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INITIAL VALUE PROBLEMS FOR NONLINEAR SECOND ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper the author uses the fixed point theory to investigate the existence of solutions of initial value problems for nonlinear second order impulsive integro-differential equations in Banach spaces.

1. Introduction. The theory of impulsive differential equations has become an important area of investigation in recent years, see [**5**]. In Section 4.3 of [**4**] and [**1**], the authors discussed the existence of solutions of boundary value problems for nonlinear second order impulsive integro-differential equations in Banach spaces E by means of Darbo's fixed point theorem. Now, under more wide conditions, see Remark 2, this paper shall also use fixed point theory to investigate the existence of solutions of initial value problems (IVP) for second order impulsive integro-differential equations in E. But we cannot obtain the results in this paper directly by means of Darbo's fixed point theorem used in [**4**] and [**1**].

Consider the IVP for impulsive integro-differential equations in a Banach space E:

$$\begin{aligned} x'' &= f(t,x,x',Tx,Sx), \quad t\in J, \quad t\neq t_k,\\ \Delta x|_{t=t_k} &= I_k(x(t_k),x'(t_k)),\\ \Delta x'|_{t=t_k} &= \bar{I}_k(x(t_k),x'(t_k)), \quad k=1,2,\dots,m,\\ x(0) &= x_0, \qquad x'(0)=x_1, \end{aligned} \tag{1.1}$$

where $f\in C[J\times E\times E\times E\times E,E]$, $J=[0,a](a>0)$, $0<t_1<t_2\cdots< t_m<a$, $I_k,\bar{I}_k\in C[E\times E,E]$, $x_0,x_1\in E$ and

$$(Tx)(t)=\int_0^t k(t,s)x(s)\,ds, (Sx)(t)=\int_0^a h(t,s)x(s)\,ds, \tag{1.2}$$

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$k \in C[D, R^1]$, $h \in C[J \times J, R^1]$, $D = \{(t,s) \in J \times J : 0 \leq s \leq t \leq a\}$, $R^1 = (-\infty, +\infty)$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e.,

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-).$$

Here $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$, respectively. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$. Let $PC^1[J, E] = \{x : x$ is a map from J into E such that $x(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+), x'(t_k^-), x'(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$. For $x \in PC^1[J, E]$, by virtue of the mean value theorem, it is easy to see that the left derivative $x'_-(t_k)$ exists and

$$x'_-(t_k) = \lim_{h \to 0^+} h^{-1}[x(t_k) - x(t_k - h)] = x'(t_k^-).$$

In IVP (1.1) and in the following, $x'(t_k)$ is understood as $x'_-(t_k)$. It is clear that $PC^1[J, E]$ is a Banach space with norm

$$\|x\|_{PC^1} = \max\Big\{ \sup_{t \in J} \|x(t)\|, \sup_{t \in J} \|x'(t)\| \Big\}.$$

Notice that $PC[J, E] = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$ is also a Banach space with norm $\|x\|_{PC} = \sup_{t \in J} \|x(t)\|$. Let $J' = J \backslash \{t_1, t_2, \ldots, t_m\}$. A map $x \in PC^1[J, E] \cap C^2[J', E]$ is called a solution of IVP (1.1) if it satisfies (1.1).

2. Some lemmas.

Lemma 1. *$x \in PC^1[J, E] \cap C^2[J', E]$ is a solution of* IVP (1.1) *if and only if $x \in PC^1[J, E]$ is a solution of the impulsive integral equation*

$$x(t) = Ax(t), \quad t \in J, \tag{2.1}$$

where

$$\begin{aligned} Ax(t) = x_0 + tx_1 &+ \int_0^t (t-s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\, ds \\ &+ \sum_{0 < t_k < t} [I_k(x(t_k), x'(t_k)) + (t - t_k)\bar{I}_k(x(t_k), x'(t_k))]. \end{aligned} \tag{2.2}$$

Proof. First suppose that $x \in PC^1[J,E] \cap C^2[J',E]$ is a solution of IVP (1.1). Evidently, for $t_k < t \le t_{k+1}$, we have

$$x(t_1) - x(0) = \int_0^{t_1} x'(s)\,ds, x(t_2) - x(t_1^+) = \int_{t_1}^{t_2} x'(s)\,ds,$$
$$\cdots$$
$$x(t_k) - x(t_{k-1}^+) = \int_{t_{k-1}}^{t_k} x'(s)\,ds, x(t) - x(t_k^+) = \int_{t_k}^{t} x'(s)\,ds.$$

By adding, we get, for $t_k < t \le t_{k+1}$,

$$x(t) - x(0) - \sum_{i=1}^{k}[x(t_i^+) - x(t_i)] = \int_0^t x'(s)\,ds,$$

that is,

$$x(t) = x(0) + \int_0^t x'(s)\,ds + \sum_{0<t_k<t}[x(t_k^+) - x(t_k)], \quad t \in J. \tag{2.3}$$

Replacing $x(t)$ by $x'(t)$ in (2.3), because of $x \in PC^1[J,E] \cap C^2[J',E]$, we have

$$x'(t) = x'(0) + \int_0^t x''(s)\,ds + \sum_{0<t_k<t}[x'(t_k^+) - x'(t_k)], \quad t \in J. \tag{2.4}$$

Then substituting (2.4) into (2.3), we can obtain (2.1).

