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ABSTRACT VOLTERRA EQUATIONS OF THE SECOND KIND

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Dedicated to Professor Philip M. Anselone

ABSTRACT. We consider the equation $x = Vx + f$ with a nonlinear Volterra operator V in a large class of spaces. We prove that it has a local solution if V is continuous and compact or (in case of a regular space) condensing.

We study the connection of local and global solutions and gain in particular an abstract extension principle and some results on global solutions for a 'nonlinear Fredholm' case and for the case of positively homogeneous operators.

Moreover, we show that compact linear Volterra operators in (not necessarily regular) ideal spaces have spectral radius zero, which generalizes a result of Zabrejko.

The algebraic definition of 'Volterra operator' in the article matches a much wider class of operators than the classical Volterra operators. This more general notion leads to new results, even when applied to the classical linear Volterra operator in L_p.

We also give some applications to differential equations in Banach spaces.

0. Introduction. We are concerned with the Volterra equation of the second kind, $x = Vx+f$ with a nonlinear Volterra operator V. Here the term *Volterra operator* is defined in an abstract way: in Section 1 we give the precise definition. Although this technical definition is of a purely algebraic nature, it 'catches' the typical behavior of Volterra operators. In Sections 2 and 3 we study the existence of 'local,' respectively 'global,' solutions of the Volterra equation. Since the results in these sections are rather involved, we summarize the most

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important special cases in Section 4. In the final Section 5 we give some applications to differential equations in Banach spaces. Because we develop a special theory for Volterra operators 'of Uryson type,' we are able to generalize Zabrejko's result [**28**] that compact linear Volterra operators in regular ideal spaces or in L_∞ have spectral radius zero to all ideal spaces. This can be considered as one of the main results of this article; a proof in a more specific setting would not be much easier. Moreover, we may generalize this result to iterates of Volterra operators in regular spaces. It is hard to see how such a generalization could be gained without our abstract approach. Hence, our algebraic definition of 'Volterra operator' not only allows us to treat the most important generalizations of the 'classical Volterra operator' in a unified way, but really yields new results, even for the classical linear Volterra operator in L_p.

1. Basic definitions. One of the main advantages of Volterra equations compared with more general integral equations is that it makes sense to speak of local solutions. We will define abstract Volterra operators in such a way that we keep this advantage. In the abstract case a *local solution* shall mean a solution in a certain subspace of the corresponding *projected equation*. Thus, we shall consider an increasing (and in some sense exhausting) chain of subspaces, which we assume to be given as the ranges of projection operators P_i.

Let Z be a linear space. Let I be a linearly ordered index set with a smallest element 0. For any $i \in I$, let $P_i : Z \to Z$ be a linear projection operator, i.e., $P_i^2 = P_i$, with the property that $P_i P_j = P_j P_i = P_i$ for $i \le j$. This means that the ranges of P_i are nondecreasing, $P_i Z \subseteq P_j Z$ for $i \le j$, and that the null spaces of P_i are nonincreasing. Moreover, $P_{ji} = P_j - P_i$, $j \ge i$, is a projection. Let $\overline{P}_i = \mathrm{id} - P_i$, $\overline{P}_{ji} = \mathrm{id} - P_{ji}$ denote the complementary projections.

We assume $P_0 = 0$ and that $x \in Z$ is characterized by its values $P_i x$, i.e., the intersection of the null spaces of P_i is trivial (this does not imply that the union of the ranges of P_i is Z, if I has no maximal element).

To get an idea, one should think of Z as a space of functions and that $P_i x$ describes the behavior of x before the time i. Then $P_{ji} x$ describes x between the times i and j.

Definition 1.1. We call a (nonlinear) operator $V : D \subseteq Z \to Z$ a 0-*Volterra operator*, with respect to the family of projections P_i, if P_iVx depends only on P_ix, i.e.,

$$P_ix = P_iy \quad \Longrightarrow \quad P_iVx = P_iVy, \quad x, y \in D. \tag{1}$$

We call V a *Volterra operator* if V additionally is *partially additive*, i.e., if

$$V(P_ix + \overline{P}_iy) + V(\overline{P}_ix + P_iy) = Vx + Vy \tag{2}$$

holds for all x, y, for which this equation makes sense, i.e., if x, y, $P_ix + \overline{P}_iy$, $\overline{P}_ix + P_iy \in D$.

Roughly speaking, a 0-Volterra operator V describes a system whose current state does not depend on the future. If we properly want to define the term *local solution of* $x = Vx + f$ *at time* $i = 0$, this can be done only if V is a 0-Volterra operator. The next proposition will show that the partial additivity is a natural assumption. In order to define the term *local solution of* $x = Vx + f$ *at time* i for any i in a sense which really deserves this name, it is necessary and sufficient that V be a Volterra operator.

A 0-Volterra operator V is characterized by the equation

$$P_iV = P_iVP_i, \quad i \in I, \tag{3}$$

where in case $P_iD \not\subseteq D$, the characterization means that we may use (3) to define P_iVP_i on D (well-defined). In the following we shall use this fact without comment.

In case P_iD, $\overline{P}_iD \subseteq D$, the partial additivity just means

$$VP_i + V\overline{P}_i = V + V0, \quad i \in I. \tag{4}$$

In case $P_iD \subseteq D$, we put $V_i = V - VP_i$. If V is a 0-Volterra operator, then (3) implies $P_iV_i = 0$. Roughly speaking, V_ix describes the behavior of Vx if we ignore (subtract) how x behaves before the time i. Now we prove the announced simpler characterization of Volterra operators.

Proposition 1.1. *If* $P_iD \subseteq D$, *then* $V : D \to Z$ *is a Volterra operator if and only if* $P_{ji}V_ix$ *depends only on* $P_{ji}x$, *i.e.,*

$$P_{ji}x = P_{ji}y \quad \Longrightarrow \quad P_{ji}V_ix = P_{ji}V_iy, \quad x, y \in D, j \geq i. \tag{5}$$

Proof. Observe that (5) is equivalent to

$$P_{ji}V_ix = P_{ji}V_i(P_{ji}x + \overline{P}_{ji}y)$$

for all $x, y \in Z$ with $x, P_{ji}x + \overline{P}_{ji}y \in D$. For a 0-Volterra operator V, we have

$$\begin{aligned}
P_{ji}[V_ix - V_i(P_{ji}x + \overline{P}_{ji}y)]& \\
&= P_{ji}[Vx - VP_ix - V(P_{ji}x + \overline{P}_{ji}y) + VP_iy] \\
&= P_j[Vx - VP_ix - VP_j(P_{ji}x + \overline{P}_{ji}y) + VP_iy] \\
&\quad - P_i[Vx - VP_ix - VP_i(P_{ji}x + \overline{P}_{ji}y) + VP_iy] \\
&= P_j[Vx - VP_ix - VP_j(\overline{P}_ix + P_iy) + VP_iy] \\
&= P_j[Vx - VP_ix - V(\overline{P}_ix + P_iy) + VP_iy].
\end{aligned}$$

Together, this implies that (5) is for a 0-Volterra operator V equivalent to

$$VP_ix + V(\overline{P}_ix + P_iy) = Vx + VP_iy. \tag{6}$$

Thus, if V is a Volterra operator, then (2) applied to P_iy instead of y shows that (6) and thus (5) is satisfied. Conversely, if (5) holds, then we find for $i = 0$ that V is a 0-Volterra operator, whence (6) is satisfied. Interchanging the roles of x and y, this implies

$$VP_iy + V(P_ix + \overline{P}_iy) = Vy + VP_ix.$$

Adding this equation to (6), we gain (2). □

One might, of course, discuss whether the partial additivity should already be included in the definition of the term *Volterra operator*. As a matter of fact, even for 0-Volterra operators which are not partially additive it is possible to define the term *local extension of a solution of*

$x = Vx + f$ *at time* i in a reasonable sense. However, the condition (5) will allow us a definition of the term *local solution of* $x = Vx + f$ *at time* i, which is easier to apply. Indeed, (5) means that V_i is a 0-Volterra operator with respect to the family $\tilde{P}_j = P_{ji}$, $j \in I_i := \{j \in I : j \geq i\}$. (Observe that $i \in I_i$ is the smallest element of I_i, whence i takes the role of 0. We remark that the values $\tilde{P}_j x$ do not characterize x but only characterize $\overline{P}_i x$; but this is inessential for our purposes.) This observation allows us for Volterra operators V to reduce the definition of the term *local solution at* i to that of *local solution at* 0, and so we get a 'homogeneous' definition.

Whenever possible we formulate our results already for (not necessarily partially additive) 0-Volterra operators V. We only break this rule for the principles in Section 3 which allow us to conclude from the existence of 'local solutions' to the existence of 'global solutions,' because we want to formulate them in a manner which is most suitable for our applications. However, by the same methods of proof the reader should have no difficulties in formulating corresponding principles for 0-Volterra operators which allow us to conclude from 'local extensibility of a solution' to the existence of 'global solutions.'

The reason why we did not use (5) as the definition of a Volterra operator is that (5) only makes sense if $P_i D \subseteq D$, otherwise V_i need not be defined. Similarly, as before, we may (even in case $P_{ji} D \not\subseteq D$) characterize (5) by

$$P_{ji} V_i = P_{ji} V_i P_{ji}, \quad j \geq i. \tag{7}$$

In other words if $P_i D \subseteq D$, then Volterra operators are characterized by the single equation (7).

Let us briefly compare our definition with the one of Gohberg and Kreĭn for linear Volterra operators. The P_i in [**10**] correspond to our complementary projections $\overline{P}_i$. In [**10**, Chapter 1] linear Volterra operators are characterized by the fact that the ranges R_i of $\overline{P}_i$ are a chain of invariant subspaces. This corresponds to our definition. If $V : Z \to Z$ is a 0-Volterra operator with $V0 = 0$, then $VR_i \subseteq R_i$. Conversely, if $V : Z \to Z$ satisfies $VR_i \subseteq R_i$ and is partially additive, then V is a 0-Volterra operator (whence even a Volterra operator). Note, however, that our definition even in the linear case is essentially different. Gohberg and Kreĭn define Volterra operators to be linear and compact with spectral radius zero (in contrast, we will prove that compact linear Volterra operators have spectral radius zero).

We remark that $\tilde{V} = \mathrm{id} - V$ is also a Volterra operator, so that in particular for abstract Volterra operators it is just a matter of taste whether we formulate our results for the equation $x = Vx + f$ of the second kind or for the Volterra equation $\tilde{V}x = f$ of the first kind. However, quite often we will assume that V is small in some sense, e.g., compact, so that for the 'classical' Volterra equation of the first kind just a few of our results are applicable (although there are some).

Our model situation is the following, still very abstract, example.

Example 1.1. Let Z be a linear space of functions $x : S \to W$ with some set S and some linear space W. Let $S = \cup S_i$ with $S_i \subseteq S_j$ for $i \leq j$, $S_0 = \varnothing$. We let $P_i x(s) = \chi_{S_i}(s)x(s)$.

We have a similar situation, if S is a measure space, all S_i are measurable, W is a Banach space, and Z consists of (classes of) (strongly Bochner) measurable functions $x : S \to W$.

V is a 0-Volterra operator if and only if the values of Vx on the set S_i depend only on the values of x on this set. V is a Volterra operator if and only if the values of $V_i x$ on the set $S_j \backslash S_i$ depend only on the values of x on this set, assuming $P_i D \subseteq D$.

The situation in this example is the appropriate setting to describe Volterra operators in ideal spaces [**27, 29**] as, e.g., in the Lebesgue (-Bochner) spaces $L_p(S, W)$ or in Orlicz spaces. We are mainly interested in this situation. However, many important results are also available in, e.g., spaces of continuous functions (we will give some examples), if we define the projections differently.

