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ON THE EFFICIENT DISCRETIZATION OF INTEGRAL EQUATIONS OF THE THIRD KIND

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ABSTRACT. A new discretization scheme for solving ill-posed integral equations of the third kind is proposed. We show that when this scheme is combined with Morozov's discrepancy principle for Landweber iteration, the resulting method is more efficient than the collocation method, in terms of the order of the number of arithmetic operations required to achieve a given accuracy in the approximate solution.

1. Introduction. In his fundamental papers on integral equations, Hilbert [**5**] introduced the notion of integral equations of the first, second and of the third kind. A linear integral equation

$$rx + Kx \equiv r(t)x(t) + \int_0^1 k(t,\tau)x(\tau)\,d\tau = y(t) \tag{1}$$

is said to be of the first kind if $r \equiv 0$, of the second kind if r is a nonzero constant, and of the third kind if r is a function with zeros in its domain (if r is never zero the equation is equivalent to an equation of the second kind). If the function r is continuous and has a finite number of zeros, then equation (1) is a special type of nonelliptic singular integral equation investigated by Prössdorf [**11**]. For functions r with known zeros approximate methods for solving integral equation (1) were proposed by Gabbasov, see, for example, [**3**]. But these methods are completely unusable if r is, for example, a characteristic function of a proper subset of positive measure. Moreover, as indicated in [**12**], if for each neighborhood V of zero the inverse $r^{-1}(V)$ of V has positive measure, then the problem of solving the equation (1) is not well posed in the sense of Hadamard and therefore regularization techniques are required for solving (1). In our opinion it makes sense to apply the regularization methods, even when the function r has a finite number of zeros with unknown locations.

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Usually, the application of a regularization method is preceded by the discretization of the problem and there is a close connection between the amount of discrete information and the choice of the regularization parameter. The aim of this paper is to discuss this connection for the approximate solution of ill-posed equations (1). Moreover, some estimate for the number of arithmetic operations required in order to reach fixed accuracy ε will be obtained.

2. The discretization scheme. Throughout this paper we shall consider the integral equations (1) with operators K acting continuously from L_2 to the Sobolev space W_2^1 and with $y \in W_2^1$ where L_2 is the Hilbert space of square-summable functions on $[0,1]$ with the usual norm $\|\cdot\|$ and the usual inner product $(\cdot,\cdot)$ and W_2^1 is the normed space of functions $f(t)$ having square-summable derivatives $f' \in L_2$. The norm on W_2^1 is

$$\|f\|_{W_2^1} = \|f\| + \left\|\frac{d}{dt}f\right\|.$$

Moreover, it will be assumed that the operators K have some additional smoothness. Namely, with $\gamma = (\gamma_1, \gamma_2, \gamma_3)$,

$$K \in K_\gamma^1 := \Bigg\{K : \|K\|_{L_2 \to W_2^1} \le \gamma_1, \\ \|K^*\|_{L_2 \to W_2^1} \le \gamma_2, \left\|\left(\frac{d}{dt}K\right)^*\right\|_{L_2 \to W_2^1} \le \gamma_3\Bigg\}$$

where $\|\cdot\|_{X \to Y}$ is the usual norm in the space of all linear bounded operators from X into Y; B^* denotes the adjoint operator of $B : L_2 \to L_2$. If the kernel $k(t,\tau)$ of the integral operator K has mixed partial derivatives and

$$\int_0^1 \int_0^1 \left[\frac{\partial^{i+j} k(t,\tau)}{\partial t^i \partial \tau^j}\right]^2 d\tau\, dt < \infty, \quad i,j = 0,1,$$

then it is easy to see that $K \in \mathcal{K}_\gamma^1$ for some γ.

