JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 14, Number 1, Spring 2002

IMPLICIT INTEGRAL EQUATIONS WITH DISCONTINUOUS NONLINEARITIES

PASQUALE CANDITO

ABSTRACT. In this paper we establish the existence of at least one solution for a class of implicit integral equations with possibly discontinuous nonlinearities, which includes the well-known Chandrasekhar equation, among others. Our approach fully depends on a very recent result on fixed points for increasing, not necessarily continuous, operators in ordered Banach space due to Bonanno and Marano; see Theorem 1 below.

Very recently, in [**6**], the following fixed point result has been established; see [**6**, Theorem 2.1].

Theorem 1. *Let* $(E, \|\cdot\|, K)$ *be an ordered Banach space with a regular cone* K*, let* $[a, b]$ *be an order interval in* E*, and let* $F : [a, b] \to [a, b]$ *be an increasing function. Then:*

A1) *The function* F *has a minimal fixed point* v_* *and a maximal fixed point* v^*.

A2) $v_* = \min\{v \in [a, b] : v \le F(v)\}$ *while* $v^* = \max\{v \in [a, b] : F(v) \le v\}$.

A3) *For continuous* F *one has* $v_* = \lim_{n\to\infty} F^n(a)$ *as well as* $v^* = \lim_{n\to\infty} F^n(b)$.

As pointed out in [**6**], due to the monotone convergence theorem, a natural framework where the above result applies successfully is given by usual Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$, $1 \le p < +\infty$, equipped with the positive cone

$$K_p := \{u \in L^p(\Omega) : u(t) \ge 0 \quad \text{a.e. in } \Omega\}. \tag{1}$$

Received by the editors on June 20, 2001 and in revised form on December 6, 2001.

In this direction, the authors obtain an existence result for a semi-linear elliptic equation in the whole space and with discontinuous nonlinear terms, see [**6**, Theorem 3.1]. Here, we investigate the following implicit integral equation with discontinuous nonlinearities

$$h(u(t)) = \varphi_0(t) + f(t, u(t)) \int_\Omega g(t, s, u(s))\, ds, \quad u \in L^p(\Omega), \tag{2}$$

where Ω is a Lebesgue measurable, not necessarily bounded, subset of $\mathbf{R}^n$, $\varphi_0 \in L^p(\Omega)$, while $f : \Omega \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$, $g : \Omega \times \Omega \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$ and $h : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ are three monotone increasing functions. Besides the Urysohn type integral equations (2) includes as a special case the well-known Chandrasekhar equation

$$u(t) = \varphi_0(t) + \lambda u(t) \int_\Omega k(t, s) u(s)\, ds, \tag{3}$$

which arises in the kinetic theory of gases and in transport theory, see for instance [**9, 10**] and the references therein.

Numerous papers are devoted to investigating (3) through a technical chiefly based on fixed point results. To be precise, the goal is frequently achieved gathering the Banach-Caccioppoli contraction principle with some classical results on bilinear maps, see [**2, 3, 9**]; we refer also to Corollary 4 and Remarks 5–7 below.

If h turns out to be the identity mapping on $\mathbf{R}_0^+$, one solution of (2) is obtained by using the Darbo Fixed Point Theorem. This approach is prevalently exploited inside the Banach algebra $C(\Omega)$, see [**4, 5, 14**] and [**15**].

In this paper we look at (2) from another point of view, which fully depends on a simple but useful consequence of Theorem 1, namely Theorem 2 below.

Here is the plan of the paper. After establishing Theorem 2, two examples and some remarks are presented. In particular, Example 1 shows that the minimal solution v_* and the maximal solution v^*, given by Theorem 2, can be different, while Example 2 deals with an application of this result to two-point boundary value problems with discontinuous nonlinearities. In Remark 3 we discuss the iterative method given by A3) of Theorem 1 also in connection with the existing literature. Next Theorem 3 shows that a meaningful special case of (2), see Remark 4, admits at least one solution whenever a suitable assumption is made. Finally, Corollary 4 treats integral equations like (3).

We start by establishing the following result, which represents our main tool for investigating (2).