Example 1.2. Let Z, S, S_i, W be as in the previous example. For $i > 0$, let $\pi_i : S \to S_i$ be mappings with $\pi_i(s) = s$ on S_i, and $c_i : S \to \mathbf{R}$ be mappings with $c_i(s) = 1$ on S_i. Moreover, assume that $\pi_i(\pi_j(s)) = \pi_i(s)$ for $j > i > 0$. Then $P_i x(s) = c_i(s)x(\pi_i(s))$, $P_0 := 0$, is a family of projections with all desired properties.

If S is a topological space, the $S_i \subseteq S$ are classed and c_i are continuous (and bounded) it may quite often be arranged that π_i is continuous on the support of c_i, i.e., that π_i retracts $\operatorname{supp} c_i$ onto S_i, see [**4**]. In case of a topological vector space W, this implies that the projection P_i maps the space of continuous (and bounded) functions into itself.

Moreover, if $S_i \subseteq \mathbf{R}^n$ are 'good,' it may additionally be arranged that $\pi_i(s)$ is 'near' to an element of S_i with smallest distance to s and that $c_i : S \to [0,1]$ vanishes outside a 'small' neighborhood of S_i so that P_i maps also, e.g., the space of continuous functions with compact support into itself. This in particular is the case in the following examples.

In the above settings we have the classical nonlinear Volterra operator. (Integrals occurring in the examples of this article are always meant in the sense of Bochner-Lebesgue, although in most cases the results carry over to other notions of integrals like Pettis integrals or the Cauchy mean value, too.)

Example 1.3. Let S be some interval, $t_0 \in \overline{S}$, $I = [0,\infty]$ and $S_i = S \cap [t_0 - i, t_0 + i]$, $S_0 := \varnothing$. Then

$$Vx(t) = \int_{t_0}^{t} g(t,s,x(s))\,ds \tag{8}$$

is a Volterra operator. It may happen that $P_i = P_j$ for $i \neq j$ in this case, which is not very natural; this can be avoided. If, e.g., $S = [t_0, t_0 + a)$ or $S = [t_0, t_0 + b]$, we may just put $I = [0, a)$ or $I = [0, b]$. Also for other intervals we may proceed similarly: If e.g. $S = [t_0 - 1, t_0 + 2)$, we may put $I = [0, 2)$; for $S = (t_0 - 1, t_0 + 2]$ we may redefine, e.g., $S_i = [t_0 - i, t_0 + 3i]$ and put $I = [0, 1)$. For the setting of Example 1.1 we could also have chosen open or half-open intervals for the definition of S_i. For the setting of Example 1.2 we define of course $\pi_i(s)$ as the element of S_i with the smallest distance to s, i.e.,

$$\pi_i(s) = \begin{cases} s & \text{if } t_0 - i \le s \le t_0 + i, \\ t_0 - i & \text{if } s < t_0 - i, \\ t_0 + i & \text{if } s > t_0 + i. \end{cases}$$

Observe that also the cases $t_0 = \pm\infty$ can be treated similarly.

However, our definition is made to deal with natural generalizations.

Example 1.4. Let $S = J \times T$ with some interval J, $t_0 \in \overline{J}$, and a σ-finite (unsigned) measure space T. Put $I = [0, \infty]$ and $S_i = S \cap ([t_0 - i, t_0 + i] \times T)$, $S_0 \neq \varnothing$. Then

$$Vx(t,\tau) = \int_{t_0}^{t} G(t,\tau,s,x(s,\cdot))\,ds \tag{9}$$

is a Volterra operator (G takes values in a Banach space W, and the last argument of G is a function $y : T \to W$). Similarly, as in Example 1.3, we may redefine I and S_i also to have $P_i \neq P_j$ for $i \neq j$ or to treat the cases $t_0 = \pm\infty$. The most important special case of (9) is that G is an integral functional,

$$Vx(t,\tau) = \int_{t_0}^{t} \int_T g(t,\tau,s,\sigma,x(s,\sigma))\,d\sigma\,ds, \tag{10}$$

since thus V is a Uryson operator, whence it is usually compact.

Be aware that if we identify functions which coincide almost everywhere, V in general is not well-defined by (9), but if it is well-defined, it is a Volterra operator. However, this problem obviously does not occur in (10).

We remark that 'formally' (10) might be considered as a special case of (8), if we assume that x and g in (8) take values in an appropriate Banach space of functions $y : T \to W$ and if we identify $x(t)(\tau) =: x(t,\tau)$. However, whether the corresponding Volterra equations really are equivalent depends on the considered spaces. For results in this direction in L_p, see, e.g., [**12**, pp. 68–70] or (for special cases in C and L_p) [**9**]; for more general ideal spaces we refer to [**27**].

Equation (10) contains in particular the generalization of Example 1.3 for $S \subseteq \mathbf{R}^N$,

$$Vx(t) = \int_{\{s \in S : \|s - t_0\| \leq \|t\|\}} g(t,s,x(s))\,ds,$$

if we put $T = \mathbf{R}^{N-1}$ (for $N = 1$ let $T = \{0\}$).

But even Example 1.4 may be generalized.

Example 1.5. With the notation of Example 1.4, the operator

$$Vx(t,\tau) = \int_{t_0}^{t} G(t,\tau,s,x(s,\cdot))\,d\mu_{t,\tau}(s) \tag{11}$$

is a Volterra operator. Here $\mu_{t,\tau}$ is a σ-finite and, without loss of generality, positive measure on $[t_0, t]$, respectively $[t, t_0]$. Equation (11) in particular includes the nonlinear partial integral operator of Volterra type

$$
\begin{aligned}
Vx(t,\tau) = {} & g_0(t,\tau,x(t,\tau)) + \int_T g_1(t,\tau,\sigma,x(t,\sigma))\,d\sigma \\
& + \int_{t_0}^t g_2(t,\tau,s,x(s,\tau))\,ds \\
& + \int_{t_0}^t \int_T g_3(t,\tau,s,\sigma,x(s,\sigma))\,d\sigma\,ds.
\end{aligned} \tag{12}
$$

To see this, let $\mu_{t,\tau}$ be the sum of the point measure concentrated at t and of the Lebesgue measure, and put

$$
G(t,\tau,s,y) = \begin{cases} g_0(t,\tau,y(\tau)) + \int_T g_1(t,\tau,\sigma,y(\sigma))\,d\sigma & \text{if } s = t, \\ g_2(t,\tau,s,y(\tau)) + \int_T g_3(t,\tau,s,\sigma,y(\sigma))\,d\sigma & \text{if } s < t. \end{cases}
$$

Again, (12) is well-defined, even if functions coinciding almost everywhere on S (with respect to the Lebesgue measure on J) are identified. For the general theory of partial integral operators, we refer to [**2, 6, 14, 22**] and the references therein, and for the linear Volterra case in particular to [**15, 30**]. Note that the partial integral operator (12) usually is not compact.

Operators such as (10) arise naturally in the theory of integro-differential equations of Barbashin type [**3, 9**], and operators such as (12) in generalized equations [**5**].

Most of our results will be applicable to operators of the form (11) of the previous example. In this connection it is important to note that this example includes all operators that occur typically in the study of functional differential equations which arise from practical problems like differential equations with memory or other delay equations.

Example 1.6. If we chose $T = \{0\}$ in the previous example, then (11) may with the obvious identification, ignoring the trivial second argument of x, be written in the form

$$
Vx(t) = \int_{t_0}^t g(t,s,x(s))\,d\mu_t(s), \tag{13}
$$

where μ_t is a measure on $[t_0, t]$, respectively $[t, t_0]$. Operators with retarded argument, as, e.g., of the form

$$Vx(t) = \sum_{k=1}^{n} f_k(t, x(\theta_k(t))) + \int_{t_0}^{t} f_0(t, s, x(s))\, ds,$$

$t_0 \leq \theta_k(t) \leq t$, are of the form (13), i.e., they are special cases of the previous example. To see this, let μ_t be the sum of the measure concentrated at the points of $\{\theta_k(t) : k = 1, \ldots, n\}$ and of the Lebesgue measure, and put

$$g(t, s, y) = \begin{cases} \sum_{k:\theta_k(t)=s} f_k(t, y) & \text{if the sum is not empty,} \\ f_0(t, s, y) & \text{otherwise.} \end{cases}$$

Observe that in all previous examples V is partially additive, since it is some sort of Uryson operator. In Section 5 we will see, however, that it is very useful that we do not restrict our attention to Uryson operators (although we are mainly interested in those).

An example of a 0-Volterra operator which is (usually) not a Volterra operator is an operator of the form

$$Vx(t) = G\biggl(\int_0^t g(t, s, x(s))\, ds\biggr). \tag{14}$$

As a final example, we give another abstract Volterra-Uryson operator in Banach spaces.

Example 1.7. Let Z be a Banach space with a (Schauder) base $(e_n)_n$. Let $I = \mathbf{N}$ and $P_i(\sum a_n e_n) = \sum_{n \leq i} a_n e_n$. Then the 'Uryson' operator

$$V\Bigl(\sum a_n e_n\Bigr) = \sum_n \Bigl(\sum_k g_{nk}(a_k)\Bigr) e_n$$

is a Volterra operator if and only if $g_{nk} = 0$ for $k > n$, i.e., if the second sum is finite and taken over the values $k = 1, \ldots, n$.

Of course, by our definition, any operator can be written as a Volterra operator with $P_0 = 0$, $P_1 = \mathrm{id}$. But, for 'reasonable' Volterra operators as in the previous examples, the projections P_{ji} become small in some sense, and it is advantageous to consider the 'local' behavior of V on the corresponding ranges, which usually is much easier to study.

2. Local solvability. Let $Y \subseteq X \subseteq Z$ be subspaces of Z, and let $V : D \subseteq X \to Y$ be a 0-Volterra operator. We shall not assume that $P_i D \subseteq D$. However, we will assume that $P_i X \subseteq X$ for all i. In the setting of Example 1.1 one should think of Z as a 'large' space like the space of all (measurable) functions and of X as a 'good' space like spaces of summable or (essentially) bounded functions; it is not excluded so far that Y and D contain only smooth functions, e.g., $V : C(S, W) \to C(S, W)$.

Since for the study of local solutions it is inessential whether x is characterized by its values $P_i x$, we drop this assumption in this section. We only require that I be nontrivial, i.e., that there is some $i > 0$.

Definition 2.1. We say that V is *locally solvable* at 0 for some $f \in X$ if there exists some $x \in D$ and some $i > 0$ such that $P_i x = P_i(Vx + f)$.

We call V *uniformly locally solvable* at 0 for some $F \subseteq X$ if there is some $i > 0$ such that $P_i x = P_i(Vx + f)$ has for each $f \in F$ a solution $x \in D$.

In the setting of Example 1.1 a solution of $P_i x = P_i(Vx + f)$ is a solution $x \in D$ on S_i. Let $\|\cdot\|$ be a norm on X (and thus also on Y).

Our key to local solvability is the following technical property.

Definition 2.2. We call V *locally invariant* at 0 for $f \in X$ if there exists some $i > 0$, some nonempty, bounded, convex and closed in X set $K \subseteq D$ and some $g \in X$ with $P_i g = P_i f$ such that $g + P_i V K \subseteq K$. Similarly, we say that V is *uniformly locally invariant* at 0 for $F \subseteq X$ if i can be chosen independently of $f \in F$.

If we know that V is locally invariant at 0, we can apply Schauder's fixed point theorem to prove that V is locally solvable at 0. Let us formulate a more general result.

We say that $B : D \to X$ has the *fixed point property*, if the fact that

$Ax = Bx + f$ (with some $f \in X$) maps a closed bounded and convex set $K \subseteq D$ into itself implies that A has a fixed point in K. In particular, if X is a Banach space, any compact and continuous B has the fixed point property by Schauder's fixed point theorem.

Proposition 2.1. *Assume that each P_iV has the fixed point property. If V is (uniformly) locally invariant at 0, then it is (uniformly) locally solvable at 0.*

Proof. Choose K as in Definition 2.2. Then the operator $Ax = P_iVx + g$ maps K into itself, whence it has a fixed point $x \in K : x = P_iVx + g$. Thus, $P_ix = P_i(P_iVx + g) = P_i(Vx + f)$. □

Note that we have even found a solution $x \in K$.