Conversely, assume that $x \in PC^1[J,E]$ is a solution of Equation (2.1). Evidently,

$$\Delta x|_{t=t_k} = I_k(x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, m.$$

Direct differentiation implies, for $t \in J$, $t \ne t_k$,

$$\begin{aligned} x'(t) = (Ax)'(t) = x_1 &+ \int_0^t f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\,ds \\ &+ \sum_{0<t_k<t} \bar{I}_k(x(t_k), x'(t_k)) \end{aligned} \tag{2.5}$$

and

$$x''(t) = f(t, x(t), x'(t), (Tx)(t), (Sx)(t)),$$

hence $x \in C^2[J', E]$ and

$$\Delta x'|_{t=t_k} = \bar{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m.$$

On the other hand, from (2.1) and (2.5) we can calculate $x(0) = x_0$ and $x'(0) = x_1$. The proof is completed. □

In the following, let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \dots, J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, a]$, $k_0 = \max\{k(t,s) : (t,s) \in D\}$ and $h_0 = \max\{h(t,s) : (t,s) \in J \times J\}$. For $H \subset PC^1[J,E]$, we denote $H' = \{x' : x \in H\} \subset PC[J,E]$, $H_k = \{x|_{J_k} : x \in H\}$, $H'_k = \{x'|_{J_k} : x \in H\}$, $H(t) = \{x(t) \in E : x \in H\} \subset E$, $H'(t) = \{x'(t) \in E : x \in H\} \subset E (t \in J)$. Similarly, we can define $(TH)(t)$, $(SH)(t)$, $(TH)'(t)$ and $(SH)'(t)$, where $k = 0, 1, 2, \dots, m$, $t \in J$. Then, using the same method as in the proof (29) of [**1**], we can get

Lemma 2. *If $H \subset PC^1[J,E]$ is bounded and the elements of H' are equicontinuous on each J_k, $k = 0, 1, 2, \dots, m$, then*

$$\alpha(H(J)) \le 2\alpha(H), \qquad \alpha(H'(J)) \le 2\alpha(H),$$

where α denotes the Kuratowski measure of noncompactness, $H(J) = \{x(s) : x \in H, s \in J\}$ and $H'(J) = \{x'(s) : x \in H, s \in J\}$.

Lemma 3. *If $H \subset PC^1[J,E]$ and the elements of H are equicontinuous on each J_k, $k = 0, 1, 2, \dots, m$, then the elements of $\overline{\mathrm{co}}\, H \subset PC^1[I,E]$ are also equicontinuous on each J_k, $k = 0, 1, 2, \dots, m$.*

Proof. For any given $\varepsilon > 0$, it is easy to show, using the conditions of Lemma 3, that there exists $\delta > 0$ such that if $t_1, t_2 \in J_k$ and $|t_1 - t_2| < \delta$, then

$$\|x(t_1) - x(t_2)\| < \frac{\varepsilon}{3} \tag{2.6}$$

holds for any $x \in H$. For any $y \in \overline{\mathrm{co}}\, H$, there exists $y_k \in \mathrm{co}\, H$ such that

$$\|y_k - y\|_{PC^1} < \frac{\varepsilon}{3}. \tag{2.7}$$

Obviously, there exists $y_i^{(k)} \in H$, $i = 1, 2, \ldots, n_k$, such that $y_k = \sum_{i=1}^{n_k} \alpha_i^{(k)} y_i^{(k)}$, where $\alpha_i^{(k)} \geq 0$ and $\sum_{i=1}^{n_k} \alpha_i^{(k)} = 1$. Thus, by (2.6) and (2.7) we know, for $t_1, t_2 \in J_k$ and $|t_1 - t_2| < \delta$,

$$\begin{aligned}\|y(t_1) - y(t_2)\| &\leq \|y(t_1) - y_k(t_1)\| + \|y_k(t_1) - y_k(t_2)\| \\ &\quad + \|y_k(t_2) - y(t_2)\| \\ &\leq 2\|y - y_k\|_{PC^1} + \sum_{i=1}^{n_k} \alpha_i^{(k)} \|y_i^{(k)}(t_1) - y_i^{(k)}(t_2)\| \\ &< \varepsilon.\end{aligned}$$

Therefore, the elements of $\overline{\text{co}}\, H$ are equicontinuous on each J_k. The proof is completed. □

On account of Theorem 1.2.2 in [**4**], we can easily show the following lemma.

