want.

Lemma 6 [**4**, p. 11]. *Let* $\mathbf{K} : L^1(R) \to L^2(R)$ *be a bounded operator. The following are equivalent*:

(1) $||\mathbf{K}f - g||_{L^2} = \inf\{||\mathbf{K}x - g||_{L^2} : x \in L^1\}$,

(2) $\mathbf{K}f = \mathbf{P}g$, *where* $\mathbf{P}$ *is an orthogonal projection from* $L^2(R)$ *onto* $\overline{\mathit{Range}\,(\mathbf{K})}$.

By combining Lemma 5 and Lemma 6, we have

Corollary 7. *Let the kernel* $k \in L^1(R)$ *be continuous and bounded. Then the least square solution of* (1) *in the space* V_m *satisfies* $KP_m f = g_{m+1}$.

Under certain conditions, we can get an estimation of $||\bar{f}_{m+1} - f||_{L^2(R)}$.

Theorem 8. *Let the kernel* $k \in L^1(R)$ *have a nonvanishing Fourier transform,* $\hat{k}(\omega) \neq 0$, *let* $\hat{\phi}(\omega)$ *be continuous on* R *and let* $1 \leq \alpha \leq \beta$ *be real numbers such that*

(1) *the kernel* $|\hat{k}(\omega)| \geq C^{-1}(1 + \omega^2)^{-\alpha/2}$,

(2) $g \in H^\beta(R)$.

Then we have the estimate

$$||\bar{f}_{m+1} - f||_{L^2(R)} \leq C||g - g_m||_{H^\beta}, \tag{19}$$

or if f *and* $\bar{f}_{m+1} \in H^\gamma(R)$, $\gamma \leq \beta - \alpha$,

$$||\bar{f}_{m+1} - f||_{H^\gamma} \leq C||g - g_m||_{H^{\beta+\gamma}}. \tag{20}$$

Proof. The calculation is simple; we may write

$$\begin{aligned}||\bar{f}_{m+1} - f||_{L^2(R)} &= \int_{-\infty}^{\infty} \left|\frac{\hat{g}(\omega) - \hat{g}_m(\omega)}{\hat{k}(\omega)}\right|^2 d\omega \\ &\le \int_{-\infty}^{\infty} |\hat{g}(\omega) - \hat{g}_m(\omega)|^2 C^2 (1+\omega^2)^{\alpha}\, d\omega \\ &\le C^2 \int_{-\infty}^{\infty} |\hat{g}(\omega) - \hat{g}_m(\omega)|^2 (1+\omega^2)^{\beta}\, d\omega,\end{aligned}$$

and the result follows for (19). A slight change will give (20). □

Remark 1. If we assume that the kernel and righthand side function satisfy (1) and (2) in Theorem 8, then the solution function f must be in $L^2(R)$. Notice that the differentiability of the Fourier transform plays no role in the convergence rate; rather, it is the differentiability of the functions in the time domain that is important. This corresponds to membership in the Sobolev spaces.

Remark 2. In order for condition (1) to hold, the kernel generally cannot be analytic on the strip containing the real line since then the Fourier transform would decay too fast. But many interesting kernels are not analytic. As an example, the bilateral kernel $W(t) = (1/2)e^{-|t|}$ with Fourier transform $\hat{W}(\omega) = (1+\omega^2)^{-1}$ satisfies (1), and has operator bound $k_m = 1 + (2^m(4/3)\pi)^2$. On the other hand, the operator bound in (16) may be increasing very fast as $m \to \infty$. For example, for the Gaussian kernel $k(t) = (1/\sqrt{2\pi})e^{-(t^2/2)}$, with $\hat{k}(\omega) = e^{-(\omega^2/2)}$, $k_m = e^{2^{\pi^2(2m-1)16/9}}$, and the hypothesis (1) is not satisfied.

Remark 3. In fact, even for k satisfying the hypotheses of Theorem 8 and $\alpha > 0$, we have

$$||\bar{f}_{m+1}||_{\infty} = \sup_{t\in R}\left|\frac{1}{2\pi}\int_{-2^m(4\pi/3)}^{2^m(4\pi/3)} \frac{\hat{g}_m(\omega)}{\hat{k}(\omega)} e^{it\omega}\, d\omega\right|;$$

therefore $\lim_{m\to\infty} ||\bar{f}_{m+1}||_{\infty} = \infty$ in many cases. This case can still be handled if we consider the function

$$\begin{aligned}\mathbf{E}_f(m,\lambda) &= ||\mathbf{P}_m(f)||_{\infty} + \lambda||g - \mathbf{P}_m(g)||_{\infty} \\ &= ||\bar{f}_{m+1}||_{\infty} + \lambda||g - \mathbf{P}_m(g)||_{\infty},\end{aligned}$$

and notice that $\lim_{m\to\infty} ||\bar{f}_{m+1}||_\infty = \infty$, but $||g - \mathbf{P}_m(g)||_\infty \to 0$, as $m \to \infty$. We can then choose the parameter, λ, such that $\mathbf{E}_f(m, \lambda)$ is minimal with respect to m, as in other regularization methods.