First, some words on the fixed point property of P_iV.

We call $V : D \to Y$ *compact*, if it maps bounded sets into precompact sets, i.e., if for any bounded sequence $x_n \in D$ the sequence Vx_n contains a Cauchy subsequence (observe that we require neither Y nor X to be complete, nor V to be continuous).

Recall that the Hausdorff measure of noncompactness γ of a set $M \subseteq X$ is defined as the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in X (if X is important, we write γ_X instead). The Kuratowski measure ψ of noncompactness is the infimum of all $\delta > 0$ such that M has a finite covering of sets with diameter less than δ. An operator $A : D \subseteq X \to Y$ is called

1. *q-condensing* (usually with $q < 1$), if

$$\gamma_Y(AM) \leq q\gamma_X(M), \quad M \subseteq D,$$

2. *strictly q-condensing*, if it is q-condensing and

$$\gamma_Y(AM) < q\gamma_X(M), \quad M \subseteq D \text{ bounded but not precompact},$$

3. *condensing*, if it is strictly 1-condensing.

Analogously, we may replace γ by ψ. If it is important, whether we consider γ or ψ, we write, e.g., *ψ-condensing* or *q-γ-condensing.*

Darbo's generalization of Schauder's fixed point theorem [**7**] states that whenever a continuous condensing operator $A : D \subseteq X \to X$ maps a closed, bounded, and convex subset $M \neq \varnothing$ of a Banach space into itself, it has a fixed point in M.

Proposition 2.2. *Let X be a Banach space. Then $P_i V$ has the fixed point property if it is continuous and condensing. This is the case if V is continuous and strictly q-condensing with $q\|P_i\| \leq 1$.*

The second part is a simple consequence of the fact that each bounded linear operator P is $\|P\|$-condensing (with respect to γ and ψ).

It is a natural question whether we may replace the completeness of X by that of Y. Let us first try to replace the completeness of X in Proposition 2.2 by the weaker condition that $P_i Y$ is contained in a complete subspace of X, i.e., that the closure $\overline{P_i Y}$ of $P_i Y$ in X is complete. If $P_i V$ is continuous and ψ-condensing, in particular, also if V is compact, this indeed is possible. $\overline{P_i Y}$ is closed in the completion of X, whence also the linear hull X_f of $\overline{P_i Y}$ and some given $f \in X$, see, e.g., [**23**, Theorem 1.42]. Thus, $X_f \subseteq X$ is a Banach space. If $Ax = P_i V x + f$ maps a nonempty closed and convex set $K \subseteq D$ into itself, we have $K \subseteq AD \subseteq X_f$. Whence, by Darbo's fixed point theorem, A has a fixed point in K, if A is continuous and condensing on K, with respect to the space X_f. For ψ this is satisfied, since $\psi_X(M) = \psi_{X_f}(M)$ for any $M \subseteq X_f$. The Kuratowski measure of noncompactness of M depends only on the metric space induced by M, not on the underlying space. However, for the Hausdorff measure γ of noncompactness this need not be true.

Example 2.1. Let $X = L_\infty([0,2])$, $X_0 = \{x \in X : x \in C([0,2]),\ x|_{[1,2]} = 0\}$, and $M = \{x \in X_0 : x([0,2]) \subseteq [0,1]\} \subseteq X_0$. Then $\gamma_X(M) \leq 1/2$ (the 1/2-net is given by the constant function $x(s) \equiv 1/2$), but in X_0 obviously there exists no finite ε-net for $\varepsilon < 1$ (for any given finite net in X_0 a function in M which is 1 on $[0, 1-\delta]$ has for sufficiently small $\delta > 0$ distance bigger than ε), whence $\gamma_{X_0}(M) \geq 1$. This example matches in the previous context for, e.g., $Y = X_0$, $P_i x(s) = \chi_{[0,i]}(s)x(s)$, since then $X_f = X_0$ for $i \geq 1$ and $f \in Y$.

We have shown that (for the Kuratowski measure ψ of noncompactness), the completeness of X in Proposition 2.2 may be replaced by the completeness of $\overline{P_iY}$. One might guess that this condition is satisfied if Y is complete (which implies $Y = \overline{Y}$), at least if P_i is bounded. The following discouraging example shows that this need not hold, even if P_i is an orthogonal projection.

Example 2.2. Let Z be a separable Hilbert space with orthonormal base $(e_n)_n$ and Y the closed linear span of the vectors $e_{2n} + ne_{2n+1}$. Define a projection by $P(\sum a_k e_k) = \sum a_{2n}e_{2n}$. Let X be the linear span of Y and PY. We claim that neither X nor $\overline{PY}$ (the closure taken in X) is complete, since both spaces are not closed in Z. Indeed, PY is a space containing each of the vectors e_{2n}, whence PY is dense in the closed linear span of these vectors. In particular, $z = \sum n^{-1}e_{2n}$ belongs to the closure of PY (and of X) in the space Z. On the other hand, z does not belong to X. Since X is the direct sum of Y and PY, it obviously suffices to check that z does not belong to PY, and this is easily verified.

For condensing or compact 0-Volterra operators V, Proposition 2.2 reduced the problem of finding a local solution at 0 to the problem of proving that V is locally invariant at 0. We show now that a compact 0-Volterra operator V usually is *automatically* locally invariant at 0. Indeed, in this case we do not have to find a *bounded* invariant subset (which of course is the crucial condition in Definition 2.2).

We write $K_r(x_0)$ for the closed ball $\{x \in X : \|x - x_0\| \leq r\}$ and let $K_r(F) = \cup_{f\in F} K_r(f)$.

Lemma 2.1. *Let $V : D \to Y$ be a 0-Volterra operator. Let $F, G \subseteq X$ and $M \subseteq D$ be convex and closed in X such that, for each $i > 0$ and each $f \in F$ there is some $g \in G$ with $P_i g = P_i f$ and $g + P_i VM \subseteq M$.*

Moreover, assume that $\|P_i\| \leq C$ and that there are $r > 0$, $B \subseteq X$, $y_0 \in X$, $N \geq 0$ such that, for $K_0 = M \cap K_r(G)$ we have $VK_0 \subseteq B$,

$$\inf_{i>0} \|P_i(y - y_0)\| \leq N, \quad y \in B, \tag{15}$$

and $C(\gamma_B(VK_0) + \|y_0\| + N) < r$. Then V is uniformly locally invariant at 0 for F (with $K = M \cap K_r(g)$, provided $K \neq \varnothing$).

Proof. We show first that, for any $K \subseteq K_0$,

$$m(K) = \inf_{i>0} \sup_{x\in K} \|P_i V x\| \le C(\gamma_B(VK) + \|y_0\| + N). \tag{16}$$

Choose $H > \gamma_B(VK)$, $N_1 > N$ arbitrary. Let $y_1, \dots, y_n \in B$ be a finite H-net for VK. By (15) there exist $i_k > 0$ with $\|P_{i_k}(y_k - y_0)\| \le N_1$. For $i = \min\{i_1, \dots, i_n\}$ this implies $\|P_i(y_k - y_0)\| \le CN_1$. Given some $x \in K$ there exists some $k \in \{1, \dots, n\}$ with $\|P_i(Vx - y_k)\| \le C\|Vx - y_k\| \le CH$, whence

$$\|P_i V x\| \le CH + \|P_i y_k\| \le CH + C\|y_0\| + CN_1.$$

Thus $m(K) \le C(H + \|y_0\| + N_1)$, which implies (16).

Now we apply (16) for $K = K_0$ and find some $i > 0$ with $\|P_i V x\| \le r$ for all $x \in K_0$. This is the desired i. Indeed, for given f and corresponding g, we have with $K = M \cap K_r(g)$ that $g + P_i V K \subseteq g + P_i V K_0 \in M \cap K_r(g) = K$. $\square$

For easier reference we write down the most important special case:

Lemma 2.2. *Let $V : D \to Y$ be a 0-Volterra operator. Let $F, G \subseteq X$ and $M \subseteq D$ be convex and closed in X such that, for each $i > 0$ and each $f \in F$, there is some $g \in G$ with $P_i g = P_i f$ and $g + P_i V M \subseteq M$.*

Moreover, assume that $\|P_i\| \le C$ and that there are $VM \subseteq B \subseteq X$, $y_0 \in X$, with

$$\inf_{i>0} \|P_i(y - y_0)\| = 0, \quad y \in B. \tag{17}$$

Let $r > C\|y_0\|$ be such that $K_0 = M \cap K_r(G)$ satisfies $C_{\gamma_B}(VK_0) < r - C\|y_0\|$. Then V is uniformly locally invariant at 0 for F (with $K = M \cap K_r(g)$, provided $K \neq \varnothing$).

The typical application of the lemma is meant for the case $D = X$ and either $B = Y$ or $B = VX$. In this case we may put $M = X$ and $G = F$. Assume that (17) holds with $y_0 = 0$. We will define a large class of spaces where this is satisfied. In other spaces this is no principal restriction, since we may rewrite the equation $x = Vx + f$ in the form

$x = (Vx - y_0) + (f + y_0)$ and consequently replace V by $V - y_0$ (this is still a 0-Volterra operator) and F by $F + y_0$; however, observe that in case $B = VX$ this transformation changes the condition (17) (and in case $D \neq X$ even more).

In this situation the lemma reads as follows. If (17) holds and $C_{\gamma_B}(VK_r(F)) < r$, then V is locally invariant at 0 for F (with $K = K_r(f)$). In particular, if each P_iV has the fixed point property there is some $i > 0$ such that, for each $f \in F$, the equation $P_ix = P_i(Vx + f)$ has a solution $x \in X$ with $\|x - f\| \leq r$.

In view of the last inequality, it is of interest that $r > 0$ may be chosen arbitrarily small for bounded $F \subseteq X$, if V is compact. But also, under different conditions, the inequality $C_{\gamma_B}(VK_r(F)) < r$ is satisfied. For example, in the important case $F = \{f\}$, we have $\gamma_X(K_r(F)) \leq r$, even with equality for infinite-dimensional X, see [**1**, Theorem 1.1.6], whence it suffices that V is strictly q-γ-condensing as a mapping from X into B, where $qC \leq 1$. Then, for $F = \{f\}$, the inequality $C_{\gamma_B}(VK_r(F)) < r$ holds for arbitrarily small $r > 0$.

If V is even q-γ-condensing with $qC < 1$, then the inequality $C_{\gamma_B}(VK_r(F)) < r$ holds for bounded F, if r is chosen sufficiently large.

The previous discussion showed that, for compact 0-Volterra operators we reduced the problem of finding local solutions to the crucial condition (17). We are going to show now that this condition is usually satisfied. We will consider two situations. Either the space X is 'nice,' or the operator V is 'of Uryson type.' The first assumption is almost trivial.

Definition 2.3. We call a set $M \subseteq X$ *regular at* 0 (with respect to the family P_i) if

$$\inf_{i>0} \|P_ix\| = 0, \quad x \in M.$$

If B is regular at 0, then (17) holds with $y_0 = 0$.

All regular ideal spaces [**27, 29**] such as $L_p(S, W)$, $1 \leq p < \infty$, over an atomic-free measure space S are regular at 0 for appropriate (nontrivial) $P_ix(s) = \chi_{S_i}(s)x(s)$ (and any M). But there are also other examples. In particular, ideal spaces in the sense of Nguyen [**20**], including the most general form of $\mathbf{R}^N$-valued Orlicz spaces [**19, 21**] fall into this class, if they are regular and the underlying measure space S is atomic-free.

In contrast to the L_p-case, in many ideal spaces there exist nontrivial sets which are regular at 0, although the whole space need not be regular in 0. This is the case, for example, for all Orlicz spaces, where the generating Young function is finite but does not satisfy a Δ_2-condition [**16**]. So it is important that in (17) we do not assume the whole space X to be regular at 0, but assume this only for B.