Let us consider the Haar orthonormal basis $\chi_1, \chi_2, \dots, \chi_m, \dots$ of piecewise constant functions, where $\chi_1(t) \equiv 1$, and for $m = 2^{k-1} + j$,

$k = 1, 2, \ldots; \ j = 1, 2, \ldots, 2^{k-1}$

$$\chi_m(t) = \begin{cases} 2^{(k-1)/2} & t \in [(j-1)/2^{k-1}, (j-1/2)/2^{k-1}), \\ -2^{(k-1)/2} & t \in [(j-1/2)/2^{k-1}, j/2^{k-1}), \\ 0 & t \notin [(j-1)/2^{k-1}, j/2^{k-1}], \end{cases}$$

and let P_m be the orthogonal projector on span $\{\chi_1, \chi_2, \ldots, \chi_m\}$, that is,

$$P_m f(t) = \sum_{i=1}^{m} (f, \chi_i)\chi_i(t).$$

It is well known that [**6**, pp. 81, 82]

$$\|I - P_m\|_{W_2^1 \to L_2} \leq cm^{-1}, \tag{2}$$

where I is the identity operator and c is some absolute constant. Moreover, if $|r'(t)| \leq d$, then for any $t \in [0, 1]$

$$|r(t) - P_m r(t)| \leq 3dm^{-1}. \tag{3}$$

To construct an efficient method for discretizing ill-posed equations (1), we shall use a specific "hyperbolic cross" approximation of the kernel function $k(t, \tau)$. This means that instead of (1) we consider now the equation

$$xP_{2^n} r + K_n x = P_{2^n} y, \tag{4}$$

where

$$\begin{aligned} K_n &= \sum_{k=1}^{n} (P_{2^k} - P_{2^{k-1}})KP_{2^{n-k}} + P_1 K P_{2^n} \\ &= \sum_{(i,j) \in \Gamma_n} \chi_i(\chi_i, K\chi_j)(\chi_j, \cdot), \end{aligned}$$

with

$$\Gamma_n = \{1\} \times [1, 2^n] \bigcup_{k=1}^{n} (2^{k-1}, 2^k] \times [1, 2^{n-k}].$$

It is obvious that K_n is the integral operator with degenerate kernel

$$k_n(t, \tau) = \sum_{(i,j) \in \Gamma_n} \hat{k}(i, j)\chi_i(t)\chi_j(\tau),$$

where $\hat{k}(i,j)$ denotes the Fourier coefficients of function $k(t,\tau)$ with respect to the Haar system, i.e.,

$$\hat{k}(i,j) = \int_0^1 \int_0^1 k(t,\tau)\chi_i(t)\chi_j(\tau)\,dt\,d\tau.$$

Let $\text{card}\,(\Gamma_n)$ be the number of Fourier coefficients $\hat{k}(i,j)$ required to construct $k_n(t,\tau)$. It is easily verified that

$$\text{card}\,(\Gamma_n) \asymp n2^n.$$

As usual, we write $T(u) \asymp S(u)$ if there are constants c and c_1 such that for all u belonging to the domain of definition $T(u)$ and $S(u)$, we have

$$cT(u) \le S(u) \le c_1 T(u).$$

Moreover, for simplicity we often use the same symbol c for possibly different constants.

If we denote by N_{disc} the number of all Fourier coefficients

$$\hat{k}(i,j) = (\chi_i, K\chi_j), \qquad \hat{r}(i) = (r,\chi_i), \qquad \hat{y}(i) = (y,\chi_i) \tag{5}$$

taking part in the definition of equation (4), then

$$N_{\text{disc}} \asymp 2^{n+1} + \text{card}\,(\Gamma_n) \asymp n2^n. \tag{6}$$

Even when we assume for the moment that the solution of (4) exists and is unique, the direct solution of (4) would take too many arithmetic operations. Instead we consider iterative methods. We are especially interested in iterations associated with regularization methods. In this paper we consider the Landweber iteration

$$\begin{gathered} x_{m,n} = x_{m-1,n} - \mu B_n^*(B_n x_{m-1,n} - P_{2^n} y), \\ m = 1,2,\dots, x_{0,n} = 0, \end{gathered} \tag{7}$$

where

$$\begin{gathered} B_n f = f P_{2^n} r + K_n f, \qquad B_n^* f = f P_{2^n} r + K_n^* f, \\ 0 < \mu < 2/\|B_n\|^2_{L_2 \to L_2}. \end{gathered}$$

Further examples of iterative methods are discussed in [**13**].

