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EXISTENCE AND STABILIZATION OF SOLUTIONS TO THE PHASE-FIELD MODEL WITH MEMORY

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ABSTRACT. A phase field model is considered when the classical Fourier law is replaced by the linearized Gurtin-Pipkin constitutive assumption for the heat flux. The resulting system of partial differential equations consists in a Volterra integro-differential equation coupled with a nonlinear parabolic inclusion. The initial and boundary value problem with homogeneous Neumann boundary conditions is investigated for a kernel of positive type. Results on the long-time behavior of solutions are obtained in a quite general setting.

1. Introduction. This paper is devoted to the study of the so-called phase-field model, see, e.g., [**6, 10, 12**], for the temperature ϑ and the phase variable χ, in the case where the classical Fourier law $\mathbf{q} = -k_0 \nabla\vartheta$, k_0 constant, is replaced by the following nonlocal condition

$$\mathbf{q}(x,t) = -\int_{-\infty}^{t} k(t-s)\ \nabla\vartheta(x,s)\,ds, \tag{1.1}$$

for a kernel $k : (0,+\infty) \to \mathbf{R}$ of positive type. Here, $x \in \Omega$ denotes the space variable and t represents the time, letting Ω be a bounded domain of $\mathbf{R}^3$ with smooth boundary Γ and t vary in $(0,+\infty)$, as the past evolution of ϑ is supposed to be a known function ϑ_P up to $t=0$,

$$\vartheta = \vartheta_P \quad \text{in } \Omega \times (-\infty, 0). \tag{1.2}$$

The relation (1.1) then states that the heat flux depends only on the temporal history of the temperature gradient and turns out to be compatible with classical thermodynamical laws whenever k is a kernel of positive type, cf. [**11**]. This is indeed a basic assumption in our approach. Two other important facts to be mentioned at once are the facts that we allow quadratic nonlinearities in the model and that we consider an initial-boundary value problem with homogeneous Neumann boundary conditions for both unknowns. Concerning the

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former, we want to notice that thus our system does not apply only to solid-liquid phase transitions but also to ferromagnetic transformations. For homogeneous Neumann boundary conditions, we just say that they allow multiple steady state solutions for the temperature and obviously this makes the analysis more delicate.

Let us then state the initial-boundary value problem which reads

$$\chi_t - \Delta\chi + \beta(\chi) \ni -\sigma'(\chi) + \lambda'(\chi)\,\vartheta \quad \text{in } Q := \Omega \times (0,+\infty), \tag{1.3}$$

$$(\vartheta + \lambda(\chi))_t - \Delta(k * \vartheta) = g \quad \text{in} \quad Q, \tag{1.4}$$

$$\frac{\partial\chi}{\partial n} = \frac{\partial(k*\vartheta)}{\partial n} = 0 \quad \text{on} \quad \Sigma := \Gamma \times (0,+\infty), \tag{1.5}$$

$$\chi(0) = \chi_0, \qquad \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega, \tag{1.6}$$

where $*$ denotes the usual convolution product with respect to time over $(0,t)$,

$$(k*\vartheta)(x,t) = \int_0^t k(t-s)\ \vartheta(x,s)\ ds, \quad (x,t) \in Q,$$

and the datum g is specified by, cf. (1.2),

$$g(x,t) = \int_{-\infty}^0 k(t-s)\Delta\vartheta_P(x,s)\,ds, \quad (x,t) \in Q,$$

in case of no heat source in the energy balance equation (1.4). The phase dynamics is described by the relationship (1.3), in which β represents a maximal monotone graph and σ and λ are two nonlinearities playing in the Ginzburg-Landau free energy, as well as the convex function $\hat\beta$ whose subdifferential $\partial\hat\beta$ is equal to β. The system (1.3)–(1.4) is actually derived from the free energy by imposing the balance of energy and satisfying the second principle of thermodynamics.

In this paper, we prove an existence result for (1.3)–(1.6) in the framework outlined above. Moreover, we investigate the long-time behavior of solutions to (1.3)–(1.6), carrying out sharp convergences and identifying the limit points for the trajectories, i.e., their ω-limit set.

Existence and uniqueness of strong and weak solutions to (1.3)–(1.6) have already been discussed in [**7**] and [**8**] in the case of a smooth kernel

k and a function λ which is either Lipschitz continuous or linear, i.e., λ' is a constant l. Thus, our first result turns out to be a generalization of some results of [**7**] (to which we refer for a more detailed presentation of the model) and [**8**] along the direction of kernels k of positive type, see (2.4) below for a precise definition. Within this setting, the problem (1.3)–(1.6) has already been investigated by Aizicovici and Barbu [**1**] in the particular situation when

$$\beta(\chi) + \sigma'(\chi) = \chi^3 - \chi = \frac{1}{4}((\chi^2 - 1)^2)'$$

and $\lambda'(\chi) = l$, for Dirichlet boundary conditions. Using semigroup techniques, it is shown in [**1**] that there exists a solution to (1.3)–(1.6) which is unique provided that k is a nonnegative decreasing and convex function. Moreover, stability and asymptotic convergences as $t \to +\infty$ are derived for $(\chi(t), \vartheta(t))$ under quite heavy assumptions on k, assuming in particular that k is of strongly positive type (a precise definition is provided in (2.8)). On the contrary, our analysis gives a complete description of the ω-limit set of (χ, ϑ), i.e., the set of cluster points of $(\chi(t), \theta(t))$ as $t \to +\infty$ in some suitable topology, when $\lambda'(\chi) = l$, β is single-valued and $k \in L^1(0, +\infty)$ is of positive type. A similar result is also obtained for a wider class of functions λ and β under stronger requirements on k, namely, assuming k of strongly positive type.

The paper is organized as follows. In the next section, we introduce the set of assumptions on the data of (1.3)–(1.6) together with some notations and state our results. Section 3 is devoted to the existence proof (of Theorem 2.3) which relies on an approximation procedure and a priori estimates. In Section 4.1 we prove the first result concerning long-time behavior, assuming that k is a kernel of positive type which belongs to $L^1(0, +\infty)$ and that λ' is a constant and β is single-valued. It is worth mentioning at this point that, since we do not assume that k is a kernel of strongly positive type, we cannot use the methods of [**1**], while the memory term prevents us from applying the classical device of [**14**]. In fact, the main difficulty encountered at this stage is to find a connection between the cluster points of $\vartheta(t)$ and $(k * \vartheta)(t)$ as $t \to +\infty$. Such a connection is derived from (1.3)–(1.4) by a careful analysis of the behavior of the trajectories starting from these cluster points. We finally deduce our second result on long-time behavior in Section 4.2. The approach used there is in the spirit of [**14**], but additional arguments from Section 4.1 are needed to handle the memory term.

2. Main results. Let Ω be a bounded and connected open subset of $\mathbf{R}^3$ with smooth boundary Γ. We put

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad W = \left\{ v \in H^2(\Omega),\ \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma \right\},$$

and denote by V' and W' the dual spaces of V and W, respectively. Notice that $W \hookrightarrow V \hookrightarrow H$ with dense and compact injections. Also, we denote by $\langle\langle ., . \rangle\rangle$ the duality pairing between W' and W and by $\langle ., . \rangle$ the duality pairing between V' and V. Moreover, $(., .)$ stands for the usual scalar product in the space H, which is identified with H', and $|.|_H$ represents the norm both in H and $L^2(\Omega; \mathbf{R}^3)$.

