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A PHASE FIELD SYSTEM WITH MEMORY: GLOBAL EXISTENCE

AMY NOVICK-COHEN

ABSTRACT. In the present paper we analyze a phase field model with memory:

$$
\text{(PFM)} \quad \begin{cases} u_t + (l/2)\phi_t = \int_{-\infty}^{t} a_1(t-s)\Delta u(s)\,ds \\ \qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,T) \\ \tau\phi_t = \int_{-\infty}^{t} a_2(t-s)[\xi^2\Delta\phi + (\phi-\phi^3)/\eta + u](s)\,ds \\ \qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,T) \\ \mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0 \quad (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x),\ \phi(x,0) = \phi_0(x) \quad x \in \Omega \end{cases}
$$

for $T > 0$, which has been recently proposed [**29**] as a phenomenological model to describe phase transitions in the presence of slowly relaxing internal variables. The system yields motion by mean curvature with memory under suitable assumptions in a sharp interface limit. In the present paper we give a proof of global existence of a weak solution $(u,\phi) \in \mathbf{C}([0,T]; L^2(\Omega) \times H^1(\Omega))$ for (PFM) assuming that Ω is a smooth bounded domain in R^n, $n = 1, 2$, or 3, the kernels $a_1, a_2 \in L^1(R^+)$ are of positive type, the initial data is in $L^2(\Omega) \times H^1(\Omega)$, and the history is in $L^1(-\infty, 0; H^2(\Omega))$ and $L^1(-\infty, 0; H^3(\Omega)) \cap L^5(-\infty, 0; L^6(\Omega))$, respectively. Our methodology combines results from the theory of Volterra integral equations with Galerkin methods and energy estimates. The results presented here were announced in [**26**].

Keywords and phrases. Phase field equations, memory, integro-differential equations, Galerkin methods, phase transitions.

1. Introduction. We establish global existence of a weak solution, $(u,\,\phi) \in \mathcal{C}([0,\,T], L^2(\Omega) \times H^1(\Omega))$, $T > 0$, for the system:

(1.1)
$$u_t + \frac{l}{2}\phi_t = a_1 * \triangle u + f_1, \quad (x,t) \in \Omega \times (0,T),$$

(1.2)
$$\tau\phi_t = a_2 * \left[\xi^2 \triangle\phi + \frac{\phi - \phi^3}{\eta} + u\right] + f_2, \quad (x,t) \in \Omega \times (0,T),$$

(1.3)
$$\mathbf{n} \cdot \nabla u = \mathbf{n} \cdot \nabla \phi = 0, \quad (x,t) \in \partial\Omega \times (0,T),$$

(1.4)
$$u(x,0) = u_0(x),\ \phi(x,\,0) = \phi_0(x), \quad x \in \Omega,$$

where $u = u(x,t)$ represents a dimensionless temperature and $\phi = \phi(x,t)$ is a nonconserved order parameter. The constant l is a dimensionless latent heat, τ is a dimensionless relaxation time, ξ is a dimensionless interaction length, and η is a dimensionless potential well depth. Equation (1.1) constitutes an energy balance equation, and equation (1.2) is a type of phase relaxation equation. The underlying constitutive assumption here is that the system responds in a delayed or time averaged fashion to thermal gradients and to deviations from equilibrium [**29**]. We shall assume Ω to be a bounded domain in R^n, $n = 1, 2$, or 3, with a sufficiently smooth boundary, and we shall furthermore assume that the initial data $\{u_0, \phi_0\}$ is prescribed in $L^2(\Omega) \times H^1(\Omega)$.

In (1.1)–(1.2), the first terms on the right hand side are convolution terms, defined by

$$(a_i * \Psi)(t) := \int_0^t a_i(t-s)\,\Psi(s)\,ds \quad i = 1 \,\text{or}\ \ 2,$$

for $0 \le t < T$, $\Psi \in L^p(0,\,T; L^p(\Omega))$, $1 \le p \le \infty$. Here a_i, $i = 1, 2$, act as "memory kernels," mediating a delayed or averaged response of the system to thermal gradients in equation (1.1) and to deviations from equilibrium in equation (1.2), respectively. We shall assume throughout that $\langle\cdot,\,\cdot\rangle$ denotes the $L^2(\Omega)$ inner product. With regard to the memory kernels, we shall assume throughout that $a_i \in L^1(R^+)$, $i = 1, 2$, and that the kernels a_i are of *positive type*.

Definition 1. A kernel a is said to be of positive type on the interval $[0,T]$ for $T>0$ if $a \in L^1(0,T)$, and

$$\int_0^T \langle \psi,\, a_i * \psi \rangle \, dt \geq 0, \quad \forall\, \psi \in L^2(0,T;\, L^2(\Omega)). \tag{1.5}$$

The terms f_1 and f_2 reflect the influence of the "history" of the system. We shall assume that $\{f_1, f_2\} \in L^1(R^+;\, L^1(\Omega) \times H^1(\Omega))$. One can assume, though it is by no means essential, that:

$$f_1(x,t) = \int_{-\infty}^0 a_1(t-s) \,\triangle u(x,\, s)\, ds, \quad (x,t) \in \Omega \times [0,T] \tag{1.6}$$

and

$$f_2(x,t) = \int_{-\infty}^0 a_2(t-s) \left[\xi^2 \triangle \phi + \frac{\phi - \phi^3}{\eta} + u \right] (x,s)\, ds \qquad (x,t) \in \Omega \times [0,T], \tag{1.7}$$

respectively, where

$$u(x,t) = u_h(x,t), \quad \phi(x,t) = \phi_h(x,t), \quad (x,t) \in \Omega \times (-\infty, 0),$$

for prescribed history functions u_h and ϕ_h such that

$$u_h(x,t) \in L^1(-\infty, 0; H^2(\Omega))$$

and

$$\phi_h(x,t) \in L^1(-\infty, 0; H^3(\Omega)) \cap L^5(-\infty, 0; L^6(\Omega)).$$

Thus (1.1)–(1.2) may be written equivalently as they appear in (PFM). We remark here that though the effects of possible body heating and boundary heating have been neglected for simplicity in (PFM), they can be included in the system (1.1)–(1.4) by incorporating appropriate forcing terms into f_1 and f_2, and the analysis which we present here may be suitably modified accordingly. See for example the discussion in [**25**], where possible boundary heating was taken into account.

Note that if the kernels are chosen as $a_i(t) = \alpha_i \delta(t)$, where α_i is a constant, then the system (PFM) reduces to the classical phase field equations as introduced by Fix [**14**] and Caginalp [**7**] and which have their roots in Landau-Ginzburg theory [**23, 22**]. See [**6**] for an overview and discussion. Classical phase field equations were designed to describe nonisothermal phase separation, and the literature treating their analysis and predictions is vast. It is relevant for our present considerations to note that global existence was proven for the classical phase field equations taking, as we do, initial data in $L^2(\Omega) \times H^1(\Omega)$, by Bates and Zheng [**3**]. For earlier existence results, see Caginalp [**7**] and later Elliott and Zheng [**13**]. In the classical phase field context, Bates and Zheng also proved uniqueness, additional regularity, as well existence of a compact global attractor and inertial sets in $L^2(\Omega) \times H^1(\Omega)$ for fixed energy levels, i.e., for $\int_\Omega \{u + (l/2)\phi\}\, dx =$ constant. Such properties as uniqueness, additional regularity and compactness in $L^2(\Omega) \times H^1(\Omega)$ cannot be expected to hold for phase field equations with memory without sufficiently strong assumptions on the kernels, a_1, a_2.

There is also a large literature concerning phase field equations in which memory effects have been included in the energy balance equation, but not in the phase relaxation equation. The rationale behind the inclusion of the memory effects in the energy balance equation is to rid the system of the anomaly of infinite speed of heat propagation, essentially incorporating a Gurtin-Pipkin type formulation [**21**] for memory effects in the energy balance equation into a phase field setting. With regard to papers which have been published treating such systems, we note that existence of solutions $(u, \phi) \in \mathcal{C}(R^+; L^2(\Omega) \times H^1(\Omega))$ was proven by Aizicovici and Barbu [**2**] taking initial conditions in $L^2(\Omega) \times H^1(\Omega)$, (thermal) memory kernels of positive type in $L^1(R^+)$, (thermal) history $f_1 \in L^2_{loc}(R^+; L^2(\Omega))$, and assuming Robin and Dirichlet boundary conditions from u and ϕ respectively, and uniqueness was demonstrated in one space dimension, $n = 1$. Under additional assumptions on the kernel uniqueness could be proven for $n = 2$ and 3, and making suitable restrictions on the kernel and on the data, further regularity and asymptotic properties were obtained. For the case of Neumann boundary conditions, initial data in $L^2(\Omega) \times H^1(\Omega)$, $f_1 \in L^1(R^+; L^2(\Omega))$, and $a_1 \in L^1_{loc}(R^+)$, existence of weak solutions was proven by Colli and Laurençot [**11**], but continuity from the initial data was not obtained. In [**12**] under additional assumptions on the kernel, they demonstrated uniqueness and characterized the ω-limit set. Further analysis of the long time behavior for the phase field model with memory in the energy balance equation has recently been undertaken by Giorgi, Grasselli and Pata [**15,16**]. In particular, we note that in [**16**], the existence of a uniform attractor was proven.

The focus of the present paper is on proving existence for (PFM). After announcing these results in [**26**], the author became aware that existence had been under independent study by Grasselli and Rotstein [**18, 17**] for (PFM), though history effects in the phase relaxation equation were not taken into account there. We prove existence of a solution which is smoother than the solution which is obtained by Grasselli in [**17**] under the same (very weak) assumptions on the memory kernels, in that continuity is achieved from initial data in $L^2(\Omega) \times H^1(\Omega)$. Our methodology differs from that of [**17**] in that we rely on weaker approximants and obtain additional (two-sided) estimates. Existence is proven by Grasselli and Rotstein in [**18**] under considerably more restrictive assumptions on the memory kernels. In [**18, 17**] uniqueness and well-posedness are also proven based on additional assumptions on the memory kernels; in the present manuscript no explicit discussion is made of either uniqueness or well-posedness.