The number of iteration steps m acts as a regularization parameter and the usual discussion of rates of convergence of iterative methods for ill-posed equations is done under the assumption that the exact solution $\bar{x}$ of (1) belongs to the range of operator $|B|^p$ for some $p \geq p_0$, where $|B|^p = (B^*B)^{p/2}$ and

$$Bf(t) = r(t)f(t) + Kf(t).$$

Therefore, from now on, we assume that the exact solution of (1) belongs to the class $\Phi^{p_0}_{\gamma,p,d}$, that is, the class of all

$$\begin{gathered} \bar{x} = |B|^p v, \quad \|v\| \leq \rho, \ p \geq p_0 \geq 1 \\ K \in \mathcal{K}^1_\gamma, \quad |r'(t)| \leq d \\ y \in W^1_{2,1} = \{f : f \in W^1_2, \|f\|_{W^1_2} \leq 1\} \end{gathered} \tag{8}$$

for given positive numbers γ, ρ, d and $p_0 \geq 1$.

In what follows, we need

Lemma 1. *Let $K \in \mathcal{K}^1_\gamma$ and $|r'(t)| \leq d$. Then*

$$\|B - B_n\|_{L_2 \to L_2} \leq cn2^{-n},$$

where the constant c depends on γ and d.

Proof. From the definition of the operator K_n, we find

$$P_{2^n}K - K_n = \sum_{k=1}^{n} (P_{2^k} - P_{2^{k-1}})K(I - P_{2^{n-k}}) + P_1 K(I - P_{2^n}).$$

Using an argument like that in the proof of Lemma 3.2 of [**8**] for $K \in \mathcal{K}^1_\gamma$, we get the estimate

$$\|(P_{2^k} - P_{2^{k-1}})K(I - P_{2^{n-k}})\|_{L_2 \to L_2} \leq c2^{-n}.$$

Then, by virtue of (2), we have

$$\begin{aligned}\|K-K_n\|_{L_2\to L_2} &\le \|(I-P_{2^n})K\|_{L_2\to L_2}+\|P_{2^n}K-K_n\|_{L_2\to L_2}\\ &\le c2^{-n}\|K\|_{L_2\to W_2^1}\\ &\quad+\sum_{k=1}^{n}\|(P_{2^k}-P_{2^{k-1}})K(I-P_{2^{n-k}})\|_{L_2\to L_2}\\ &\quad+\|P_1K(I-P_{2^n})\|_{L_2\to L_2}\\ &\le c2^{-n}\gamma_1+cn2^{-n}+\|(I-P_{2^n})K^*\|_{L_2\to L_2}\\ &\le c2^{-n}(\gamma_1+\gamma_2)+cn2^{-n}\\ &\le cn2^{-n}.\end{aligned}$$

Using this bound and (3), we obtain the estimate

$$\begin{aligned}\|B-B_n\|_{L_2\to L_2} &\le \max_{0\le t\le 1}|r(t)-P_{2^n}r(t)|+\|K-K_n\|_{L_2\to L_2}\\ &\le 3d2^{-n}+cn2^{-n}\le cn2^{-n},\end{aligned}$$

as claimed.

An appropriate discretization (4) and the number of iteration steps m depending on a predetermined order of accuracy $O(\varepsilon)$ for $\|\bar{x}-x_{m,n}\|$ have to be chosen. One of the most widely used strategies for choosing the regularization parameter m, which are also called "stopping rules" in the literature, is Morozov's discrepancy principle. We shall consider this discrepancy principle in the form tailored to the discretized version of Landweber iteration (7) for equations (1) from $\Phi^{p_0}_{\gamma,\rho,d}$:

Let $d_1>1$. A stopping rule for (7) is given by choosing the first integer m such that $m\le m_{\max}\asymp\varepsilon^{-2/p_0}$ and

$$\|P_{2^n}f-B_nx_{m,n}\|\le d_1\varepsilon^{(p_0+1)/p_0}. \tag{9}$$

If there is no $m\le m_{\max}$ such that (9) holds, then choose $m=[m_{\max}]+1$ with $[m_{\max}]$ denoting the largest integer which is not greater than $m_{\max}\asymp\varepsilon^{-2/p_0}$.