Concerning the data, we assume the following.

$$\text{(2.1)} \qquad \begin{cases} \beta \text{ is a maximal monotone graph in } \mathbf{R}^2 \\ \text{with domain } D(\beta) \text{ such that int}\,(D(\beta)) \\ \text{is nonempty and } 0 \in \beta(0). \text{ We denote by} \\ \hat{\beta} \text{ a lower semicontinuous and proper convex} \\ \text{function such that } \beta = \partial \hat{\beta}. \end{cases}$$

$$\text{(2.2)} \qquad \sigma, \lambda \in \mathcal{C}^2(\mathbf{R}) \quad \text{with} \quad \sigma'', \lambda'' \in L^\infty(\mathbf{R}).$$

$$\text{(2.3)} \qquad \text{The function } \hat{\beta} + \sigma \text{ is nonnegative.}$$

$$\text{(2.4)} \qquad \begin{cases} k \in L^1(0,T) \text{ for each } T \in (0,+\infty) \\ \text{and is of positive type, i.e.,} \\ \int_0^T (v(t), (k * v)(t))\, dt \geq 0, \\ v \in L^2(0,T;H),\ T \in (0,+\infty). \end{cases}$$

$$\text{(2.5)} \qquad \chi_0 \in V, \quad \hat{\beta}(\chi_0) \in L^1(\Omega), \quad \vartheta_0 \in H.$$

$$g \in L^1(0,T;H) \quad \text{for each} \quad T \in (0,+\infty). \tag{2.6}$$

Note that the datum g accounts for the history of ϑ up to $t = 0$, namely,

$$g(x,t) = \int_{-\infty}^{0} k(t-s)\Delta\vartheta_P(x,s)\,ds, \quad (x,t) \in \Omega \times (0,+\infty),$$

and ϑ_P is a known function.

In the remainder of the paper, and in particular for the study of long-time behavior, we shall use some of the following assumptions.

$$\begin{cases} g \in L^1(0,+\infty;H) \quad \text{and} \quad k \in L^1(0,+\infty) \\ \qquad\qquad \text{with} \quad \int_0^\infty k(s)\,ds \neq 0. \end{cases} \tag{2.7}$$

$$\begin{cases} k \in W^{1,1}(0,+\infty) \text{ and } k \text{ is of strongly positive type, i.e.} \\ \text{there exists } \eta > 0 \text{ such that} \\ t \mapsto k(t) - \eta e^{-t} \text{ is of positive type.} \end{cases} \tag{2.8}$$

Under this last condition on k, it turns out that the operator $v \mapsto k * v$ enjoys the following additional property, see e.g. [**16**, Lemma 4.2].

Lemma 2.1. *Assume that k fulfills* (2.8). *If $v \in L^1(0,T;H)$ for each $T \in (0,+\infty)$, the following holds*

$$\int_0^T |(k*v)(s)|_H^2\,ds \leq C_1 \int_0^T (v(s),(k*v)(s))\,ds, \qquad T \in (0,+\infty), \tag{2.9}$$

where C_1 is a constant which depends only on $|k|_{W^{1,1}(0,+\infty)}$ and η.

Definition 2.2. A solution to (1.3)–(1.6) is a set of functions (χ,ξ,ϑ) such that, for each $T \in (0,+\infty)$,

$$\chi \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega)), \tag{2.10}$$
$$\vartheta \in \mathcal{C}([0,T];W') \cap L^\infty(0,T;H), \quad (\subset \mathcal{C}([0,T];V')), \tag{2.11}$$
$$\vartheta_t - g \in L^2(0,T;W'), \quad k*\vartheta \in \mathcal{C}([0,T];H), \tag{2.12}$$
$$\xi \in L^2(0,T;L^{3/2}(\Omega)), \tag{2.13}$$

and

$$\chi_t - \Delta\chi + \xi + \sigma'(\chi) = \lambda'(\chi)\vartheta \quad \text{a.e. in} \quad \Omega \times (0,T), \tag{2.14}$$

$$\langle\langle(\vartheta + \lambda(\chi))_t(t), v\rangle\rangle - \int_\Omega (k * \vartheta)(t)\Delta v \, dx = \int_\Omega g(t) v \, dx, \tag{2.15}$$

for all $v \in W$ and almost every $t \in (0,T)$,

$$\xi \in \beta(\chi) \quad \text{a.e. in} \quad \Omega \times (0,T), \tag{2.16}$$

$$\frac{\partial\chi}{\partial n} = 0 \quad \text{a.e. on} \quad \Gamma \times (0,T), \tag{2.17}$$

$$\chi(0) = \chi_0 \quad \text{in } H, \qquad \vartheta(0) = \vartheta_0 \quad \text{in } W'. \tag{2.18}$$

Observe that the terms $(\lambda'(\chi)\vartheta)$ and $(\lambda(\chi))_t$ in (2.14)–(2.15) belong to the space $L^2(0,T;L^{3/2}(\Omega))$, due to (2.10)–(2.11) and Sobolev embeddings. We may now state our results, first studying the existence of solutions to (1.3)–(1.6).

Theorem 2.3. (i) *Assume that* (2.1)–(2.6) *are fulfilled. Then* (1.3)–(1.6) *has at least a solution* (χ, ξ, ϑ) *in the sense of Definition* 2.2, *satisfying in addition that, for each* $t \in (0, +\infty)$,

$$\begin{aligned} |\nabla\chi(t)|_H^2 + |\vartheta(t)|_H^2 + \int_0^t |\chi_t(s)|_H^2 \, ds \\ \leq C_2(1 + |\chi_0|_V^2 + |\vartheta_0|_H^2 + |\hat{\beta}(\chi_0)|_{L^1(\Omega)} + |g|_{L^1(0,t;H)}^2), \end{aligned} \tag{2.19}$$

where the constant C_2 *depends only on* Ω *and* σ.

(ii) *Assume that* (2.1)–(2.6) *and* (2.8) *are fulfilled. Then* (1.3)–(1.6) *has one solution* (χ, ξ, ϑ) *satisfying*

$$k * \vartheta \in L^2(0,T;V), \quad T \in (0,+\infty) \tag{2.20}$$

and, for each $t \in (0, +\infty)$,

$$\begin{aligned} |\nabla\chi(t)|_H^2 + |\vartheta(t)|_H^2 + \int_0^t (|\chi_t(s)|_H^2 + |(k * \nabla\vartheta)(s)|_H^2) \, ds \\ \leq C_3(1 + |\chi_0|_V^2 + |\vartheta_0|_H^2 + |\hat{\beta}(\chi_0)|_{L^1(\Omega)} + |g|_{L^1(0,t;H)}^2), \end{aligned} \tag{2.21}$$

where C_3 depends only on Ω, σ, $|k|_{W^{1,1}(0,+\infty)}$ and η.

The triplets (χ, ξ, ϑ) given by Theorem 2.3 will be obtained as limits of solutions to a regularized problem whose well-posedness is studied in [**3**].

Remark 2.4. Assuming further that k is a smooth function and that λ' is bounded, we may infer from [**7, 8**] and Theorem 2.3 that (1.3)–(1.6) has a unique solution.