Before turning to the details of the existence proof, we give some words of introduction to (PFM) as it is a model which has only been proposed quite recently. The present formulation essentially constitutes a phenomenological extension of the classical phase field equations in which memory effects are taken into account both in the energy balance equation and in the phase relaxation equation. The rationale for including memory effects in the phase relaxation equation is to take into account in an averaged way the presence of slowly relaxing "internal variables" which are troublesome to represent explicitly. Such internal variables could represent, for example, configurational degrees of freedom which are important in polymer melts during phase transition. Equation (1.2) can be seen to have the structure

$$\tau\phi_t = -\int_{\infty}^{t} a_2(t-s)\frac{\delta\mathcal{F}(u,\phi)}{\delta\phi}(s)\,ds, \tag{1.8}$$

where $\mathcal{F}(u,\phi)$ is an appropriately defined free energy, as opposed to classical phase relaxation which has the form

$$\tau\phi_t = -\frac{\delta\mathcal{F}(u,\phi)}{\delta\phi}.$$

While the effects of delayed or averaged response in the energy balance equation are predicted to be noticeable under extreme thermal conditions, such as very high temperatures or very low temperatures, the effects of delayed or averaged response in the phase relaxation equation should not require such extreme conditions to be influential. A lengthier discussion of the derivation and the implications of (1.8) is given in [**29**]. For conditions which guarantee thermodynamic consistency of (PFM), see [**19**]. We remark that it is also feasible to consider phase relaxation with memory in the context of a conserved order parameter in analogy with the conserved phase field equations introduced by Caginalp [**8**] or the conserved phase field model with memory in the energy balance equation as proposed by the author in [**25**].

To gain intuition into our expectations from the system (PFM), it is helpful to consider the qualitative long time behavior, which should include a description of such features as coarsening. Such qualitative descriptions should be viewed as a focusing on the behavior in the neighborhood of some specific region of the global attractor. For the classical phase field equations, the work on global attractors of Bates and Zheng [**3**], see also [**4**], is complemented by the work by Caginalp and Fife [**10**] where it was formally demonstrated that under suitable scaling assumptions, the limiting motion as $t \to \infty$ is described by a Stefan problem which includes Gibbs-Thomson effects. More recently, Caginalp and Chen [**9**] have demonstrated rigorously that depending on which particular distinguished is limit considered, the limiting motion may be given by the classical Stefan problem, a type of surface tension model, with or without attachment kinetics, a two phase Hele-Shaw model, or motion by mean curvature. In particular, we note that motion by mean curvature is predicted when $\tau = \mathcal{O}(\varepsilon)$, $\xi = \mathcal{O}(\varepsilon^{1/2})$, $\eta = \mathcal{O}(\varepsilon)$, and $l = \mathcal{O}(\varepsilon)$, for $0 < \varepsilon \ll 1$. For the standard phase field model with memory, where memory effects are included in the energy balance equation but not in the phase relaxation equation, while considerable analytical work has been undertaken regarding certain aspects of the long time behavior, the qualitative behavior has for the most part yet to be studied.

With regard to the system (PFM), we remark that certain interesting sharp interface limiting motions have been worked out formally. For example, if the distinguished limit $\tau = \mathcal{O}(\varepsilon)$, $\xi = \mathcal{O}(\varepsilon^{1/2})$, $\eta = \mathcal{O}(\varepsilon)$, $l = \mathcal{O}(\varepsilon^2)$ is considered, where $0 < \varepsilon \ll 1$, and the kernels are taken to be exponential functions, then the (scaled) limiting motion is given by [**27, 29**]

$$V_t + \gamma V(1 - V^2) = \kappa(1 - V^2), \tag{1.9}$$

where V denotes the normal velocity of the front, κ denotes the mean curvature and γ is the rate of exponential decay of the phase memory kernel, or if the distinguished limit $\tau = \mathcal{O}(\varepsilon)$, $\xi = \mathcal{O}(\varepsilon^{1/2})$, $\eta = \mathcal{O}(\varepsilon)$, $l = \mathcal{O}(\varepsilon^2)$ is considered, and the kernels are taken to be approximately "weakly singular;" i.e., $a_i = b_i\gamma_i \exp(-\gamma_i t)$ where $b_i = \mathcal{O}(1)$ and $\gamma_i = \mathcal{O}(\varepsilon^{-3/2})$, then the (scaled) limiting motion:

$$\varepsilon^{3/2} V_t + V = \kappa \tag{1.10}$$

[**1**] is predicted. A crystalline algorithm was constructed to study equations such as (1.9) and (1.10) in [**27**]. Implementing the crystalline algorithm for initially convex polygonal phase boundaries with vanishing initial velocity, it could be seen that while monotone melting occurred for regular polygonal initial conditions, for sufficiently asymmetric convex polygonal initial conditions two-dimensional damped oscillations appeared [**28, 29**]. For further discussion of equations (1.9), (1.10) and their generalizations, see [**1, 30**].

The outline of our paper is as follows. In Section 2 we rescale equations (1.1)–(1.4) for simplicity, summarize our assumptions on kernels and present some technical lemmas and inequalities to be employed in the sequel. In Section 3 we state and prove our main existence result, Theorem 1 which we paraphrase below as Theorem A.

Theorem A. *Suppose that* $\{u_0, \phi_0\} \in L^2(\Omega) \times H^1(\Omega)$ *and* $\{f_1, f_2\} \in L^1(0, T; L^2(\Omega) \times H^1(\Omega))$*, then there exists a solution to* (1.1)–(1.4) *such that* $\{u, \phi\} \in \mathcal{C}([0, T]; L^2(\Omega) \times H^1(\Omega))$ *and* $\{u_t, \phi_t\} \in L^\infty(0, T; H^{-2}(\Omega) \times H^{-1}(\Omega)) + L^1(0, T; L^2(\Omega) \times H^1(\Omega))$.

We remark that if $u_h \in L^1(0, T; H^2(\Omega))$ and $\phi_h \in L^1(0, T; H^3(\Omega)) \cap L^5(0, T; L^6(\Omega))$, then $\{f_1, f_2\}$ satisfy the assumptions stated in Theorem A, see Lemma 2.3. Also, the solution which we obtain is too weak to satisfy (1.3) in any strong sense; hence, (1.3) should be understood as being satisfied in the sense of approximating sequences.

The proof of Theorem A in Section 3 proceeds by making Galerkin approximations for u and ϕ and demonstrating that the Galerkin coefficients are uniquely determined and differentiable in L^1 on some maximal interval $[0, T_N)$ (Lemma 3.1), then obtaining an energy inequality which yields that $T_N = \infty$ (Lemma 3.2, Lemma 3.3) and gives some a priori bounds (Lemma 3.4). Afterwards, subsequences are taken and a solution $\{u, \phi\}$ is obtained, $\{u, \phi\} \in \mathcal{C}([0, T]; H^{-2}(\Omega) \times H^{-1}(\Omega))$ (Lemmas 3.5–3.7). Lastly in Lemmas 3.8 and 3.9, using weak continuity and energy estimates we demonstrate that in fact $\{u, \phi\} \in \mathcal{C}([0, T]; L^2(\Omega) \times H^1(\Omega))$; i.e., that the solution is continuous in the spaces is which the initial data was assumed to be taken. Section 4 contains a few comments.

Our method of proof is similar to that of [**17**] in that there also Galerkin approximants are employed; however, the approximants used in [**17**] are more regular than those which we employ here, being based on a Coleman-Gurtin type regularization of the Gurtin-Pipkin heat law and $\mathcal{C}([0, T]; L^2(\Omega))$ approximants of the history. Moreover, in [**17**] two sequential limiting processes are needed to obtain a solution, and history in the phase relaxation equation is neglected. Thus our methodology is seemingly more straightforward. We note also that Lemma 3.9 given here in Section 3 can be adapted without undo difficulty to demonstrate that the solution obtained in Theorem 2.1 of [**17**] in fact does lie in $\mathcal{C}([0, T]; L^2(\Omega) \times H^1(\Omega))$. Similarly the results achieved in [**11**] in the context of the classical phase field model with memory may be strengthened to yield a solution $\{u, \phi\} \in \mathcal{C}([0, T]; L^2(\Omega) \times H^1(\Omega))$ by suitably adapting Lemma 3.9 given here to the context of [**11**].

2. Preliminaries. It is easy to verify that by appropriate rescaling of the parameters, functions, and variables in (1.1)–(1.4), it is possible, by identifying

$$f_1(x, t) = \int_{-\infty}^{0} a_1(t - s) \triangle u(s) ds, \quad (x, t) \in \Omega \times [0, T]$$

and

$$f_2(x, t) = \int_{-\infty}^{0} a_2(t - s)[\xi^2 \triangle \phi + \phi - \phi^3 + u](s) ds, \quad (x, t) \in \Omega \times [0, T],$$

to write (1.1)–(1.4) in the form:

(2.1)
$$u_t + \frac{l}{2}\phi_t = a_1 * \triangle u + f_1, \quad (x,\, t) \in \Omega \times (0,T),$$

(2.2)
$$\phi_t = a_2 * [\xi^2 \triangle \phi + \phi - \phi^3 + u] + f_2, \quad (x,t) \in \Omega \times (0,T),$$

(2.3)
$$\mathbf{n} \cdot \nabla u = \mathbf{n} \cdot \nabla \phi = 0, \quad (x,t) \in \partial\Omega \times (0,T),$$

(2.4)
$$u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x), \quad x \in \Omega.$$

We shall give our analysis in the sequel in terms of the system (2.1)–(2.4), for the sake of simplicity.

