JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 11, Number 4, Winter 1999

NONLINEAR BOUNDARY INTEGRAL EQUATIONS FOR HARMONIC PROBLEMS

M. GANESH AND O. STEINBACH

ABSTRACT. Novel first kind Steklov-Poincaré and hypersingular operator boundary integral equations with nonlinear perturbations are proposed to solve harmonic problems in two and three dimensional Lipschitz domains with nonlinear boundary conditions. The equivalence and regularity of the solutions of the formulations are described. To initiate computational procedures for the solution of nonlinear boundary integral equations, a standard Newton scheme is analyzed and corresponding convergence results are given.

1. Introduction. In this work we are interested in computing an isolated harmonic solution u of the nonlinear boundary value problem described by

$$\Delta u(x) = 0 \quad \text{for } x \in \Omega \subset \mathbf{R}^n, \quad n = 2, 3 \tag{1.1}$$

and the nonlinear boundary condition

$$\frac{\partial}{\partial n_x} u(x) + g(x, u(x)) = f(x) \quad \text{for } x \in \Gamma, \tag{1.2}$$

where Ω is a bounded domain with a Lipschitz boundary Γ. In (1.2), n_x is the outer normal unit vector defined almost everywhere for $x \in \Gamma$ and $f : \Gamma \to \mathbf{R}$, $g : \Gamma \times \mathbf{R} \to \mathbf{R}$ are given functions.

We assume that the following hold:

(A0) $f \in L^2(\Gamma)$.

(A1) (1.1) and (1.2) have an isolated solution $u \in H^{1+s}(\Omega)$ for some $s \geq 1/2$.

Received by the editors on October 30, 1998, and in revised form on May 28, 1999.
AMS *Mathematics Subject Classification.* 45B05, 65R20.
Key words and phrases. Boundary integral equations, nonlinear boundary conditions, Newton method.

(A2) For all $x \in \Gamma$, $g(x, \cdot) : \mathbf{R} \to \mathbf{R}$ is twice differentiable and the derivatives are locally bounded, i.e., for every finite interval $[a, b]$, there exist a constant $M_{[a,b]}$ such that

$$\left| \frac{\partial^i g(x, \alpha)}{\partial^i \alpha} \right| \leq M_{[a,b]} \quad \text{for } x \in \Gamma,\ a \leq \alpha \leq b,\ i = 1, 2.$$

We use the standard notation $H^s(\Gamma)$ for the usual Sobolev space on Γ with its dual $H^{-s}(\Gamma)$ and the norm $\| \cdot \|_s$. With $\langle \cdot , \cdot \rangle$ we denote the duality pairing in $L^2(\Gamma)$. We also use the standard notion of *isolated* solution in (A1), which plays an important role throughout the paper: *u is an isolated solution of* (1.1) *and* (1.2) *if it has a neighborhood containing no other solutions of* (1.1) *and* (1.2).

The boundary value problem (1.1) and (1.2) has many applications in applied science and engineering, see, for example, [**3, 12, 18, 20**] and further references cited in these literature.

The existence, uniqueness and numerical aspects of (1.1) and (1.2) for the smooth boundary case using a direct boundary integral formulation with the single and double layer potentials and the corresponding Galerkin discretization were first initiated in [**23**] (with global monotonicity, Lipschitz and linear growth conditions on the nonlinear function required for both existence theory and numerical treatment). This problem was further investigated in [**2, 9, 10, 11, 22**] using Galerkin, collocation or Nystrom methods and variants of these techniques for the smooth boundary case and in [**8**] using the standard collocation method for the polygonal boundary case.

In this paper we derive some novel nonlinear boundary integral equations which are equivalent to (1.1) and (1.2) and having excellent computational advantages; for example, the nonlinearity does not appear as a density of boundary integral operators. Our minimal assumption (A2) on the nonlinear function allows us to consider a wider class of nonlinear problems. For example, we may consider a test model problem (1.1) and (1.2) with $g(x, \alpha) = \exp(\alpha)$ and $f(x) = c$, a nonnegative constant. In this test case (A0)–(A2) hold with $u(x) = \log(c)$. By *wider class*, we mean allowing nonlinear functions which do not satisfy strong conditions required in [**23**] for analysis of our numerical methods, having established existence of an isolated solution by some technique. Finally, the symmetric and elliptic structures present in our formulations have marked advantages in both analysis and computational procedures.

The first reformulation approach is based on a boundary integral representation of the Steklov-Poincaré operator realizing the Dirichlet-Neumann map. This formulation involves the nonlinear term in its simplest form as in (1.2) (and it provides the best possible orders of convergence for a Galerkin scheme). However, due to the complicated structure of the Steklov-Poincaré operator, its realization is costly to compute. The second approach is based on using an indirect method, and we reformulate (1.1) and (1.2) as an equivalent hypersingular-double layer nonlinear boundary integral equation with integrands not involving the nonlinear term. This new formulation is computationally efficient, but the solution has some regularity restrictions compared with the Steklov-Poincaré operator formulation.

Our first step in solving the nonlinear boundary integral equations is to use a Newton scheme to approximate the continuous nonlinear problems by a sequence of perturbed linear first kind boundary integral equations involving either the Steklov-Poincaré operator or the hypersingular integral operator. We show that each of these continuous linear problems is stable and the solutions converge to an isolated solution of the associated nonlinear boundary integral equation.

This paper is mainly devoted to the formulation of nonlinear boundary integral equations related to (1.1) and (1.2) and to initiate computational procedures by first proposing and analyzing the Newton methods. Our novel formulations and the linearization process itself, we believe, will have spin-off in many applications. Due to involved computational procedures and the corresponding analysis, we describe Galerkin boundary element methods and preconditioners for the numerical solution of the linearized equations in our sequel work [**13**]. Combining the Steklov-Poincaré operator formulation and the indirect hypersingular integral formulation, in [**13**] we propose a hybrid solution strategy yielding almost optimal results with respect to the order of convergence and the amount of numerical work.

We organize the rest of the paper as follows. In Section 2 we reformulate the nonlinear boundary value problem (1.1) and (1.2) as equivalent nonlinear boundary integral equations using the Steklov-Poincaré operator, an indirect and a direct hypersingular integral formulation. In Section 3 we introduce an iterative scheme to linearize the continuous nonlinear problems and prove the existence, uniqueness and convergence results.