Now we turn to the other condition, which implies (17). To get an idea, observe that the condition (17) for $B = VM$ reads $\inf_{i>0} \|P_i(Vx - y_0)\| = 0$, which in case $P_i M \subseteq D$ can be rewritten as $\inf_{i>0} \|P_i(VP_i x - y_0)\| = 0$. If $\|P_i\|$ is uniformly bounded, it thus suffices that

$$\inf_{i>0} \|VP_i x - y_0\| = 0, \quad x \in M. \tag{18}$$

This motivates the following definition. For the remainder of this section we will restrict ourselves to 0-Volterra operators V as described in the setting of Example 1.1 (or of Example 1.2) with a topological space W.

Definition 2.4. We call $V : D \to Y$ of *Uryson type* at 0, if there is some $y_0 \in Y$ such that for any $x \in D$ there exists some sequence $i_1 \geq i_2 \geq \cdots > 0$ with the following property. For all, respectively almost all, $t \in S$, we have

$$\lim_{n\to\infty} VP_{i_n} x(t) = y_0(t).$$

Example 2.3. Consider the setting of Example 1.1. If $P_i D \subseteq D$, the operators of Examples 1.3 and 1.4 are of Uryson type at 0 with $y_0 = V0$; the same holds for Example 1.5, if we either may identify functions coinciding almost everywhere, or if $\{t_0\} \cap J$ is a null set with respect to $\mu_{t,\tau}$.

It suffices to prove this for the most general Example 1.5. To consider Bochner integrals, we assume here that $(W, |\cdot|)$ is a Banach space. Fix $x \in D$. If $\mu_{t,\tau}(\{t_0\} \cap J) \neq 0$, we may assume $x(t_0, \cdot) = 0$ without loss of generality. For all (almost all) $(t, \tau) \in S$, we have that $Vx(t,\tau)$ and $y_0(t,\tau) = V0(t,\tau)$ is defined, whence $m(s) = G(t,\tau,s,x(s,\cdot)) - G(t,\tau,s,0)$ is integrable with respect to $\mu_{t,\tau}$. Let E_n be defined by $S_{1/n} = E_n \times T$, i.e., E_n is the first component of $S_{1/n}$. Since E_n is a decreasing sequence with $\cap E_n = \{t_0\} \cap J$, Lebesgue's dominated convergence theorem (with dominating function $|m|$) implies

$$\begin{aligned}|VP_{1/n}x(t,\tau) - y_0(t,\tau)|& \\ &\leq \int_{E_n} |G(t,\tau,s,x(s,\cdot)) - G(t,\tau,s,0)|\, d\mu_{t,\tau}(s) \\ &\longrightarrow \int_{\{t_0\}\cap J} |G(t,\tau,s,x(s,\cdot)) - G(t,\tau,s,0)|\, d\mu_{t,\tau}(s) \\ &= 0 \quad \text{as } n \to \infty.\end{aligned}$$

For the last equality we used that either $x(t_0,\cdot) = 0$ or $\mu_{t,\tau}(\{t_0\} \cap J) = 0$.

We remark that an analogous result holds for the setting of Example 1.2 only under restrictive additional conditions on V and D, even in the case of a classical linear operator V. For this reason the following results are of practical interest only in the setting of Example 1.1 (although they hold more general).

Definition 2.5. We call the norm in Y *compatible with pointwise convergence* if, for any Cauchy sequence $x_n \in Y$ which converges pointwise, respectively almost everywhere to 0, we have $\|x_n\| \to 0$.

All 'natural' normed function spaces are compatible with pointwise convergence. Typical spaces of measurable functions like $L_p(S,W)$ or Orlicz spaces or, more generally, all ideal spaces [**27, 29**] or ideal spaces in the sense of Nguyen [**20**] have this property. Other examples are $C(S,W)$, $C^m(S,W)$ or $W^m(S,W)$, and its various generalizations. Moreover, the norm in Y is compatible with pointwise convergence if it is a subspace of a space with this property. However, there exist pathological spaces without this property. An example is given by the preideal space $Y = X$ of all measurable functions $x : [0,1] \to \mathbf{R}$ with finite norm $\|x\| = \int_0^1 |x(s)|\, ds + \operatorname{ess\,lim\,sup}_{s\to 0} |x(s)|$, the norm of the Cauchy sequence $x_n = \chi_{(0,n^{-1}]}$ does not tend to 0.

Lemma 2.3. *Let* $\|P_i\| \le C$*,* $M, P_iM \subseteq D \subseteq X$*,* $V : D \to Y$ *be of Uryson type at* 0 *and the norm in* Y *be compatible with pointwise convergence. If* V *is compact, then* (17) *is satisfied with* $B = VM$*.*

Proof. Fix $x \in D$ and choose i_n as in Definition 2.4. By the compactness of V, the sequence $y_n = VP_{i_n}x$ contains a Cauchy subsequence y_{n_k}. Since $y_{n_k}(t) \to y_0(t)$ for all (almost all) t, we have $\|y_{n_k} - y_0\| \to 0$. Thus (18) holds, which implies (17) as shown above. □

Although Lemma 2.3 is almost the best we can say for abstract Volterra operators, in nonregular spaces, for many concrete operators the compactness is not needed for (17).

Example 2.4. Consider the 'classical' Volterra operator (8) in $X = L_\infty(S, W)$. Observe that X is as 'nonregular' as possible. It seems natural to assume that, for any $x \in D$, we have $Vx(t) \to 0$ essentially as $t \to t_0$. For the P_i of Example 1.3, this already implies that $\|P_iVx\| \to 0$ for $i \to 0$, i.e., (17) is satisfied with $y_0 = 0$ (and $B = VD$).

However, even the linear Volterra operator $Vx(t) = t^{-1}\int_0^t x(s)\,ds$ fails to satisfy (17) in $X = L_\infty$, observe that thus Lemma 2.3 already implies that V is not compact in X.

3. Global solvability. For global solvability of $x = Vx + f$ it is usually not enough that V is compact and continuous, contrasting with the case of local solvability. In typical cases, the solvability of $x = Vx + f$ for all $f \in D$ implies that V may not grow too fast near ∞.

Example 3.1. Let $S = [0, 1]$, $W = \mathbf{R}$ and D contain the constant functions. Let V be the autonomous Volterra operator

$$Vx(t) = \int_0^t g(x(s))\,ds.$$

Assume that g is continuous and on some interval $J = [y_0, \infty)$ positive with

$$\int_J \frac{du}{g(u)} < \infty.$$

Then $x = Vx + f$ is not solvable for some $f \in D$. Indeed, choose $y > y_0$ with $\int_y^\infty g(u)^{-1}\,du < 1$, and put $f(s) \equiv y$. Then $x = Vx + f$ has no solution. By the continuity of g, such an x would be continuously differentiable and solve the initial value problem $x' = g(x)$, $x(0) = y$. Then $x(t) \geq y$ for all t (if t_0 is the infimum of all t with $x(t) < y$, we have $x(t_0) = y$, $x'(t_0) = g(x(t_0)) = g(y) > 0$, a contradiction). Thus $x(t) \in J$ implies $x'(t) > 0$, whence x is strictly increasing, and so

$$1 = \int_0^t \frac{x'(s)}{g(x(s))}\,ds = \int_y^{x(1)} \frac{du}{g(u)},$$

which is not possible.

A similar result for the nonlinear Abel operator $Vx(t) = \int_0^t (t - s)^{\alpha-1} g(x(s))\,ds$ under more restrictive conditions on g can be found in [**11**, Chapter 13, Theorem 5.4].

On the other hand, it is well known that linearly bounded differential equations have global solutions.

We present some principles that allow us to prove the existence of global solutions. As we already remarked, these principles could be formulated more general for 0-Volterra operators which are not necessarily partially additive (including, e.g., operators like (14)), but for simplicity we intend to use the following notion of local solutions.

Recall that if V is a Volterra operator and $P_i D \subseteq D$, then $V_i = V - VP_i$ is a 0-Volterra operator with respect to the sequence $\tilde{P}_j = P_{ji}$ (for the index set $I_i = \{j : j \geq i\}$; observe that i is the smallest element in this set, i.e., in our previous notation we have $i = 0$). If this operator is locally solvable at 0, we call V locally solvable at i.

Definition 3.1. The Volterra operator $V : D \to Z$ with $P_i D \subseteq D$ is *locally solvable* at i with respect to f_0 and a set $E \subseteq D$ if, for any $f \in f_0 + VP_i E$ there exists some $j > i$ such that $P_{ji} x = P_{ji}(V_i x + f)$ has a solution $x \in E$ or if $i = \max I$.

Similarly we say V has *at most one local solution* at i if, for any $f \in f_0 + VP_i E$ and corresponding two local solutions $x, y \in E$ at i there exists some $j > i$ such that $P_{ji} x = P_{ji} y$, or if $i = \max I$.

If $x = Vx + f$ has a solution x, then V is locally solvable at each i with respect to each $E \supseteq \{x\}$. However, if $x \notin E$ it is not necessarily true that V is locally solvable at i.

For differential equations it is well known that under weak assumptions the existence of local solutions implies the existence of a maximal solution (on a maximal interval of existence). For abstract Volterra equations we need some notions whose meaning is obvious for differential equations.

Definition 3.2. We call a set M *full for* D if, for any family $x_i \in D$, $i \in I$, with $P_i x_i = P_i x_j$, $i \leq j$, there exists some $x \in M$ with $P_i x = P_i x_i$.

We call D *decomposable in* M if, for any $x, y \in D$, $i \in I$, there is some $z \in M$ with $P_j z = P_j(P_i x + \overline{P}_i y)$, $j \geq i$.

We say that i_0 is a *closure point* of $I_0 \subseteq I$ if $x, y \in Z$ and $P_i x = P_i y$, $i \in I_0$, implies $P_{i_0} x = P_{i_0} y$.

We call $I_0 \subseteq I$ *interval* if, $i, j \in I_0$ and $i < i_0 < j$ implies $i_0 \in I_0$.

We call P_i *dense* if, for each interval $0 \in I_0 \subsetneq I$ there exists $i_0 = \sup I_0$ and i_0 is a closure point of I_0.

There exists a set M full for D if and only if Z is full for D.

This means that Z should be chosen 'sufficiently large.' In Example 1.1 the space Z of all (measurable) functions is full for any $D \subseteq Z$. For $S = \mathbf{R}$, $I = [0, \infty]$, $S_i = [-i, i]$ intervals are usual intervals, and the closure points of I_0 are the points in the least closed interval containing I_0 and 0. In all our examples the family P_i is dense. One should have this case in mind, now.

Proposition 3.1 (Extension principle). *Assume the axiom of choice. Let D be decomposable in D, $P_i D \subseteq D$ and M be full for D. Let V be locally solvable at any i with respect to f and $E = D$. Then there exists a maximal solution $x \in M$ of $x = Vx + f$, i.e., there is some interval $I_0 \subseteq I$, $I_0 \supsetneq \{0\}$ with the property that, for all $i \in I_0$, the element $P_i x$ belongs to D and is a solution of $P_i x = P_i(Vx + f)$. Either $I_0 = I$ or I_0 is maximal in the sense that if $i_0 = \sup I_0$ exists and is a closure* *point of* I_0*, then* $P_{i_0}x \notin D$*, i.e., there is no* $y \in Z$*,* $P_iy = P_ix$ *for* $i \in I_0$ *with* $P_{i_0}y \in D$*.*

Proof. Let X be the set of all pairs (P_ix, i) with $x \in M$, $i > 0$, such that $P_ix \in D$ is a solution of $P_ix = P_i(Vx + f)$. X is partially ordered by the relation $(x, i) \leq (y, j)$ for $i \leq j$ and $P_ix = P_iy$. Applying the local solvability for $i = 0$, X is nonempty. By Hausdorff's maximality principle, see, e.g., [**24**, Appendix], there exists a maximal linearly ordered subset $H \subseteq X$. Let I_0 be the union of all i for which there is some $x_i \in Z$ with $(P_ix_i, i) \in H$. We have $P_ix_i \leq P_ix_j$ for $i \leq j$, whence there is some $x \in M$ with $P_ix = P_ix_i$ for $i \in I_0$. In particular, $H = \{(P_ix, i) : i \in I_0\}$.