4. Kernels whose Fourier transform contains single zeroes. In the last section the kernels had nonvanishing Fourier transform. However some kernels of interest may have Fourier transforms with isolated zeros. In this section we consider the case of single zeros which, we may assume without loss of generality, is located at zero. We will look for the solution in scaling subspaces first and then in wavelet subspaces.

4.1. The solution in scaling subspaces. We start with the simplest case: the Fourier transform of the kernel only has a simple zero at the origin. In this case, we may write

$$\hat{k}(\omega) = \omega s(\omega), \quad |s(\omega)| > 0.$$

Formally, equation (10) can be written as

$$\omega s(\omega)\hat{f}(\omega) = \hat{g}(\omega),$$

or

$$\omega \hat{f}(\omega) = \frac{\hat{g}(\omega)}{\hat{s}(\omega)}. \tag{21}$$

By the same argument as in the last section, $f' \in V_{m+1}$. This is because

$$\hat{f}'(\omega) = -i\omega \hat{f}(\omega),$$

also $\hat{f}'(0) = 0$, so that $\int_{-\infty}^{\infty} f'(s)\, ds = 0$. Hence

$$f(t) = \int_{-\infty}^{t} f'(s)\, ds = \int_{t}^{\infty} f'(s)\, ds.$$

If $\hat{f}' \in C^p(R)$, $p > 1$, then, since it has compact support, $f'(t) = O(|t|^{-p})$ as in the last section and $f(t) = O(|t|^{-p+1})$ and therefore belongs to $L^2(R)$.

Furthermore, the support of $\hat{f}$ is the same as that of $\hat{f}'$ and by a similar argument of Lemma 3, it follows that $f \in V_{m+2}$.

If the zero of $\hat{k}(\omega)$ is of higher order, then the same argument applies except for requiring the condition $\hat{f}^{(r)} \in C^p(R)$, $p > r$. This gives us

Theorem 9. *Let $g \in V_m$, and let the kernel $k \in L^1(R)$ satisfy the condition that, except for a single zero of multiplicity r at 0, $\hat{k}(\omega) \neq 0$. Suppose $\hat{g}, \hat{\phi}, \hat{k} \in C^p(R)$, $p > r$. Then* (1.1) *has a solution in V_{m+r+1}.*

4.2. The solution in wavelet subspaces. We observe that any Meyer wavelet has the property that its Fourier transform vanishes in an interval neighborhood of the origin although the length of the interval is decreasing to zero as the resolution level $m \to -\infty$. On the other hand, we notice that the approach in the previous subsection uses only the fact that $\hat{k}(\omega) \neq 0$, for $\omega \neq 0$. In virtue of this zero property of Meyer wavelets around the origin, we can weaken the hypothesis to allow $\hat{k}$ to be zero in an interval. We now are looking for possible solutions in wavelet subspaces other than in the scaling subspaces and suppose $g \in W_m$, the mth wavelet subspace. We have the following lemma:

Lemma 10. *If $g \in W_m$, the convolution equation* (1) *has a solution in $W_{m-1} \oplus W_m \oplus W_{m+1}$ satisfying*

$$||f^+||^2_{L^2} \leq ||g||^2_{L^2(R)} \sup_{2^{m+1}(\pi/3) \leq |\omega| \leq 2^{m+3}(\pi/3)} \{|\hat{k}(\omega)|^{-1}\}. \tag{22}$$

In general, if $g \in \oplus_{m_1}^{m_2} W_k$, $m_1 \leq m_2$, (1) *has a solution in $\oplus_{m_1-1}^{m_2+1} W_k$ satisfying*

$$||f||_{L^2} \leq ||g||^2_{L^2(R)} \sup_{2^{m_1+1}(\pi/3) \leq |\omega| \leq 2^{m_2+3}(\pi/3)} \{|\hat{k}(\omega)|^{-1}\}. \tag{23}$$

Proof. Suppose $g \in W_m$, then we have:

$$\hat{g}(\omega) = b^g_m(\omega) \hat{\psi}\left(\frac{\omega}{2^m}\right),$$

and therefore $\hat{g}$ has support in $[2^m(2\pi/3), 2^{m+1}(4\pi/3)] \cup [-2^{m+1}(4\pi/3), -2^m(2\pi/3)]$.