Throughout the paper, by c we will denote a general constant which may have different values at different occurrences.

2. Nonlinear boundary integral formulations. To reformulate the nonlinear boundary value problem (1.1) and (1.2) as an equivalent nonlinear boundary integral equation, one may use either a direct or an indirect approach. In a direct boundary integral method both Cauchy data $[u(x), (\partial/\partial n_x)u(x)]\big|_{x\in\Gamma}$ are linked via two basic boundary integral equations. Based on these equations one can derive a boundary integral representation of the Dirichlet-Neumann map $(\partial/\partial n_x)u(x) = (Su)(x)$ for $x \in \Gamma$ using the Steklov-Poincaré operator S. In addition to the direct formulations we will use an indirect double layer potential ansatz to get a nonlinear boundary integral equation related to the nonlinear boundary condition (1.2).

From a theoretical point of view all formulations considered in this paper are of the same structure involving a pseudodifferential operator of order one with a nonlinear perturbation, and hence we can use similar techniques to analyze them. However, these equations exhibit different behavior when we consider Galerkin schemes to compute numerical solutions, for example, with respect to the order of convergence and with respect to the amount of computational costs involved. In [**13**] we will design a numerical algorithm yielding almost optimal results with respect to both aspects. This algorithm is a hybrid solution strategy based on different formulations discussed in this section.

2.1. Boundary integral operators. The fundamental solution of the Laplacian given by

$$U^*(x,y) = \begin{cases} -\dfrac{1}{\omega_n} \log|x-y| & \text{for } n=2; \\ \dfrac{1}{\omega_n |x-y|} & \text{for } n=3, \end{cases}$$

where $\omega_2 = 2\pi$ and $\omega_3 = 4\pi$. We first consider some standard boundary integral operators. For $x \in \Gamma$, the single layer potential is

$$(Vt)(x) = \int_\Gamma U^*(x,y)t(y)\,ds_y, \tag{2.1}$$

the double layer potential operator is

$$(Ku)(x) = \int_\Gamma u(y)\frac{\partial}{\partial n_y}U^*(x,y)\,ds_y \tag{2.2}$$

and the hypersingular integral operator is given by

$$(Du)(x) = -\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}U^*(x,y)\,ds_y. \tag{2.3}$$

Using the Green's identity and the jump relation of the double layer potential, see, for example, [**2, 25**], boundary integral equations related to the partial differential equation (1.1) are

$$(Vt)(x) = [\sigma(x)I + K]u(x) \quad \text{for } x \in \Gamma \tag{2.4}$$

or

$$(Du(x) = [(1-\sigma(x))I - K']t(x) \quad \text{for } x \in \Gamma, \tag{2.5}$$

with $\sigma(x) = \alpha(x)/\omega_n$ where $\alpha(x)$ denotes the interior solid angle at $x \in \Gamma$, K' is the adjoint operator of K, given by the normal derivative of the single layer potential and $t(x) = (\partial/\partial n_x)u(x)$ is the exterior normal derivative of the potential u defined for $x \in \Gamma$ almost everywhere.

For a bounded domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, with a Lipschitz boundary Γ, all boundary integral operators introduced above are bounded for $s \in [-(1/2), (1/2)]$, see [**4**]:

$$\begin{aligned} V :& H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma);\\ K :& \ H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma);\\ K' :& H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma);\\ D :& \ H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma). \end{aligned}$$

The single layer potential V is self-adjoint and $H^{-1/2}(\Gamma)$-elliptic, i.e.,

$$\langle Vt, t\rangle \geq c \cdot \|t\|^2_{-1/2} \quad \text{for all } t \in H^{-1/2}(\Gamma).$$

Note that for $n = 2$ we require $\operatorname{diam}\Omega < 1$ to ensure the positive definiteness of V, see [**16**]. However, this condition will be needed only in the Steklov-Poincaré operator formulation described in Section 2.2. The hypersingular integral operator D is self-adjoint and $H^{1/2}(\Gamma)$-semi-elliptic, see, for example, [**7**],

$$\langle Du, u\rangle \geq c \cdot \|u\|_{1/2}^2 \quad \text{for all } u \in H_0^{1/2}(\Gamma)$$

with

$$H_0^{1/2}(\Gamma) := \left\{ v \in H^{1/2}(\Gamma) : \int_\Gamma v(x)\, ds_x = 0 \right\}.$$

The operators

$$\begin{aligned} (1-\sigma(x))I - K &: H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \\ (1-\sigma(x))I - K' &: H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \end{aligned}$$

are bijective [**16**]. From this, together with Theorem 3 of [**4**], we get that the operator

$$(1-\sigma(x))I - K : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma)$$

is bijective for all $s \in [0, (1/2)]$.

In addition to the standard boundary integral operators introduced above, we define the Nemytskii operator

$$(Nu)(x) = g(x, u(x)) \quad \text{for } x \in \Gamma. \tag{2.6}$$

In the two-dimensional case with a polygonal bounded domain Ω, the double layer potential has extra smoothness [**5, 6**]. Let J denote the number of corner points of the polygon Γ with open straight lines Γ^j, $j = 1, \dots, J$, and let α_j denote the interior angles between Γ^j and Γ^{j+1}. If we define

$$\sigma_0 := \min_{j=1,\dots,J}\{\sigma_j\}, \quad \sigma_j := \min\left\{\frac{\pi}{\alpha_j}, \frac{\pi}{2\pi - \alpha_j}\right\}, \tag{2.7}$$

then the double layer potential $K : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is bounded for all $s \in (-\sigma_0, \sigma_0)$.

Now we are in a position to formulate nonlinear boundary integral equations using (2.4), (2.5), the nonlinear boundary condition (1.2) and definition (2.6).