Following our remarks in the introduction, we shall formulate our assumptions on the kernels a_i, $i = 1$ or 2 as

(**H1**) $\quad a_i \in L^1(R^+)$,

(**H2**) $$\int_0^T \langle \psi, a_i * \psi \rangle \, dt \geq 0 \quad \forall \psi \in L^2(0,T; L^2(\Omega)), \quad \forall T > 0.$$

In our analysis we shall frequently make use of *Young's inequality for convolutions*:

Lemma 2.1. *If $a \in L^1(J)$ and $\varphi \in L^p(J)$, $p \in [1,\infty]$ where $J = R$, R^+ or $[0,T]$, $0 < T < \infty$, then $a * \varphi \in L^p(J)$, and*

(2.5)
$$\|a * \varphi\|_{L^p(J)} \leq \|a\|_{L^1(J)} \cdot \|\varphi\|_{L^p(J)}.$$

Proof: This result is standard, see, e.g., [**20**, Theorem 2.2]. □

From Lemma 2.1 and Fubini's theorem, one easily obtains

Lemma 2.2. *If $a \in L^1(J)$ and $\phi \in L^p(J; L^p(\Omega))$, $p \in [1,\infty]$ where $J = R$, R^+ or $[0,T]$, $0 < T < \infty$, then $a * \phi \in L^p(J;\, L^p(\Omega))$, and*

(2.6)
$$\|a * \phi\|_{L^p(J;\, L^p(\Omega))} \leq \|a\|_{L^1(J)} \cdot \|\phi\|_{L^p(J;\, L^p(\Omega))}.$$

We present below for the reader's convenience a number of embedding results to be used in the sequel. Note that the Gagliardo-Nirenberg inequality:

$$\|D^j v\|_{L^p(\Omega)} \leq C_1 \|D^m v\|^a_{L^r(\Omega)} \|v\|^{1-a}_{L^q(\Omega)} + C_2 \|v\|_{L^q(\Omega)}$$

for

$$\frac{j}{m} \leq a \leq 1 \quad \text{and} \quad \frac{1}{p} \geq \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

where $C_i = C_i(\Omega)$, implies in particular that for $\Omega \subset R^n$, $n = 1, 2, 3$,

(2.7)
$$||\phi||_{L^4(\Omega)} \leq C_3 ||\nabla\phi||^a_{L^2(\Omega)} ||\phi||^{1-a}_{L^2(\Omega)} + C_4 ||\phi||_{L^2(\Omega)}, \quad \forall a \in \left[\frac{3}{4}, 1\right],$$

(2.8)
$$||\phi||_{L^4(\Omega)} \leq C_5 ||\phi||_{H^s(\Omega)} + C_6 ||\phi||_{L^2(\Omega)}, \quad \forall s \geq \frac{3}{4},$$

(2.9)
$$||\phi||_{L^6(\Omega)} \leq C_7 ||\nabla\phi||_{L^2(\Omega)} + C_8 ||\phi||_{L^2(\Omega)},$$

(2.10)
$$||\phi||_{L^8(\Omega)} \leq C_9 ||D^3\phi||^{1/16}_{L^2(\Omega)} ||\phi||^{15/16}_{L^6(\Omega)} + C_{10} ||\phi||_{L^6(\Omega)},$$

(2.11)
$$||\nabla\phi||_{L^4(\Omega)} \leq C_{11} ||D^3\phi||^{3/8}_{L^2(\Omega)} ||\phi||^{5/8}_{L^6(\Omega)} + C_{12} ||\phi||_{L^6(\Omega)},$$

where $C_i = C_i(\Omega)$. It is useful to note in particular that (2.7) and (2.9) imply that constants C_{13} and C_{14} exist such that

$$||\phi||_{L^4(\Omega)} \leq C_{13} ||\phi||_{H^1(\Omega)}, \tag{2.12}$$

and

$$||\phi||_{L^6(\Omega)} \leq C_{14} ||\phi||_{H^1(\Omega)}. \tag{2.13}$$

In the sequel, C_i shall denote constants which depend on Ω, and possibly on ξ, l, the history, and the initial conditions.

Below we prove the lemma mentioned in the introduction.

Lemma 2.3. *If $a_1, a_2 \in L^1(R^+)$, $u_h \in L^1(-\infty, 0;\, H^2(\Omega))$, and $\phi_h \in L^1(-\infty, 0;\, H^3(\Omega)) \cap L^5(-\infty, 0;\, L^6(\Omega))$, then f_1, f_2 as defined in* (1.6) *and* (1.7) *satisfy $f_1 \in L^1(-\infty, 0;\, L^2(\Omega))$ and $f_2 \in L^1(-\infty, 0; H^1(\Omega))$.*

Proof: It follows from (1.6) that

$$||f_1||_{L^1(-\infty,0;\, L^2(\Omega))} \le \int_0^T \int_{-\infty}^0 |a_1(t-s)|\, ||\triangle u_h(x,s)||_{L^2(\Omega)}\, ds\, dt.$$

Interchanging the order of integration and recalling assumption (**H1**)

$$||f_1||_{L^1(-\infty,0;\, L^2(\Omega))} \le ||a_1||_{L^1(R+)} ||\triangle u_h||_{L^1(-\infty,0;\, L^2(\Omega))}.$$

Thus if $u_h \in L^1(-\infty, 0;\, H^2(\Omega))$, then $f_1 \in L^1(-\infty, 0;\, L^2(\Omega))$.

With regard to f_2, we have similarly that

$$\begin{aligned} ||f_2||_{L^1(-\infty,0;\, H^1(\Omega))} &\le ||a_2||_{L^1(R+)} ||\xi^2 \triangle \phi_h^3 + \phi_h - \phi_h^3 \\ &\qquad + u_h||_{L^1(-\infty,0;\, H^1(\Omega))} \\ &\le ||a_2||_{L^1(R+)} \Big\{ \xi^2 ||\triangle \phi_h||_{L^1(-\infty,0;\, H^1(\Omega))} \\ &\qquad + ||\phi_h||_{L^1(-\infty,0;\, H^1(\Omega))} \\ &\quad + ||\phi_h^3||_{L^1(-\infty,0;\, H^1(\Omega))} + ||u_h||_{L^1(-\infty,0;\, H^1(\Omega))} \Big\}. \end{aligned}$$

Noting that

$$\begin{aligned} &\int_{-\infty}^0 \left[\int_\Omega |\nabla \phi^3|^2 + |\phi^3|^2 \right]^{1/2} dt \\ &\qquad \le C_{15} \int_{-\infty}^0 \Big\{ ||\phi||^3_{L^6(\Omega)} + ||\phi||^2_{L^8(\Omega)} ||\nabla \phi||_{L^4(\Omega)} \Big\}\, dt, \end{aligned}$$

and using the interpolation inequalities (2.10), (2.11), one finds that

$$\begin{aligned} ||f_2||_{L^1(-\infty,0;\, H^1(\Omega))} &\le C_{16}\, ||a_2||_{L^1(R+)} \Big\{ ||u_h||_{L^1(-\infty,0;\, H^1(\Omega))} \\ &\quad + ||\phi_h||_{L^1(-\infty,0;\, H^3(\Omega))} + ||\phi_h||_{L^5(-\infty,0;\, L^6(\Omega))} \Big\}. \end{aligned}$$

Thus the assumptions on u_h and ϕ_h guarantee that $f_2 \in L^1(-\infty, 0; H^1(\Omega))$, as claimed. $\square$

3. Existence. We now turn to prove our main theorem:

Theorem 1. *Suppose that* $\{u_0, \phi_0\} \in L^2(\Omega) \times H^1(\Omega)$ *and* $\{f_1, f_2\} \in L^1(0, T; L^2(\Omega) \times H^1(\Omega))$*, then there exists a global solution to (2.1)–(2.4) in the sense of Definition* 2.

Definition 2. We shall say that $\{u, \phi\}$ constitutes a solution to (2.1)–(2.4) on the interval $[0, T]$, $0 < T < \infty$, if

$$\{u, \phi\} \in \mathcal{C}([0, T];\, L^2(\Omega) \times H^1(\Omega))$$

$$\{u_t, \phi_t\} \in L^\infty(0, T;\, H^{-2}(\Omega) \times H^{-1}(\Omega)) + L^1(0, T;\, L^2(\Omega) \times H^1(\Omega)),$$

$\{u, \phi\}$ satisfy the initial conditions (2.4), and

$$\int_0^T \int_\Omega y(x,t) \left[u_t + \frac{l}{2}\phi_t - f_1 \right](x,t)\, dx\, dt \\ - \int_0^T \int_\Omega \triangle y(x,t)\, (a_1 * u)(x,t)\, dx\, dt = 0,$$

$$\int_0^T \int_\Omega z(x,t)\, [\phi_t - a_2 * (\phi - \phi^3 + u) - f_2](x,t)\, dx\, dt \\ + \int_0^T \int_\Omega \nabla z(x,t) \cdot a_2 * \nabla \phi(x,t)\, dx\, dt = 0,$$

for any $y \in L^1(0, T;\, H^2(\Omega)) \cap L^\infty(0, T;\, L^2(\Omega))$ and $z \in L^1(0, T;\, H^1(\Omega)) \cap L^\infty(0, T;\, H^{-1}(\Omega))$.

Proof: Our method of proof of existence relies on a Galerkin approximation based on the eigenfunctions of the linear operator $\mathcal{A} : L^2(\Omega) \to H^{-2}(\Omega)$,

$$\mathcal{A}\Psi = -\Delta\Psi, \quad x \in \Omega, \quad \mathbf{n} \cdot \nabla\psi = 0, \quad x \in \partial\Omega. \tag{3.1}$$

Let $\{\Psi_i\}$ denote an $L^2(\Omega)$-orthonormal sequence of eigenfunctions of the linear operator $\mathcal{A}$ which are ordered sequentially so that the associated eigenvalues λ_i satisfy

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots ,$$

and note that $\Psi_0 = |\Omega|^{-\frac{1}{2}}$.