Lemma 4. *If $H \subset PC^1[J, E]$ is bounded and the elements of H are equicontinuous on each J_k, then $\alpha(H(t))$ is continuous on each J_k and*

$$\alpha\left(\left\{ \int_J x(t)\, dt : x \in H \right\}\right) \leq \int_J \alpha(H(t))\, dt,$$

where $k = 0, 1, 2, \ldots, m$.

In the following, let $r > 0$, $T_r = \{x \in E : \|x\| \leq r\} \subset E$. Then we list for convenience the following assumptions:

(H_1) *For any T_r, f is uniformly continuous on $J \times T_r \times T_r \times T_r \times T_r$, and there exist nonnegative constants L_i, $i = 1, 2, 3, 4$, such that*

$$\tag{2.8} \begin{gathered}\alpha(f(t, B_1, B_2, B_3, B_4)) \leq \sum_{i=1}^{4} L_i \alpha(B_i), \\ \forall\, t \in J, \quad \text{bounded } B_i \subset E, \quad i = 1, 2, 3, 4.\end{gathered}$$

(H_2) *For any T_r, I_k and $\bar{I}_k$, $k = 1, 2, \ldots, m$, are bounded on $T_r \times T_r$, and for any bounded $H \subset PC^1[J, E]$, $I_k(H(t_k), H'(t_k))$ and $\bar{I}_k(H(t_k), H'(t_k))$, $k = 1, 2, \ldots, m$, are relatively compact sets in E.*

Lemma 5. *Suppose that the assumptions* (H_1) *and* (H_2) *are fulfilled. Then the operator* A *defined by* (2.2) *is a continuous and bounded map from* $PC^1[J,E]$ *into* $PC^1[J,E]$, *and there exist a positive integer* n_0 *and a constant* $0 \le \tau < 1$ *such that, for any bounded* $H \subset PC^1[J,E]$,

$$\alpha(\tilde{A}^{n_0}(H)) \le \tau\alpha(H) \tag{2.9}$$

holds, where

$$\tilde{A}^1(H) = A(H), \qquad \tilde{A}^n(H) = A(\overline{\text{co}}(\tilde{A}^{n-1}(H))), \\ n = 2, 3, \dots . \tag{2.10}$$

Proof. It is easy to see that the uniform continuity of f on $J \times T_r \times T_r \times T_r \times T_r$ implies the boundedness of f on $J \times T_r \times T_r \times T_r \times T_r$. Then, by (2.2) and (2.5) we know that A is a bounded and continuous operator from $PC^1[J,E]$ into $PC^1[J,E]$. Obviously, in view of the boundedness of $H \subset PC^1[J,E]$ and by (2.2), $A(H) \subset PC^1[J,E]$ is bounded and it is easy to show using (2.2) and (2.5) that the elements of $A(H)$ and $(A(H))'$ are equicontinuous on each J_k. Noticing (2.10) again, we have, for any fixed n, $n = 1, 2, \dots$, that $\tilde{A}^n(H)$ and $(\tilde{A}^n(H))'$ are also bounded on $PC^1[J,E]$ and that the elements of $\tilde{A}^n(H)$ and $(\tilde{A}^n(H))'$ are also equicontinuous on each J_k. Hence, Lemma 4.3.11 in [**4**] or Lemma 3 in [**1**] implies

$$\alpha(\tilde{A}^n(H)) = \max\Big\{\sup_{t\in J}\alpha(\tilde{A}^n(H)(t)), \sup_{t\in J}\alpha((\tilde{A}^n(H))'(t))\Big\}. \tag{2.11}$$

By virtue of (2.2), (2.8), (2.10), (H_2) and in view of Lemma 2, we have

$$\begin{aligned}
\alpha((\tilde{A}^1(H))(t)) &= \alpha((A(H))(t)) \\
&\le t\alpha(\overline{\text{co}}\{(t-s)f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) \mid \\
&\qquad\qquad x \in H, s \in J\}) \\
&\quad + \sum_{0<t_k<t}[\alpha(I_k(H(t_k), H'(t_k))) \\
&\qquad\qquad + (t-t_k)\alpha(\bar{I}_k(H(t_k), H'(t_k)))] \\
&\le t^2[L_1\alpha(H(J)) + L_2\alpha(H'(J)) \\
&\qquad + ak_0L_3\alpha(H(J)) + \alpha h_0L_4\alpha(H(J))] \\
&\le 2t^2[(L_1 + ak_0L_3 + ah_0L_4)\alpha(H) + L_2\alpha(H'(J))] \\
&\le Lt^2\alpha(H),
\end{aligned} \tag{2.12}$$

where $L = 2(L_1 + L_2 + ak_0L_3 + ah_0L_4)$. Similarly, by (2.5), (2.8), (2.10), (H_2) and Lemma 2, we can get

$$\alpha((\tilde{A}^1(H))'(t)) \leq Lt\alpha(H), \tag{2.13}$$

which, together with (2.12), implies

$$\max\{\alpha((\tilde{A}^1(H))(t)), \alpha((\tilde{A}^1(H))'(t))\} \leq L(a+1)t\alpha(H). \tag{2.14}$$