2.2. Nonlinear boundary integral equations.

The Steklov-Poincaré operator formulation. From (2.4) and (2.5) we can find a boundary integral representation of the Dirichlet-Neumann map

$$t(x) := \frac{\partial}{\partial n_x} u(x) = (Su)(x) \quad \text{for } x \in \Gamma \tag{2.8}$$

using the Steklov-Poincaré operator

$$(Su)(x) := V^{-1}[\sigma(x)I + K]u(x) \tag{2.9}$$
$$= [D + [\sigma(x)I + K']V^{-1}[\sigma(x)I + K]]u(x). \tag{2.10}$$

Note that (2.9) corresponds to (2.4) while (2.10) is based on (2.5) using (2.9).

From the properties of all boundary integral operators involved, it follows that S is self-adjoint, bounded and $H^{1/2}(\Gamma)$ semi-elliptic, i.e.,

$$\langle Su, u\rangle \geq c \cdot \|u\|^2_{1/2} \quad \text{for all } u \in H^{1/2}_0(\Gamma).$$

Inserting the Dirichlet-Neumann map (2.8) into the nonlinear boundary condition (1.2), we get the nonlinear boundary integral equation

$$(Su)(x) + (Nu)(x) = f(x) \quad \text{for } x \in \Gamma. \tag{2.11}$$

This formulation has the immediate advantage of the nonlinearity occurring in its simplest form, as well as the boundary integral operator having excellent properties.

In this paper we will use the Steklov-Poincaré operator S in its symmetric representation (2.10), which is well suited for a Galerkin discretization scheme yielding a symmetric stiffness matrix. Alternatively, one may use also the nonsymmetric representation (2.9), which can be discretized using either a Galerkin or collocation scheme for the boundary integral equation (2.4) yielding a nonsymmetric stiffness matrix, where some appropriate conditions have to be satisfied to ensure stability [**24**]. Note that the difference in using (2.9) or (2.10) is reflected only in the approximation scheme to discretize the Steklov-Poincaré operator S. The analysis to solve the nonlinear boundary integral equation (2.11) is hence independent of the used representation of the Steklov-Poincaré operator.

For a bounded domain Ω with a Lipschitz boundary Γ we get the following result:

Lemma 2.1. *The nonlinear boundary value problem* (1.1) *and* (1.2) *has a solution* $u \in H^{1+s}(\Omega)$ *for some* $s \in [0,(1/2)]$ *if and only if the nonlinear boundary integral equation* (2.11) *has a solution* $u^* \in H^{1/2+s}(\Gamma)$.

Proof. Let $u \in H^{1+s}(\Omega)$ be a solution of the nonlinear boundary value problem (1.1) and (1.2). We define $u^* = \mathrm{trace}\,(u)$ and use the trace theorem [**4, 14**] to get $u^* \in H^{1/2+s}(\Gamma)$. If we define $t^* = (\partial/\partial n_x)u(x)$ as the unique solution of the boundary integral equation

$$(Vt^*)(x) = [\sigma(x)I + K]u^*(x), \quad x \in \Gamma,$$

then for $x \in \Gamma$,

$$f(x) - (Nu^*)(x) = \frac{\partial}{\partial n_x}u(x) = t^*(x) = (Su^*)(x)$$

due to (1.2) and the definition (2.9) of the Steklov-Poincaré operator S. Therefore $u^* \in H^{1/2+s}(\Gamma)$ is a solution of the nonlinear boundary integral equation (2.11).

For the converse case, we take $u^* \in H^{1/2+s}(\Gamma)$ to be a solution of (2.11). If we let $t^*(x) = (Su^*)(x)$ for $x \in \Gamma$ and define

$$u(x) = \int_\Gamma U^*(x,y)t^*(y)\,ds_y - \int_\Gamma u^*(y)\frac{\partial}{\partial n_y}U^*(x,y)\,ds_y, \quad x \in \Omega,$$

then u satisfies the partial differential equation (1.1). Taking the limit $x \to \Gamma$ and using the jump relation of the double layer potential, see, for example, [**2, 25**],

$$u(x) = (Vt^*)(x) + (1 - \sigma(x))u^*(x) - (Ku^*)(x), \quad x \in \Gamma$$

and hence using (2.9) and the definition of t^*, we get $u(x) = u^*(x)$ for $x \in \Gamma$. Again, from the definition of t^*,

$$t^*(x) = (Su^*)(x) = (Su)(x) = \frac{\partial}{\partial n_x} u(x).$$

Therefore, using (2.11), $u(x)$ satisfies the boundary condition (1.2). Applying the inverse trace theorem [**14, 21**] gives $u \in H^{1+s}(\Omega)$. $\square$

In the case of a piecewise C^∞ boundary Γ, which is given as union of locally smooth parts Γ^k, we can give a more general result:

Corollary 2.1. *Let* Γ *be a* (*piecewise*) C^∞ *boundary. The nonlinear boundary value problem* (1.1) *and* (1.2) *has a solution* $u \in H^{1+s}(\Omega)$ *for some* $s \geq 0$ *if and only if the nonlinear boundary integral equation* (2.11) *has a solution* $u^* \in H^{1/2+s}(\Gamma)$.

Note that the solution u^* of the nonlinear boundary integral equation (2.11) is the trace of the solution u of the boundary value problem (1.1) and (1.2) and is therefore independent of the properties of boundary potentials used in the representation formula.

The indirect hypersingular formulation. Using the double layer potential ansatz, the solution $u(x)$ of the partial differential equation (1.1) can be represented as

$$u(x) = -\int_\Gamma v(y) \frac{\partial}{\partial n_y} U^*(x,y)\, ds_y \quad \text{for } x \in \Omega \tag{2.12}$$

with an unknown density v. Taking the normal derivative and the limit to a point $x \in \Gamma$, we get

$$t(x) := \frac{\partial}{\partial n_x} u(x) = (Dv)(x) \quad \text{for } x \in \Gamma, \tag{2.13}$$

with the hypersingular integral operator D as defined in (2.3). The jump relation of the double layer potential gives, on the other hand,

$$u(x) = [(1-\sigma(x))I - K]v(x) \quad \text{for } x \in \Gamma. \tag{2.14}$$

Combining (2.13) and (2.14) with the nonlinear boundary condition (1.2) yields a nonlinear boundary integral equation in v:

$$(Dv)(x) + [N((1-\sigma(x))I - K)v](x) = f(x) \quad \text{for } x \in \Gamma. \tag{2.15}$$

Remark 2.1. We may also derive (2.15) using the direct formulation (2.11). Indeed, using $KV = VK'$, we have

$$\begin{aligned} S[(1-\sigma(x))I - K] &= V^{-1}(\sigma(x)I + K)[(1-\sigma(x))I - K] \\ &= V^{-1}[(1-\sigma(x))I - K](\sigma(x)I + K) \\ &= [(1-\sigma(x))I - K']V^{-1}(\sigma(x)I + K) \\ &= D. \end{aligned} \tag{2.16}$$

Now (2.14) and (2.16) yield (2.15).