Let $i_0 = \sup I_0$ be a closure point of I_0. If $y = P_{i_0}x \in D$, then $P_iy = P_i(Vy + f)$ for $i \in I_0$ implies $P_{i_0}y = P_{i_0}(Vy + f)$, whence $(y, i_0) \in X$. Since $(y, i_0) \geq (P_ix, i)$ for all $i \in I_0$, the maximality of H implies $(y, i_0) \in H$.

If $i_0 = \max I$, we thus have $I_0 = I$ and are done. Otherwise, since V is locally solvable at i_0, there exists some $j > i_0$ and some $y_0 \in D$ with $P_{ji_0}y_0 = P_{ji_0}(V_{i_0}y_0 + VP_{i_0}x + f)$. Since D is decomposable in D and P_jy_0, $P_{i_0}x \in D$, also $z = P_{ji_0}y_0 + P_{i_0}x$ belongs to D. Moreover, $P_{ji_0}z = P_{ji_0}(V_{i_0}z + VP_{i_0}x + f) = P_{ji_0}(V_{i_0}z + VP_{i_0}z + f) = P_{ji_0}(Vz + f)$ and $P_{i_0}z = P_{i_0}y = P_{i_0}(Vy + f) = P_{i_0}(Vz + f)$, which together implies $P_jz = P_j(Vz + f)$, whence $(z, j) = (P_jz, j) \in X$. Since $(z, j) \geq (y, i_0) \geq (P_ix, i)$ for all $i \in I_0$, the maximality of H implies $(z, j) \in H$, whence $j \in I_0$, contradicting $j > i_0 = \sup I_0$. □

The axiom of choice is not needed if $P_ix = P_i(Vx + f)$ has for each i at most one solution in P_iM. In this case the order on X in the proof is linear, and so the choice $H = X$ is possible. This uniqueness is of interest anyway.

Lemma 3.1. *Let* $V : D \to Z$ *with* $P_iD \subseteq D$ *have at most one local solution at* i *with respect to* f *and* $E = D$*. Assume that* P_i *is dense. Then* $P_ix = P_i(Vx + f)$ *has for each* i *at most one solution in* P_iZ*.*

Proof. Let $x, y \in P_iZ$ be different solutions of $P_ix = P_i(Vx + f)$. Let I_0 be the interval of all j such that $P_jx = P_jy$ and $i_0 = \sup I_0$. Since

i_0 is a closure point of I_0, we have $z = P_{i_0}x = P_{i_0}y$ (and thus $i_0 < i$). Both x and y are solutions of the equation $P_{ii_0}x = P_{ii_0}(V_{i_0}x + Vz + f)$ since $P_ix = P_i(Vx + f) = P_i(V_{i_0}x + Vz + f)$ and analogously $P_iy = P_i(V_{i_0}y + Vz + f)$. Since V has at most one local solution at i_0, we thus have $P_{ji_0}x = P_{ji_0}y$ for some $j > i_0$, which implies $P_jx_0 = P_jy_0$, contradicting the maximality of i_0. ◻

Now we turn to a principle that yields global solutions, which is somewhat related to the extension principle. In some applications it may be hard to say something a priori about $P_{i_0}x$ if $i_0 = \sup I_0$ is a closure point of I_0 and $P_ix = P_i(Vx + f)$, $i \in I_0$. However, it might be possible to say something useful, if we know even more that $P_ix = P_i(Vx + f)$, $i \in I_0$, has a solution for each f in a set F.

Definition 3.3. Let $V : D \to Z$ be a 0-Volterra operator. Let P_i be dense and (A_i) be statements with the following property. Whenever (A_i) is true for all i in some interval $0 \in I_0 \subsetneq I$, then (A_{i_0}) is true for $i_0 = \sup I_0$.

We say that V is of *Fredholm type* for $F \subseteq Z$ if there exist such statements (A_i) and sets $E_i \subseteq D$ such that, for each i the following alternative holds. Either

1. (A_i) holds and $P_ix = P_i(Vx + f)$ has for each $f \in F$ a solution in E_i, or

2. (A_i) fails and $P_ix = P_i(Vx + f)$ has for some $f \in F$ no solution in D.

A typical example of such a statement (A_i) is (let $f_0, y_0 \in Z$ be fixed for all i): Each solution of $P_ix = P_i(Vx + f_0)$ satisfies $P_ix = P_iy_0$.

In the case $f_0 = y_0 = 0$ and linear $V : X \to X$, X being a linear space with $P_iX \subseteq X$, the alternative becomes the classical Fredholm alternative for $\mathrm{id} - P_iV$ with $F = E_i = X$; we show a refinement (for bounded F and E_i) in the proof of Theorem 5.2.

Definition 3.4. The Volterra operator $V : D \to Z$ with $P_iD \subseteq D$ is *uniformly locally solvable at* i with respect to F and a set $E_i \subseteq D$ if, for any $f \in F + VE_i$, there exists some $j > i$ such that $P_{ji}x = P_{ji}(V_i x + f)$ has a solution, or if $i = \max I$.

Proposition 3.2 (First uniform principle). *Let V be a Volterra operator of Fredholm type for F. Let P_i be dense, D decomposable in D and $P_i D \subseteq D$. Assume that V is uniformly locally solvable at each i with respect to F and the set E_i for each i.*

Then for each $f \in F$ and each i, the equation $P_i x = P_i(Vx + f)$ has a solution in E_i.

Proof. Let I_0 be the set of all i for which the first alternative applies. We have to prove that $I_0 = I$.

Assume the contrary. $0 \in I_0 \subsetneq I$ is an interval. If $i \leq j$, $j \in I_0$, then $i \in I_0$. Indeed, for each $f \in F$, the equation $P_j x = P_j(Vx + f)$ has a solution $x \in E_j$ whence $P_i x \in D$ is a solution of $P_i x = P_i(Vx + f)$.

The Fredholm condition implies $i_0 \in I_0$ for $i_0 = \sup I_0$. Indeed, since (A_i) is true for all $i \in I_0$, also (A_{i_0}) is true.

Now choose $j > i_0$ such that $P_{ji_0}x = P_{ji_0}(V_{i_0}x + Vz + f)$ has for each $z \in E_{i_0}$, $f \in F$, a solution $x \in D$. Since $P_{i_0}x = P_{i_0}(Vx + f)$ has a solution $z \in E_{i_0}$, the same argument as in the proof of the extension principle shows that $y = P_{ji_0}x + P_{i_0}z$ is a solution of $P_j x = P_j(Vx + f)$, contradicting the maximality of i_0. □

Observe that if $\max I$ exists, then the conclusion says that $x = Vx + f$ has a solution in D. If $\max I$ does not exist, but Z is full for D, then we still have a (unique) $x \in Z$ such that $P_i x \in D$ solves $P_i x = P_i(Vx + f)$, provided that either

1. V is uniquely locally solvable at each i, for all $f \in F$, or that

2. in the first case of the Fredholm alternative, we always have a unique solution of $P_i x = P_i(Vx + f)$ in $P_i Z$.

Now we turn to yet another principle. For a differential equation the idea is just to partition the interval I into a finite, or countable, number of subintervals and to solve the equation on each subinterval. However, for abstract Volterra equations, where I may be 'large,' there is no reason to restrict to a countable number of subintervals. The number of intervals can be any ordinal number.

Definition 3.5. A *partition* U is a mapping from an ordinal number α into I with $U(k) \leq U(m)$ for $k \leq m$, $U(0) = 0$, such that $x \in Z$ is characterized by the values $P_{U(k)}x$, i.e., the intersection of the null spaces of $P_{U(k)}$ is trivial.

Given a partition U, we write $Q_k = P_{U(k+1)U(k)} = P_{U(k+1)} - P_{U(k)}$.

In the situation of Example 1.1, we may define $\sum Q_k x_k$ pointwise without any notion of convergence since all except one term vanish. Generalizing this idea yields

Definition 3.6. If $U : \alpha \to I$ is a partition and $x_k \in D$, then there exists at most one $x \in Z$ with $Q_k x = Q_k x_k$ for all k. We write $\sum Q_k x_k := x$.

If, for any $x_k \in D$, we have that $\sum Q_k x_k$ is defined and belongs to M, we call D *summable* in M.

Since x is characterized by $P_{U(k)}x$, uniqueness is trivial.

Let us briefly address the question, when D is summable in M. For a finite partition, i.e., $\alpha < \omega$, it obviously is sufficient that D is decomposable in M; for a countable partition $\alpha = \omega$, it suffices that M additionally is full for D, and in general, a transfinite induction shows that it suffices that additionally M is full for M.

Now assume $P_i D \subseteq D$. For a set $E \subseteq D$ and a partition $U : \alpha \to I$, let $U[E]$ denote the set of all sums of the form

$$\sum_{k<k_0} Q_k x_k, \quad k_0 \in \alpha, x_k \in E,$$

(we assume that these sums exist). Observe that the inequality $k < k_0$ is strict (in case $k_0 = 0$, we let the empty sum be 0, as usual). We assume that $U[E] \subseteq D$. In the most important case of a finite or countable partition $\alpha \leq \omega$, our assumptions are satisfied if D is decomposable in D.

Definition 3.7. We call V *uniformly locally solvable* with respect to f_0 and some set $E \subseteq D$ if there exists a partition U with the following property. For any k and any $f \in f_0 + VU[E]$ there exists a solution $x \in E$ of the equation $Q_k x = Q_k(V_{U(k)}x + f)$.

If $x = Vx + f$ has a solution x, then V is uniformly locally solvable with each $E \supseteq \{x\}$. It turns out that also the converse is true.

We now require additionally that E be summable in some M and that, for each sum x, each projection $P_{U(k)}x$ belongs to D. For a finite partition this is satisfied, for $M = D$, if E is decomposable in D. For a countable partition it suffices that additionally E is summable in M, e.g., additionally M is full for E.

Proposition 3.3 (Second uniform principle). *Let V be uniformly locally solvable with respect to f and some $E \neq \varnothing$. Then there is some $x \in M$ such that $P_i x \in D$ is a solution of $P_i x = P_i(Vx + f)$ for all $i \leq U(k)$, $k \in \alpha$.*

Proof. At first we define $x_k \in E$ by transfinite induction. Assume x_k is already defined for all $k < k_0$. Then we put

$$f_{k_0} = f + V\left(\sum_{k<k_0} Q_k x_k\right),$$

and let $x_{k_0} \in E$ be a solution of

$$Q_{k_0} x_{k_0} = Q_{k_0}(V_{U(k_0)} x_{k_0} + f_{k_0}).$$

The desired solution is now given by $x = \sum Q_k x_k$. Indeed, as is easily checked, $P_{U(k_0)}x$ has the properties required in the definition of $\sum_{k<k_0} Q_k x_k$, i.e.,

$$P_{U(k_0)}x = \sum_{k<k_0} Q_k x_k.$$

Putting $y = P_{U(K)}x$, $K \in \alpha$ arbitrary, this implies

$$\begin{aligned} Q_{k_0} y &= Q_{k_0}[Vy - VP_{U(k_0)}y + (f + VP_{U(k_0)}y)] \\ &= Q_{k_0}(Vy + f) \end{aligned}$$

for all $k_0 < K$, and thus our claim is obvious. □

Observe that the proof tacitly used the α-axiom of dependent choices, see, e.g., [**13**], where α is the ordinal number used as the domain of U. We needed this for the assumption that x_k is a family (we have to recursively 'choose' one solution x_k for each k). But this is no severe restriction. The (countable) axiom of dependent choices is usually assumed in analysis anyway, and if one uses uncountable α, one will probably assume even the axiom of choice (in its most general form).