As before we rewrite $\hat{f}(\omega)$ as

$$\hat{f}(\omega) = \frac{\hat{g}(\omega)}{\hat{k}(\omega)} = \frac{b_m^g(\omega)\hat{\psi}(\omega/2^m)}{\hat{k}(\omega)},$$

where by Lemma 3, $f \in V_{m+2}$. However, $\hat{f}(\omega) = 0$ for $|\omega| \leq 2^{m-1}(4\pi/3)$. Thus $\mathbf{P}_k f = 0$, for all $k \leq m-1$. Hence, since $V_{m+2} = \oplus_{k=-\infty}^{m+2} W_k$, we have $f \in W_{m-1} \oplus W_m \oplus W_{m+1}$. We denote this solution by f^+.

To get the bound (22), we calculate,

$$\begin{aligned} ||f_m^+||_{L^2}^2 &= \frac{1}{2\pi}\left\{ \int_{2^{m+1}(\pi/3)}^{2^{m+3}(\pi/3)} + \int_{-2^{m+3}(\pi/3)}^{-2^{m+1}(\pi/3)} \right\} \left| \frac{\hat{g}(\omega)}{\hat{k}(\omega)} \right|^2 d\omega \\ &\leq ||\hat{g}||_{L^2}^2 \sup_{(2^{m+1}\pi/3) \leq |\omega| \leq (2^{m+3}\pi/3)} \{|k(\omega)|^{-2}\}. \end{aligned}$$

In the case that $g \in \oplus_{m_1}^{m_2} W_k$, we write g as,

$$g(t) = g^{m_1}(t) + g^{m_1+1}(t) + \cdots + g^{m_2}(t),$$

where $g^k = \mathbf{P}^k g \in W_k$, $m_1 \leq m_2$. Consider the convolution equations

$$k * f = g^k, \quad m_1 \leq k \leq m_2. \tag{24}$$

Each of the equations has a solution in $W_{k-1} \oplus W_k \oplus W_{k+1}$; denote this solution by $\tilde{f}_{k-1} + \tilde{f}_k + \tilde{f}_{k+1}$, $\tilde{f}_{k+i} \in W_{k+i}$, $i = -1, 0, 1$. By the linearity of the convolution operator, we have, by combining the $\tilde{f}_k$, that

$$f = f_{m_1-1}^+ + f_{m_1}^+ + f_{m_1+1}^+ + \cdots + f_{m_2+1}^+, \quad f_k^+ \in W_k,$$

is a solution of (1) (here we have relabeled each of the components of f). The result will follow from a routine induction argument. □

As in Section 3, we consider the case when the righthand side function $g \notin \oplus_{m_1}^{m_2} W_k$. In this case, we replace the function g by its projection on the subspace $\oplus_{m_1}^{m_2} W_k$, denoted by $g_{m_1}^{m_2}$.

Since
$$V_{m_2+1} = V_{m_1} \oplus \left(\bigoplus_{m_1}^{m_2} W_k \right),$$
it follows that $g_{m_1}^{m_2} = g_{m_2+1} - g_{m_1}$, where g_k is the projection of g onto V_k.

We then consider the convolution equation:

$$k * f(t) = g_{m_1}^{m_2}(t). \tag{25}$$

By Lemma 10, (25) has a solution $f_{m_1}^{m_2} \in \oplus_{m_1-1}^{m_2+1} W_k$. Recall $L^2(R) = \oplus_{k=-\infty}^{\infty} W_k$, and for any $f \in L^2(R)$ we have the expansions of the projections onto V_{m_2+1},

$$\begin{aligned} f_{m_2+1}(t) = \mathbf{P}_{m_2+1} f(t) &= \sum_{n=-\infty}^{\infty} (f, \phi_{m_2+1,n}) \phi_{m_2+1,n}(t) \\ &= \sum_{k=-\infty}^{m_2+1} \sum_{n=-\infty}^{\infty} (f, \psi_{k,n}) \psi_{k,n}(t). \end{aligned}$$

Therefore

$$f_{m_2+1}(t) - f_{m_1-2}(t) = \sum_{k=m_1-1}^{m_2+1} \sum_{n=-\infty}^{\infty} (f, \psi_{k,n}) \psi_{k,n}(t) = \sum_{k=m_1-1}^{m_2+1} f^k(t),$$

where

$$f^k(t) = \sum_{n=-\infty}^{\infty} (f, \psi_{k,n}) \psi_{k,n}(t) \in W_k, \quad m_1 - 1 \le k \le m_2 + 1.$$