For a bounded domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, with a Lipschitz boundary $\Gamma = \partial\Omega$, we have the following result:

Lemma 2.2. *The nonlinear boundary value problem* (1.1) *and* (1.2) *has a solution* $u \in H^{1+s}(\Omega)$ *for some* $s \in [0, (1/2)]$ *if and only if the nonlinear boundary integral equation* (2.15) *has a solution* $v^* \in H^{1/2+s}(\Gamma)$.

Proof. Let $u \in H^{1+s}(\Omega)$ be a solution of (1.1) and (1.2). For $s \in [0, (1/2)]$, $(1-\sigma(x))I - K : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is a continuous bijective map. Hence, we can define

$$v^*(x) = [(1-\sigma(x))I - K]^{-1}u(x), \quad x \in \Gamma. \tag{2.17}$$

We claim that u satisfies the representation formula (2.12), with $v = v^*$. Indeed, if

$$\tilde{u}(x) = -\int_\Gamma v^*(x)\frac{\partial}{\partial n_y}U^*(x,y)\,ds_y \quad \text{for } x \in \Omega,$$

then using the jump relation of the double layer potential and (2.17),

$$\tilde{u}(x) = [(1-\sigma(x))I - K]v^*(x) = u(x), \quad x \in \Gamma,$$

i.e., $\tilde{u}$ is a solution of the Dirichlet problem

$$\Delta\tilde{u}(x) = 0 \quad \text{for } x \in \Omega, \quad \tilde{u}(x) = u(x) \quad \text{for } x \in \Gamma.$$

The uniqueness of the solution of the Dirichlet problem implies that $u = \tilde{u}$ in Ω. Hence we have

$$\frac{\partial}{\partial n_x} u(x) = (Dv^*)(x) \quad \text{for } x \in \Gamma$$

and the nonlinear boundary condition (1.2) implies that

$$(Dv^*)(x) + (Nu)(x) = f(x) \quad \text{for } x \in \Gamma.$$

Substituting u by (2.17) yields $v^* \in H^{1/2+s}(\Gamma)$ is a solution of (2.15). For the converse case, using the above arguments, it is easy to show using the properties of the double layer potential that u defined by (2.12), with $v = v^*$, is in $H^{1+s}(\Omega)$ and it satisfies (1.1) and (1.2). □

Remark 2.2. In the case of a polygonal bounded domain $\Omega \subset \mathbf{R}^2$ the operator $(1 - \sigma(x))I - K : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is bounded and bijective for all $s \in (-\sigma_0, \sigma_0)$ with σ_0 as defined in (2.7). From this we conclude that the nonlinear boundary value problem (1.1) and (1.2) has a solution $u \in H^{1+s}(\Omega)$ for some $s \in [0, \sigma_0)$ if and only if the nonlinear boundary integral equation (2.15) has a solution $v \in H^{1/2+s}(\Gamma)$.

Note that the solution u of the nonlinear boundary value problem (1.1) and (1.2) and therefore the trace $u^* = u_{|\Gamma}$ may be more regular than described in Remark 2.2. However, since v^* is the solution of the nonlinear boundary integral equation (2.15), the regularity of v^* is restricted by the properties of the double layer potential.

When using the indirect hypersingular integral formulation (2.15), the regularity result stated in Lemma 2.2 and in Remark 2.2 can be responsible for a slower convergence of the Galerkin solutions compared with the Steklov-Poincaré operator formulation (2.11), see Corollary 2.1. However, the solution v of (2.15) can be computed significantly faster than the solution u of (2.11), as can be seen from the simple representation of the hypersingular operator compared to the involved representation of the Steklov-Poincaré operator in (2.10). In [**13**] we will describe a hybrid solution strategy combining the fast solution process through (2.15) with the better convergence results obtained using (2.11).

The direct hypersingular integral formulation. Instead of using either the Steklov-Poincaré operator formulation (2.11) or the indirect hypersingular integral formulation (2.15), one may use any other boundary integral equation derived from (2.4) and (2.5) in combination with the nonlinear boundary condition (1.2).

If we use

$$t(x) = f(x) - (Nu)(x)$$

in (2.4) or (2.5), we get the nonlinear boundary integral equations

$$[\sigma(x)I + K]u(x) + [V(Nu)](x) = (Vf)(x), \quad x \in \Gamma, \tag{2.18}$$

or

$$\begin{aligned}(Du)(x)+[(1-\sigma(x))I-K'](Nu)(x) \\ = [(1-\sigma(x))I-K']f(x), \quad x \in \Gamma,\end{aligned} \tag{2.19}$$

respectively. Equation (2.18) is the standard boundary integral formulation related to nonlinear boundary value problem (1.1) and (1.2) considered already in [**2, 8, 9, 10, 11, 22, 23**].

In both the formulations (2.18) and (2.19), the nonlinearity Nu appears as a density of some boundary integral operators, which requires either a costly discretization scheme or an additional approximation of Nu yielding slower convergence in most cases.

In addition to our novel formulations (2.11) and (2.15), in this paper we will include the formulation (2.19) only in our analysis. This is due to the property that equation (2.19), as equations (2.11) and (2.15), is of the form pseudodifferential operator of order one plus a nonlinear perturbation. However, we do not give a detailed analysis of (2.19) as this can be concluded easily from formulations (2.11) and (2.15) which will be discussed in detail. Note that one can easily apply Lemma 2.1 to get equivalent solutions of the nonlinear boundary value problem (1.1) and (1.2) and the nonlinear boundary integral equation (2.19). This is mainly due to the bounded and bijective mapping $(1 - \sigma(x))I - K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ yielding the equivalence of (2.19) with (2.11).