To apply this principle, it is usually a good idea to consider (refinement) sequences of partitions U_n and to check the assumptions for sufficiently large n. Let us give two sample examples.

For simplicity, let $D = X \subseteq Z$ be a Banach space with $P_i X \subseteq X$, $Y \subseteq X$ be a subspace and $V : X \to Y$ a Volterra operator.

In the following we will consider (refinement) sequences of partitions U_n. We will assume that these partitions are either finite or countable. In the first case we put $M = D$, in the second we assume that $M \subseteq Z$ is full for D. We say that x is a *global solution* for $x = Vx + f$ with respect to the given partition sequence, if $x \in M$, and if there is one partition $U = U_n$, such that $P_i x \in X$ solves $P_i x = P_i(Vx + f)$ for all $i \leq U(k)$. The second uniform principle implies that $x = Vx + f$ has a global solution if V is uniformly locally solvable with respect to f and $E = X$ for some partition $U = U_n$.

We say that V is F-*onto* with respect to a partition sequence U_n if the fact that $Q_k^{(n)} x = Q_k^{(n)}(Vx + f)$ has a solution in X for each $f \in F$ implies that it even has a solution in X for each $f \in \lambda F$, $\lambda \geq 1$. For example, if V satisfies $V(\lambda x) = \lambda V x$ for all $\lambda > 0$, then V is F-onto for any F and any partition sequence U_n.

Similarly, as in Lemma 2.2, we can now prove

Theorem 3.1. *Let in the above situation V be $K_\delta(0)$-onto for some $\delta > 0$ and $\|Q_k^{(n)}\| \leq C$. Assume, moreover, that*

$$\lim_{n\to\infty} \sup_k \|Q_k^{(n)} y\| = 0, \quad y \in Y. \tag{19}$$

Let V be continuous and q-γ-condensing as a mapping from $D = X$ into Y, where $q < 1/(C + C^2)$. Then $x = Vx + f$ has a global solution for each $f \in Z$ with $P_i f \in X$, $i \in I$.

Proof. Choose $r > 0$ with $qC(1+C)r + \delta < r$ and $\varepsilon > 0$ with $q(1+C)r < \varepsilon$ such that $\rho = r - (C\varepsilon + \delta) > 0$. Put $G = VK_r(0) - VK_{Cr}(0)$. We have $\gamma_Y(G) \le \gamma_Y(VK_r(0)) + \gamma_Y(VK_{Cr}(0)) \le q(\gamma_X(K_r(0)) + \gamma_X(K_{C_r}(0))) = q(1+C)r < \varepsilon$. Thus there exists a finite ε-net $y_m \in Y$ for G. For sufficiently large n we have $\|Q_k^{(n)} y_m\| \le \rho$ for all k and m. We claim that each such partition $U = U_n$ the solution $Q_k x = Q_k(Vx + f)$ has for each $f \in K_\delta(0)$ (whence even for each $f \in X$) a solution in X. By the second uniform principle for $E = X$, this proves the statement.

Indeed, let $f \in K_\delta(0)$ and k be given. Then $Ax = Q_k V_{U(k)} x + f$ maps $K_r(0)$ into itself. Given $x \in K_r(0)$, we have $V_{U(k)} x \in G$. Whence there exists some m with $\|V_{U(k)} x - y_m\| \le \varepsilon$. Then $\|Q_k(V_{U(k)} x - y_m)\| \le C\varepsilon$, whence

$$\|Ax\| \le C\varepsilon + \|Q_k y_m\| + \|f\| \le C\varepsilon + \rho + \delta = r.$$

Next observe that A is continuous in X and condensing (even $q(C + C^2)$-condensing) since $\gamma_X(AK) \le C\gamma_Y(V_{U(k)}K) \le C\ (\gamma_Y(VK) + \gamma_Y(VP_{U(k)}K)) \le Cq(\gamma_X(K) + \gamma_X(P_{U(k)}K)) \le qC(1+C)\gamma_X(K)$. By Darbo's fixed point theorem A has a fixed point $x \in K_r(0)$, i.e., $Q_k x = Q_k Ax = Q_k(V_{U(k)} x + f)$. □

Roughly speaking, we replaced the condition that B is regular in X in Lemma 2.2 by the 'uniform regularity' condition (19) and adopted the proof of Lemma 2.2 to show that V is uniformly locally solvable with respect to f and some appropriate partition $U = U_n$.

Observe that if $\max U_n$ $(= \max I)$ exists for all n, a global solution x of $x = Vx + f$ is indeed a solution in X of this equation. In many cases there exist such partitions U_n satisfying (19). Consider the setting of Example 1.1 with a σ-finite atomic-free measure space, S. If Y is a regular ideal space (in the usual sense [**27, 29**] or, in the sense of Nguyen [**20**]) it is easy to check that, for 'sufficiently dense' P_i (in particular for the P_i used in Examples 1.3–1.5 with $I = [0, \infty]$) there exist even finite partitions U_n satisfying (19). For example, for $S = \mathbf{R}$, $S_i = [t_0 - i, t_0 + i]$, we may let U_n take the values $0, 2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \ldots, n, \infty$.

Sadly, by the method of proof in the previous theorem, it is not possible to consider spaces which are not regular. The difficulty is that in transferring the condition (15), respectively (17), from the single operator V to the family $V_{U_n(k)}$, the ranges of these operators differ with n and k (in the previous proof, we just included them straightforwardly in Y).

However, by a refinement of our technique, it is also possible to transfer the condition (15), if we additionally assume that V is compact and the family $Q_k^{(n)} V_{U_n(k)}$ is equicontinuous. The idea is the simple observation that if a compact operator A maps a bounded set H into itself, it also maps the precompact set $H \cap AH$ into itself. And in compact sets, the pointwise limit (20) becomes a uniform limit due to the equicontinuity of the family $Q_k^{(n)} V_{U_n(k)}$.

Theorem 3.2. *Consider the situation described in front of Theorem* 3.1. *Assume there is a precompact set $G \subseteq X$ such that each $VK_r(0)$ is contained in some positive multiple of G. Let V be F-onto, where F is the closed and convex hull of $G, 0$, and some given $f \in X$. Let $\|Q_k^{(n)}\| \le C$, and assume that the family $Q_k^{(n)} V_{U_n(k)}$ is equicontinuous on compact sets. Assume, moreover, that*

$$\limsup_{n\to\infty} \sup_k \|Q_k^{(n)} V_{U_n(k)} x\| \le N < \infty, \quad x \in X. \tag{20}$$

Then $x = Vx + f$ has a global solution.

Proof. By the second uniform principle, it suffices to prove that V is uniformly locally solvable with respect to f and $E = X$ for each partition $U = U_n$ with sufficiently large n. At first we determine those n:

1. Since F is compact, we have $F \subseteq K_r(0)$ for some $r > 0$. Choose $\delta > 0$, $N_1 > N$ and $R \ge Cr + N_1 + \delta$. The set $H = \overline{\operatorname{conv}}\,(VK_R(0) - VK_{CR}(0) + F)$ is compact. By assumption, there exists some $e > 0$ such that

$$\|x-y\| \le \varepsilon \quad \Longrightarrow \quad \|Q_k^{(n)} V_{U_n(k)} x - Q_k^{(n)} V_{U_n(k)} y\| \le \delta \qquad x, y \in H. \tag{21}$$

Moreover, H contains a finite ε-net $x_m \in H$. For all $U = U_n$ with sufficiently large n, we have $\|Q_k V_{U(k)} x_m\| \le N_1$ for all k and m, whence

$$\sup_k \|Q_k V_{U(k)} x\| \le N_1 + \delta, \quad x \in H. \tag{22}$$

2. Fix $g \in F$. The operator $Ax = Q_k(V_{U(k)}x + g)$ maps the set $K = K_R(0) \cap Q_k H$ into itself: let $x \in K$. On the one hand, $x \in K_R(0)$ implies $Ax \in Q_k H$. On the other hand, there is some $y \in H$ with $x = Q_k y$ whence $Ax = Q_k(V_{U(k)}y + g)$, and (22) implies $\|Ax\| \leq N_1 + \delta + C\|g\| \leq R$, i.e., $Ax \in K_R(0)$. Together we have $Ax \in K$ as stated.

3. Since K is nonempty, $0 \in K$, convex and compact (because H is compact), Schauder's fixed point theorem implies that A has a fixed point $x \in K$. Thus $Q_k x = Q_k Ax = Q_k(V_{U(k)}x + g)$.

4. We have shown that for each $g \in F$ the equation $Q_k x = Q_k(V_{U(k)}x + g)$ has a solution $x \in X$. Since V is F-onto, the equation even has a solution for each $g \in \lambda F$, $\lambda \geq 1$. Since $VX \subseteq \cup_{\lambda>0}\lambda G$, we have $\cup_{\lambda \geq 1}\lambda F \supseteq f + VX$. In particular, the equation $Q_k x = Q_k(V_{U(k)}x + g)$ has for each $g \in f + VX$ a solution in X, which we had to prove. □

The first two conditions of Theorem 3.2 are satisfied if V is compact and $V(\lambda x) = \lambda Vx$, $\lambda > 0$. In this case one may, e.g., choose G as the convex hull of 0 and $VK_1(0)$.

The equicontinuity assumption in Theorem 3.2 is satisfied if V is uniformly continuous on balls. Indeed, by $\|Q_k^{(n)}\| \leq C$, the family $VQ_k^{(n)}$ and thus also $Q_k^{(n)}VQ_k^{(n)}$ is equicontinuous on compact sets, and it remains to observe that $Q_k^{(n)}V_{U_n(k)} = Q_k^{(n)}V_{U_n(k)}Q_k^{(n)} = Q_k^{(n)}(VQ_k^{(n)} - V0)$.

We emphasize that Theorem 3.2 is not the most general theorem that we may obtain by this method of proof. Observe in particular that $\delta > 0$ was chosen arbitrarily in the proof, so that we did not need the full strength of both, the compactness of H and of the equicontinuity of $Q_k^{(n)}V_{U_n(k)}$ for the estimate (21). If we can gain such an estimate for all y of a finite ε-net on H by other means, e.g., by an appropriate combination of a Lipschitz or Hölder condition and some condensing condition for V, we could proceed similarly also for other V, in particular for certain noncompact V (in the third step of the proof we then will have to assume some fixed point property and replace K by $\overline{K}$; observe that $AK \subseteq K$ implies $A\overline{K} \subseteq \overline{K}$ for continuous A).

Similarly, as (15), the condition (20) is usually even satisfied with

$N = 0$; this is the case, of course, if VX is 'uniformly regular,' i.e., if (19) holds. But this is also satisfied for Volterra operators, which are 'uniformly of Uryson type,' as can be seen analogously to Lemma 2.3.

Lemma 3.2. *Consider the setting of Example* 1.1 *with a topological space W in the above situation. Let the norm in Y be compatible with pointwise convergence and $\|Q_k^{(n)}\| \le C$. Assume for each $x \in X$ we have for all, respectively almost all, $t \in S$, that*

$$\lim_{n\to\infty} VQ_k^{(n)}x(t) = V0(t), \quad \textit{uniformly in } k. \tag{23}$$

If V is compact, then (20) *holds with $N = 0$.*

Proof. Assume the contrary. Then there are $x \in X$, $\delta > 0$ and, for each n some k_n such that $\|Q_{k_n}^{(n)} V_{U_n(k_n)} x\| > \delta$. Since $y_n = Q_{k_n}^{(n)}(x)$ is bounded, the sequence $z_n = V_{U_n(k_n)} y_n = V_{y_n} - V0$ has a Cauchy subsequence. Since, by assumption, z_n converges pointwise to 0, this subsequence must converge to 0 in norm. But by $\|Q_k^{(n)}\| \le C$ this implies that also a subsequence of $Q_{k_n}^{(n)} z_n = Q_{k_n}^{(n)} V_{U_n(k_n)} x$ converges to 0 in norm, a contradiction. □

Similarly, as in Example 2.3, one may check that the crucial condition (23) holds for the operators of the Examples 1.3 and 1.4, $I = [0, \infty]$, with an appropriate sequence of (even finite) partitions U_n. We may, e.g., let U_n take the values $0, 2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \dots, n, \infty$.