Theorem 2. *Let Ω be a nonempty Lebesgue measurable subset of $\mathbf{R}^n$; let a, b and φ_0 belong to $L^p(\Omega)$, $1 \leq p < +\infty$ with $a \leq b$ and $\varphi_0 \geq 0$; let $f : \Omega \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$, $g : \Omega \times \Omega \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$, and let $h : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ be three functions. Assume that:*

B1) *For almost every $t \in \Omega$, $f(t, \cdot)$ is increasing and sup-measurable.*

B2) *For each measurable $u : \Omega \to \mathbf{R}$, the function $(t, s) \to g(t, s, u(s))$ is measurable in $\Omega \times \Omega$.*

B3) *h is a one-to-one function with h^{-1} strictly increasing and sup-measurable.*

B4) *For almost every $(t, s) \in \Omega \times \Omega$, $g(t, s, \cdot)$ is increasing and $g(t, \cdot, b(\cdot))$ lies in $L^1(\Omega)$.*

B5) *For almost every $t \in \Omega$, the following result*

$$h(a(t)) \leq \varphi_0(t) + f(t, a(t)) \int_\Omega g(t, s, a(s))\, ds,$$

$$\varphi_0(t) + f(t, b(t)) \int_\Omega g(t, s, b(s))\, ds \leq h(b(t)).$$

Then equation (2) admits the minimal solution v_* and the maximal solution v^* belonging to order interval $[a, b]$.

Proof. We first reduce (2) to a fixed point problem through the function $F : [a, b] \to [a, b]$ defined by putting

$$F(u)(t) := h^{-1}\left(\varphi_0(t) + f(t, u(t)) \int_\Omega g(t, s, u(s))\, ds\right) \tag{4}$$

for all $u \in [a, b]$ and $t \in \Omega$. Clearly, each fixed point of F is a solution to (2) and vice versa. Let us now apply Theorem 1, with $E = L^p(\Omega)$,

$K = K_p$, where K_p is the cone given by (1) and F as above. To this end, we note that, because of B1)–B4), the function F is well defined and increasing. Indeed, it is easily seen that $F(u)$ is measurable provided the function

$$t \longrightarrow \int_{\Omega} g(t, s, u(s))\, ds \tag{5}$$

enjoys the same property, which immediately follows from Theorem 8.8 (a) of [**16**]. Moreover, by B5), we have $F(a) \geq a$ as well as $F(b) \leq b$. Since F satisfies all the assumptions of Theorem 1, the proof is complete. □

Remark 1. The monotonicity condition requested in assumptions B1)–B4) doesn't guarantee the sup-measurability; see, for instance, [**1**, page 218].

Remark 2. We explicitly observe that the minimal solution v_* and the maximal solution v^* given by Theorem 2 can be different, as the following simple example shows.

Example 1. Consider the quadratic integral equation

$$u(t) = \frac{6}{5} + \frac{u(t)}{5} \int_0^1 u(s)\, ds, \quad u \in L^1(\Omega) \tag{6}$$

and define, for every $t \in [0, 1]$,

$$a(t) := \frac{6}{5}, \quad b(t) := 3.$$

It is a simple matter to verify that all the assumptions of Theorem 2 are satisfied. Furthermore, since the constant functions $u \equiv 2$ and $u \equiv 3$ are solutions to (6), v_* and v^* must be different.

The following example shows that Theorem 2 can be applied successfully in solving two-point boundary value problems with discontinuous nonlinearities.

Example 2. Let $[\beta, \gamma]$ be a compact real interval. Consider the following two-point boundary value problem

$$\begin{cases} u''(t) + \Psi(u(t)) = 0 & \text{a.e. in }]\beta, \gamma[\\ u(\beta) = u(\gamma) = 0, \end{cases}$$

where $\Psi : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is a possibly discontinuous, increasing and sup-measurable function with

$$\operatorname*{ess\,inf}_{\mathbf{R}_0^+} \Psi > 0.$$

Clearly a solution $u \in W^{2,p}[\beta, \gamma]$ to this problem is obtained by solving the nonlinear integral equation

$$u(t) = \int_\beta^\gamma k(t, s)\Psi(u(s))\, ds, \tag{7}$$

where $k : [\beta, \gamma] \times [\beta, \gamma] \to \mathbf{R}_0^+$ denotes the Green function, namely,

$$k(t, s) := \begin{cases} (\gamma - t)(s - \beta)/(\gamma - \beta) & \text{if } \beta \le s \le t \le \gamma \\ (\gamma - s)(t - \beta)/(\gamma - \beta) & \text{if } \beta \le t \le s \le \gamma. \end{cases} \tag{8}$$