Using (A1), Lemmas 2.1 and 2.2, we conclude that the nonlinear boundary integral equations (2.11) and (2.19) have an isolated solution $u^* \in H^1(\Gamma)$ and (2.15) has an isolated solution $v^* \in H^1(\Gamma)$. If we compute the solution v^* of (2.15), then the solution u of (1.1) and (1.2) can be computed using (2.12). On the other hand, if we compute u^* of (2.11) or (2.19), and let $t^* = f - Nu^*$, then the solution u of (1.1) and (1.2) can be computed using the representation formula for $x \in \Omega$:

$$u(x) = \int_\Gamma U^*(x,y)t^*(y)\,ds_y - \int_\Gamma u^*(y)\frac{\partial}{\partial n_y}U^*(x,y)\,ds_y. \tag{2.20}$$

3. Convergence analysis of the Newton scheme. In this section we apply the standard Newton iteration scheme to the nonlinear boundary integral equations (2.11) and (2.15). Hence we have to ensure the solvability of the linearized equations as well as the convergence of the iterate solutions to the exact one.

Firstly, we analyze the mapping properties of the Fréchet derivative $N'(\cdot)$ of the Nemytskii operator N defined in (2.6).

Using (A1), let us denote by $u^* = u|_\Gamma \in H^1(\Gamma)$ the trace of the isolated solution u of the nonlinear boundary value problem (1.1) and (1.2). For any $\rho > 0$, we denote by

$$\mathcal{U}_\rho(u^*) = \{\varphi \in H^{1/2}(\Gamma) : \|\varphi - u^*\|_{1/2} \le \rho\},$$

a ball in $H^{1/2}(\Gamma)$ with center u^* and radius ρ.

Lemma 3.1. *Let* (A0)–(A2) *hold. For all* $\varphi \in \mathcal{U}_\rho(u^*)$, $\rho > 0$, *the Fréchet derivative* $N'(\varphi)$ *exists as a bounded linear operator on* $L^2(\Gamma)$.

Proof. Since $g(x,\alpha)$ is twice differentiable with respect to α, we have for all $\varphi \in \mathcal{U}_\rho(u^*)$,

$$N'(\varphi)v(x) = g_\alpha(x,\varphi(x))v(x), \quad v \in H^{1/2}(\Gamma). \tag{3.1}$$

The local boundedness in (A2) implies that, for any $\rho > 0$,

$$|g_\alpha(x,\varphi(x))| \le c(\rho,u^*) \quad \text{for all } \varphi \in \mathcal{U}_\rho(u^*). \tag{3.2}$$

To show that for all $\varphi \in \mathcal{U}_\rho(u^*)$, $N'(\varphi)$ maps $L^2(\Gamma)$ into itself and is bounded, we let $v \in L^2(\Gamma)$. Using (3.1) and (3.2),

$$\|N'(\varphi)v\|_0^2 = \int_\Gamma |g_\alpha(x,\varphi(x))|^2 |v(x)|^2 \, ds_x \le [c(\rho,u^*)]^2 \|v\|_0^2. \qquad \square$$

To prove the next lemma, we introduce an additional assumption:

(A3) $g_\alpha(x,\alpha) > 0$ for all $(x,\alpha) \in \Gamma \times \mathbf{R}$.

Lemma 3.2. *Let assumptions* (A0)–(A3) *hold. Then, for any* $\rho > 0$, *the homogeneous linear boundary integral equation*

$$(Su)(x) + N'(\varphi)u(x) = 0, \quad x \in \Gamma, \quad \varphi \in \mathcal{U}_\rho(u^*) \tag{3.3}$$

has only the trivial solution in $H^{1/2}(\Gamma)$.

Proof. Let $\varphi \in \mathcal{U}_\rho(u^*)$ be fixed. We first observe that the homogeneous Robin problem

$$\Delta u(x) = 0 \quad \text{for } x \in \Omega, \tag{3.4}$$

$$\frac{\partial}{\partial n_x} u(x) + g_\alpha(x,\varphi(x))u(x) = 0 \quad \text{for } x \in \Gamma \tag{3.5}$$

has only the trivial solution. Indeed, if u satisfies (3.4) and (3.5), using the Green's identity and (A3), we have

$$0 \le \int_\Omega |\nabla u|^2 \, dx = \int_\Gamma u(x) \frac{\partial}{\partial n_x} u(x) \, ds_x = -\int_\Gamma u^2(x) g_\alpha(x,\varphi(x)) \, ds_x \le 0.$$

Hence,

$$\nabla u = 0 \quad \text{on } \Omega$$

and

$$u^2(x) g_\alpha(x,\varphi(x)) = 0 \quad \text{for } x \in \Gamma.$$

Since $g_\alpha > 0$, we have $u = 0$ on Γ and $u = \text{constant}$ on Ω, yielding $u = 0$ on $\overline{\Omega}$ by continuous extension.

Now we show that (3.3) has only the trivial solution. Let u be any solution of the homogeneous integral equation (3.3). If we let $t(x) = (Su)(x)$ for $x \in \Gamma$, then the function $\tilde{u}$ defined as

$$\tilde{u}(x) = \int_\Gamma U^*(x,y)t(y)\,ds_y - \int_\Gamma u(y)\frac{\partial}{\partial n_y}U^*(x,y)\,ds_y \quad \text{for } x \in \Omega$$

satisfies $\Delta\tilde{u}(x) = 0$ for $x \in \Omega$. Moreover, using the definition of S and the jump relation of the double layer potential,

$$\begin{gathered} \tilde{u}(x) = u(x), \\ t(x) = (Su)(x) = (S\tilde{u})(x) = \frac{\partial}{\partial n_x}\tilde{u}(x) \quad \text{for } x \in \Gamma. \end{gathered}$$

Hence, by (3.3), $\tilde{u}$ is a solution of the homogeneous Robin problem (3.4) and (3.5). Therefore, $\tilde{u}(x) = 0$ for all $x \in \overline{\Omega}$, i.e., $u(x) = 0$ for $x \in \Gamma$.