Although Theorems 3.1 and 3.2 seem to be applicable mainly for just linear operators, we may also use them for certain nonlinear operators.

Example 3.2. Let X be an ideal space of real functions over some (bounded or unbounded) interval S. Let

$$Vx(t) = \int_{t_0}^{t} k(t,s)|x(s)|\, ds$$

act in X, the cases $t_0 = \pm\infty$ are not excluded. Assume that V is compact. Alternatively, assume that V is q-γ-condensing with $q < 1/2$

as a mapping from X into a regular ideal subspace $Y \subseteq X$. Then, for any $f \in X$, there exists some $x \in X$ with $x = Vx + f$.

Indeed, if V acts in X, also the linear integral operator $Ax(t) = \int_{t_0}^{t} k(t,s)x(s)\,ds$ acts in X (writing $x = x^+ - x^-$ with $x^+, x^- \geq 0$ we have $Ax = Vx^+ - Vx^-$). This implies that A is bounded [**27**] (for $X = L_p$, see, e.g., [**17**]). Whence, V is Lipschitz continuous with constant $\|A\|$. Now apply Theorem 3.2, respectively 3.1.

4. Summary of main results. Let us first summarize the most important special cases that guarantee local solutions (at 0). Let $V : D \to Y$ be a 0-Volterra operator, where Y is a subspace of some normed linear space $X \supseteq D$. Assume $P_iX \subseteq X$ and, moreover, that the linear projections $P_i : X \to X$ are bounded. By Lemma 2.2, with $G := F$ for simplicity, we have for regular spaces:

Theorem 4.1. *Let $Y \subseteq X$ be regular at* 0 *(Definition* 2.3*). Let X be a Banach space. Let $M \neq \varnothing$ be convex and closed in X with $f + P_iVM \subseteq M$, $i > 0$. Assume that V is continuous and strictly q-γ-condensing as a mapping from $D \subseteq X$ into Y where $q\|P_i\| \leq 1$.*

Then $x = Vx + f$ has a local solution at 0 *in M, i.e., $P_ix = P_i(Vx + f)$, for some $i > 0$.*

If even $q\|P_i\| \leq c < 1$, then V is uniformly locally solvable at 0 *for any bounded $F \subseteq X$, i.e., i may be chosen independently of $f \in F$, provided that $F + P_iVM \subseteq M$, $i > 0$.*

In case $D = X$ one may, of course, choose $M = X$.

If VD is not regular at 0, we assume that V is of Uryson type at 0, Definition 2.4. Recall that all Volterra operators in Examples 1.3–1.5, for Example 1.5 with the restriction given in Example 2.3, have this property.

Then Lemma 2.2 and Lemma 2.3 imply:

Theorem 4.2. *Let V be of Uryson type at* 0*, continuous and compact. Assume that $P_iD \subseteq D$ and that X is a Banach space. Let $M \neq \varnothing$ be convex and closed in X with $F + P_iVM \subseteq M$ for some* *bounded $F \subseteq X$. Let $\|P_i\| \le C$ be uniformly bounded and the norm in Y be compatible with pointwise convergence.*

Then V is uniformly locally solvable at 0 for F.

For the classical Volterra operator in ideal spaces the previous theorems mean the following:

Corollary 4.1. *Let X be a (vector-valued) ideal space over some interval S, e.g., $X = L_p(S, W)$, $1 \le p \le \infty$, and $t_0 \in \overline{S}$, $t_0 = \pm\infty$ is not excluded. Assume that the Volterra-Uryson operator*

$$Vx(t) = \int_{t_0}^{t} g(t, s, x(s))\, ds \tag{24}$$

acts from X into X and is compact and continuous. If X is regular, $p < \infty$, V need not be compact but only γ-condensing.

Then, for any $f \in X$ there are a neighborhood of t_0 and $x \in X$ such that $x(t) = Vx(t) + f(t)$ holds for almost all $t \in S$ in this neighborhood. Moreover, if either X is regular and V is q-γ-condensing with $q < 1$, or if V is compact, the neighborhood may be chosen independently of f in bounded subsets of X.

In the space $X = C(S, W)$ of bounded continuous functions with the sup-norm we assume that the range of V is even contained in the subspace $Y = C_0(S, W)$ of functions vanishing at t_0, i.e., $x(t) \to 0$ for $t \to t_0$. In case $t_0 \in S$ this already follows from $V : X \to X$, if one understands the definition of V in the sense that $Vx(t)$ is given by (24) for any t (but observe that sometimes it may be convenient to weaken this convention by allowing Vx to be modified on, e.g., null sets to get a continuous image; in this case $VX \subseteq Y$ is an additional requirement, of course). Since, for the projections of Example 1.2, the set $Y = C_0(S, W)$ is regular at 0, we have

Corollary 4.2. *Let, with the notation of the previous corollary $X = C(S, W)$ or $X = C_0(S, W)$ and $V : X \to C_0(S, W)$ be γ-condensing. Then, for any $f \in X$ there are a neighborhood of t_0 and $x \in X$ such that $x(t) = Vx(t) + f(t)$ holds for all $t \in S$ in this neighborhood. Moreover,* *if V is q-γ-condensing with $q < 1$, the neighborhood may be chosen independently of f in bounded subsets of X.*

Analogous results hold of course also for the more general Volterra operators of Examples 1.4 and 1.5, in particular for (10).

To our knowledge, Corollary 4.1 is new in this generality. In the literature it only had been observed that certain compactness criteria give invariant subsets for the application of Schauder's fixed point theorem for Volterra equations, see, e.g., [**26**] or the proofs of [**11**, Chapter 12, Theorems 4.1–4.4]. But we showed that already the compactness itself is the reason for this. The reader may find it enlightening trying to prove Corollary 4.2 straightforwardly by Darbo's fixed point theorem. To find an invariant subset for the operator $Ax = P_i(Vx + f)$ one has to employ the fact that the image of V is 'almost equicontinuous at t_0' on bounded subsets. This is what we have done implicitly by Lemma 2.2.

Concerning global solvability, we consider now the extension principle. Let us assume now that $P_iD \subseteq D$. The crucial assumption is of course that $V_i = V - VP_i$ must be locally solvable at any i. For this, we may use our previous results, if we replace V by V_i (and the projections P_i by P_{ji}). Thus, we define corresponding to Definition 2.3,

Definition 4.1. We call a set $M \subseteq X$ *regular* (with respect to the family P_i) if, for each $i \in I$, $i \neq \max I$,

$$\inf_{j>i} \|P_{ji}x\| = 0, \quad x \in M.$$

Each regular ideal space (in the usual sense [**27, 29**], or in the sense of Nguyêñ [**20**]) over some $S \subseteq \mathbf{R}^n$ is regular in the sense of Definition 4.1 for the projections used in Examples 1.3–1.5.

Theorem 4.3. *Let Y be regular. Let X be a Banach space. Suppose that, for each $y \in VD$ and any $i \neq \max I$ there is some $M \neq \varnothing$ (convex and closed in X) with $P_jV_iM + y + f \subseteq M$, $j > i$. Assume that each V_i is continuous and strictly q-γ-condensing as a mapping from $D \subseteq X$ into Y, where $q\|P_{ji}\| \leq 1$.*

Then V is locally solvable at each i with respect to f and $E = D$.

If even $q\|P_{ji}\| \le c_i < 1$ and E_i, $F \subseteq X$ are bounded such that, for any $i \ne \max I$ there is some M with $P_j V_i M + F + V P_i E_i \subseteq M$, $i < j$, then V is uniformly locally solvable (*Definition* 3.4) *at each i with respect to F and E_i.*

Similarly we say that V is of *Uryson type*, if each V_i, $i \ne \max I$, is of Uryson type at 0 with respect to the family $\tilde{P}_j = P_{ji}$. As in Example 2.3, it can be checked that all operators of Examples 1.3–1.5 are of Uryson type, for Example 1.5 under the restriction that functions coinciding almost everywhere are identified.

Theorem 4.4. *Let V be of Uryson type, continuous and compact. Assume that X is a Banach space, $P_{ji}D \subseteq D$, $\|P_i\| \le C$ is uniformly bounded, and the norm in Y is compatible with pointwise convergence. Suppose that, for each $y \in VD$ and any $i \ne \max I$ there is some $M \ne \varnothing$* (*convex and closed in X*) *with $P_j V_i M + y + f \subseteq M$, $j > i$.*

Then V is locally solvable at each i with respect to f and $E = D$. If $E_i, F \subseteq X$ are bounded such that, for any $i \ne \max I$ there is some M with $P_j V_i M + F + V P_i E_i \subseteq M$, $i < j$, then V is uniformly locally solvable at each i with respect to F and E_i.

To apply the extension principle, we will assume that Z is full for D and that the P_i are dense (Definition 3.2). This is in particular satisfied for the projections occurring in Examples 1.3–1.5. We will also assume that D is decomposable in D (Definition 3.2) which is natural in these examples, if we let $D \subseteq X$ be the natural domain of definition.

Theorem 4.5. *Let, in the above situation, V be locally solvable at each i with respect to $f \in X$ and $E = D$. Assume the axiom of choice.*

Assume that we know a priori that, for any $x \in Z$ with the property that $P_i x \in D$ is a solution of $P_i x = P_i(Vx+f)$ for all i in some interval $0 \in I_0 \subsetneq I$, we have for $i_0 = \sup I_0$ that $P_{i_0} x \in D$.

Then $x = Vx + f$ has a (*global*) *solution $x \in Z$, i.e., $P_i x \in D$ solves $P_i x = P_i(Vx + f)$ for all i. If $\max I$ exists, $x = Vx + f$ has a solution $x \in D$.*

If V is even uniquely locally solvable, then the axiom of choice is not needed and, moreover, the solution is unique in Z.

In Example 1.3 and $S = [0, \infty)$ the a priori assumption is much more restrictive, if $I = [0, \infty]$ than if $I = [0, \infty)$. Observe that in the second case $\max I$ does not exist!

In many cases, e.g., for ideal spaces, X is full for bounded subsets of X. In these cases the a priori assumption is satisfied, if for any interval $0 \in I_0 \subsetneq I$ and any $x \in Z$ with $P_i x \in D$, $P_i x = P_i(Vx + f)$ for $i \in I_0$, we have that $\|P_i x\|$ is uniformly bounded for $i \in I_0$. In particular, a priori estimates lead to global existence results (as for differential equations).

But we emphasize that, in contrast to differential equations, a priori estimates are usually not the best way to prove global existence results for abstract Volterra equations. Quite often it is more convenient to use one of the two uniform principles instead, cf. also Theorems 3.1 and 3.2.

5. Applications. At first we give some applications to differential equations in Banach spaces. Although these applications are well known in principle, cf., e.g., [**8**], we are able to drop some conditions, which usually are imposed. The proofs might throw some light on our abstract approach.

By a solution of the initial value problem

$$x'(t) = g(t, x(t)), \qquad x(t_0) = y \in W \tag{25}$$

in a Banach space W on some interval $S \ni t_0$, we mean by definition a *continuous* solution of the Volterra equation $x = Vx + f$ on S, where

$$Vx(t) = \int_{t_0}^{t} g(s, x(s))\, ds, \qquad f(t) \equiv y.$$

Theorem 4.5 implies together with Theorem 4.3, respectively 4.4,

Theorem 5.1. *Let S be compact and V act from $X = L_p(S, W)$, $1 \leq p \leq \infty$, into $Y = C(S, W)$. Let $V : X \to X$ be continuous and* *each* $V_i : X \to X$ *be* γ*-condensing. In case* $p = \infty$*, let* $V : X \to X$ *even be compact. Assume the axiom of choice.*

If we know a priori that each continuous local solution of (25) *is* (*uniformly*) *bounded on its interval of existence, then* (25) *has a global continuous solution.*

Proof. Since a solution of $x = Vx + f$ by the acting property of V automatically belongs to Y, we only have to look for solutions of $x = Vx + f$ in X. In view of Theorem 4.3, respectively 4.4, it remains to check the a priori condition of Theorem 4.5. Here we have the slight difficulty that we only know something about *continuous* local solutions of $x = Vx + f$.