We next turn to the investigation of the long-time behavior of the solution to (1.3)–(1.6) given by Theorem 2.3. More precisely, if (χ, ϑ) is a solution to (1.3)–(1.6), we wish to identify the cluster points as $t \to +\infty$ of $(\chi(t), \vartheta(t))$ in some suitable topology. In our setting, cf. (2.10)–(2.11), it turns out that a suitable topology for the study of the long-time behavior is that of $H \times V'$. We then define the ω-limit set $\omega(\chi, \vartheta)$ of (χ, ϑ) in $H \times V'$ by

(2.22)
$$\omega(\chi,\vartheta) = \left\{ \begin{array}{l} (\chi_\infty, \vartheta_\infty) \in H \times V' \text{ such that there exists} \\ \text{a sequence of positive real numbers } \{t_n\} \text{ with} \\ t_n \nearrow +\infty \text{ and } (\chi(t_n), \vartheta(t_n)) \to (\chi_\infty, \vartheta_\infty) \text{ in } H \times V'. \end{array} \right\}$$

Our first result then reads

Theorem 2.5. *Assume that* (2.1)–(2.6) *and* (2.7) *are fulfilled, and let* (χ, ξ, ϑ) *be a solution to* (1.3)–(1.6) *given by Theorem* 2.3 (i). *Assume further that*

$$\lambda'(r) = l \neq 0, \quad r \in \mathbf{R} \quad \text{and} \quad \beta \text{ is single-valued.} \tag{2.23}$$

Then $\omega(\chi, \vartheta)$ is a nonempty compact and connected subset of $H \times V'$, and if $(\chi_\infty, \vartheta_\infty)$ belongs to $\omega(\chi, \vartheta)$, one has

$$\chi_\infty \in W, \quad \vartheta_\infty = \text{const.} = M_0 - \frac{l}{|\Omega|} \int_\Omega \chi_\infty(x)\, dx, \tag{2.24}$$

$$-\Delta\chi_\infty + \beta(\chi_\infty) + \sigma'(\chi_\infty) = l\vartheta_\infty, \tag{2.25}$$

where

$$M_0 = \frac{1}{|\Omega|}\bigg(\int_\Omega (\vartheta_0(x) + l\chi_0(x))\,dx + \int_0^\infty \int_\Omega g(x,s)\,dx\,ds\bigg). \tag{2.26}$$

Let us first notice that the problem (2.24)–(2.26) also arises in the study of the long-time behavior of solutions to the standard phase-field model, see, e.g., [**9**]. We next mention that though the above theorem identifies the possible cluster points of $(\chi(t),\vartheta(t))$ as $t\to+\infty$, it does not indicate whether $(\chi(t),\vartheta(t))$ has a limit or not as $t\to+\infty$. This last question may be answered in some cases by the study of the solutions to (2.24)–(2.26). Indeed, if one can show that the number of solutions to (2.24)–(2.26) is finite, the connectedness of $\omega(\chi,\vartheta)$ yields that $\omega(\chi,\vartheta)$ is a singleton, i.e., $(\chi(t),\vartheta(t))$ has a limit in $H\times V'$ as $t\to+\infty$. Results in that direction may be found in [**9**] for the special case $(\beta+\sigma')(r) = r^3 - r$.

If we strengthen the assumptions on k, we may relax the assumption (2.23) on λ and β and still obtain a result similar to Theorem 2.5.

Theorem 2.6. *Assume that* (2.1)–(2.6) *and* (2.7)–(2.8) *are fulfilled and let* (χ,ξ,ϑ) *be a solution to* (1.3)–(1.6) *given by Theorem* 2.3 (ii). *Then* $\omega(\chi,\vartheta)$ *is a nonempty compact and connected subset of* $H\times V'$, *and if* $(\chi_\infty,\vartheta_\infty)\in\omega(\chi,\vartheta)$, *one has*

$$\chi_\infty \in W, \quad \vartheta_\infty = \text{const.} = M_0' - \frac{1}{|\Omega|}\int_\Omega \lambda'(\chi_\infty)(x)\,dx, \tag{2.27}$$

$$-\Delta\chi_\infty + \beta(\chi_\infty) + \sigma'(\chi_\infty) \ni \lambda'(\chi_\infty)\vartheta_\infty, \tag{2.28}$$

where

$$M_0' = \frac{1}{|\Omega|}\bigg(\int_\Omega (\vartheta_0(x) + \lambda(\chi_0)(x))\,dx + \int_0^\infty \int_\Omega g(x,s)\,dx\,ds\bigg). \tag{2.29}$$

3. Existence of solutions. Let $(g_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(0,+\infty;H)$ such that, for each $T\in(0,+\infty)$,

$$\lim_{\varepsilon\to 0} |g_\varepsilon - g|_{L^1(0,T;H)} = 0. \tag{3.1}$$

For $\varepsilon > 0$, we consider the following regularized problem.

Problem $(\mathbf{P}_\varepsilon)$. Find (χ, ξ, ϑ) satisfying, for each $T \in (0, +\infty)$,

$$\chi \in W^{1,2}(0,T;H) \cap \mathcal{C}([0,T];V) \cap L^2(0,T;W), \quad \chi(0) = \chi_0, \tag{3.2}$$

$$\vartheta \in W^{1,2}(0,T;V') \cap L^2(0,T;V), \quad \vartheta(0) = \vartheta_0, \tag{3.3}$$

$$\chi_t - \Delta\chi + \xi + \sigma'(\chi) = \lambda'(\chi)\vartheta \quad \text{a.e. in } \Omega \times (0,T), \tag{3.4}$$

$$\begin{aligned} \langle(\vartheta + \lambda(\chi))_t(t), v\rangle &+ \int_\Omega (\varepsilon\nabla\vartheta(t) + (k * \nabla\vartheta)(t))\nabla v \, dx \\ &= \int_\Omega g_\varepsilon(t) v \, dx, \end{aligned} \tag{3.5}$$

for all $v \in V$ and almost every $t \in (0,T)$,

$$\xi \in \beta(\chi) \quad \text{a.e. in} \quad \Omega \times (0,T), \qquad \xi \in L^2(0,T;H). \tag{3.6}$$

Note that here $(\lambda'(\chi)\vartheta)$ and $(\lambda(\chi))_t$ stay in $L^2(0,T;H)$. Thanks to the results of [**3**], we can infer this statement.

Proposition 3.1. *Assume that* (2.1)–(2.5) *are fulfilled. For each* $\varepsilon > 0$*, Problem* (P_ε) *has a unique solution* $(\chi_\varepsilon, \vartheta_\varepsilon)$ *with the further property that* $\hat{\beta}(\chi_\varepsilon) \in L^\infty(0,T;L^1(\Omega))$ *for each* $T \in (0,+\infty)$*. Moreover, there is a constant* C_4 *such that, for all* $t \in [0,+\infty)$,

$$\begin{aligned} &|\nabla\chi_\varepsilon(t)|_H^2 + |\vartheta_\varepsilon(t)|_H^2 + \int_0^t (|\chi_{\varepsilon,t}(s)|_H^2 + \varepsilon|\nabla\vartheta_\varepsilon(s)|_H^2) \, ds \\ &+ \int_0^t \int_\Omega \nabla\vartheta_\varepsilon(s) \cdot \nabla(k * \vartheta_\varepsilon)(s) \, dx \, ds \\ &\qquad \leq C_4(1 + |\chi_0|_V^2 + |\vartheta_0|_H^2 + |\hat{\beta}(\chi_0)|_{L^1(\Omega)} + |g_\varepsilon|^2_{L^1(0,t;H)}), \end{aligned} \tag{3.7}$$

where C_4 *depends only on* Ω *and* σ.