By the uniform continuity of k, h and f on $J \times T_r \times T_r \times T_r \times T_r$, we can easily see, for any bounded $B \subset PC^1[I, E]$ that, if the elements of B' are equicontinuous on each J_k, then the elements of TB, SB and $f(t, B(t), B'(t), (TB)(t), (SB)(t))$ are also equicontinuous on each J_k. Therefore, by the boundedness and the equicontinuity of the elements of $\tilde{A}^1(H)$ and $(\tilde{A}^1H))'$ on each J_k, it is easy to show from Lemma 3 that

$$f(s, (\overline{\text{co}}\,\tilde{A}^1(H))(s), (\overline{\text{co}}\,\tilde{A}^1(H))'(s), (T\overline{\text{co}}\,\tilde{A}^1(H))(s), (S\overline{\text{co}}\,\tilde{A}^1(H))(s)) \tag{2.15}$$

is bounded and that the elements of (2.15) are equicontinuous on each J_k. Thus, by (2.2), (2.8), (2.14), (H_2) and in view of Lemma 4, we get

$$\begin{aligned}
\alpha((\tilde{A}^2(H))(t)) &\leq \alpha\bigg(\int_0^t (t-s) f(s, (\overline{\text{co}}\,\tilde{A}^1(H))(s), (\overline{\text{co}}\,\tilde{A}^1(H))'(s), \\
&\qquad (T\overline{\text{co}}\,\tilde{A}^1(H))(s), (S\overline{\text{co}}\,\tilde{A}^1(H))(s))\, ds\bigg) \\
&\quad + \sum_{0<t_k<t} \Big[\alpha(I_k((\overline{\text{co}}\,\tilde{A}^1(H))(t_k), (\overline{\text{co}}\,\tilde{A}^1(H))'(t_k))) \\
&\qquad + (t-t_k)\alpha(\bar{I}_k((\overline{\text{co}}\,\tilde{A}^1(H))(t_k), (\overline{\text{co}}\,\tilde{A}^1(H))'(t_k)))\Big] \\
&\leq \int_0^t \alpha(t-s)(f(s, (\overline{\text{co}}\,\tilde{A}^1(H))(s), (\overline{\text{co}}\,\tilde{A}^1(H))'(s), \\
&\qquad (T\overline{\text{co}}\,\tilde{A}^1(H))(s), (S\overline{\text{co}}\,\tilde{A}^1(H))(s)))\, ds \\
&\leq t\int_0^t \Big[(L_1 + ak_0L_3 + ah_0L_4)\alpha((\tilde{A}^1(H))(s)) \\
&\qquad + L_2\alpha((\tilde{A}^1(H))'(s))\Big]\, ds
\end{aligned}$$

$$
\begin{aligned}
&\le Lt \int_0^t \max\{\alpha((\tilde{A}^1(H))(s)), \alpha((\tilde{A}^1(H))'(s))\}\, ds \\
&\le L^2(a+1)t\alpha(H) \int_0^t s\, ds \\
&= \frac{L^2(a+1)t^3}{2}\alpha(H).
\end{aligned}
$$

In the same way, by (2.5), (2.8), (2.14), (H_2) and on account of Lemma 4, we can show that

$$
\alpha((\tilde{A}^2(H))'(t)) \le \frac{L^2(a+1)t^2}{2}\alpha(H),
$$

thus,

$$
\max\{\alpha((\tilde{A}^2(H))(t)), \alpha((\tilde{A}^2(H))'(t))\} \le \frac{L^2(a+1)^2t^2}{2!}\alpha(H).
$$

Using induction on n, we have, for $n = 1, 2, \dots,$

$$
\max\{\alpha((\tilde{A}^n(H))(t)), \alpha((\tilde{A}^n(H))'(t))\} \le \frac{L^n(a+1)^nt^n}{n!}\alpha(H). \tag{2.16}
$$

(2.11) and (2.16) imply

$$
\alpha(\tilde{A}^n(H)) \le \frac{L^n(a+1)^na^n}{n!}\alpha(H). \tag{2.17}
$$

Clearly, $(L^n(a+1)^na^n/n!) \to 0$, $n \to \infty$, hence it follows from (2.17) that there exist $0 < \tau < 1$ and a positive integer n_0 such that $\alpha(\tilde{A}^{n_0}(H)) \le \tau\alpha(H)$. The proof is completed. $\square$

Remark 1. The assumptions of Lemma 5 are automatically satisfied when E is finite dimensional.