We shall now seek approximations for u and ϕ based on the ordered sequence of eigenfunctions $\{\Psi_i\}$. More specifically, we shall seek approximations of the form:

$$u_N(x,t) := \sum_{i=0}^{N} c_{Ni}(t)\Psi_i(x) \tag{3.2}$$

$$\phi_N(x,t) := \sum_{i=0}^{N} d_{Ni}(t)\Psi_i(x). \tag{3.3}$$

Let $Sp(N) := \operatorname{span}\{\Psi_0, \dots , \Psi_N\}$. We denote by $P^i : L^2(\Omega) \to Sp(N)$ the projection of $L^2(\Omega)$ onto Ψ_i and by $P_N : L^2(\Omega) \to Sp(N)$ the projection of $L^2(\Omega)$ onto the span of the first $N+1$ modes, i.e., $P_N = \sum_{i=0}^{N} P^i$.

The functions $\{u_N, \phi_N\}$ shall constitute an approximation to a solution $\{u, \phi\}$ of (PFM) in that they shall be required to satisfy

$$\int_0^T b_{1i}\, \langle \Psi_i, u_{Nt} + \frac{l}{2}\phi_{N_t} - a_1 * \Delta u_N - P_N f_1\rangle \, dt = 0 \tag{3.4}$$

$$\int_0^T b_{2i}\, \langle \Psi_i, \phi_{N_t} - a_2 * [\xi^2 \Delta\phi_N + \phi_N - P_N(\phi_N)^3 + u_N] - P_N f_2\rangle \, dt = 0 \tag{3.5}$$

$$\langle \Psi_i, u_N\rangle = P^i u_0, \quad \langle \Psi_i, \phi_N\rangle = P^i \phi_0, \tag{3.6}$$

for $i = 0, 1, \dots , N$ and for all $b_{1i}, b_{2i} \in L^\infty(0,T)$. Equations (3.4)–(3.6) imply that $\{c_{Ni}, d_{Ni}\}$ satisfy

$$c_{Ni_t} + \frac{l}{2}\, d_{Ni_t} = -\lambda_i \xi^2 a_1 * c_{Ni} + P^i f_1 \tag{3.7}$$

$$d_{Ni_t} = -\lambda_i \xi^2 a_2 * d_{Ni} + a_2 * (c_{Ni} + d_{Ni}) + g_{Ni}(d_{N0}, \dots , d_{NN}) + P^i f_2 \tag{3.8}$$

$$c_{Ni}(0) = P^i u_0, \quad d_{Ni}(0) = P^i \phi_0, \tag{3.9}$$

for $i = 0, 1, \dots, N$, in the $L^1(0,T)$ sense, where g_{Ni} denotes a nonlinear term which can be written explicitly as:

$$g_{Ni} := g_{Ni}(d_{N0}, d_{N1}, \dots, d_{NN}) = -\langle \Psi_i, a_2 * P_N(\phi_N)^3 \rangle.$$

We now consider existence, uniqueness and regularity of solutions to (3.7)–(3.9).

Lemma 3.1. *For $(u_0, \phi_0) \in L^2(\Omega) \times L^2(\Omega))$ and $f_1, f_2 \in L^1(0, \infty; L^2(\Omega))$, there exists a unique solution $\{c_{Ni}, d_{Ni}\} \in (\mathbf{C}[0, T_N))^{2(N+1)}$ to the system (3.7)–(3.9) for $N = 0, 1, \dots,$ where the interval $(0, T_N)$ is maximal, i.e., either $T_N = \infty$ or else the solution becomes unbounded as $t \uparrow T_N$. Moreover, the solution is differentiable with values in $L^1(0, T_N)$.*

Proof: We first show that there exists a unique solution

$$\{c_{Ni}, d_{Ni}\} \in (\mathbf{C}[0, T_N))^{2(N+1)}$$

to (3.7)–(3.9) for $N = 0, 1, 2, \dots,$ where $T_N > 0$ is maximal and afterwards we return to demonstrate the desired differentiability properties.

Let us substitute (3.8) into (3.7). Integrating the resultant equations over the interval $(0, t)$ for $t > 0$, and using the initial conditions (3.9) yields

$$\begin{aligned} c_{Ni}(t) &= 1 * a_1 * (-\lambda_i \xi^2 c_{Ni}) + 1 * P^i f_1 \\ &\quad - \frac{1}{2}\Big[1 * a_2 * (-\lambda_i \xi^2 + 1)\, d_{Ni} + 1 * a_2 * c_{Ni} \\ &\quad + 1 * g_{Ni} + 1 * P^i f_2\Big] + c_{Ni}(0) + \frac{l}{2}\, d_{Ni}(0), \end{aligned} \tag{3.10}$$

$$\begin{aligned} d_{Ni}(t) &= 1 * a_2 * (-\lambda_i \xi^2 + 1)\, d_{Ni} + 1 * a_2 * c_{Ni} \\ &\quad + 1 * g_{Ni} + 1 * P^i f_2 + d_{Ni}(0). \end{aligned} \tag{3.11}$$

We shall now see that all terms on the right hand side of the system (3.10)–(3.11) which do not depend on the history or the initial condition are the form

$$1 * a_j * h(c_{N0}, \dots, c_{NN}, d_{N0}, \dots, d_{NN}), \quad j = 1 \text{ or } 2, \tag{3.12}$$

where h is a continuously differentiable function of its variables. For the terms which are of the form $const. \cdot 1 * a_j * c_{Ni}$ or $const. \cdot 1 * a_j * d_{Ni}$, this is obvious. That terms proportional to $1 * g_{Ni}$ are of this form can be readily seen by noting that

$$g_{Ni} = -\langle \Psi_i, a_2 * \sum_{j=0}^{N} \langle \Psi_j, (\phi_N)^3 \rangle \Psi_j \rangle = -a_2 * \langle \Psi_i, (\phi_N)^3 \rangle,$$

for $i = 0, \dots, N$ and by recalling the definition of ϕ_N.

Consider now an arbitrary term of the form (3.12). By formally exchanging the order of integration

$$1 * a_i * h = \int_0^t \alpha_i(t-s) h(c_{N0}(s), \dots, c_{NN}(s), d_{N0}(s), \dots, d_{NN}(s))\, ds \tag{3.13}$$

where $\alpha_i(t) = \int_0^t a_i(\tau)\, d\tau$. Thus, $1 * a_i * h$ may be written as

$$1 * a_i * h = \int_0^t b(t, s, c_{N0}(s), \dots, c_{NN}(s), d_{N0}(s), \dots, d_{NN}(s))\, ds, \tag{3.14}$$

where $b = b(t, s, c_{N0}, \dots, c_{NN}, d_{N0}, \dots, d_{NN})$ is a continuous function of its arguments.

Note that terms which depend on the history are of the form

$$const. \cdot 1 * P^i f_j, \quad j = 1 \text{ or } 2. \tag{3.15}$$

Since by assumption $f_1, f_2 \in L^1(0, \infty; L^2(\Omega))$, these terms constitute continuous functions of t. Thus, having formally exchanged the order of integration, (3.10)–(3.11) can be expressed as

$$x(t) = \int_0^t \tilde{b}(t, s, x(s))\, ds + \tilde{f}(t) + x(0), \tag{3.16}$$

where

$$x = (c_{N0}, \dots, c_{NN}, d_{N0}, \dots, d_{NN}),$$

$\tilde{b}, \tilde{f} \in R^{2(N+1)}$ and $\tilde{b} = \tilde{b}(t, s, x)$ and $\tilde{f} = \tilde{f}(t)$ depend continuously on their arguments. Thus written, standard theorems on Volterra integral equations of the second kind can be invoked. In particular, by [**24**, Theorems 1.1 and 2.2], there exists a continuous solution $x(t)$ to (3.16) on a maximal interval. The continuity of the solution to (3.16) now allows us to re-exchange the order of integration, yielding a continuous solution $\{c_{Ni}, d_{Ni}\}_{i=0,\ldots,N}$ to (3.10)–(3.11).

With regard to uniqueness, note by (3.13) that (3.14) may be written more specifically as

$$x(t) = \alpha_1 * \tilde{h}_1(x) + \alpha_2 * \tilde{h}_2(x) + \tilde{f}(t) + x(0), \tag{3.17}$$

where $\alpha_j(t), \tilde{h}_j(x)$, and $\tilde{f}(t)$ are continuous functions of their arguments. This implies, see, e.g., [**24**, Theorem 2.3], that the solution to (3.16) is unique. This can readily be seen to imply in turn that $\{c_{Ni}, d_{Ni}\}_{i=0,\ldots,N}$ constitute a unique continuous solution to (3.10)–(3.11).

With regard to the differentiability of $\{c_{Ni}, d_{Ni}\}_{i=0,\ldots,N}$, we proceed as follows. Formally differentiating either of the integral terms in (3.17) with respect to time and recalling that $\alpha_j = 1 * a_j$ yields a term of the form

$$\int_0^t a_j(t-s)\tilde{h}_j(x(s))\, ds,$$

where $\tilde{h}_j(x)$ is continuous and $j = 1$ or 2. Upon changing variables, this can be written as

$$\int_0^t a_j(q)\tilde{h}_j(x(t-q))\, dq. \tag{3.18}$$

Since $a_j \in L^1(R^+)$ and $\tilde{h}_j$ is continuous with respect to the variable x whose components $c_{N0}, \ldots, c_{NN}, d_{N0}, \ldots, d_{NN}$ are continuous with respect to t, such terms are clearly continuous with respect to time.