Now we can state the main result.

Theorem 2. *Let $n2^{-n}\asymp\varepsilon^{(p_0+1)/p_0}$, and let the number of iteration steps m in* (7) *be chosen according to the discrepancy principle* (9). *If equation* (1) *belongs to the class $\Phi^{p_0}_{\gamma,\rho,d}$, $p_0\ge 1$, then*

$$\|\bar{x}-x_{m,n}\|=O(\varepsilon).$$

Proof. The regularization method (7) is generated by the function

$$g_m(\lambda) = \lambda^{-1}[1 - (1-\mu\lambda)^m], \quad \lambda > 0.$$

Namely, $x_{m,n} = R_{m,n} P_{2^n} y$, where

$$R_{m,n} = g_m(B_n^* B_n) B_n^*.$$

We put $S_{m,n} = I - R_{m,n} B_n$. From [**10**] one sees that

$$\begin{gathered} \|R_{m,n}\|_{L_2\to L_2} \le c_1 m^{1/2}, \qquad \|S_{m,n}\|_{L_2\to L_2} \le c_2, \\ \|I - B_n R_{m,n}\|_{L_2\to L_2} \le 1, \qquad \|S_{m,n}|B_n|^p\|_{L_2\to L_2} \le c_{1,p} m^{-p/2}, \\ \|B_n S_{m,n}|B_n|^p\|_{L_2\to L_2} \le c_{2,p} m^{-(p+1)/2}. \end{gathered} \tag{10}$$

Using (2), (10) and Lemma 1, we find from the definition $x_{m,n}$

$$\begin{aligned} \|\bar{x} - x_{m,n}\| &\le \|S_{m,n}\bar{x} + R_{m,n}[(I - P_{2^n})y - (B - B_n)\bar{x}]\| \\ &\le \|S_{m,n}\bar{x}\| + cm^{1/2}(\|(I - P_{2^n})y\| + \|B - B_n\|_{L_2\to L_2}) \\ &\le \|S_{m,n}\bar{x}\| + cm_{\max}^{1/2}(2^{-n} + n2^{-n}) \\ &\le \|S_{m,n}\bar{x}\| + c\varepsilon^{-1/p_0}\varepsilon^{(p_0+1)/p_0} \\ &\asymp \|S_{m,n}\bar{x}\| + \varepsilon. \end{aligned} \tag{11}$$

Let us estimate $\|S_{m,n}\bar{x}\|$. Using the inequality

$$\||B|^p - |B_n|^p\|_{L_2\to L_2} \le c\|B - B_n\|_{L_2\to L_2}^{\min\{1,p\}} |\ln(\|B - B_n\|_{L_2\to L_2})|,$$

see [**14**, p. 93], (8), (10) and Lemma 1 for $p \ge p_0 \ge 1$ we have

$$\begin{aligned} \|S_{m,n}\bar{x}\| &\le \|S_{m,n}|B_n|^p v\| + \|S_{m,n}(|B|^p - |B_n|^p)v\| \\ &\le c_{1,p}\rho m^{-p/2} + c_2\rho n^2 2^{-n} \\ &\le c(m^{-p/2} + \varepsilon^{(p_0+1)/p_0}\ln(1/\varepsilon)). \end{aligned} \tag{12}$$

If $m > \varepsilon^{-2/p}$, the assertion of the theorem follows from (11) and (12).

Assume now that $m < m_1 = [\varepsilon^{-2/p}] + 1$. With an argument like that in the proof of Theorem 3.3 of [**10**], we get the estimate

$$\|S_{m,n}\bar{x}\|^2 \le c(\|S_{m_1,n}\bar{x}\|^2 + m_1\|B_n S_{m,n}\bar{x}\|^2). \tag{13}$$