Proof. The existence and uniqueness of a solution $(\chi_\varepsilon, \vartheta_\varepsilon)$ to (P_ε) follows at once from [**3**, Theorem 2.1]. It remains to derive (3.7). For that purpose, we choose $v = \vartheta_\varepsilon$ in (3.5) and take the scalar product of

(3.4) with $\chi_{\varepsilon,t}$. Summing the resulting identities and integrating over $(0,t)$, we obtain, thanks to [**4**, Lemma III.3.3],

$$\begin{aligned}
\int_0^t |\chi_{\varepsilon,t}(s)|_H^2\,ds &+ \frac{1}{2}|\nabla\chi_\varepsilon(t)|_H^2 \\
&+ \int_\Omega (\hat{\beta}(\chi_\varepsilon(t)) + \sigma(\chi_\varepsilon(t)))\,dx + \frac{1}{2}|\vartheta_\varepsilon(t)|_H^2 \\
&+ \int_0^t \left(\varepsilon|\nabla\vartheta_\varepsilon(s)|_H^2 + \int_\Omega \nabla\vartheta_\varepsilon(s)\cdot\nabla(k*\vartheta_\varepsilon)(s)\,dx\right) ds \\
\le \frac{1}{2}|\chi_0|_V^2 &+ \int_\Omega (\hat{\beta}(\chi_0) + \sigma(\chi_0))\,dx + \frac{1}{2}|\vartheta_0|_H^2 \\
&+ \int_0^t \int_\Omega g_\varepsilon(x,s)\vartheta_\varepsilon(x,s)\,dx\,ds.
\end{aligned}$$

From (2.2), (2.3) and (2.5) we easily recover

$$\begin{aligned}
\frac{1}{2}(|\nabla\chi_\varepsilon(t)|_H^2 + |\vartheta_\varepsilon(t)|_H^2) &+ \int_0^t |\chi_{\varepsilon,t}(s)|_H^2\,ds \\
&+ \int_0^t \left(\varepsilon|\nabla\vartheta_\varepsilon(s)|_H^2 + \int_\Omega \nabla\vartheta_\varepsilon(s)\cdot\nabla(k*\vartheta_\varepsilon)(s)\,dx\right) ds \\
&\le C_4(1 + |\chi_0|_V^2 + |\vartheta_0|_H^2 + |\hat{\beta}(\chi_0)|_{L^1(\Omega)}) \\
&\quad + \int_0^t |g_\varepsilon(s)|_H(|\nabla\chi_\varepsilon(s)|_H^2 + |\vartheta_\varepsilon(s)|_H^2)^{1/2}\,ds.
\end{aligned}$$

The above inequality and [**4**, Lemma A.5] then yield (3.7). □

We next deduce the estimates on $(\chi_\varepsilon, \vartheta_\varepsilon)$ that we will need to pass to the limit as $\varepsilon \searrow 0$.

Lemma 3.2. *For each $T \in (0,+\infty)$, there is a constant $C_5(T)$ depending on Ω, σ, $|\chi_0|_V$, $|\vartheta_0|_H$, $|\hat{\beta}(\chi_0)|_{L^1(\Omega)}$, T, $|k|_{L^1(0,T)}$ and $\sup\{|g_\varepsilon|_{L^1(0,T;H)}, \varepsilon > 0\}$ such that*

$$\begin{aligned}
(3.8)\quad |\chi_\varepsilon|_{L^\infty(0,T;V)} + |\chi_{\varepsilon,t}|_{L^2(0,T;H)} &+ |\vartheta_\varepsilon|_{L^\infty(0,T;H)} \\
&+ \varepsilon^{1/2}|\vartheta_\varepsilon|_{L^2(0,T;V)} \le C_5(T),
\end{aligned}$$

$$|\xi_\varepsilon|_{L^2(0,T;L^{3/2}(\Omega))} + |\chi_\varepsilon|_{L^2(0,T;W^{2,3/2}(\Omega))} \leq C_5(T), \tag{3.9}$$

$$|\vartheta_{\varepsilon,t} - g_\varepsilon|_{L^2(0,T;W')} \leq C_5(T). \tag{3.10}$$

Proof. Let $T > 0$. Since k is of positive type, (3.8) is a straightforward consequence of (2.5), (3.1) and (3.7). Next, for almost every $t \in (0,T)$, $\chi_\varepsilon(t)$ is a solution to

$$\begin{gathered} \chi_\varepsilon(t) - \Delta\chi_\varepsilon(t) + \xi_\varepsilon(t) = f_\varepsilon(t) \quad \text{in} \quad \Omega, \\ \frac{\partial \chi_\varepsilon(t)}{\partial n} = 0 \quad \text{in} \quad \Gamma, \end{gathered} \tag{3.11}$$

$$\xi_\varepsilon(t) \in \beta(\chi_\varepsilon(t)), \tag{3.12}$$

where $f_\varepsilon = \chi_\varepsilon - \sigma'(\chi_\varepsilon) - \chi_{\varepsilon,t} + \lambda'(\chi_\varepsilon)\vartheta_\varepsilon$ satisfies

$$|f_\varepsilon|_{L^2(0,T;L^{3/2}(\Omega))} \leq C_5(T). \tag{3.13}$$

Note that (2.2) and (3.8) yield that $\lambda'(\chi_\varepsilon)$ is bounded in $L^\infty(0,T;L^6(\Omega))$. Since β is a maximal monotone graph, a monotonicity argument applied to (3.11) entails, see, e.g., [**5**],

$$|\xi_\varepsilon(t)|_{L^{3/2}(\Omega)} \leq |f_\varepsilon(t)|_{L^{3/2}(\Omega)}, \quad t \in (0,T). \tag{3.14}$$

Combining (3.11), (3.14) and classical elliptic estimates, we further obtain

$$|\chi_\varepsilon(t)|_{W^{2,3/2}(\Omega)} \leq C_\Omega |f_\varepsilon(t)|_{L^{3/2}(\Omega)}, \quad t \in (0,T), \tag{3.15}$$

for some constant C_Ω depending on Ω. Then, (3.9) follows from (3.13)–(3.15) after integration of (3.14)–(3.15) over $(0,T)$.

Finally, let $v \in W$. Since $W \subset V$, we deduce from (3.5) that

$$\langle\langle \vartheta_{\varepsilon,t} - g_\varepsilon, v\rangle\rangle = -\varepsilon \int_\Omega \nabla\vartheta_\varepsilon \cdot \nabla v \, dx + \int_\Omega (k * \vartheta_\varepsilon)\Delta v \, dx,$$

$$|\langle\langle \vartheta_{\varepsilon,t} - g_\varepsilon, v\rangle\rangle| \leq (\varepsilon \, |\nabla\vartheta_\varepsilon|_H + |k * \vartheta_\varepsilon|_H)|v|_W,$$

whence

$$|\vartheta_{\varepsilon,t} - g_\varepsilon|_{W'} \leq (\varepsilon \, |\vartheta_\varepsilon|_V + |k * \vartheta_\varepsilon|_H).$$

The Young inequality for convolution product then implies

$$\begin{aligned} |\vartheta_{\varepsilon,t} - g_\varepsilon|_{L^2(0,T;W')} \leq \varepsilon |\vartheta_\varepsilon|_{L^2(0,T;V)} & \\ + |k|_{L^1(0,T)} |k * \vartheta_\varepsilon|_{L^2(0,T;H)}&, \end{aligned}$$

and (3.10) follows from (3.8) and the above estimate. □

We are now ready to prove Theorem 2.3. Indeed, let $T > 0$. We infer from Lemma 3.2 and [**15**, Corollary 4] that there are a subsequence of $\{\chi_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon\}$ (not relabeled) and some functions

$$\begin{gathered} \chi \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega)), \\ \xi \in L^2(0,T;L^{3/2}(\Omega)), \\ \vartheta \in \mathcal{C}([0,T];W') \cap L^\infty(0,T;H), \quad \vartheta_t - g \in L^2(0,T;W'), \end{gathered}$$

such that (hereafter, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ stand for weak and weak star convergences, respectively)

$$\chi_\varepsilon \to \chi \quad \text{in } \mathcal{C}([0,T];H) \cap L^2(0,T;V), \tag{3.16}$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(0,T;L^{3/2}(\Omega)), \tag{3.17}$$

$$\vartheta_\varepsilon \overset{*}{\rightharpoonup} \vartheta \quad \text{in } L^\infty(0,T;H), \tag{3.18}$$

$$\psi_\varepsilon \to \psi \quad \text{in } \mathcal{C}([0,T];V'), \tag{3.19}$$

where $\psi_\varepsilon = \vartheta_\varepsilon - 1 * g_\varepsilon$ and $\psi = \vartheta - 1 * g$. Since V is continuously embedded into $L^3(\Omega) = L^{3/2}(\Omega)'$ and β induces a maximal monotone operator from $L^2(0,T;L^3(\Omega))$ in $L^2(0,T;L^{3/2}(\Omega))$, it follows from [**2**, Lemma II.1.3], (3.16) and (3.17) that

$$\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \quad \text{a.e. in} \quad \Omega \times (0,T). \tag{3.20}$$