To verify the differentiability of the history contribution to (3.17), $\tilde{f}(t)$, we note that differentiating terms of the form (3.15) yields terms of the form $const.P^i f_j$ where $P^i f_j$ can be expressed as

$$\langle \Psi_i, f_j(t)\rangle. \tag{3.19}$$

Since, by assumption, $f_1, f_2 \in L^1(0,\infty; L^2(\Omega))$, by Cauchy-Schwartz

$$\begin{aligned}\int_0^{T_N} |\langle \Psi_i, f_j(t)\rangle|\, dt &\le \|\Psi_i\|_{L^2(\Omega)} \cdot \|f_j\|^2_{L^1(-\infty,0;L^2(\Omega))} \\ &= \|f_j\|^2_{L^1(-\infty,0;L^2(\Omega))}.\end{aligned} \tag{3.20}$$

Hence terms of the form (3.19) belong to $L^1(0,T_N)$. Combining this result with the continuity of the terms of the form (3.18), we find that $c_{Ni_t}, d_{Ni_t} \in L^1(0,T_N)$, for $i = 0,1,\dots,N$.

Returning to the original form of the problem (3.10)–(3.11), by differentiating and subtracting, the claim of the lemma is proven. ◻

Lemma 3.2. *If* $u_0(x) \in L^2(\Omega)$, $\phi_0 \in H^1(\Omega)$, $f_1 \in L^1(0,\infty;L^2(\Omega))$, *and* $f_2 \in L^1(0,\infty;H^1(\Omega))$, *then* $T_N = \infty$ *for any* $N = 0,1,2,\dots$, *where* T_N *denotes the maximal interval of existence of the solution* $\{c_{Ni}, d_{Ni}\}_{i=0,\dots,N}$ *of* (3.7)–(3.9).

Proof: By Lemma 3.1, for any $N = 0,1,\dots$, there exists a unique continuous solution $\{c_{Ni}, d_{Ni}\}_{i=0,\dots,N}$ to (3.7)–(3.9) with time derivatives in L^1 on the interval $[0,T_N)$. By (3.2)–(3.3), this implies that there exists a unique set of approximants $\{u_N(x,t), \phi_N(x,t)\}$ for $N = 0,1,2,\dots$, which are arbitrarily smooth in space and possess time derivatives in L^1 on the interval $[0,T_N)$ and satisfy equations (3.4)–(3.6) on the interval $[0,T_N)$. The proof of Lemma 3.2 is based on obtaining an a priori estimate which is uniform in N and T. The a priori estimate is derived below as Lemma 3.3.

Lemma 3.3. *If* $\{c_{Ni}, d_{Ni}\}_{i=0,\dots,N}$ *denotes the solution of* (3.7)–(3.9) *with initial conditions and history as prescribed in Lemma* 3.2, *then*

$$\sum_{i=0}^{N} |c_{Ni}(T)|^2 \le C \quad \textit{and} \quad \sum_{i=0}^{N} |d_{Ni}(T)|^2 \le C, \tag{3.21}$$

for any $0 < T < T_N$, *where* C *depends on the initial conditions, the history,* Ω, *and the parameters* l *and* ξ, *but is independent of* N *and* T.

Proof: Let us now multiply (3.7) by $(2/l)c_{Ni}$ and (3.8) by $\lambda_i \xi^2 d_{Ni} - d_{Ni} + \langle \Psi_i, P_N(\phi_N^3)\rangle - c_{Ni}$. Summing over i, $i = 0,1,\dots,N$, recalling

(3.2)–(3.3), and adding together the two resultant expressions yields

(3.22)
$$\Big\langle -[\xi^2\Delta\phi_N+\phi_N-P_N(\phi_N)^3],\phi_{N_t}\Big\rangle+\Big\langle\frac{2}{l}u_N,u_{N_t}\Big\rangle$$
$$=\frac{2}{l}\Big\langle u_N,\int_0^t a_1(t-s)\Delta u_N(s)\,ds+f_1\Big\rangle$$
$$-\Big\langle[\xi^2\Delta\phi_N+\phi_N-P_N(\phi_N)^3+u_N],$$
$$\int_0^t a_2(t-s)[\xi^2\Delta\phi_N+\phi_N-P_N(\phi_N)^3+u_N](s)\,ds+f_2\Big\rangle.$$

Since $\phi_{N_t}\in Sp(N)$ with coefficients in $L^1(0,T)$,

$$\langle P_N(\phi_N)^3,\phi_{N_t}\rangle=\langle(\phi_N)^3,\phi_{N_t}\rangle.$$

Integrating both sides of (3.22) between 0 and T, for $0<T<T_N$, and invoking Lemma A.1, see Appendix A, we obtain

$$\int_\Omega\left[\frac{\xi^2}{2}|\nabla\phi_N|^2-\frac{1}{2}|\phi_N|^2+\frac{1}{4}\phi_N^4+\frac{1}{l}u_N^2\right]\Big|_0^T$$
$$=-\frac{2}{l}\int_0^T\langle u_N(t),\int_0^t a_1(t-s)\Delta u_N(s)\,ds+f_1\rangle\,dt$$
$$-\int_0^T\langle[\xi^2\Delta\phi_N+\phi_N-P_N(\phi_N)^3+u_N](t),$$
$$\int_0^t a_2(t-s)[\xi^2\Delta\phi_N+\phi_N-P_N(\phi_N)^3+u_N](s)\,ds+f_2\rangle\,dt.$$

By construction u_N is smooth and satisfies the Neumann boundary condition (2.3). Splitting the integrals on the right hand side of the above expression into contributions which depend on $u_N(t),\phi_N(t)$, for $0\le t\le T$, and into contributions which depend on the history as well, then integrating by parts the term on the right hand side which depends on u_N only, we obtain

$$\int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 - \frac{1}{2}|\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](T) \tag{3.23}$$
$$= \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 - \frac{1}{2}\phi_N^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](0)$$
$$- \frac{2}{l}\int_0^T \langle\psi_1(s), \int_0^t a_1(t-s)\psi_1(s)\,ds\rangle\,dt$$
$$- \int_0^T \langle\psi_2(s), \int_0^t a_2(t-s)\psi_2(s)\,ds\rangle\,dt$$
$$+ \frac{2}{l}\int_0^T \langle u_N, f_1(x,t)\rangle\,dt - \int_0^T \langle\psi_2(s), f_2(x,t)\rangle\,dt,$$

where

$$\psi_1 = \nabla u_N \quad \text{and} \quad \psi_2 = \xi^2\Delta\phi_N + \phi_N - P_N(\phi_N)^3 + u_N. \tag{3.24}$$

We now estimate the various terms in (3.23). Let us first treat the right hand side of (3.23). Note that the first term on the right hand side of (3.23) is bound by

$$C_0 = \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_0|^2 + \frac{1}{4}\phi_0^4 + \frac{1}{l}u_0^2\right].$$

The second and third terms on the right hand side are nonpositive by virtue of the assumption that a_1 and a_2 are kernels of positive type. In order to estimate the last two terms on the right hand side, we proceed as follows. We write the first of these latter terms as

$$\frac{2}{l}\int_0^T \langle u_N, f_1\rangle\,dt \le \frac{2}{l}\int_0^T \|u_N\|_{L^2(\Omega)}\|f_1\|_{L^2(\Omega)}\,dt \tag{3.25}$$
$$\le \int_0^T \left\{\frac{1}{l}\|u_N\|^2_{L^2(\Omega)} + \frac{1}{l}\right\}$$
$$\times \{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\,dt$$
$$= \int_0^T \frac{1}{l}\|u_N\|^2_{L^2(\Omega)}\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\,dt$$
$$+ \frac{1}{l}\{\|f_1\|_{L^1(0,T;L^2(\Omega))} + \|f_2\|_{L^1(0,T;H^1(\Omega))}\}.$$

The treatment of the very last term is slightly technical. We break up this term into four parts by writing:

$$
\begin{aligned}
&-\int_0^T \langle \xi^2\Delta\phi_N + \phi_N - P_N\phi_N^3 + u_N, f_2\rangle\, dt \\
&= -\int_0^T \langle \xi^2\Delta\phi_N, f_2\rangle\, dt + \int_0^T \langle \phi_N, f_2\rangle\, dt - \int_0^T \langle P_N\phi_N^3, f_2\rangle\, dt \\
&\qquad\qquad + \int_0^T \langle u_N, f_2\rangle\, dt,
\end{aligned}
$$

and we estimate each of the parts separately. With regard to the first part, we recall that ϕ_N is smooth and satisfies the Neumann boundary condition (2.3) by construction. Hence, we may integrate by parts and use the Cauchy-Schwartz inequality to obtain

$$
\begin{aligned}
\int_0^T \langle \xi^2\Delta\phi_N, f_2\rangle\, dt &\le \int_0^T \xi^2 \|\nabla\phi_N\|_{L^2(\Omega)} \|\nabla f_2\|_{L^2(\Omega)}\, dt \\
&\le \int_0^T \frac{\xi^2}{2}\{\|\nabla\phi_N\|^2_{L^2(\Omega)} + 1\} \\
&\quad \times \{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\, dt \\
&= \int_0^T \frac{\xi^2}{2}\|\nabla\phi_N\|^2_{L^2(\Omega)}\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\, dt \\
&\quad + \frac{\xi^2}{2}\{\|f_1\|_{L^1(0,T;L^2(\Omega))} + \|f_2\|_{L^1(0,T;H^1(\Omega))}\}.
\end{aligned}
\tag{3.26}
$$