On the other hand, from (2), (9), (10) and Lemma 1, we know that

$$\begin{aligned}\|B_n S_{m,n}\bar{x}\| &\le \|P_{2^n}y - B_n x_{m,n}\| \\ &\qquad + \|I - B_n R_{m,n}\|_{L_2\to L_2}(\|(B_n - B)\bar{x}\| + \|y - P_{2^n}y\|) \\ &\le d_1\varepsilon^{(p_0+1)/p_0} + c(n2^{-n} + 2^{-n}) \\ &\asymp \varepsilon^{(p_0+1)/p_0}.\end{aligned}$$

Moreover, using the inequality (12), we obtain

$$\|S_{m_1,n}\bar{x}\|^2 \le c(m_1^{-p/2} + \varepsilon^{(p_0+1)/p_0}\ln(1/\varepsilon))^2 \le c\varepsilon^2.$$

Thus, from (13) one sees that for $m < m_1$ and $p \ge p_0$,

$$\|S_{m,n}\bar{x}\|^2 \le c\varepsilon^2 + m_1\varepsilon^{(2(p_0+1)/p_0)} \le c(\varepsilon^2 + \varepsilon^{2-(2/p)+(2/p_0)}) \asymp \varepsilon^2.$$

Combining this estimate and (11) for $m < m_1$, we have

$$\|\bar{x} - x_{m,n}\| = O(\varepsilon).$$

The theorem is proved. □

Corollary 3. *Let $N_{\rm disc}$ be an amount of discrete information* (5) *required to construct an approximate solution $x_{m,n}$. From Theorem* 2 *and* (6) *it follows that within the framework of discretization scheme* (4) *we can guarantee on the class $\Phi^{p_0}_{\gamma,\rho,d}$ the order of accuracy ε in the case when*

$$N_{\rm disc} \asymp \left(\frac{1}{\varepsilon}\right)^{(p_0+1)/p_0} \ln^2\frac{1}{\varepsilon}.$$

3. Complexity of the algorithm. Let us estimate the number $N_{\rm op}$ of arithmetic operations on the values of Fourier coefficients (5) required to construct an approximate solution $x_{m,n}$.

Proposition 4. *Let*

$$g(t) = \sum_{i=1}^{2^n} g_i\chi_i(t)$$

be an arbitrary element of subspace span $\{\chi_1, \chi_2, \dots, \chi_{2^n}\}$. *To represent an element*

$$f(t) = g(t)P_{2^n} r(t) \in \operatorname{span}\{\chi_1, \chi_2, \dots, \chi_{2^n}\}$$

in the standard form

$$f(t) = \sum_{i=1}^{2^n} f_i \chi_i(t) \tag{14}$$

it suffices to perform no more than $c2^n$ arithmetic operations on the coefficients g_i and $\hat{r}(i)$.

Proof. Note that $g(t)$, $P_{2^n} r(t)$ and $f(t)$ are the constants on the dyadic intervals

$$\Delta_{n,i} = \left(\frac{i-1}{2^n}, \frac{i}{2^n} \right), \quad i = 1, 2, \dots, 2^n.$$

Keeping in mind that, see [**6**, p. 78]

$$P_{2^n} \varphi(t) = 2^n \int_{\Delta_{n,i}} \varphi(\tau)\, d\tau, \quad t \in \Delta_{n,i},$$

for any $t \in \Delta_{n,i}$, $i = 1, 2, \dots, 2^n$, we have

$$\begin{aligned} P_{2^n} f(t) &= 2^n \int_{\Delta_{n,i}} g(\tau) P_{2^n} r(\tau)\, d\tau \\ &= 2^n \int_{\Delta_{n,i}} g(\tau)\, d\tau \cdot 2^n \int_{\Delta_{n,i}} P_{2^n} r(\tau)\, d\tau \\ &= P_{2^n} g(t) P_{2^n} r(t) \\ &= g(t) P_{2^n} r(t) \\ &= f(t). \end{aligned} \tag{15}$$

Thus $f(t) = P_{2^n} f(t) \in \operatorname{span}\{\chi_1, \chi_2, \dots, \chi_{2^n}\}$.