From the identity $S[(1-\sigma(x))I - K] = D$, see (2.16), we get the following corollary:

Corollary 3.1. *Let assumptions* (A0)–(A3) *hold. Then the homogeneous boundary integral equation*

$$\begin{gathered} (Dv)(x) + (N'(\varphi)[(1-\sigma(x))I - K]v)(x) = 0 \\ \text{for } x \in \Gamma,\ \varphi \in \mathcal{U}_\rho(u^*) \end{gathered} \tag{3.6}$$

has only the trivial solution in $H^{1/2}(\Gamma)$.

Remark 3.1. We use the assumption (A3) just to prove that the Robin boundary value problem (3.4) and (3.5) has only the trivial solution. If this result holds due to some other technique, then we do not need the assumption (A3) at all. Therefore, we assume throughout the paper that Lemma 3.2 holds. Further, for boundary element analysis in our sequel work [**13**], we do not need (A3).

The following result is crucial to apply the Newton iterative method to solve (2.11) as well as for the solution of (2.15):

Theorem 3.1. *Let assumptions* (A0)–(A2) *hold. For each* $\varphi \in \mathcal{U}_\rho(u^*)$*, the bounded linear map*

$$S + N'(\varphi) : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$$

is invertible.

Proof. Let $L : H^{1/2}(\Gamma) \to C^\infty(\Gamma)$ be defined by

$$(Lu)(x) = \int_\Gamma u(y)\, ds_y \quad \text{for } u \in H^{1/2}(\Gamma),\ x \in \Gamma. \tag{3.7}$$

Then $S + L : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ satisfies

$$\langle (S+L)v, v\rangle \geq c \cdot \|v\|_{1/2}^2 \quad \text{for all } v \in H^{1/2}(\Gamma), \tag{3.8}$$

see for example [**15**] and hence $S + L$ is invertible and has a bounded inverse. Since

$$S + N'(\varphi) = (S+L)\,\underbrace{\{I + (S+L)^{-1}[N'(\varphi) - L]\}}_{=:T}, \tag{3.9}$$

if we show that the operator T is bijective as an operator on $L^2(\Gamma)$, then we are through. In this case the bounded inverse

$$[S + N'(\varphi)]^{-1} : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$$

is given by

$$[S + N'(\varphi)]^{-1} = \{I + (S+L)^{-1}[N'(\varphi) - L]\}^{-1}(S+L)^{-1}. \tag{3.10}$$

Using the boundedness of L, $N'(\varphi)$ on $L^2(\Gamma)$, the compact imbedding of $L^2(\Gamma)$ into $H^{-1/2}(\Gamma)$ [**21**] and the boundedness of $(S+L)^{-1} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, we get

$$(S+L)^{-1}[N'(\varphi) - L] : L^2(\Gamma) \to L^2(\Gamma) \tag{3.11}$$

is a compact linear map. So, to show the invertibility of T, it is enough to show that T is injective on $L^2(\Gamma)$.

Let $w \in L^2(\Gamma)$ with

$$\{I + (S+L)^{-1}[N'(\varphi) - L]\}w = 0.$$

This implies

$$0 = [(S+L) + N'(\varphi) - L]w = [S + N'(\varphi)]w. \tag{3.12}$$

Using (3.12) and Lemma 3.2, we get $w = 0$ and the injectivity of T follows. Thus,

$$[S + N'(\varphi)] : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$$

has a bounded inverse given by (3.10). □

Using the arguments as in the proof of Theorem 3.1, one can show:

Corollary 3.2. *Let assumptions* (A0)–(A2) *hold. For each* $\varphi \in \mathcal{U}_\rho(u^*)$ *the linear map*

$$D + N'(\varphi)[(1-\sigma(x))I - K] : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$$

is invertible, and its inverse operator is given by

$$\begin{aligned} &\{D + N'(\varphi)[(1-\sigma(x))I - K]\}^{-1} \\ &\quad = \{I + (D+L)^{-1}[N'(\varphi)((1-\sigma(x))I - K) - L]\}^{-1}(D+L)^{-1}. \end{aligned} \tag{3.13}$$

Returning back to compute either a solution of (2.11) or of (2.15), we first use the standard Newton method. Considering first the formulation (2.11), as in the case of any Newton scheme, we start with an initial guess u^0 sufficiently close to the solution u^*, i.e., we assume $u^0 \in \mathcal{U}_{\tilde{\rho}}(u^*)$ for some $0 < \tilde{\rho} < 1$. We compute the iterates

$$u^{k+1} = u^k - [F'(u^k)]^{-1}F(u^k) \quad \text{for } k = 0, 1, 2, \dots, \tag{3.14}$$

where F is the nonlinear operator

$$F(v) = Sv + Nv - f. \tag{3.15}$$

For any $\varphi \in \mathcal{U}_\rho(u^*)$ using Lemma 3.1 the Fréchet derivative $F'(\varphi) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ exists and is given by

$$F'(\varphi)v = Sv + N'(\varphi)v \quad \text{for } v \in H^{1/2}(\Gamma). \tag{3.16}$$

Using Theorem 3.1, $[F'(\varphi)]^{-1}$ exists for any $\varphi \in \mathcal{U}_\rho(u^*)$.

Now we can formulate the Newton algorithm for (2.11) using (3.15) and (3.16). The iteration formula (3.14) can be written as

$$[S + N'(u^k)]u^{k+1} = f + N'(u^k)u^k - Nu^k. \tag{3.17}$$

To check the convergence of the Newton iterates, we need to compute the residuals

$$r^{k+1} = Su^{k+1} + Nu^{k+1} - f \quad \text{for } k \geq 0.$$

Since the evaluation of the Steklov-Poincaré operator can be expensive, we use the fact that u^{k+1} is a solution of (3.17), i.e.,

$$Su^{k+1} = f + N'(u^k)(u^k - u^{k+1}) - Nu^k,$$

and compute the residual using the formula

$$r^{k+1} = N'(u^k)(u^k - u^{k+1}) + Nu^{k+1} - Nu^k. \tag{3.18}$$

So our algorithm is:

1. Let $u^0 \in \mathcal{U}_{\tilde{\rho}}(u^*)$ be given; Compute $r^0 = Su^0 + Nu^0 - f$. Let $k = 0$.

2. If the residuum is small enough, stop.

3. Else solve the linear problem (3.17) to compute the new iterate u^{k+1}.

4. Compute r^{k+1} via (3.18), set $k := k + 1$ and go to step 2.

To compute the righthand side in (3.17) and (3.18), one has to compute Nu^k, $N'(u^k)u^k$ and $N'(u^k)u^{k+1}$ once per iteration step. Moreover, we can use u^k as an initial guess in an iteration process to solve the linear system (3.17).