Similarly,

$$
\begin{aligned}
&\int_0^T \langle \phi_N, f_2\rangle\, dt \\
&\quad \le \int_0^T \|\phi_N\|_{L^2(\Omega)} \|f_2\|_{L^2(\Omega)}\, dt \\
&\quad \le |\Omega|^{1/4}\int_0^T \|\phi_N\|_{L^4(\Omega)} \|f_2\|_{L^2(\Omega)}\, dt \\
&\quad \le \frac{|\Omega|^{1/4}}{4}\int_0^T \|\phi_N\|^4_{L^4(\Omega)}\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\, dt \\
&\qquad + \frac{3|\Omega|^{1/4}}{4}\{\|f_1\|_{L^1(0,T;L^2(\Omega))} + \|f_2\|_{L^1(0,T;H^1(\Omega))}\},
\end{aligned}
\tag{3.27}
$$

and

$$\begin{aligned}(3.28)\qquad \int_0^T \langle u_N, f_1\rangle\, dt \\ &\leq \int_0^T \frac{1}{2}\|u_N\|^2_{L^2(\Omega)}\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\, dt \\ &\quad + \frac{1}{2}\{\|f_1\|_{L^1(0,T;L^2(\Omega))} + \|f_2\|_{L^1(0,T;H^1(\Omega))}\}.\end{aligned}$$

To estimate the remaining term, using Hölder's inequality and Jensen's inequality, we write

$$\begin{aligned}\int_0^T \langle P_N\phi_N^3, f_2\rangle\, dt &\leq \int_0^T \|\phi_N\|^3_{L^4(\Omega)}\|f_2\|_{L^4(\Omega)}\, dt \\ &\leq \frac{3}{4}\int_0^T \|\phi_N\|^4_{L^4(\Omega)}\|f_2\|_{L^4(\Omega)}\, dt \\ &\quad + \frac{1}{4}\|f_2\|_{L^1(0,T;L^4(\Omega))}.\end{aligned}$$

The embedding inequality (2.12) allows us to conclude that

$$\begin{aligned}(3.29)\qquad \int_0^T \langle P_N\phi_N^3, f_2\rangle\, dt \\ &\leq \frac{3}{4}C_{13}\int_0^T \|\phi_N\|^4_{L^4(\Omega)}\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\, dt \\ &\quad + \frac{1}{4}C_{13}\{\|f_1\|_{L^1(0,T;L^2(\Omega))} + \|f_2\|_{L^1(0,T;H^1(\Omega))}\}.\end{aligned}$$

Combining the estimates (3.25)–(3.29), we obtain

$$\begin{aligned}(3.30)\qquad \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 - \frac{1}{2}\phi_N^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](T) \\ &\leq \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_0|^2 + \frac{1}{4}\phi_0^4 + \frac{1}{l}u_0^2\right] \\ &\quad + C_{17}\int_0^T \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right] \\ &\quad \times \left\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\right\} dt \\ &\quad + C_{18}\left\{\|f_1\|_{L^1(0,T;L^2(\Omega))} + \|f_2\|_{L^1(0,T;H^1(\Omega))}\right\},\end{aligned}$$

where C_{17} and C_{18} depend on l, ξ and Ω only. Finally noting that, by Young's inequality,

$$-\int_\Omega \frac{1}{8}\phi_N^4 - \frac{1}{2}|\Omega| \le \int_\Omega -\frac{1}{2}\phi_N^2, \tag{3.31}$$

and recalling the assumed L^1-integrability of f_1 and f_2 in $L^2(\Omega)$ and $H^2(\Omega)$ respectively, (3.30) can be written as

$$\begin{aligned}
&\int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{8}\phi_N^4 + \frac{1}{l}u_N^2\right](T) \\
&\quad \le C_{19} + C_{20}\int_0^T \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{8}\phi_N^4 + \frac{1}{l}u_N^2\right] \\
&\qquad\qquad \times \left\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\right\} dt,
\end{aligned}$$

where C_{19} depends on the initial conditions, the history, as well as on l, ξ and Ω, and C_{20} depends on l, ξ and Ω only. By Gronwall's inequality, we obtain now that

$$\int_\Omega \left[\frac{1}{2}|\nabla\phi_N|^2 + \frac{1}{8}\phi_N^4 + \frac{1}{l}u_N^2\right](T) \le C_{21}, \tag{3.32}$$

where C_{21} is independent of N and T. It follows from (3.32) that

$$\|u_N\|_{L^\infty(0,T;L^2(\Omega))} \le (lC_{21})^{1/2} \quad \text{and} \quad \|\phi_N\|_{L^\infty(0,T;L^4(\Omega))} < (8C_{21})^{1/4}.$$

This implies in turn that

$$\|u_N\|_{L^\infty(0,T;L^2(\Omega))} < (lC_{21})^{1/2} \quad \text{and} \quad \|\phi_N\|_{L^\infty(0,T;L^2(\Omega))} < (8\Omega C_{21})^{1/4},$$

and therefore by the assumed form of u_N and ϕ_N and the orthonormality of the eigenfunctions $\{\Psi_i\}$,

$$\sum_{i=0}^N |c_{Ni}(T)|^2 \le C \quad \text{and} \quad \sum_{i=0}^N |d_{Ni}(T)|^2 \le C,$$

where C is independent of N and T. This completes the proof of Lemma 3.3. $\square$

Since by (3.21) the functions c_{Ni}, d_{Ni} are uniformly bounded in time, this implies that $T_N = \infty$, for $N = 0, 1, 2, \ldots$, and completes the proof of Lemma 3.2. $\square$

To guarantee the existence of a solution to (PFM) in the sense indicated in Definition 2, we must ascertain convergence of a subsequence of the approximants $\{u_N, \phi_N\}$ in an appropriate sense. For this purpose, we state

Lemma 3.4. *There exists a constant $\tilde{C}$ which may depend on the initial conditions, the history l, ξ and Ω but is independent of N and T, such that*

$$\|u_N\|_{L^\infty(0,T;L^2(\Omega))} \le \tilde{C}, \tag{3.33}$$

$$\|\phi_N\|_{L^\infty(0,T;H^1(\Omega))} \le \tilde{C}, \tag{3.34}$$

$$\|u_{N_t} - f_{1_N} + f_{2_N}\|_{L^\infty(0,T;H^{-2}(\Omega))} \le \tilde{C}, \tag{3.35}$$

$$\|u_{N_t}\|_{L^\infty(0,T;H^{-2}(\Omega))+L^1(0,T;L^2(\Omega))} \le \tilde{C}, \tag{3.36}$$

$$\|\phi_{N_t} - f_{2_N}\|_{L^\infty(0,T;H^{-1}(\Omega))} \le \tilde{C}, \tag{3.37}$$

$$\|\phi_{N_t}\|_{L^\infty(0,T;H^{-1}(\Omega))+L^1(0,T;H^1(\Omega))} \le \tilde{C}, \tag{3.38}$$

$$\|a_1 * u_N\|_{L^\infty(0,T;L^2(\Omega))} \le \tilde{C}, \tag{3.39}$$

$$\|a_2 * \phi_N\|_{L^\infty(0,T;H^1(\Omega))} \le \tilde{C}, \tag{3.40}$$

and

$$\|a_2 * \phi_N^3\|_{L^\infty(0,T;L^2(\Omega))} \le \tilde{C}. \tag{3.41}$$

Proof: Let $\tilde{C}$ denote initially a generic constant whose value may change from line to line; afterwards it can be taken to denote the maximum among the bounds obtained in this fashion. The a priori estimate (3.32) obtained earlier implies (3.33),

$$\|\phi_N\|_{L^\infty(0,T;L^4(\Omega))} \le \tilde{C}, \tag{3.42}$$

and

$$\|\nabla\phi_N\|_{L^\infty(0,T;L^2(\Omega))} \le \tilde{C}. \tag{3.43}$$

Recalling (3.31), (3.34) is obtained from (3.42) and (3.43). Note also that (3.34) together with the embedding inequality (2.13) imply that

$$\|\phi_N^3\|_{L^\infty(0,T;L^2(\Omega))} \le \tilde{C}. \tag{3.44}$$

To obtain the estimates on u_{N_t} and ϕ_{N_t}, we refer back to the equations (3.4) and (3.5) which are satisfied by ϕ_{N_t} and u_{N_t}. With regard to ϕ_{N_t}, (3.37) and (3.38) follow from (3.5), using (2.6), (H1), the estimates (3.33), (3.34) and (3.44) and our assumptions on the history. Turning now to (3.4), we see that the assumptions on the history and on a_1 together with (2.6) and the estimates (3.33), (3.37) and (3.38) yield (3.35)–(3.36). The estimates (3.39) and (3.40) follow from (2.6) and (3.33), (3.34). Finally, (3.41) follows from (3.44) and (2.6). ◻

From Lemma 3.4, it readily follows that

Lemma 3.5. *For any $T > 0$, there exist functions u, ϕ, χ_0, χ_1 and χ_2 and a subsequence $\{u_{N'}, \phi_{N'}\}$, denoted for simplicity again as $\{u_N, \phi_N\}$ such that the following convergences hold*

$$u_N \overset{*}{\rightharpoonup} u \ \ in L^\infty(0,T;L^2(\Omega)), \tag{3.45}$$

$$\phi_N \overset{*}{\rightharpoonup} \phi \ \ in L^\infty(0,T;H^1(\Omega)), \tag{3.46}$$

$$u_{N_t} - f_{1_N} + f_{2_N} \overset{*}{\rightharpoonup} u_t - f_1 + f_2 in\ L^\infty(0,T;H^{-2}(\Omega)), \tag{3.47}$$

$$\phi_{N_t} - f_{2_N} \overset{*}{\rightharpoonup} \phi_t - f_2 in\ L^\infty(0,T;H^{-1}(\Omega)), \tag{3.48}$$

$$a_1 * u_N \overset{*}{\rightharpoonup} \chi_0 \ \ in L^\infty(0,T;L^2(\Omega)), \tag{3.49}$$

$$a_2 * \phi_N \overset{*}{\rightharpoonup} \chi_1 \ \ in L^\infty(0,T;H^1(\Omega)), \tag{3.50}$$

$$a_2 * \phi_N^3 \overset{*}{\rightharpoonup} \chi_2 \ \ in L^\infty(0,T;L^2(\Omega)). \tag{3.51}$$