In order to get the rate of convergence, we need following lemma [**7**]:

Lemma 11. *Let $f \in H^\alpha$. Then $||f_m - f||_{H^\beta} \le C_{\alpha\beta} ||f||_{H^\alpha} 2^{(\beta-\alpha)m}$, where $\alpha > \beta \ge 0$ and $C_{\alpha\beta}$ is a constant independent of m and f.*

Proof. We calculate

$$
\begin{aligned}
||f_m - f||^2_{H^\beta} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}_m(\omega) - \hat{f}(\omega)|^2(1+\omega^2)^\beta\, d\omega \\
&= \frac{1}{2\pi}\bigg\{\int_{(|\omega|/2^m)\ge(5\pi/4)} + \int_{(3\pi/4)\le(|\omega|/2^m)\le(5\pi/4)} \\
&\qquad\qquad + \int_{(|\omega|/2^m)\le(3\pi/4)}\bigg\}\, d\omega \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Then we have $I_3 = 0$ and

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{(|\omega|/2^m)\ge(5\pi/4)} |\hat{f}_m(\omega) - \hat{f}(\omega)|^2(1+\omega^2)^\beta\, d\omega \\
&= \frac{1}{2\pi}\int_{(|\omega|/2^m)\ge(5\pi/4)} |\hat{f}(\omega)|^2(1+\omega^2)^\beta\, d\omega \\
&= \frac{1}{2\pi}\int_{(|\omega|/2^m)\ge(5\pi/4)}^{\infty} |\hat{f}(\omega)|^2(1+\omega^2)^\alpha \frac{(1+\omega^2)^\beta}{(1+\omega^2)^\alpha}\, d\omega \\
&\le \frac{1}{2\pi}\left(\frac{4}{5\pi}\right)^{2(\alpha-\beta)} 2^{-2m(\alpha-\beta)}||f||^2_{H^\alpha}.
\end{aligned}
$$

A similar calculation will lead to the bound:

$$
I_2 \le \frac{1}{\pi}\left(\frac{4}{3\pi}\right)^{2(\alpha-\beta)} 2^{-2m(\alpha-\beta)}||f||^2_{H^\alpha}.
$$

Therefore, we have

$$
||f_m - f||_{H^\beta} \le C_{\alpha\beta}||f||_{H^\alpha}2^{-m(\alpha-\beta)},
$$

where $C_{\alpha\beta}$ is a constant only dependent on α and β. $\square$

Now we are in the position to state

Theorem 12. *Let $k \in L^1(R)$ and $\hat{k}(\omega) \ne 0$, except at $\omega = 0$; let $\hat{\phi}$ be continuous on R; let $\delta \ge 0$, $\gamma \ge 0$ and let*

(1) $|\omega|^{-\delta}|\hat{k}(\omega)| \geq C^1$, *as* $\omega \to 0$ *and*

(2) $\omega|^{\gamma}|\hat{k}(\omega)| \geq C_1$, *as* $|\omega| \to \infty$.

Then if

(3) $\hat{g}(\omega) = O(|\omega|^{\rho})$, *as* $\omega \to 0$, $\rho \geq \delta$ *and*

(4) $g \in H^{\alpha}(R)$, $\alpha \geq \gamma$,

there is a solution $f \in H^{\beta}(R)$ *such that*

$$(26) \quad ||f - f_{m_1}^{m_2}||^2_{L^2(R)} \leq C_{\alpha,\beta,\gamma}[2^{m_1} + 2^{-2(\alpha-\beta)m_2}]^2 \cdot \max\{2^{-2m_1\delta}, 2^{2m_2\gamma}\}$$

where $C_{\alpha,\beta,\gamma}$ *is a constant independent of* m_1 *and* m_2. *If* $\alpha - \beta > 1$, $\delta < 1$, $\gamma < 1$, *then*

$$||f - f_{m_1}^{m_2}||^2_{H^\beta} \to 0, \quad as\ m_2 \to \infty, \quad m_1 \simeq -m_2.$$