For the Newton scheme to solve the nonlinear boundary integral equation (2.15) in the indirect hypersingular integral formulation and

(2.19) in the direct hypersingular integral formulation, we may derive similar algorithms: Instead of (3.17), for the indirect hypersingular integral formulation we have to solve the linear equation

$$[D+N'(u^k)[(1-\sigma(x))I-K]]v^{k+1}=f+N'(u^k)u^k-Nu^k \tag{3.19}$$

with $u^k=[(1-\sigma(x)]I-K)v^k$ and an initial guess v^0 such that

$$u^0=[(1-\sigma(x))I-K]v^0\in\mathcal{U}_{\tilde\rho}(u^*)$$

for some $0<\tilde\rho<1$. For the direct hypersingular integral formulation, the linearized equation is given by

$$\begin{aligned}[D+[(1-\sigma(x))I-K']N'(u^k)]u^{k+1}\\ =f+[(1-\sigma(x))I-K'][N'(u^k)u^k-Nu^k]\end{aligned} \tag{3.20}$$

with an initial guess $u^0\in\mathcal{U}_{\tilde\rho}(u^*)$ for some $0<\tilde\rho<1$.

Note that one can derive and write the corresponding algorithms similar to compute the residuals r^{k+1} as in (3.18).

Now we can prove the uniqueness and regularity of the solution of (3.17) and establish the convergence of the Newton iterates u^k to solve the nonlinear boundary integral equation (2.11).

Theorem 3.2. *Let assumptions* (A0)–(A2) *hold. Let* $u^0\in\mathcal{U}_{\tilde\rho}(u^*)$ *be given for some* $0<\tilde\rho<1$. *For all* $k\ge 0$, (3.17) *has a unique solution* $u^{k+1}\in\mathcal{U}_{\rho_{k+1}}(u^*)$ *for some* $\rho_{k+1}>0$ *with*

$$\|u^{k+1}\|_1\le c\cdot\|f+N'(u^k)u^k-Nu^k\|_0 \tag{3.21}$$

and

$$\|u^{k+1}-u^*\|_{1/2}\le c\cdot\|u^k-u^*\|^2_{1/2}. \tag{3.22}$$

Proof. Let $k=0$ and $\rho_0=\tilde\rho$. First we show that the righthand side of (3.17) is in $L^2(\Gamma)$. Since $u^k\in\mathcal{U}_{\rho_k}(u^*)$ using Lemma 3.1, $N'(u^k)u_k\in L^2(\Gamma)$. Since $Nu^*=f-(\partial/\partial n_x)u^*(x)$ on Γ, using (A0) and (A1), $Nu^*\in L^2(\Gamma)$. Further, since $Nu^k=[Nu^k-Nu^*]+Nu^*$, to show $Nu^k \in L^2(\Gamma)$, it is enough to show $[Nu^k - Nu^*] \in L^2(\Gamma)$. Since $u^k \in \mathcal{U}_\rho(u^*)$, using (A2), for $x \in \Gamma$,

$$\begin{aligned}|(Nu^k)(x) - (Nu^*)(x)| &= |g(x, u^k(x)) - g(x, u^*(x))| \\ &\leq c|u^k(x) - u^*(x)|.\end{aligned}$$

Hence $Nu^k - Nu^* \in L^2(\Gamma)$, implying that $Nu^k \in L^2(\Gamma)$. This, together with (A0), implies that

$$f + N'(u^k)u^k - Nu^k \in L^2(\Gamma). \tag{3.23}$$

Using Theorem 3.1,

$$[S + N'(u^k)]^{-1} : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$$

exists and is bounded and hence (3.17) has a unique solution $u^{k+1} \in H^{1/2}(\Gamma)$. Further, using (3.23) and Lemma 3.1 in (3.17),

$$Su^{k+1} = f + N'(u^k)u^k - Nu^k - N'(u^k)u^{k+1} \in L^2(\Gamma).$$

Hence, by Theorem 3 in [**4**], we get $u^{k+1} \in H^1(\Gamma)$ and the inequality (3.21).

Now we show the convergence of the Newton iterates. Equation (3.17) is equivalent to the Newton scheme (3.14) to solve the original equation

$$F(u) = 0 \tag{3.24}$$

in $\mathcal{U}_\gamma(u^*) \subset H^{1/2}(\Gamma)$ for some $\gamma > 0$. (A1) and Lemma 2.1 imply that (3.24) has an isolated solution $u^* \in H^1(\Gamma)$ and Theorem 3.1 yields that $F'(u^k)$ is invertible. Further, (A2) and the boundedness of the Steklov-Poincaré operator imply that there exists a constant $\gamma > 0$ such that

$$F' : \mathcal{U}_\gamma(u^*) \longrightarrow BL(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$$

is Lipschitz continuous, where $BL(X, X)$ denotes the space of all bounded linear operators in a Banach space X. So we have

(a.) (3.24) has an isolated solution u^*;

(b.) $F' : \mathcal{U}_\gamma(u^*) \to BL(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$ is Lipschitz continuous;

(c.) $F'(u^*)$ is nonsingular.

Hence the standard assumptions needed for the convergence of the Newton iterates are satisfied. Following standard arguments, see, for example, Theorem 5.1.2 in [**19**], (3.22) holds. Hence, $u^{k+1} \in \mathcal{U}_{\rho_{k+1}}(u^*)$ with $\rho_{k+1} = c\rho_k^2$. Repeating the above arguments, we get the result for all $k \geq 0$. $\square$

Remark 3.2. As a conclusion of Theorem 3.2, if we choose $\rho_0 = \tilde{\rho}$ sufficiently small so that $c\tilde{\rho}^2 \leq \tilde{\rho}$, then

$$\|u^{k+1} - u^*\|_{1/2} \leq c \cdot \tilde{\rho}^2 < \tilde{\rho},$$

implies $u^{k+1} \in \mathcal{U}_{\tilde{\rho}}(u^*)$ for all $k \geq 0$.