On the other hand, (3.1) entails that $(1 * g_\varepsilon)$ converges to $(1 * g)$ in $\mathcal{C}([0,T];H)$. Combining this fact and (3.19) yields that

$$\vartheta_\varepsilon \to \vartheta \quad \text{in} \quad \mathcal{C}([0,T];V'). \tag{3.21}$$

Finally, we infer from (3.18) that

$$k * \vartheta_\varepsilon \overset{*}{\rightharpoonup} k * \vartheta \quad \text{in} \quad L^\infty(0,T;H). \tag{3.22}$$

Owing to the convergences (3.16)–(3.19), (3.21)–(3.22) and to Lemma 3.2, we may pass to the limit as $\varepsilon \searrow 0$ in (P_ε) and find that (χ, ϑ) is a solution to (1.3)–(1.6). Indeed, note that, thanks to (3.16), (3.18) and (2.2), $(\lambda'(\chi_\varepsilon)\vartheta_\varepsilon)$ converges to $(\lambda'(\chi)\vartheta)$ in the weak topology of $L^2(0,T;L^{3/2}(\Omega))$. Moreover, since k is of positive type, (3.7) yields (2.19) and the proof of Theorem 2.3 (i) is complete.

Next, if k satisfies the additional assumption (2.8), we infer from (3.7) and Lemma 2.1 that $(k * \vartheta_\varepsilon)$ is bounded in $L^2(0,T;V)$. Recalling (3.22), we obtain that $k * \vartheta \in L^2(0,T;V)$ and (2.21) follows from (3.7) and Lemma 2.1. □

4. Long-time behavior.

4.1 *Proof of Theorem* 2.5. Let (χ, ϑ) be a solution to (1.3)–(1.6) given by Theorem 2.3 (i) and define $\psi = \vartheta - 1 * g$. In the following, we denote by C_i, $i \geq 6$, any nonnegative real number depending only on Ω, σ, $|\chi_0|_V$, $|\vartheta_0|_H$, $|\hat{\beta}(\chi_0)|_{L^1(\Omega)}$, $|g|_{L^1(0,+\infty;H)}$, $|k|_{L^1(0,+\infty)}$ and l. The dependence of the C_i's upon additional parameters will be indicated explicitly.

Lemma 4.1. *The following holds*

$$\chi \in L^\infty(0,+\infty;V), \quad \chi_t \in L^2(0,+\infty;H), \tag{4.1}$$

$$\vartheta \in L^\infty(0,+\infty;H), \quad k * \vartheta \in L^\infty(0,+\infty;H), \tag{4.2}$$

$$\psi \in L^\infty(0,+\infty;H), \quad \psi_t \in L^2(0,+\infty;W') + L^\infty(0,+\infty;W'). \tag{4.3}$$

Proof. We first infer from (2.19) that

$$\begin{gathered} \nabla\chi \in L^\infty(0,+\infty;H)^3, \quad \vartheta \in L^\infty(0,+\infty;H), \\ \chi_t \in L^2(0,+\infty;H). \end{gathered} \tag{4.4}$$

We next take $v = 1$ in (2.15) and obtain, thanks to (2.23),

$$\left|\int_\Omega \chi(x,t)\,dx\right| \le C_6\left(|\vartheta(t)|_H + |\vartheta_0|_H + |\chi_0|_H + \int_0^t |g(s)|_H\,ds\right),$$
$$t > 0.$$

It follows from (4.4) and (2.7) that the righthand side of the above inequality lies in $L^\infty(0,+\infty)$. Consequently, $t \mapsto \int_\Omega \chi(x,t)\,dx$ belongs to $L^\infty(0,+\infty)$ and (4.4) and a generalized Poincaré inequality ensure that $\chi \in L^\infty(0,+\infty;V)$, whence (4.1) is verified.

Next, since $k \in L^1(0,+\infty)$ by (2.7), one sees that (4.4) and the Young inequality yield (4.2). We also infer from (4.4) and (2.7) that $\psi \in L^\infty(0,+\infty;H)$ and then deduce (4.3) from (2.15), (4.1) and (4.2). $\square$

A first consequence of Lemma 4.1 is that the set $\{(\chi(t),\vartheta(t)), t \ge 0\}$ is bounded in $V \times H$ and thus is relatively compact in $H \times V'$. Therefore, $\omega(\chi,\vartheta)$ is a nonempty compact subset of $H \times V'$. In addition, since $(\chi,\vartheta) \in \mathcal{C}([0,+\infty);H \times V')$, a classical argument from the theory of dynamical systems ensures that $\omega(\chi,\vartheta)$ is connected in $H \times V'$, see, e.g., [**13**, p. 12]. Another consequence of Lemma 4.1 is the following result.

Lemma 4.2. *For each $T > 0$, there is a constant $C_7(T)$ such that*

$$\sup_{t\ge 0} |\psi_t|_{L^2(t,t+T;W')} \le C_7(T), \tag{4.5}$$

$$\sup_{t\ge 0} |(k*\vartheta)_t|_{L^1(t,t+T;W')} \le C_7(T). \tag{4.6}$$

Proof. Owing to (4.3), we can set

$$\psi_t = f_2 + f_\infty,$$

where $f_2 \in L^2(0,+\infty;W')$ and $f_\infty \in L^\infty(0,+\infty;W')$. Then it is clear that

$$\int_t^{t+T} |\psi_t|^2_{W'}\,ds \le 2\int_t^{t+T} |f_2|^2_{W'}\,ds + 2\int_t^{t+T} |f_\infty|^2_{W'}\,ds$$
$$\le 2\,|f_2|^2_{L^2(0,+\infty;W')} + 2T|f_\infty|^2_{L^\infty(0,+\infty;W')},$$

hence (4.5) is satisfied.

We also have $\vartheta_t = g + f_2 + f_\infty$ and $g \in L^1(0,+\infty;W')$. Since $k \in L^1(0,+\infty)$ by (2.7), the Young inequality yields

$$k * \vartheta_t \in L^1(0,+\infty;W') + L^2(0,+\infty;W') + L^\infty(0,+\infty;W').$$

A computation similar to the one above then gives

$$\int_t^{t+T} |k * \vartheta_t|_{W'}\, ds \le C_7(T).$$

As $(k * \vartheta)_t = k\vartheta_0 + k * \vartheta_t$, thanks to (2.5) and (2.7) we finally obtain (4.6). $\square$

Now, let $(\chi_\infty, \vartheta_\infty)$ be in $\omega(\chi, \vartheta)$ and $\{t_n\}$ be a sequence of positive real numbers such that $t_n \nearrow +\infty$ and

$$(\chi(t_n), \vartheta(t_n)) \longrightarrow (\chi_\infty, \vartheta_\infty) \quad \text{in} \quad H \times V'. \tag{4.7}$$

For $n \ge 1$ and $t \ge 0$, we define

$$\begin{aligned} &\chi_n(t) = \chi(t_n + t), && \xi_n(t) = \xi(t_n + t),\\ &\vartheta_n(t) = \vartheta(t_n + t), && \psi_n(t) = \psi(t_n + t),\\ &\zeta_n(t) = (k * \vartheta)(t_n + t). \end{aligned}$$

Note that $\zeta_n \ne k * \vartheta_n$. As a consequence of the previous two lemmata, we may derive some estimates for χ_n, ξ_n, ϑ_n, ψ_n and ζ_n in various norms which are uniform with respect to $n \ge 1$.