3. Main theorem. In the following, let

$$
\begin{aligned}
\overline{\lim_{\|x\|+\|y\|+\|z\|+\|w\|\to\infty}} \left(\sup_{t\in J} \frac{\|f(t,x,y,z,w)\|}{\|x\|+\|y\|+\|z\|+\|w\|} \right) &= \beta, \\
\overline{\lim_{\|x\|+\|y\|\to\infty}} \frac{\|I_k(x,y)\|}{\|x\|+\|y\|} &= \beta_k, \quad k = 1, 2, \dots, m, \\
\overline{\lim_{\|x\|+\|y\|\to\infty}} \frac{\|\bar{I}_k(x,y)\|}{\|x\|+\|y\|} &= \bar{\beta}_k, \quad k = 1, 2, \dots, m.
\end{aligned}
$$

So $0 \leq \beta, \beta_k, \bar{\beta}_k \leq \infty$.

Theorem 1. *Let the assumptions of Lemma* 5 *be satisfied. Suppose that*

$$\eta = \max\left\{a^2\beta(2 + ak_0 + ah_0) + 2\sum_{k=1}^{m}[\beta_k + (a - t_k)\bar{\beta}_k],\ a\beta(2 + ak_0 + ah_0) + 2\sum_{k=1}^{m}\bar{\beta}_k\right\} < 1. \tag{3.1}$$

Then IVP (1.1) *has at least one solution in* $PC^1[J, E] \cap C^2[J', E]$.

Proof. By (3.1), we can choose $\beta' > \beta$, $\beta'_k > \beta_k$ and $\bar{\beta}'_k > \bar{\beta}_k$, $k = 1, 2, \dots, m$, such that

$$\eta' = \max\left\{a^2\beta'(2 + ak_0 + ah_0) + 2\sum_{k=1}^{m}[\beta'_k + (a - t_k)\bar{\beta}'_k],\ a\beta'(2 + ak_0 + ah_0) + 2\sum_{k=1}^{m}\bar{\beta}'_k\right\} < 1. \tag{3.2}$$

On account of the definition of β and β', there exists $N > 0$ such that

$$\|f(t, x, y, z, w)\| < \beta'(\|x\| + \|y\| + \|z\| + \|w\|)$$

for $t \in J$, $\|x\| + \|y\| + \|z\| + \|w\| \geq N$. So

$$\|f(t, x, y, z, w)\| \leq \beta'(\|x\| + \|y\| + \|z\| + \|w\|) + M, \quad t \in J, x, y \in E, \tag{3.3}$$

where $M = \sup_{t \in J, \|x\|+\|y\|+\|z\|+\|w\| \leq N} \|f(t, x, y, z, w)\| < \infty$. Similarly, we have

$$\|I_k(x, y)\| \leq \beta'_k(\|x\| + \|y\|) + M_k, \quad x, y \in E, k = 1, 2, \dots, m, \tag{3.4}$$

$$\|\bar{I}_k(x, y)\| \leq \bar{\beta}'_k(\|x\| + \|y\|) + \bar{M}_k, \quad x, y \in E, k = 1, 2, \dots, m, \tag{3.5}$$

where $M_k, \bar{M}_k$, $k = 1, 2, \dots, m$, are positive constants. Now (2.2) and (3.3)–(3.5) imply

$$
\begin{aligned}
\|(Ax(t)\| \le{}& \|x_0\| + a\|x_1\| \\
&+ a\int_0^a [\beta'(\|x(s)\| + \|x'(s)\| + \|(Tx)(s)\| + \|(Sx)(s)\|) + M]\, ds \\
&+ \sum_{0<t_k<t} \big\{[\beta'_k(\|x(t_k)\| + \|x'(t_k)\|) + M_k] \\
&\qquad + (t-t_k)[\bar{\beta}'_k(\|x(t_k)\| + \|x'(t_k)\|) + \bar{M}_k]\big\} \\
\le{}& \left\{a^2\beta'(2 + ak_0 + ah_0) + 2\sum_{k=1}^m [\beta'_k + (a-t_k)\bar{\beta}'_k]\right\}\|x\|_{PC^1} + C_1 \\
\le{}& \eta'\|x\|_{PC^1} + C_1,
\end{aligned} \tag{3.6}
$$

where η' is defined by (3.2) and C_1 is a positive constant. Similarly, from (2.5) and (3.3)–(3.5), we can get