Using the above arguments, one can prove results similar to Theorem 3.2 for the indirect and the direct hypersingular integral formulation. We describe this result below in the case of the indirect hypersingular integral formulation.

Corollary 3.2. *Let assumptions* (A0)–(A2) *hold. Let* $u^0 = [(I - \sigma(x))I - K]v^0 \in \mathcal{U}_{\tilde{\rho}}(u^*)$ *for some* $0 < \tilde{\rho} < 1$. *For all* $k \geq 0$, *equation* (3.19) *has a unique solution* $v^{k+1} \in H^1(\Gamma)$ *with* $[(I-\sigma(x))I-K]v^{k+1} \in \mathcal{U}_{\rho_{k+1}}(u^*)$ *for some* $\rho_{k+1} > 0$ *and*

$$\|v^{k+1}\|_1 \leq c \cdot \|f + N'(u^k)u^k - Nu^k\|_0, \tag{3.25}$$

$$\|v^{k+1} - v^*\|_{1/2} \leq c \cdot \|v^k - v^*\|_{1/2}^2. \tag{3.26}$$

As a conclusion, to compute an isolated solution of the nonlinear boundary value problem (1.1) and (1.2), based on our results in this paper it is enough to describe and analyze suitable computational procedures to solve the sequence of linear boundary integral equations given by (3.17) or (3.19). In our sequel work [**13**] we will describe and analyze boundary element methods (with preconditioning strategies) to solve these formulations. We will also substantiate in [**13**] the need to have these different formulations by proposing a hybrid method, yielding optimal results.

Acknowledgments. Part of this work was carried out while the second author was a Visiting Fellow in the School of Mathematics, UNSW, under an Australian Research Council grant. The support of the Australian Research Council is gratefully acknowledged by both authors. We would like to thank Dr. W. McLean and Dr. T. Tran for valuable discussions. We would like to thank the referees for valuable comments.

REFERENCES

1. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, Cambridge, 1997.

2. K.E. Atkinson and G.A. Chandler, *BIE methods for solving Laplace's equation with nonlinear boundary conditions*: *The smooth boundary case*, Math. Comp. **55** (1991), 451–457.

3. R. Bialecki and A.J. Nowak, *Boundary value problems in heat conduction with nonlinear material and nonlinear boundary conditions*, Appl. Math. Modelling **5** (1981), 417–421.

4. M. Costabel, *Boundary integral operators on Lipschitz domains*: *Elementary results*, SIAM J. Math. Anal. **19** (1988), 613–626.

5. M. Costabel and E.P. Stephan, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, in *Mathematical models and methods in mechanics*, Banach Centre Publ., Vol. 15, PWN, Warsaw, 1985.

6. ———, *On the convergence of collocation methods for boundary integral equations on polygons*, Math. Comp. **49** (1987), 461–478.

7. R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, in *Integral equations and numerical methods*, Vol. 4, Springer-Verlag, Berlin, 1990.

8. R.L. Doucette, *A collocation method for the numerical solution of Laplace's equation with nonlinear boundary conditions on a polygon*, SIAM J. Numer. Anal. **30** (1993), 717–732.

9. M.A. Efendiev, H. Schmitz and W.L. Wendland, *On some nonlinear potential problems*, Electron. J. Differential Equations **18** (1999), 17 pp.

10. P.P.B. Eggermont and J. Saranen, *L^p estimates of boundary integral equations for some nonlinear boundary value problems*, Numer. Math. **58** (1990), 465–478.

11. M. Ganesh, *A BIE method for a nonlinear BVP*, J. Comput. Appl. Math. **45** (1993), 299–308.

12. M. Ganesh, I.G. Graham and J. Sivaloganathan, *A pseudospectral three-dimensional boundary integral method applied to a nonlinear model problems from finite elasticity*, SIAM J. Numer. Anal. **31** (1994), 1378–1414.

13. M. Ganesh and O. Steinbach, *Boundary element methods for potential problems with nonlinear boundary conditions*, UNSW Appl. Math. preprint AMR98/17, 1998. (`http://www.maths.unsw.edu.au/appweb/preprints/amr98`.)

14. P. Grisvard, *Elliptical problems in nonsmooth domains*, Pitman, Boston, 1985.

15. M. Hahne and E.P. Stephan, *Schwarz iterations for the efficient solution of screen problems with boundary elements*, Computing 56 (1996), 61–85.

16. G.C. Hsiao, E.Schnack and W.L. Wendland, *A hybrid coupled finite-boundary element method in elasticity*, Comput. Methods Appl. Mech. Engrg. **173** (1999), 287–316.

17. G.C. Hsiao and W.L. Wendland, *A finite element method for some integral equations of the first kind*, J. Math. Appl. **58** (1977), 449–481.

18. D.B. Ingham, P.J. Higgs and M. Manzoor, *Boundary integral equation solution of non-linear plane potential problem*, IMA J. Numer. Anal. **1** (1981), 415–426.

19. C.T. Kelley, *Iterative methods for linear and nonlinear equations*, SIAM, Philadelphia, PA, 1995.

20. M.A. Kelmanson, *Solution of nonlinear elliptic equations with boundary singularities by an integral equation method*, J. Comp. Phys. **56** (1984), 244–258.

21. J.L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer, New York, 1972.

22. K. Ruotsalainen and J. Saranen, *On the collocation method for a nonlinear boundary integral equation*, J. Comput. Appl. Math. **28** (1989), 339–348.

23. K. Ruotsalainen and W.L. Wendland, *On the boundary element method for some nonlinear boundary value problems*, Numer. Math. **53** (1988), 299–314.

24. O. Steinbach, *Mixed approximations for boundary elements*, submitted.

25. W.L. Wendland, *Die Behandlung von Randwertaufgaben im R_3 mit Hilfe von Einfach- und Doppelschichtpotentialen*, Numer. Math. **11** (1968), 380–404.

School of Mathematics, University of New South Wales, Sydney 2052, Australia
E-mail address: ganesh@maths.unsw.edu.au

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail address: steinbach@mathematik.uni-stuttgart.de