Proof. From conditions (1), and (3), we know that $\hat{g}(\omega)/\hat{k}(\omega)$ is bounded near 0, and from conditions (2) and (4) it belongs to $H^{\beta}(R)$. Hence (1) has a solution f in $H^{\beta}(R)$. We denote by $f_{m_1}^{m_2}$ the solution to (25) and by $d\beta(\omega) = (\omega^2+1)^{\beta}\, d\omega$. We then write

$$\begin{aligned}(27) \quad ||f - f_{m_1}^{m_2}||^2 &\leq \frac{1}{2\pi}\int_{-\infty}^{-2^{m_2+4}(\pi/3)} |\hat{f}|^2\, d\beta \\ &\quad + \frac{1}{2\pi}\int_{2^{m_1}(\pi/3)}^{2^{m_1}(\pi/3)} |\hat{f} - \hat{f}_{m_1}^{m_2}|^2\, d\beta \\ &\quad + \frac{1}{2\pi}\int_{2^{m_2+4}(\pi/3)}^{\infty} |\hat{f}|^2\, d\beta \\ &\quad + \frac{1}{2\pi}\int_{2^{m_1}(\pi/3)\leq|\omega|\leq 2^{m_2+4}(\pi/3)} |\hat{f} - \hat{f}_{m_1}^{m_2}|^2\, d\beta \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Then we have, since $f \in H^{\alpha-\gamma}$ as well, that

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{-\infty}^{-2^{m_2+4}(\pi/3)} |\hat{f}(\omega)|^2 (1+\omega^2)^{\beta}\, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{-2^{m_2+4}(\pi/3)} |\hat{f}(\omega)|^2 (1+\omega^2)^{(\alpha-\gamma)} \frac{1}{(1+\omega^2)^{(\alpha-\gamma-\beta)}}\, d\omega \\
&\le \frac{1}{2\pi} \cdot \frac{1}{2^{2(\alpha-\gamma-\beta)(m_2+4)}(\pi/3)^{2\beta}} ||f||^2_{H^{(\alpha-\gamma)}}.
\end{aligned} \tag{28}
$$

Similarly, we have,

$$
I_3 \le \frac{1}{2\pi} \cdot \frac{1}{2^{2(\alpha-\gamma-\beta)(m_2+4)}(\pi/3)^{2\beta}} ||f||^2_{H^{(\alpha-\gamma)}}. \tag{29}
$$

For I_2, we have,

$$
I_2 = \frac{1}{2\pi} \int_{-2^{m_1}(\pi/3)}^{2^{m_1}(\pi/3)} |\hat{f}|^2 d\beta \le \frac{1}{3} 2^{m_1} ||f||^2_{H^{\beta}}. \tag{30}
$$

By using the fact that

$$
g_{m_1}^{m_2} = g_{m_2+1} - g_{m_1},
$$

and the result from Lemma 11 that $||g - g_m||^2_{H^\beta} = O(2^{-2m(\alpha-\beta)})$, we get

$$
\begin{aligned}
I_4 &= \frac{1}{2\pi} \int_{2^{m_1}(\pi/3) \le |\omega| \le 2^{m_2+4}(\pi/3)} \left| \frac{\hat{g}(\omega) - \hat{g}_{m_1}^{m_2}(\omega)}{\hat{k}(\omega)} \right|^2 (1+\omega^2)^{\beta}\, d\omega \\
&\le \frac{1}{2\pi} \sup \frac{1}{|\hat{k}(\omega)|^2} \Bigg[(||g - g_{m_2+1}||_{H^\beta} \\
&\qquad + \left\{ \int_{(\pi/3) \le |\omega/2^{m_1}| \le (4\pi/3)} |\hat{g}_{m_1}|^2\, d\beta \right\}^{1/2} \Bigg]^2,
\end{aligned} \tag{31}
$$

where the sup is taken over the interval $2^{m_1}(\pi/3) \le |\omega| \le 2^{m_1}(4\pi/3)$. This leads to the bounds

$$
I_4 \le \max(2^{-2m_1\delta}, 2^{2(\gamma-\beta)m_2}) \cdot (C'' 2^{-m_2\alpha} + C' 2^{\beta m_1})^2.
$$

By combining these bounds with (27) to (31), we obtain (26). The final result follows by taking $m_1 = -m_2 = -m$, and letting $m \to \infty$. This last inequality then becomes

$$I_4 \leq \max(2^{2m\delta}, 2^{2(\gamma-\beta)m}) \cdot (C'' 2^{-m\alpha} + C' 2^{-\beta m})^2,$$

which completes the proof. $\square$

Remark 4. The results in this paper have been worked out only for the Meyer type wavelets. We have used the fact that integration and differentiation is possible in the multiresolution subspaces associated with such wavelets and generally will change the scale by at most one. This property does not hold for any other wavelets, e.g., if we differentiate a Daubechies scaling function, the derivative will not be an element in any of the subspaces in the multiresolution analysis. Thus a different approach is needed in these cases. This will be the subject of a future work.

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