$$
\|(Ax)'(t)\| \le \eta'\|x\|_{PC^1} + C_2, \quad t \in J, \tag{3.7}
$$

where C_2 is a positive constant. The relations (3.6) and (3.7) imply

$$
\|Ax\|_{PC^1} \le \eta'\|x\|_{PC^1} + C, \quad x \in PC^1[J, E], \tag{3.8}
$$

where $\eta' < 1$ is defined by (3.2) and $C = \max\{C_1, C_2\} = \text{const}$. Letting $R > C/(1-\eta')$ and $B_R = \{x \in PC^1[J, E] : \|x\|_{PC^1} \le R\} \subset PC^1[J, E]$, by (3.8) we have $A(B_R) \subset B_R$. Thus it is clear from Lemma 5 that A is a bounded and continuous operator from B_R into B_R. Using Lemma 5 again, we know that there exist a positive integer n_0 and $0 < \tau < 1$ such that (2.9) holds for any $H \subset B_R$, where $\tilde{A}^n$ is defined by (2.10). Let

$$
B_0 = B_R, \qquad B_1 = \overline{\text{co}}\,(\tilde{A}^{n_0}(B_0)), \qquad B_n = \overline{\text{co}}\,(\tilde{A}^{n_0}(B_{n-1})), \quad n = 2, 3, \dots .
$$

We shall show

(i) $B_0 \supset B_1 \supset B_2 \cdots \supset B_n \supset \cdots$;

(ii) $\lim_{n\to\infty} \alpha(B_n) = 0$.

In fact, by (2.10) and $B_1 = \overline{\text{co}}(\tilde{A}^{n_0}(B_0)) \subset B_R = B_0$, we have $\tilde{A}^{n_0}(B_1) \subset \tilde{A}^{n_0}(B_0)$. Thus,

$$B_2 = \overline{\text{co}}(\tilde{A}^{n_0}(B_1)) \subset \overline{\text{co}}(\tilde{A}^{n_0}(B_0)) = B_1.$$

Using induction on n, we get $B_n \subset B_{n-1}$, $n = 1, 2, \dots$, so (i) is proved. Then by (2.9) and the properties of the Kuratowski measure of noncompactness, we have

$$\alpha(B_n) = \alpha(\overline{\text{co}}\tilde{A}^{n_0}(B_{n-1})) \leq \tau\alpha(B_{n-1}) \leq \cdots \leq \tau^n\alpha(B_0) \longrightarrow 0, \\ n \to \infty;$$

thus (ii) is fulfilled. Therefore, it follows from Lemma 5.2 of Chapter II in [**3**] that $\tilde{B} = \cap_{n=0}^{\infty} B_n$ is a nonempty, convex and compact set in B_R.

In the following, we need only to show $A\tilde{B} \subset \tilde{B}$. By $\tilde{A}^1(B_0) = A(B_0) \subset B_0$, we have $\overline{\text{co}}(\tilde{A}^1(B_0)) \subset B_0 = B_R$ and, by (2.10),

$$\begin{aligned} \tilde{A}^2(B_0) &= A(\overline{\text{co}}(\tilde{A}^1(B_0))) \subset A(B_0) = \tilde{A}^1(B_0), \\ \tilde{A}^3(B_0) &= A(\overline{\text{co}}(\tilde{A}^2(B_0))) \subset A(\overline{\text{co}}(\tilde{A}^1(B_0))) = \tilde{A}^2(B_0), \\ &\cdots \\ \tilde{A}^{n_0}(B_0) &= A(\overline{\text{co}}(\tilde{A}^{n_0-1}(B_0))) \subset A(\overline{\text{co}}(\tilde{A}^{n_0-2}(B_0))) = \tilde{A}^{n_0-1}(B_0); \end{aligned}$$

hence,

$$B_1 = \overline{\text{co}}(\tilde{A}^{n_0}(B_0)) \subset \overline{\text{co}}(\tilde{A}^{n_0-1}(B_0));$$

thus,

$$A(B_1) \subset A(\overline{\text{co}}(\tilde{A}^{n_0-1}(B_0))) = \tilde{A}^{n_0}(B_0) \subset \overline{\text{co}}(\tilde{A}^{n_0}(B_0)) = B_1.$$

In the same way, we can get $AB_n \subset B_n$, $n = 2, 3, \dots$. So

$$AB_n \subset B_n, \quad n = 0, 1, 2, \dots \tag{3.9}$$

holds. The relation (3.9) implies $A\tilde{B} = \cap_{n=0}^{\infty} AB_n \subset \cap_{n=0}^{\infty} B_n = \tilde{B}$. Then, noting that $\tilde{B}$ is a nonempty, convex and compact set in B_R, it is easy to show using Schauder's fixed point theorem that A has a fixed point in $\tilde{B} \subset B_R \subset PC^1[J, E]$, and the theorem is proved. □

Remark 2. In order to use Darbo's fixed point theorem to investigate similar problems, the references [**4**] and [**1**] placed strict restrictions on the constants in the compactness type conditions, see 4.3.111 and 4.3.114 in [**4**], (21) and (24) in [**1**]. In the same way, from (2.11)–(2.13), it is not difficult to see that if we also use Darbo's fixed point theorem to consider IVP (1.1), we have to add a restricted condition, i.e.,

$$\begin{aligned}\tau = \max\{&2(L_1 + L_2 + ak_0L_3 + ah_0L_4)a,\\ &2(L_1 + L_2 + ak_0L_3 + ah_0L_4)a^2\} < 1.\end{aligned} \tag{3.10}$$

But, in this paper, we delete any restriction on the constants L_i, $i = 1, 2, 3, 4$, in (2.8). Thus we cannot obtain the results of the paper by the same method, that is, Darbo's fixed point theorem, as in [**4**] or [**1**].