Let us denote by h_{kj}, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, 2^{k-1}$, the Haar functions $\chi_2, \chi_3, \dots, \chi_{2^n}$, labeled by two indices. Namely, for $m = 2^{k-1} + j$,

$$\chi_m(t) = h_{k,j}(t).$$

Then, for any $\varphi \in L_2$, we put $\varphi(k,j) = \hat{\varphi}(2^{k-1}+j)$. We also introduce the averages $\bar{\varphi}(k,j)$ of $\varphi(t)$ on $\Delta_{k,j}$ as

$$\bar{\varphi}(k,j) = |\Delta_{k,j}|^{-1} \int_{\Delta_{k,j}} \varphi(\tau)\, d\tau,$$

where $|\Delta_{k,j}|$ denotes the length of $\Delta_{k,j}$. It is well known, see, for example, [**6**, p. 78], that

$$\begin{aligned} \bar{\varphi}(1,1) &= \hat{\varphi}(1) + \varphi(1,1), \\ \bar{\varphi}(1,2) &= \hat{\varphi}(1) - \varphi(1,1), \end{aligned} \tag{16}$$

and further

$$\begin{aligned} \bar{\varphi}(m,2j-1) &= \bar{\varphi}(m-1,j) + 2^{(m-1)/2}\varphi(m,j), \\ \bar{\varphi}(m,2j) &= \bar{\varphi}(m-1,j) - 2^{(m-1)/2}\varphi(m,j), \\ m &= 2,3,\dots,n; \quad j = 1,2,\dots,2^{m-1}. \end{aligned} \tag{17}$$

It is easy to see that, using (16) and (17), we can compute the averages $\bar{g}(n,i)$ and $\bar{r}(n,i)$, where $i = 1,2,\dots,2^n$, of functions $g(t)$ and $r(t)$, and evaluating the whole set of these averages requires no more than $c2^n$ arithmetic operations on the coefficients g_j, $\hat{r}(j)$.

If the averages $\bar{g}(n,i)$ and $\bar{r}(n,i)$ are known, then by virtue of (15),

$$\bar{f}(n,i) = \bar{g}(n,i)\bar{r}(n,i), \quad i = 1,2,\dots,2^n,$$

and evaluating the whole set of $\bar{f}(n,i)$ requires 2^n multiplications. Now, according to the method for calculating the Haar coefficients [**1**] the rest of the averages $\bar{f}(m,j)$ and the Fourier coefficients $f(m,j)$ can be computed for $m = n, n-1, \dots, 2$; $j = 1,2,\dots,2^{m-1}$, from the formulas

$$\begin{aligned} \bar{f}(m-1,j) &= (\bar{f}(m,2j-1) + \bar{f}(m,2j))/2, \\ f(m,j) &= 2^{-(m+1)/2}(\bar{f}(m,2j-1) - \bar{f}(m,2j)), \\ f(1,1) &= (\bar{f}(1,1) - \bar{f}(1,2))/2, \\ \hat{f}(1) &= (\bar{f}(1,1) + \bar{f}(1,2))/2. \end{aligned}$$

One can see that evaluating the whole set of averages and Fourier coefficients requires $2^{n+1} - 2$ additions and 2^{n+1} multiplications. To complete the proof it only remains for us to note that in the representation (14) we have $f_1 = \hat{f}(1)$ and

$$f_i = f(k,j), \quad k = 1,2,\dots,n, \quad j = 1,2,\dots,2^{k-1}$$

for $i = 2^{k-1} + j$. With an argument like that in the proof of Lemma 18.2 of [**9**, p. 300], we get the following assertion.

Proposition 5. *Let $g(t)$ be an arbitrary element of* span $\{\chi_1, \chi_2, \dots, \chi_{2^n}\}$. *To represent the elements $K_n g$, $K_n^* g \in$ span $\{\chi_1, \chi_2, \dots, \chi_{2^n}\}$ in the standard form* (14) *it suffices to perform no more than $cn2^n$ arithmetic operations on the coefficients $\hat{g}(i)$ and $\hat{k}(i,j)$.*

Theorem 6. *Under the assumptions of Theorem* 2 *one can guarantee on the class $\Phi^{p_0}_{\gamma,\rho,d}$ the order of accuracy ε with*

$$N_{\rm op} = O\biggl(\Bigl(\frac{1}{\varepsilon}\Bigr)^{(p_0+3)/p_0} \ln^2 \frac{1}{\varepsilon}\biggr). \tag{18}$$