However, if we have a solution $x \in X$ of $P_i x = P_i(Vx + f)$ for all i in some interval I_0, the acting property of V implies that x is continuous in the interior of the corresponding existence interval. Thus, our a priori assumption yields that x is essentially bounded on this interval, and thus that $P_{i_0} x \in X$ for $i_0 = \sup I_0$. □

We could, of course, also have replaced X by any other regular (for compact V even general) ideal space that contains $C(S, W)$.

Observe that, under our weak assumptions, the existence of even local solutions is far from being trivial for general W since g might be very badly behaved, in particular, $\sup_{|u| \le |x(t)|} |g(t, u)|$ need not even by integrable.

If we want to restrict the domain D of V, for $X = L_\infty(S, W)$, say, we may not use Theorem 4.3, respectively 4.4, without severe technical difficulties. In this case it is usually more convenient to assume the local invariance a priori, e.g.,

$$|Vx(t) - Vx(s)| \longrightarrow 0 \quad \text{for } t \to s$$

locally uniformly in $x \in D$ for open D (then $Ax = f_0 + P_{ji} V_i x$ maps for $j > i$, j sufficiently near to i, a ball $K_r(f_0)$ into itself, when $f_0 \in D$), and to apply a fixed point theorem directly. Observe that D is decomposable in D, if D is the natural domain of V, i.e., if D consists of all functions $x \in X$ such that $Gx(s) = g(s, x(s))$ is almost everywhere defined and integrable.

We remark that compactness, condensing and continuity conditions for $V : X \to C(S, W)$ have been considered by the author in [**25**].

It should be observed that the continuity of g with respect to both variables is not enough for Theorem 5.1 to hold. This is sufficient, of course, if W has finite dimension (apply the Arzela-Ascoli criterion). But otherwise we not only have the problem that V need not be compact, even if g is continuous and locally Lipschitz continuous (whence the problem is locally uniquely solvable) it may happen that the local solution x has a (uniformly) bounded extension to a maximal interval $[t_0, T)$ but that $x(t)$ does not tend to a limit for $t \to T$, see [**8**, Example 1.1]. This is due to the fact that g need not map bounded sets into bounded sets, and thus V need not act from X into Y.

A standard application of Gronwall's lemma yields

Example 5.1. Let g be continuous, and assume that there exist continuous functions a, b with $|g(s, u)| \leq a(s) + b(s)|u|$. Assume, moreover, that the mapping $V : L_p(S, W) \to L_p(S, W)$, $1 \leq p \leq \infty$, is continuous for each compact interval S and that V_i is condensing. Assume the axiom of choice. Then the initial value problem (25) has a continuous global solution.

For $p = \infty$ we only need that V_i is condensing this time, and we may also allow V to be ψ-condensing, since by our growth estimate local solutions are no problem (as sketched above, we have local invariance, and so we just need to apply Darbo's fixed point theorem).

Example 5.1 should be contrasted with Example 3.1.

We are going to show now that, even in the linear case, we have new results. At first we obtain some generalizations of Zabrejko's result, that compact linear (classical) Volterra operators in regular ideal spaces, in L_∞, or in C have a trivial spectrum [**28**]. In particular, we generalize this to *arbitrary* ideal spaces. Using the second uniform principle we already found related results in Theorems 3.1 and 3.2. But we will give yet another approach, using the first uniform principle.

For the remainder of this section we will assume that X is a Banach space, $P_i : X \to X$ is bounded, and that $V : X \to Y \subseteq X$ is a linear continuous Volterra operator. We define the spectral radius $r(V)$ of V

by the Gel'fand formula

$$r(V) = \lim_{n\to\infty} \|V^n\|^{1/n} = \inf_n \|V^n\|^{1/n}.$$

Observe that this definition also makes sense, if X is real.

Theorem 5.2. *Let Y be regular and V be q-γ-condensing where $\|P_i\|q < 1$ for all i. Let each V_i be q_i-γ-condensing with $q_i\|P_{ji}\| \le c_i < 1$. Assume that P_i is dense and* $\max I$ *exists. Then $r(V) < 1$.*

Proof. If X is real, it is straightforward to check that all assumptions are satisfied for the complexification, cf., e.g., [**1**, Section 2.6.1], of X and V. Thus, without loss of generality, we will assume that X is a complex Banach space. If the statement is false, $\mathrm{id} - \lambda V$ is not one-to-one or onto for some $0 < |\lambda| \le 1$. Since $\tilde{V} = \lambda V$ satisfies all assumptions, too, we may assume $\lambda = 1$ and have to prove that $\mathrm{id} - V$ is one-to-one and onto.

It remains to prove that $x = Vx + f$ has for each $f \in X$ a unique solution $x \in X$. Observe that $P_i V$ is $\|P_i\|q$-γ-condensing. By [**1**, Theorem 2.3.7], $\mathrm{id} - P_i V$ thus is either one-to-one and onto, or neither one-to-one nor onto. In the first case, we put $E_i = (\mathrm{id} - P_i V)^{-1} F$ ($F = K_\rho(0)$ with fixed $\rho > 0$; by the open mapping theorem, E_i is bounded), and in the second case we choose arbitrary bounded E_i. Then V is of Fredholm type, Definition 3.3, for F, E_i and the statements $(A_i) =$ 'id $- P_i V$ is one-to-one'.

Indeed, let (A_i) hold for all i in some interval $0 \in I_0 \subsetneq I$ and $i_0 = \sup I_0$. If (A_{i_0}) fails there is some $x \ne 0$ with $x = P_{i_0} V x$. Then $P_{i_0} x = P_{i_0} V x \ne 0$ implies that there is some $i \in I_0$ with $P_i x \ne 0$, since P_i is dense. But then $P_i x \ne 0$ solves $x = P_i V x$, whence (A_i) fails, a contradiction.

Also the alternative holds. If (A_i) is true, then $\mathrm{id} - P_i V$ is onto, whence for each $f \in F$ there is some solution in E_i. Conversely, if (A_i) fails, then there is some $f \in X$ (since $P_i V$ is homogeneous also some $f \in F$) such that $x = P_i V x + f$ has no solution.

By the first uniform principle it thus suffices to prove that V is uniformly locally solvable at each i with respect to F and E_i. But this is a consequence of Theorem 4.3. □

Since $\gamma_Y(V_iK) \leq \gamma_Y(VK) + \gamma_Y(VP_iK) \leq q\gamma_X(K) + q\gamma_X(P_iK) \leq q(1+C)\gamma(K)$, the previous theorem implies:

Corollary 5.1. *Let Y be regular and V be q-γ-condensing, where $q < 1/(C+C^2)$ and $\|P_{ji}\| \leq C$. Assume that P_i is dense and* $\max I$ *exists. Then $r(V) < 1$.*

This result should be compared with Theorem 3.1.

If Y is not regular, we use Theorem 4.4 instead of Theorem 4.3 (for $Y = VX$):

Theorem 5.3. *Let V be of Uryson type and the norm in VX be compatible with pointwise convergence. Let V be compact and $\|P_i\| \leq C$ be uniformly bounded. Assume that P_i is dense and* $\max I$ *exists. Then $r(V) = 0$.*

The proof is completely analogous to the previous one (now with $\lambda \neq 0$). This theorem should be compared with Theorem 3.2.

Summarizing special cases of the previous results, we state the announced generalization of Zabrejko's result (observing that if λV has spectral radius less than 1 for any $\lambda > 0$, then V has spectral radius 0):

Corollary 5.2. *Assume that P_i is dense,* $\max I$ *exists and $\|P_i\| \leq C$ is uniformly bounded. Let $V : X \to X$ be compact. Then $r(V) = 0$, if either*

1. *VX is regular, or*
2. *V is of Uryson type, and the norm in VX is compatible with pointwise convergence.*

Even for the classical linear Volterra operator in nonregular ideal spaces, this is a new result.

Example 5.2. Let X be any ideal space of (classes of) functions $x : S \to W$ with a Banach space W and a (bounded or unbounded) interval S, and $t_0 \in \overline{S}$ (the cases $t_0 = \pm\infty$ are allowed). If the 'classical' linear Volterra integral operator

$$Vx(t) = \int_{t_0}^{t} k(t,s)x(s)\,ds \tag{26}$$

acts in X and is compact, then it has spectral radius $r(V) = 0$. An analogous result holds for the more general operator of Example 1.5 with linear $G(t,\tau,s,y) = K(t,\tau,s)y$,

$$Vx(t,\tau) = \int_{t_0}^{t} K(t,\tau,s)x(s,\cdot)\,d\mu_{t,\tau}(s),$$

provided it is well-defined on the ideal space X (here X consists of classes of functions $x : S \to W$ where $S = J \times T$).

We emphasize that cases like $X = L_p([t_0,\infty))$ are included, which is also new to our knowledge.

For the spaces $X = C(S,W)$ and $X = C_0(S,W)$ we assume that $VX \subseteq C_0(S,W)$ (which in case $t_0 \in S$ is not really an additional assumption; recall the remarks in front of Corollary 4.2). Observing that $Y = C_0(S,W)$ is regular for the projections of Example 1.2, we have

Example 5.3. Let, with the notation of the previous example, either $X = C(S,W)$ or $X = C_0(S,W)$. Assume that V acts in X and $VX \subseteq C_0(S,W)$. If $V : X \to X$ is compact, then it has spectral radius $r(V) = 0$.

But Corollary 5.2 is even more powerful. It is clear that, if some iterate V^n of V has spectral radius $r(V^n) = 0$, then V has spectral radius $r(V) = 0$, too. Thus, if V^n is a compact Volterra operator then $r(V) = 0$. If we define the term 'Volterra operator' in the classical sense (26), then the condition that V^n again be a Volterra operator is rather involved. Even in $X = L_2([0,1])$ the iterate of an integral operator need not be an integral operator again [**18**, Chapter 4] (an exception are the 'regular' integral operators [**17**]).

However, for our abstract definition of Volterra operators, we may avoid this difficulty.

Lemma 5.1. *If V is an additive Volterra operator and W is a Volterra operator, then $Z = VW$ is a Volterra operator. Moreover, $Z_i = VW_i = V_iW_i$ (if V_i and W_i are defined on the same set as V, respectively W, and if $V0 = 0$).*

Proof. Trivially, Z is a 0-Volterra operator, and if W is partially additive, VW must also be partially additive. For the second statement, observe that $P_iW_i = P_i(W - WP_i) = P_iW - P_iWP_i = 0$, hence $Z_i = Z - ZP_i = V(W - WP_i) = VW_i = (V_i + VP_i)W_i = V_iW_i + VP_iW_i = V_iW_i + V0$. We remark that this equality again implies (5). □

In particular, any iterate V^n of a linear Volterra operator V is again a linear Volterra operator (and $(V^n)_i = (V_i)^n$). Thus,

Theorem 5.4. *Assume that P_i is dense,* $\max I$ *exists and $\|P_i\| \le C$ is uniformly bounded. Let V be a linear Volterra operator with a compact iterate V^n. Assume that some iterate of V has regular range V^mX. Then $r(V) = 0$.*

Proof. The operator V^k, $k = \max\{m, n\}$, is compact, and $V^kX \subseteq V^mX$ is regular. Thus, $r(V^k) = 0$ by Corollary 5.2. □

If V^mX is not regular, the situation is more difficult. It is not clear whether the iterate of a Volterra operator of Uryson type must again be of Uryson type.

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