Moreover,

$$f_{1_N} \to f_1 \ \ in\ L^1(0,T;L^2(\Omega)), \tag{3.52}$$

$$f_{2_N} \to f_2 \ \ in\ L^1(0,T;H^1(\Omega)), \tag{3.53}$$

$$u_{N_t} \rightharpoonup u_t \ \ in\ L^1(0,T;H^{-2}(\Omega)), \tag{3.54}$$

$$\phi_{N_t} \rightharpoonup \phi_t \ \ in\ L^1(0,T;H^{-1}(\Omega)), \tag{3.55}$$

and

$$u_N \to u \text{ in } L^p(0,T;H^s(\Omega)), \quad 1 \le p < \infty,\ -1 \le s < 0, \tag{3.56}$$

$$\phi_N \to \phi \text{ in } L^p(0,T;H^s(\Omega)), \quad 1 \le p < \infty,\ 0 \le s < 1. \tag{3.57}$$

Proof: The weak-star convergences indicated in (3.45), (3.46), (3.49)–(3.51), (3.54)–(3.55) follow from the uniform estimates given in Lemma 3.4. The convergences in (3.52)–(3.53) are true by construction, and together with the estimates (3.35)– (3.37), yield (3.54)–(3.55). Finally the strong convergence stated in (3.56)–(3.57) follows from the compactness results of Simon [**31**, Corollary 4]. □

From Lemma 3.5 and from the equations satisfied by u_N and ϕ_N, it follows that

$$0 = \int_0^T \langle y, u_t + \frac{l}{2}\phi_t - \Delta\chi_0 - f_1\rangle_1 \, dt, \tag{3.58}$$

and

$$0 = \int_0^T \langle z, \phi_t - \chi_1 - \chi_0 + \chi_2 - \Delta\chi_1 - f_2\rangle_2 \, dt \tag{3.59}$$

for all $y \in \tilde{X}_1$ and $z \in \tilde{X}_2$ where $\langle\cdot,\cdot\rangle_i$ indicates the bilinear functional from $\tilde{X}_i \times X_i$ to R defined by

$$\langle f, g\rangle_i = \int_\Omega f(x)g(x)\, dx, \quad \forall f \in \tilde{X}_i, g \in X_i, \quad i = 1 \text{ or } 2,$$

where

$$\tilde{X}_1 = L^1(0,\infty;H^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$
$$X_1 = L^\infty(0,\infty;H^{-2}(\Omega)) + L^1(0,T;L^2(\Omega)),$$

and

$$\tilde{X}_2 = L^1(0,\infty;H^1(\Omega)) \cap L^\infty(0,T;H^{-1}(\Omega)),$$
$$X_2 = L^\infty(0,\infty;H^{-1}(\Omega)) + L^1(0,T;H^1(\Omega)).$$

From (3.58), (3.59) and Lemma 1.1 in Temam [**32**, Chapter 3], we conclude

Lemma 3.6. *The functions* u *and* ϕ *given in Lemma* 3.5 *satisfy*

$$u \in \mathbf{C}([0,T]; H^{-2}), \tag{3.60}$$

and

$$\phi \in \mathbf{C}([0,T]; H^{-1}). \tag{3.61}$$

The results obtained in Lemma 3.6 fall short of guaranteeing continuity from initial data in $L^2(\Omega) \times H^1(\Omega)$ as claimed in Theorem 1. Also, to guarantee the existence of a solution in the sense of Definition 2, it is necessary to be able to identify the limiting functions χ_0, χ_1 and χ_2. We address this latter point first.

Lemma 3.7. *For almost every* $(x,t) \in \Omega \times (0,T)$,

$$\chi_0 = a_1 * u, \tag{3.62}$$

$$\chi_1 = a_2 * \phi, \tag{3.63}$$

$$\chi_2 = a_2 * \phi^3. \tag{3.64}$$

Proof: We know by Lemmas 2.2 and 3.5 that χ_0 and $a_1 * u$ belong to $L^\infty(0,T; H^{-1}(\Omega))$ and that χ_1 and $a_2 * \phi$ belong to $L^\infty(0,T; L^2(\Omega))$. From (2.6), (3.56)–(3.57) and weak lower semi-continuity of norms, it follows therefore that

$$\begin{aligned}
&\|\chi_0 - a_1 * u\|_{L^2(0,T;H^{-1}(\Omega))} \\
&\qquad \le \lim_{N\to\infty} \|a_1 * (u_N - u)\|_{L^2(0,T;H^{-1}(\Omega))} \\
&\qquad \le \|a_1\|_{L^1(R^+)} \lim_{N\to\infty} \|u_N - u\|_{L^2(0,T;H^{-1}(\Omega))} = 0,
\end{aligned}$$

$$\|\chi_1 - a_2 * \phi\|_{L^2(0,T;L^2(\Omega))} \leq \lim_{N\to\infty} \|a_2 * (\phi_N - \phi)\|_{L^2(0,T;L^2(\Omega))} \leq \|a_2\|_{L^1(R^+)} \lim_{N\to\infty} \|\phi_N - \phi\|_{L^2(0,T;L^2(\Omega))} = 0,$$

which imply (3.62) and (3.63).

To prove (3.64), note that by (3.51) that χ_2 and $a_2 * \phi^3$ both belong to $L^\infty(0,T;L^2(\Omega))$. Hence to identify the limit, it suffices to prove that the two functions coincide in the weaker space $L^{4/3}(0,T;L^{4/3}(\Omega))$. Therefore to complete the proof of the lemma, by weak lower semicontinuity it suffices to demonstrate that

$$\lim_{N\to\infty} \|a_2 * (\phi_N^3 - \phi^3)\|_{L^{4/3}(0,T;L^{4/3}(\Omega))} = 0. \tag{3.65}$$

Recalling Lemma 2.2,

$$\|a_2 * (\phi_N^3 - \phi^3)\|_{L^{4/3}(0,T;L^{4/3}(\Omega))} \leq \|a_2\|_{L^1(R^+)} \|\phi_N^3 - \phi^3\|_{L^{4/3}(0,T;L^{4/3}(\Omega))},$$

and we shall now show that

$$\lim_{N\to\infty} \|\phi_N^3 - \phi^3\|_{L^{4/3}(0,T;L^{4/3}(\Omega))} = 0. \tag{3.66}$$

This can be accomplished by noting that

(3.67)

$$\|\phi_N^3 - \phi^3\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))} = \int_0^T \int_\Omega (\phi_N - \phi)^{\frac{4}{3}} (\phi_N^2 + \phi_N\phi + \phi^2)^{\frac{4}{3}} dt$$

$$\leq \int_0^T \left[\int_\Omega (\phi_N - \phi)^4 \right]^{1/3} \left[\int_\Omega (\phi_N^2 + \phi_N\phi + \phi^2)^2 \right]^{\frac{2}{3}} dt$$

(3.68)

$$\leq 3 \sup_{t\in[0,T]} \left\{ \left[\int_\Omega \phi_N^4 \right]^{\frac{2}{3}} + \left[\int_\Omega \phi^4 \right]^{\frac{2}{3}} \right\} \int_0^T \left[\int_\Omega (\phi_N - \phi)^4 \right]^{\frac{1}{3}}.$$

Recalling (3.42) and (3.46), we see that (3.68) implies that

$$\|\phi_N^3 - \phi^3\|^{4/3}_{L^{4/3}(0,T;L^{4/3}(\Omega))} \leq C_{22} \|\phi_N - \phi\|_{L^{4/3}(0,T;L^4(\Omega))}.$$

From the embedding inequality (2.8), it follows that

$$\begin{aligned}\|\phi_N - \phi\|_{L^{4/3}(0,T;L^4(\Omega))} \le{}& C_5\|\phi_N - \phi\|_{L^{4/3}(0,T;H^s(\Omega))}\\ &+ C_6\|\phi_N - \phi\|_{L^{4/3}(0,T;L^2(\Omega))}\end{aligned}$$

for any $s \in [3/4, 1)$ and by Lemma 3.5,

$$\lim_{N\to\infty} \|\phi_N - \phi\|_{L^p(0,T;H^s(\Omega))} = 0$$

for any $1 \le p < \infty$ and $0 \le s < 1$. Therefore $\lim_{N\to\infty} \|\phi_N^3 - \phi^3\|_{L^{4/3}(0,T;L^{4/3}(\Omega))} = 0$ which completes the proof. $\square$

To complete the proof of Theorem 1, it remains to prove the desired continuity. With this end in mind, we first prove an auxiliary lemma, namely,

Lemma 3.8. $\phi \in \mathbf{C}([0,T]; L^2(\Omega))$.