Lemma 4.3. *Let $T > 0$. There is a constant $C_8(T)$ such that*

$$|\chi_n|_{L^\infty(0,T;V)} + |\chi_{n,t}|_{L^2(0,T;H)} \le C_8(T), \tag{4.8}$$

$$|\vartheta_n|_{L^\infty(0,T;H)} + |\psi_n|_{L^\infty(0,T;H)} + |\psi_{n,t}|_{L^2(0,T;W')} \le C_8(T), \tag{4.9}$$

$$|\zeta_n|_{L^\infty(0,T;H)} + |\zeta_{n,t}|_{L^1(0,T;W')} \le C_8(T), \tag{4.10}$$

$$|\xi_n|_{L^2(0,T;H)} + |\chi_n|_{L^2(0,T;W)} \le C_8(T). \tag{4.11}$$

Proof. First, (4.8)–(4.10) are straightforward consequences of Lemmata 4.1 and 4.2. We next infer from (1.3), (2.2), (2.23), (4.8) and (4.9) that, for almost every $t \in (0,T)$, $\chi_n(t)$ is a solution to

$$\chi_n(t) - \Delta\chi_n(t) + \xi_n(t) = F_n(t), \qquad \frac{\partial \chi_n(t)}{\partial n} = 0,$$
$$\xi_n(t) = \beta(\chi_n(t)),$$

and $F_n = l\vartheta_n - \chi_{n,t} - \sigma'(\chi_n) + \chi_n$ obeys

$$|F_n|_{L^2(0,T;H)} \leq C_8(T).$$

We now proceed as in the derivation of (3.9) and obtain (4.11). □

Similarly as in [**14**], we notice that we may identify the limit of (χ_n) as $n \to +\infty$. More precisely, we have the following item.

Proposition 4.4. *For each $T > 0$, the following holds*

$$\lim_{n\to+\infty} (|\chi_n - \chi_\infty|_{\mathcal{C}([0,T];H)} + |\chi_{n,t}|_{L^2(0,T;H)}) = 0. \tag{4.12}$$

Proof. For $n \geq 1$, we have that

$$|\chi_{n,t}|_{L^2(0,T;H)} \leq |\chi_t|_{L^2(t_n,+\infty;H)} \tag{4.13}$$

and the righthand side of the above inequality goes to zero as $n \to +\infty$ by virtue of (4.1).

Next, for $t \in (0,T)$ and $n \geq 1$, one checks that

$$\begin{aligned}|\chi_n(t) - \chi_\infty|_H &\leq |\chi_n(t) - \chi_n(0)|_H + |\chi(t_n) - \chi_\infty|_H \\ &\leq T^{1/2}\ |\chi_{n,t}|_{L^2(0,T;H)} + |\chi(t_n) - \chi_\infty|_H\end{aligned}$$

Owing to (4.7) and (4.13), the righthand side of the above estimate goes to zero as $n \to +\infty$, uniformly with respect to $t \in [0,T]$. The proof of Proposition 4.4 is thus complete. □

The next lemma is devoted to an estimate of the difference between $k * \vartheta_n$ and ζ_n in the H-norm and it will be needed later to recover a relationship between the possible limits of $\{k * \vartheta_n\}$ and $\{\zeta_n\}$.

Lemma 4.5. *For $t > 0$ and $n \geq 1$, the following holds*

$$|\zeta_n(t) - (k * \vartheta_n)(t)|_H \leq C_9 |k|_{L^1(t,+\infty)}. \tag{4.14}$$

Proof. Observe that

$$\begin{aligned} |\zeta_n(t) - (k * \vartheta_n)(t)|_H &\leq \bigg| \int_0^{t_n+t} k(s)\vartheta(., t_n + t - s)\, ds \\ &\qquad - \int_0^t k(s)\vartheta(., t_n + t - s)\, ds \bigg|_H \\ &\leq \bigg| \int_t^{t_n+t} k(s)\vartheta(., t_n + t - s)\, ds \bigg|_H \\ &\leq |\vartheta|_{L^\infty(0,+\infty;H)} \int_t^{t_n+t} |k(s)|\, ds. \end{aligned}$$

We then use (4.2) to obtain (4.14). □

After these preliminaries, we are ready to prove Theorem 2.5. It follows from Lemma 4.3, Proposition 4.4 and [**15**, Corollary 4] that there are a subsequence of $\{\chi_n, \xi_n, \psi_n, \zeta_n\}$ (not relabeled) and some functions

$$\begin{gathered} \bar{\xi} \in L^2(0, T; H) \quad \text{for each} \quad T > 0, \\ \bar{\psi} \in L^\infty(0, T; H) \cap W^{1,2}(0, T; W') \quad \text{for each} \quad T > 0, \\ \bar{\zeta} \in L^\infty(0, T; H) \quad \text{for each} \quad T > 0, \end{gathered}$$

such that

$$\begin{cases} \chi_n \to \chi_\infty & \text{in} \quad L^2(0, T; V), \\ \xi_n \rightharpoonup \bar{\xi} & \text{in} \quad L^2(0, T; H), \\ \psi_n \overset{*}{\rightharpoonup} \bar{\psi} & \text{in} \quad L^\infty(0, T; H), \\ \psi_n \to \bar{\psi} & \text{in} \quad \mathcal{C}([0, T]; V'), \\ \zeta_n \overset{*}{\rightharpoonup} \bar{\zeta} & \text{in} \quad L^\infty(0, T; H), \\ \zeta_n \to \bar{\zeta} & \text{in} \quad L^2(0, T; W'), \\ \zeta_n(t) \to \bar{\zeta}(t) & \text{in} \quad W' \quad \text{for a.e.} \quad t \in (0, T), \end{cases} \tag{4.15}$$

for each $T \in (0, +\infty)$. Note that $\chi_\infty \in W$ because of (4.11). Since β is a single-valued maximal monotone graph, we infer from (4.12), (4.15), (2.16) and [**2**, Lemma II.1.3] that

$$\bar{\xi}(t) = \xi_\infty := \beta(\chi_\infty) \quad \text{a.e. in} \quad \Omega, \tag{4.16}$$

for almost every $t \in (0, +\infty)$. Consequently, $\bar{\xi}$ does not depend on time.

The remainder of the proof may now be split into three steps. The first step will be devoted to prove the convergence of $\{\vartheta_n\}$ to some $\bar{\vartheta}$. In the second step, a system of partial differential equations satisfied by $(\chi_\infty, \bar{\vartheta}, \bar{\zeta})$ is derived and this is used in the third step to identify $(\bar{\vartheta}, \bar{\zeta})$ in terms of $(\chi_\infty, \vartheta_\infty)$.

Step 1. For $T > 0$, the following holds

$$\begin{cases} \vartheta_n \to \bar{\vartheta} & \text{in} \quad \mathcal{C}([0,T]; V'), \\ \vartheta_n \overset{*}{\rightharpoonup} \bar{\vartheta} & \text{in} \quad L^\infty(0,T;H), \\ \psi_{n,t} \rightharpoonup \bar{\vartheta}_t & \text{in} \quad L^2(0,T;W'), \end{cases} \tag{4.17}$$

where

$$\bar{\vartheta} = \bar{\psi} + G \quad \text{and} \quad G(x) = \int_0^\infty g(x,s)\, ds, \quad x \in \Omega. \tag{4.18}$$

Indeed, $G \in H$ and, cf. (2.7),

$$\lim_{t \to +\infty} |G - (1 * g)(t)|_H = 0. \tag{4.19}$$

Recalling that $\psi = \vartheta - 1 * g$, (4.17) follows from (4.9), (4.15), (4.18) and (4.19).