Proof. By virtue of (7) for any $m = 1,2,\dots,$ we have

$$\begin{aligned} x_{m,n} &= x_{m-1,n} - \mu\delta_{m-1}P_{2^n}r - \mu K_n^*\delta_{m-1}, \\ \delta_{m-1} &= x_{m-1,n}P_{2^n}r + K_n x_{m-1,n} - P_{2^n}y. \end{aligned}$$

From the definition of operator K_n and (15), one sees that $x_{m,n} \in$ span $\{\chi_1, \chi_2, \dots, \chi_{2^n}\}$ for any m. Let card (AO) be the number of arithmetic operations required for the passage from $x_{m-1,n}$ to $x_{m,n}$. From Theorem 2 and Propositions 4 and 5, it follows that

$$\operatorname{card}(AO) \le cn2^n \asymp \Bigl(\frac{1}{\varepsilon}\Bigr)^{(p_0+1)/p_0} \ln^2 \frac{1}{\varepsilon}.$$

On the other hand, within the framework of stopping rule (9),

$$\begin{aligned} N_{\rm op} &\le m_{\max}\operatorname{card}(AO) \asymp \varepsilon^{-2/p_0}\operatorname{card}(AO) \\ &\asymp \Bigl(\frac{1}{\varepsilon}\Bigr)^{(p_0+3)/p_0} \ln^2 \frac{1}{\varepsilon}, \end{aligned}$$

as claimed. $\square$

Remark. Let us assume that equation (1) belongs to $\Phi^{p_0}_{\gamma,\rho,d}$ but the function $r(t)$ has a finite number of known zeros. In this case the collocation method proposed in [**3**] can be applied. Within the framework of this method finding the approximate solution x_n of (1) reduces to solving a system of $O(n)$ linear algebraic equations. Moreover, from Theorem 1 of [**3**], it follows that

$$\|\bar{x} - x_n\| = O\left(\frac{1}{\sqrt{n}}\right).$$

Then, for guaranteeing accuracy ε, it is necessary to solve the system consisting of $n \asymp \varepsilon^{-2}$ algebraic equations. To solve this system, for example, by Gaussian elimination, it is necessary to perform $N_1 \asymp n^3 \asymp \varepsilon^{-6}$ arithmetic operations. When N_1 is compared with estimation (18), it is apparent that for the class $\Phi^{p_0}_{\gamma,\rho,d}$ the scheme (4) and (7) with stopping rule (9) is more economical than the collocation method of [**3**].

4. Differential equations and integral equations of the third kind. Integral equations of the third kind are closely related to some singular problems in differential equations.

4.1. *Volterra equations.* Let A, B be (n,n)-matrices with entries a_{jk}, b_{jk} and c an n-vector with entries c_j, which are continuous, respectively differentiable real or complex functions.

The system of linear ordinary differential equations

$$Ay' = By + c$$

is a system of differential-algebraic equations, see, e.g., [**4**], if the matrix A is singular. On the other hand, since

$$(Ay)' = A'y + Ay',$$

we have

$$(Ay)(t) = \int_0^t (A'(\tau) + B(\tau))y(\tau)\,d\tau + c(t).$$

This is a system of Volterra equations of the third kind.

In the special case $n = 1$, it is well known, see [**7**, p. 34], that a Volterra integral equation of the first kind is equivalent to a Volterra integral equation of the second kind if the kernel function does not vanish on the diagonal. If the kernel function has zeros on the diagonal, then this equation is equivalent to a Volterra equation of the third kind.

4.2. *Fredholm equations.* Let L be a linear differential operator with a continuous inverse T, let A, B, c be as above. Then the boundary problem

$$L(Ay) = By + c$$

is equivalent to the system of integral equations of the third kind

$$Ay = TBy + Tc.$$

In the case $n = 1$ and if a_{11} has zeros, then we have boundary value problems with "regular" and with "irregular" singularities, see, e.g., [**2**, p. 299].

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