4. An example.

Example. Consider the IVP of infinite system for nonlinear second order integrodifferential equations

$$\begin{aligned} x_n'' &= \frac{t^4}{12n}\left[\frac{1}{8}\,x_{2n}^3 + \left(\int_0^t e^{-ts}\sin(t-2s)x_{2n+1}(s)\,ds\right)^3\right]^{1/3}\\ &\qquad + \frac{1}{7}\left(x_n - \frac{3}{2}\,x_n' + \frac{3}{2}\int_0^1 \cos^{2/3}\pi(t-s)x_n(s)\,ds\right),\\ &\qquad\qquad 0 \le t \le 1,\ t \ne \frac{1}{2}, \qquad\qquad (4.1)\\ \Delta x_n|_{t=1/2} &= \frac{1}{30}\left[\frac{1}{\sqrt{n}}\,x_{n+1}^{3/5}\left(\frac{1}{2}\right) - \frac{1}{n}\,x_{2n-1}'\left(\frac{1}{2}\right)\right],\\ \Delta x_n'|_{t=1/2} &= \frac{1}{15\sqrt{n}}\,x_n^{1/3}\left(\frac{1}{2}\right)\left[x_{2n}'\left(\frac{1}{2}\right)\right]^{1/5},\\ x_n(0) &= \frac{1}{n^2}, \qquad x_n'(0) = 0, \quad n = 1, 2, \dots . \end{aligned}$$

Conclusion. IVP (4.1) has at least one continuous and differentiable solution $x^*(t) = (x_1^*(t), \dots, x_n^*(t), \dots)$ on $[0, 1/2) \cup (1/2, 1]$ such that $x_n(t) \to 0$ as $n \to \infty$ for $0 \le t \le 1$.

Proof. Let $E = c_0 = \{x = (x_1, \dots, x_n, \dots) : x_n \to 0\}$, with norm $\|x\| = \sup_n |x_n|$. Then (4.1) can be regarded as an IVP of the form (1.1) in E, where $J = [0,1]$, $x_0 = (1, (1/2^2), \dots, (1/n^2), \dots)$, $x_1 = (0, \dots, 0, \dots)$, $k(t,s) = e^{-ts}\sin(t-2s), h(t,s) = \cos^{2/3}\pi(t-s)$, $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$, $z = (z_1, \dots, z_n, \dots)$, $w = (w_1, \dots, w_n, \dots)$, $f = (f_1, \dots, f_n, \dots)$, with

$$\begin{aligned} f_n(t,x,y,z,w) = {} & \frac{t^4}{12n}\left(\frac{1}{8}\,x_{2n}^3 + z_{2n+1}^3\right)^{1/3} \\ & + \frac{1}{7}\left(x_n - \frac{3}{2}\,y_n + \frac{3}{2}\,3w_n\right), \end{aligned} \tag{4.2}$$

$y_n = x_n'(s)$, $z_n = \int_0^t k(t,s)x_n(s)\,ds$, $w_n = \int_0^1 h(t,s)x_n(s)\,ds$ and $m = 1$, $t_1 = (1/2)$, $I_1 = (I_{11}, \dots, I_{1n}, \dots)$, $\bar{I}_1 = (\bar{I}_{11}, \dots, \bar{I}_{1n}, \dots)$ with

(4.3)

$$I_{1n}(x,y) = \frac{1}{30}\left(\frac{1}{\sqrt{n}}\,x_{n+1}^{3/5} - \frac{1}{n}\,y_{2n-1}\right), \qquad \bar{I}_{1n}(x,y) = \frac{1}{15\sqrt{n}}\,x_n^{1/3}y_{2n}^{1/5}.$$