Step 2. We next claim that, for each $T > 0$,

$$-\Delta\chi_\infty + \xi_\infty + \sigma'(\chi_\infty) = l\bar{\vartheta}, \quad \text{a.e. in} \quad \Omega \times (0,T), \tag{4.20}$$

$$\int_0^T \langle\langle \bar{\vartheta}_t(s), v(s) \rangle\rangle\, ds = \int_0^T \int_\Omega \bar{\zeta}(s) \Delta v(s)\, dx\, ds, \qquad \forall\, v \in L^2(0,T;W), \tag{4.21}$$

To get (4.20), it suffices to pass to the (weak) limit in

$$\chi_{n,t} - \Delta\chi_n + \xi_n + \sigma'(\chi_n) = l\vartheta_n,$$

which is (2.14) restricted to $\Omega \times (t_n, t_n + T)$, as $n \to +\infty$ with the help of (4.12), (4.15), (2.2) and (4.17). On the other hand, from (2.15) we deduce that

$$\int_0^T \langle\langle \psi_{n,t}(s) + l\chi_{n,t}(s), v(s)\rangle\rangle \, ds = \int_0^T \int_\Omega \zeta_n(s)\Delta v(s) \, dx \, ds$$

and, taking the limit, on account of (4.12), (4.15) and (4.17) we obtain (4.21).

Step 3. Owing to (4.20), it turns out that $\bar\vartheta$ does not depend on t. Since $\bar\vartheta$ belongs to $\mathcal{C}([0,+\infty); V')$ and $\bar\vartheta(0) = \lim \vartheta(t_n) = \vartheta_\infty$ by (4.7), we finally obtain

$$\bar\vartheta(t) = \vartheta_\infty, \quad t \in [0,+\infty). \tag{4.22}$$

In particular, ϑ_∞ is in H because $\bar\vartheta \in L^\infty(0,+\infty; H)$ by (4.17). Also, plugging (4.22) into (4.20) and (4.21) yields that χ_∞ satisfies (2.25), while $\bar\zeta$ fulfills, for each $T > 0$,

$$\int_0^T \int_\Omega \bar\zeta(s)\Delta v(s) \, dx \, ds = 0, \quad \forall\, v \in L^2(0,T; W). \tag{4.23}$$

This last condition ensures that there is a function $\zeta : (0,+\infty) \to \mathbf{R}$ which belongs to $L^2(0,T)$ for each $T > 0$ and such that, for almost all $t \in (0,+\infty)$,

$$\bar\zeta(.,t) = \zeta(t) \quad \text{a.e. in} \quad \Omega. \tag{4.24}$$

Let us now identify ζ in terms of ϑ_∞ with the help of Lemma 4.5. Let $t \in (0,+\infty)$. For $n \geq 1$, we have

$$\begin{aligned} |\zeta(t) - (k * \vartheta_\infty)(t)|_{W'} \leq\; & |\zeta(t) - \zeta_n(t)|_{W'} + |\zeta_n(t) - (k * \vartheta_n)(t)|_{W'} \\ & + |(k * \vartheta_n)(t) - (k * \vartheta_\infty)(t)|_{W'}. \end{aligned}$$

Using the Young inequality and (4.14), we obtain

$$\begin{aligned} |\zeta(t) - (k * \vartheta_\infty)(t)|_{W'} \leq\; & |\zeta(t) - \zeta_n(t)|_{W'} + C_9 |k|_{L^1(t,+\infty)} \\ & + |k|_{L^1(0,t)} |\vartheta_n - \vartheta_\infty|_{\mathcal{C}([0,t];W')}. \end{aligned}$$

We then let $n \to +\infty$ in the above estimate and use (4.15), (4.17), (4.22) and (4.24) to deduce

$$|\zeta(t) - (k * \vartheta_\infty)(t)|_{W'} \leq C_9 |k|_{L^1(t,+\infty)} \quad \text{for a.a.} \quad t \in (0, +\infty).$$

Hence, since ϑ_∞ does not depend on t,

$$\operatorname*{ess\,lim}_{t \to +\infty} \left| \zeta(t) - \left(\int_0^t k(s)\, ds \right) \vartheta_\infty \right|_{W'} = 0. \tag{4.25}$$

We then claim that (4.25) implies that

$$\vartheta_\infty = \text{const}. \tag{4.26}$$

Indeed, consider $\phi \in H$ with $\int_\Omega \phi(x)\, dx = 0$. For $\varepsilon > 0$, we denote by ϕ_ε the solution to

$$\phi_\varepsilon \in W, \quad \phi_\varepsilon - \varepsilon \Delta \phi_\varepsilon = \phi \quad \text{a.e. in} \quad \Omega.$$

Then, ϕ_ε obeys $\int_\Omega \phi_\varepsilon(x)\, dx = 0$ and ζ, being independent of x,

$$\langle\langle \zeta(t) - \left(\int_0^t k(s)\, ds \right) \vartheta_\infty, \phi_\varepsilon \rangle\rangle = - \left(\int_0^t k(s)\, ds \right) \int_\Omega \vartheta_\infty(x) \phi_\varepsilon(x)\, dx,$$

whence, thanks to (4.25), we infer that

$$0 = - \left(\int_0^\infty k(s)\, ds \right) \int_\Omega \vartheta_\infty(x) \phi_\varepsilon(x)\, dx.$$

Since $\int_0^\infty k(s)\, ds \neq 0$ by (2.7), we obtain

$$\int_\Omega \vartheta_\infty(x) \phi_\varepsilon(x)\, dx = 0, \quad \varepsilon > 0.$$

Recalling that $\vartheta_\infty \in H$, we let $\varepsilon \to 0$ and find

$$\int_\Omega \vartheta_\infty(x) \phi(x)\, dx = 0,$$

this equality holding for any $\phi \in H$ with $\int_\Omega \phi(x)\, dx = 0$. Then (4.26) is verified.

Finally, taking $v = 1$ in (2.15) and integrating over $(0, t_n)$, we obtain

$$\int_\Omega (\vartheta(t_n) + l\chi(t_n))\,dx = \int_\Omega (\vartheta_0 + l\chi_0)\,dx + \int_0^{t_n}\int_\Omega g(x,s)\,dx\,ds.$$

Here, we let $n \to +\infty$ and use (4.12), (4.17), (4.22), (4.26) and (2.7) to obtain (2.24). Recalling (2.25), this completes the proof of Theorem 2.5. □

4.2 *Proof of Theorem* 2.6. Let (χ, ϑ) be a solution to (1.3)–(1.6) given by Theorem 2.3 (ii). Such a solution enjoys the following properties.

Lemma 4.6. *The following holds*

$$\chi \in L^\infty(0,+\infty;V), \quad \chi_t \in L^2(0,+\infty;H), \tag{4.27}$$

$$\vartheta \in L^\infty(0,+\infty;H), \quad k*\vartheta \in L^\infty(0,+\infty;H), \tag{4.28}$$
$$\nabla(k*\vartheta) \in L^2(0,+\infty;H)^3,$$

$$\vartheta_t \in L^2(0,+\infty;V') + L^1(0,+\infty;H), \tag{4.29}$$

$$(k*\vartheta)_t \in L^2(0,+\infty;V') + L^1(0,+\infty;H). \tag{4.30}$$

Proof. First, (4.27) and (4.28) are derived exactly in the same way as (4.1) and (4.2) in Lemma 4.1. The fact that $\nabla(k*\vartheta)$ belongs to $L^2(0,+\infty;H)^3$ is a consequence of (2.21). In view of (2.2) and (4.27), we have that

$$(\lambda(\chi))_t = \lambda'(\chi)\chi_t \in L^2(0,T;V').$$

Then, by comparison in (2.15) and using (2.7), we achieve (4.29). Finally, we remark that

$$(k*\vartheta)_t = k*\vartheta_t + k\vartheta_0$$

and, therefore, (4.29), the Young inequality and (2.7) yield (4.30). □

In the following, we denote by C_i, $i \geq 10$, any nonnegative real number depending only on Ω, σ, λ, $|\chi_0|_V$, $|\vartheta_0|_H$, $|\hat\beta(\chi_0)|_{L^1(\Omega)}$, $|g|_{L^1(0,+\infty;H)}$,

$|k|_{W^{1,1}(0,+\infty)}$ and η in (2.8). The dependence of the C_i's upon additional parameters will be indicated explicitly.