Proof: The proof is roughly analogous to the proof of Lemma 1.2 in Temam [**32**, Chapter 3], but we give the details for the sake of completeness. According to Lemma 3.6, ϕ is continuous from $[0,T]$ to $H^{-1}(\Omega)$ hence, by Lemma 1.4 in [**32**, Chapter 3] and (3.46), ϕ is weakly continuous from $[0,T]$ into $L^2(\Omega)$. Therefore, for any $t_0 \in [0,T]$,

$$\begin{aligned}\lim_{t\to t_0} \|\phi(t) - \phi(t_0)\|^2_{L^2(\Omega)} &= \lim_{t\to t_0} \|\phi(t)\|^2_{L^2(\Omega)}\\ &\quad - 2\lim_{t\to t_0} (\phi(t), \phi(t_0))_{L^2(\Omega),L^2(\Omega)}\\ &\quad + \|\phi(t_0)\|^2_{L^2(\Omega)}\\ &= \lim_{t\to t_0} \|\phi(t)\|^2_{L^2(\Omega)} - \|\phi(t_0)\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.69}$$

By (3.46) and (3.55)

$$\phi \in L^\infty(0,T; H^1(\Omega)),$$

and

$$\phi_t \in L^1(0,T; H^{-1}(\Omega)).$$

Hence it follows from Lemma A.1, given in Appendix A, that

$$\|\phi(t)\|^2_{L^2(\Omega)} = \|\phi(t_0)\|^2_{L^2(\Omega)} + 2\int_{t_0}^{t} \langle \phi(s), \phi_t(s)\rangle_{\{H^1(\Omega),H^{-1}(\Omega)\}}\, ds.$$

Therefore,

$$\lim_{t\to t_0} \|\phi(t)\|^2_{L^2(\Omega)} - \|\phi(t_0)\|^2_{L^2(\Omega)} = 0.$$

In view of (3.69) the proof is completed. □

We now turn to prove:

Lemma 3.9. *$u \in \mathbf{C}([0,T];L^2(\Omega))$ and $\phi \in \mathbf{C}([0,T];H^1(\Omega))$.*

Proof: The technique used here is somewhat reminiscent of that employed in [**5**, Section 3]. By Lemma 3.6 and Lemma 3.8, $u \in \mathbf{C}([0,T];H^{-2}(\Omega))$ and $\phi \in \mathbf{C}([0,T];L^2(\Omega))$ and hence u and ϕ are weakly continuous in $H^{-2}(\Omega)$ and $L^2(\Omega)$ respectively. From the weak continuity which has been demonstrated for u and for ϕ and since by Lemma 3.5,

$$u \in L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \phi \in L^\infty(0,T;H^1(\Omega)), \tag{3.70}$$

one may conclude from Lemma 1.4 in [**32**, Chapter 3] that u is weak continuous in $L^2(\Omega)$ and ϕ is weakly continuous in $H^1(\Omega)$. Since by (2.12) and (3.70)

$$\phi \in L^\infty(0,T;L^4(\Omega)),$$

we see that $\phi^2 \in L^\infty(0,T;L^2(\Omega))$ and $\phi^2 \in \mathbf{C}([0,T];L^1(\Omega))$. Therefore we may similarly conclude that ϕ^2 is weakly continuous in $L^2(\Omega)$.

By the weak continuity of ϕ in $H^1(\Omega)$, and since by assumption $\phi_0 \in H^1(\Omega)$,

$$\begin{aligned}
0 &\le \liminf_{T\to 0} \int_\Omega |\nabla\phi(T) - \nabla\phi_0|^2 \\
&= \liminf_{T\to 0} \{\|\nabla\phi(T)\|^2_{L^2(\Omega)} - 2\langle\nabla\phi(T),\nabla\phi_0\rangle + \|\nabla\phi_0\|^2_{L^2(\Omega)}\} \\
&= \liminf_{T\to 0} \|\nabla\phi(T)\|^2_{L^2(\Omega)} - \|\nabla\phi_0\|^2_{L^2(\Omega)}.
\end{aligned}$$

Hence,

$$\int_\Omega |\nabla\phi_0|^2 \le \liminf_{T\to 0} \int_\Omega |\nabla\phi(T)|^2. \tag{3.71}$$

The weak continuity which has been proven for u yields similarly that

$$\int_\Omega u_0^2 \le \liminf_{T\to 0} \int_\Omega u^2(T). \tag{3.72}$$

By treating analogously the expression

$$0 \le \liminf_{T\to 0} \int_\Omega |\phi^2(T) - \phi_0^2|^2,$$

and relying on the weak continuity which has been demonstrated for ϕ^2, we find that

$$\int_\Omega \phi_0^4 \le \liminf_{T\to 0} \int_\Omega \phi^4(T). \tag{3.73}$$

Recalling that $\phi_N \to \phi$ in $\mathbf{C}([0,T]; L^2(\Omega))$,

$$\int_\Omega \phi_0^2 = \lim_{T\to 0} \int_\Omega \phi^2(T). \tag{3.74}$$

By (3.71)–(3.73),

$$\begin{aligned} &\int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_0|^2 - \frac{1}{2}|\phi_0|^2 + \frac{1}{4}\phi_0^4 + \frac{1}{l}u_0^2\right] \\ &\qquad \le \liminf_{T\to 0} \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi|^2 - \frac{1}{2}|\phi_0|^2 + \frac{1}{4}\phi^4 + \frac{1}{l}u^2\right](T). \end{aligned} \tag{3.75}$$

To obtain an estimate in the opposite direction, we note that using the estimates (3.25)–(3.29) in (3.23) and the notation from (3.30)

$$\begin{aligned} &\int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](T) \\ &\quad \le \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](0) + \frac{1}{2}\int_\Omega [\phi_N^2(T) - \phi_N^2(0)] \\ &\quad + C_{17}\int_0^T \left\{\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right\}\{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^2(\Omega)}\}\, dt \\ &\quad + C_{18}\int_0^T \{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\}\, dt, \end{aligned} \tag{3.76}$$

where the coefficients C_{17} and C_{18} depend on l, ξ and Ω, but are independent of N and T. By Lemma A.1, we may write

$$\frac{1}{2}\int_\Omega [\phi_N^2(T) - \phi_N^2(0)] = \int_0^T \langle \phi_N, \phi_{N_t}\rangle_{\{H^1(\Omega), H^{-1}(\Omega)\}}\, dt,$$

and hence relying on the uniform estimates of Lemma 3.4,

$$\begin{aligned}\frac{1}{2}\int_\Omega [\phi_N^2(T) - \phi_N^2(0)] &= \int_0^T \langle \phi_N, (\phi_{N_t} - f_{2_N}) + f_{2_N}\rangle\, dt \\ &\le \int_0^T \|\phi_N\|_{H^1(\Omega)}\|\phi_{N_t} - f_{2_N}\|_{H^{-1}(\Omega)}\, dt \\ &\quad + \int_0^T \|\phi_N\|_{H^{-1}(\Omega)}\|f_{2_N}\|_{H^1(\Omega)} \\ &\le C_{23}\int_0^T \{1 + \|f_2\|_{H^1(\Omega)}\}\, dt.\end{aligned}$$

This allows us to write (3.76) as

$$\begin{aligned}\int_\Omega &\left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](T) \\ &\le \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right](0) \\ &\quad + C_{24}\int_0^T \{1 + \|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^1(\Omega)}\} \\ &\quad + C_{25}\int_0^T \left\{\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\right\} \\ &\qquad\qquad \times \{\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^2(\Omega)}\}\, dt.\end{aligned} \tag{3.77}$$

Adding C_{24}/C_{25} to both sides of the above equation

$$\begin{aligned}\int_\Omega &\left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2 + \frac{C_{24}}{C_{25}}\right](T) \\ &\le \int_\Omega \left[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2 + \frac{C_{24}}{C_{25}}\right](0) \\ &\quad + C_{25}\int_0^T \left\{\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2 + \frac{C_{24}}{C_{25}}\right\} \\ &\qquad\qquad \times \{1 + \|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^2(\Omega)}\}\, dt.\end{aligned} \tag{3.78}$$

Applying Gronwall's inequality to (3.78) yields

$$\begin{aligned}\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2+\frac{1}{4}\phi_N^4+\frac{1}{l}u_N^2+\frac{C_{24}}{C_{25}}\Big](T) \\ \le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2+\frac{1}{4}\phi_N^4+\frac{1}{l}u_N^2+\frac{C_{24}}{C_{25}}\Big](0) \\ \times e^{C_{25}\int_0^T\{1+\|f_1\|_{L^2(\Omega)}+\|f_2\|_{H^1(\Omega)}\}\,dt}.\end{aligned} \tag{3.79}$$

Applying weak lower semi-continuity to the lefthand side of (3.79) and by completion of the norms (semi-norms) on the right hand side, we obtain

$$\begin{aligned}\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi|^2+\frac{1}{4}\phi^4+\frac{1}{l}u^2+\frac{C_{24}}{C_{25}}\Big](T) \\ \le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi|^2+\frac{1}{4}\phi^4+\frac{1}{l}u^2-\frac{C_{24}}{C_{25}}\Big](0) \\ \times e^{C_{25}\int_0^T\{1+\|f_1\|_{L^2(\Omega)}+\|f_2\|_{H^1(\Omega)}\}\,dt}.\end{aligned}$$

From the above expression, it follows that

$$\begin{aligned}\limsup_{T\to 0}\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi|^2+\frac{1}{4}\phi^4+\frac{1}{l}u^2+\frac{C_{24}}{C_{25}}\Big](T) \\ \le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi|^2+\frac{1}{4}\phi^4+\frac{1}{l}u^2+\frac{C_{24}}{C_{25}}\Big](0).\end{aligned} \tag{3.80}$$

Combining (3.74), (3.75) and (3.80) we see that $\{u,\phi\}\in C([0,T];L^2(\Omega)\times H^1(\Omega))$ as claimed. □

Combining the results of (3.58), (3.59), Lemmas 3.5, 3.7 and 3.9, the proof of Theorem 1 is now completed. □

4. Some concluding remarks. Further properties of (PFM) are presently under active study, and (PFM) is well on its way to being on sound analytical and thermodynamical grounds. Additional goals are to connect the phase field model with memory to specific physical systems and to justify rigorously the limiting geometric motions.

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APPENDIX

We give below a technical lemma which was used in proving Lemma 3.3, Lemma 3.8 and Lemma 3.9.

Lemma A.1. *Let V, H, V' be three Hilbert spaces*

$$V \subset H \equiv H' \subset V',$$

where H' is the dual of H, V' is the dual of V and V is dense and continuously injected in H. If a function v belongs to $L^\infty(0,T;V)$ and its time derivative v_t belongs to $L^1(0,T;V')$, then v is almost everywhere equal to a function which is continuous from $[0,T]$ and we have the following equality, which holds in the scalar distribution sense on $(0,T)$:

$$\frac{d}{dt}\|v\|_{L^2(\Omega)} = 2\langle v, v_t\rangle_{\{V,V'\}}. \tag{5.81}$$

Proof: The proof is analogous to the proof of Lemma 1.2 which appears in [**32**, Chapter 3]. □

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Department of Mathematics, Technion-IIT, Haifa, Israel 32000
Email address: amync@techunix.technion.ac.il