Evidently, $f \in C[J \times E \times E \times E \times E, E]$, $I_1, \bar{I}_1 \in C[E \times E, E]$ and, for any $r > 0$, f is uniformly continuous on $J \times T_r \times T_r \times T_r \times T_r$, I_1 and $\bar{I}_1$ are bounded on $T_r \times T_r$. By (4.2), we have

$$\begin{aligned} |f_n(t,x,y,z,w)| &\le \frac{1}{12n}\left(\frac{1}{8}\|x\|^3 + \|z\|^3\right)^{1/3} + \frac{1}{7}\left(\|x\| + \frac{3}{2}\|y\| + \frac{3}{2}\|w\|\right) \\ &\le \frac{1}{12n}\left(\frac{1}{2}\|x\| + \|z\|\right) + \frac{1}{7}\left(\|x\| + \frac{3}{2}\|y\| + \frac{3}{2}\|w\|\right) \\ &\le \left(\frac{1}{24n} + \frac{1}{7}\right)\|x\| + \frac{3}{14}\|y\| + \frac{1}{12n}\|z\| + \frac{3}{14}\|w\|, \end{aligned}$$

thus

$$\|f(t,x,y,z,w)\| \le \frac{31}{168}\|x\| + \frac{3}{14}\|y\| + \frac{1}{12}\|z\| + \frac{3}{14}\|w\|. \tag{4.4}$$

It is easy to show using (4.4) that (H_1) in Theorem 1 is fulfilled. By (4.3), we know

$$\begin{aligned} |I_{1n}(x,y)| &\le \frac{1}{30\sqrt{n}}\,\|x\|^{3/5} + \frac{1}{30n}\,\|y\|, \\ |\bar{I}_{1n}| &\le \frac{1}{15\sqrt{n}}\,\|x\|^{1/3}\,\|y\|^{1/5}. \end{aligned} \tag{4.5}$$

By virtue of (4.5) and the diagonal method, we have, for any bounded $U, V \subset E = c_0$ and $t \in J = [0,1]$ that $I_1(U,V)$ and $\bar{I}_1(U,V)$ are relatively compact, hence (H_2) in Theorem 1 is fulfilled. From (4.4) and (4.5) it follows that $\beta \leq (3/14)$, $\beta_1 \leq (1/30)$, $\bar{\beta}_1 \leq (1/15)$, and therefore (3.1) is satisfied since $\eta < 1$. Thus, our conclusion follows from Theorem 1.

Finally we need to point out that the restricted condition (3.10) is not satisfied in the above example. Let

$$\begin{aligned} g_n^{(1)}(t,x,y,z,w) &= \frac{t^4}{12n}\left(\frac{1}{8}\,x_{2n}^3 + z_{2n+1}^3\right)^{1/3}, \\ g_n^{(2)}(t,x,y,z,w) &= \frac{1}{7}\,x_n, \\ g_n^{(3)}(t,x,y,z,w) &= -\frac{3}{14}\,y_n, \\ g_n^{(4)}(t,x,y,z,w) &= \frac{3}{14}w_n \end{aligned}$$

and $g^{(i)} = (g_1^{(i)}, \dots, g_n^{(i)}, \dots)$, $i = 1,2,3,4$. Then $f_n = g_n^{(1)} + g_n^{(2)} + g_n^{(3)} + g_n^{(4)}$, $n = 1,2,\dots$, and $f = g^{(1)} + g^{(2)} + g^{(3)} + g^{(4)}$. It is obvious that, for any bounded $B_i \subset E = c_0$, $i = 1,2,3,4$, and $t \in J = [0,1]$, $\alpha(g^{(2)}(t,B_1,B_2,B_3,B_4)) = \alpha(B_1)/7, \alpha(g^{(3)}(t,B_1,B_2,B_3,B_4)) = 3\alpha(B_2)/14$, $\alpha(g^{(4)}(t,B_1,B_2,B_3,B_4)) = 3\alpha(B_4)/14$. Using the diagonal method again, we have, for any bounded $B_i \subset E$, $i = 1,2,3,4$, and $t \in J = [0,1]$ that $g^{(1)}(t,B_1,B_2,B_3,B_4)$ is relatively compact, hence $\alpha(g^{(1)}(t,B_1,B_2,B_3,B_4)) = 0$. By the above discussions, we get

$$\begin{gathered} \alpha(f(t,B_1,B_2,B_3,B_4)) \leq \sum_{i=1}^{4} L_i\alpha(B_i), \\ \forall\, t \in J, \quad \text{bounded } B_i \subset E, \quad i = 1,2,3,4, \end{gathered}$$

where $L_1 = (1/7)$, $L_2 = (3/14)$, $L_3 = 0$, $L_4 = (3/14)$. Observing $a = 1$ and $k_0 = h_0 = 1$, we get

$$\begin{aligned} \tau &= \max\{2(L_1 + L_2 + ak_0L_3 + ah_0L_4)a, \\ &\qquad\quad 2(L_1 + L_2 + ak_0L_3 + ah_0L_4)a^2\} \\ &= \frac{8}{7} > 1. \end{aligned}$$

Therefore, the assumption (3.10) is not satisfied. Thus we cannot use Darbo's fixed point theorem to study IVP (1.1).

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