We now consider $(\chi_\infty, \vartheta_\infty)$ in $\omega(\chi, \vartheta)$ and $\{t_n\}$ to be a sequence of positive real numbers such that $t_n \nearrow +\infty$ and

$$(\chi(t_n), \vartheta(t_n)) \longrightarrow (\chi_\infty, \vartheta_\infty) \quad \text{in} \quad H \times V'. \tag{4.31}$$

We then put, for $t \geq 0$ and $n \geq 1$,

$$\begin{aligned} &\chi_n(t) = \chi(t_n + t), \quad \xi_n(t) = \xi(t_n + t), \\ &\vartheta_n(t) = \vartheta(t_n + t), \quad \zeta_n(t) = (k * \vartheta)(t_n + t). \end{aligned}$$

In this case, we can derive at once strong convergence also for ϑ_n.

Proposition 4.7. *For each $T > 0$, the following holds*

$$\lim_{n\to+\infty} (|\chi_n - \chi_\infty|_{\mathcal{C}([0,T];H)} + |\chi_{n,t}|_{L^2(0,T;H)}) = 0, \tag{4.32}$$

$$\lim_{n\to+\infty} (|\vartheta_n - \vartheta_\infty|_{\mathcal{C}([0,T];V')} + |\vartheta_{n,t}|_{L^2(0,T;V')+L^1(0,T;H)}) = 0, \tag{4.33}$$

and $\vartheta_\infty \in H$.

Proof. The proof of (4.32) is the same as that of Proposition 4.4. Owing to (4.29) and (4.31), we may proceed in an analogous way for ϑ_n and obtain (4.33). Finally, $\vartheta_\infty \in H$ is a consequence of (4.33) and (4.28). □

Lemma 4.8. *For each $T > 0$,*

$$|\zeta_n(0)|_H = |(k * \vartheta)(t_n)|_H \leq C_{10}, \tag{4.34}$$

$$|\xi_n|_{L^2(0,T;L^{3/2}(\Omega))} + |\chi_n|_{L^2(0,T;W^{2,3/2}(\Omega))} \leq C_{11}(T). \tag{4.35}$$

Proof. Since $k * \vartheta \in \mathcal{C}([0, +\infty); H)$, (4.34) follows from (4.28). Concerning (4.35), we can invoke (2.14), (4.27), (4.28), (2.2) and a monotonicity argument. □

It follows from Lemma 4.8 that we may extract a subsequence of $\{t_n\}$ (not relabeled) such that, for each $T > 0$, one has

$$\zeta_n(0) \longrightarrow \zeta_\infty \quad \text{in} \quad V', \tag{4.36}$$

$$\xi_n \rightharpoonup \bar{\xi} \quad \text{in} \quad L^2(0, T; L^{3/2}(\Omega)), \tag{4.37}$$

for some functions $\zeta_\infty \in V'$ and $\bar{\xi} \in L^2(0, T; L^{3/2}(\Omega))$.

Owing to (4.30) and (4.36), we may now proceed as for the derivation of (4.32)–(4.33) and obtain

$$\lim_{n \to +\infty} |\zeta_n - \zeta_\infty|_{\mathcal{C}([0,T];V')} = 0 \quad \text{for each} \quad T > 0. \tag{4.38}$$

We also infer from (4.27) and (4.32) that χ_n converges to χ_∞ in $L^2(0, T; L^3(\Omega))$ for each $T > 0$, which yields, together with (4.37) and [**2**, Lemma II.1.3] that

$$\bar{\xi}(t) \in \beta(\chi_\infty) \quad \text{a.e. in} \quad \Omega, \tag{4.39}$$

for almost every $t \in (0, +\infty)$. At this point, we cannot conclude (as in the proof of Theorem 2.5) that $\bar{\xi}$ does not depend on t, since β is multivalued.

Now, as (χ, ϑ) solves (2.10)–(2.18), the quadruple $(\chi_n, \xi_n, \vartheta_n, \zeta_n)$ satisfies, for each $T > 0$ and $n \geq 1$,

$$\begin{aligned}
\int_0^T \int_\Omega \nabla\chi_n \cdot \nabla v \, dx \, ds \\
+ \int_0^T \int_\Omega (\xi_n + \sigma'(\chi_n) - \lambda'(\chi_n)\vartheta_n) v \, dx \, ds \\
= - \int_0^T \int_\Omega \chi_{n,t} v \, dx \, ds, \quad \forall\, v \in L^2(0, T; V),
\end{aligned}$$

$$\begin{aligned}
\int_0^T \langle\langle (\vartheta_n + \lambda(\chi_n))_t(s), v \rangle\rangle \, ds \\
= \int_0^T \int_\Omega \zeta_n \Delta v \, dx \, ds \\
+ \int_0^T \int_\Omega g(. + t_n) v \, dx \, ds, \quad \forall\, v \in L^\infty(0, T; W).
\end{aligned}$$

We then let $n \to +\infty$ in the above two equations and use (4.27), (4.32), (4.33), (2.2), (4.37) and (2.7) to obtain

$$\int_0^T \int_\Omega \nabla\chi_\infty \cdot \nabla v \, dx \, ds + \int_0^T \int_\Omega (\bar{\xi}(t) + \sigma'(\chi_\infty) - \lambda'(\chi_\infty)\vartheta_\infty) v \, dx \, ds = 0, \quad \forall\, v \in L^2(0,T;V),$$

$$\int_0^T \int_\Omega \zeta_\infty \Delta v \, dx \, ds = 0, \quad \forall\, v \in L^\infty(0,T;W).$$

The latter equality implies that

$$\zeta_\infty = \text{const.} \tag{4.40}$$

while the former ensures that, for almost every $t \in (0,+\infty)$, the following holds

$$\bar{\xi}(t) = \Delta\chi_\infty - \sigma'(\chi_\infty) + \lambda'(\chi_\infty)\vartheta_\infty \quad \text{in} \quad V'. \tag{4.41}$$

Recalling (4.35), (4.37) and (4.39), we realize that χ_∞ is a solution to

$$\begin{cases} -\Delta\chi_\infty + \beta(\chi_\infty) \ni -\sigma'(\chi_\infty) + \lambda'(\chi_\infty)\vartheta_\infty & \text{a.e. in } \Omega, \\ \partial\chi_\infty/\partial n = 0 \quad \text{a.e. on } \Gamma. \end{cases} \tag{4.42}$$

Finally, we infer from Lemma 4.5, (4.33), (4.38) and the continuity of the embedding of H in V' that, for $t > 0$,

$$|\zeta_\infty - (k * \vartheta_\infty)(t)|_{V'} \le C_{12}\, |k|_{L^1(t,+\infty)}.$$

Letting $t \to +\infty$ in the above inequality yields, thanks to (2.7),

$$\zeta_\infty = \left(\int_0^\infty k(s)\, ds\right)\vartheta_\infty,$$

which in turn gives that $\vartheta_\infty = \text{const.}$, due to (4.40). This last fact and (4.42) lead to $\chi_\infty \in W$ by arguing as in the derivation of (4.11).

In order to complete the proof of Theorem 2.6, it remains to check (2.29). But this may be done in the same way as for (2.26). ◻

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