Now, due to Theorem 2, it is a simple matter to see that (7) has at least one nontrivial generalized solution provided there exists a positive constant ϱ such that

$$\frac{\Psi(\varrho)}{\varrho} \le \frac{4}{(\gamma - \beta)^2}.$$

Remark 3. It is worth noting that if in Theorem 2 we also assume that h^{-1} is continuous together with f and g continuous in the second and third variable, respectively, then, the conclusion of this result can be improved as follows:

Equation (2) admits the minimal solution v_* and the maximal solution v^* in the order interval $[a, b]$. Moreover, one has

$$v_* = \lim_{n \to \infty} F^n(a) \quad \text{as well as} \quad v^* = \lim_{n \to \infty} F^n(b) \tag{9}$$

where F is given by (4).

Indeed, let $\{v_n\}$ be a sequence in $[a,b]$ such that $v_n \le v_{n+1}$, $n \in \mathbf{N}$ and $\lim_{n\to\infty} v_n = v$. Then one has $F(v_n) \le F(v)$, $n \in \mathbf{N}$ and, taking into account both the regularity of the cone K_p and the fact that $\{F(v_n)\}$ now converges to $F(v)$ almost everywhere in Ω, we obtain $\lim_{n\to\infty} F(v_n) = F(v)$. Arguing in a standard way, it is easy to verify that the same conclusion still holds when $v_n \ge v_{n+1}$, $n \in \mathbf{N}$, results. Thus (9) is achieved once we note that, due to Remark 2.3 of [**6**], the continuity assumption on F in A3) of Theorem 1 can be replaced by the less restrictive one:

A_3^*) *For each monotone sequence* $\{v_n\} \subseteq [a,b]$, *one has*

$$\lim_{n\to\infty} v_n = v \Longrightarrow \lim_{n\to\infty} F(v_n) = F(v).$$

As classical works on this subject and as general references on monotone operators in partially ordered sets, we refer to [**11–13**] and [**7, 17, 18**], respectively. In particular, we point out that, here, in contrast to [**11**] and [**13**], the functions F can be discontinuous.

Let us now investigate some special cases of the nonlinear integral equations (2) under continuity assumptions. As usual, we denote by p' the conjugate exponent of p.

Theorem 3. *Let* Ω *be a nonempty Lebesgue measurable subset of* $\mathbf{R}^n$ *with* $m(\Omega) < +\infty$; *let* c, d, r *and* q *be four real nonnegative constants with* c, d *positive*; *let* $k : \Omega \times \Omega \to \mathbf{R}_0^+$ *and* $\omega_0 \in L^p(\Omega)$ *be two functions such that* $k \ne 0$ *and* $\omega_0 \ge 0$. *Assume that*:

C1) *For almost every* $t \in \Omega$, $k(t,\cdot)$ *is measurable and lies in* $L^{p'}(\Omega)$.

C2)
$$\alpha = \operatorname*{ess\,sup}_{t\in\Omega} \|k(t,\cdot)\|_{p'} < +\infty.$$

C3) There exists $\varrho \in (c^*, +\infty)$ *such that*

$$\alpha \|\omega_0\|_p \le \frac{\varrho^d - c^*}{\varrho^{r+q}}$$

where $c^* = \max\{c, c^{1/d}\}$.

Then the integral equation

$$u(t)^d = c + u(t)^r \int_\Omega k(t,s)\omega_0(s)u(s)^q\,ds, \quad u \in L^p(\Omega), \tag{10}$$

admits the minimal solution v_* and the maximal solution v^* in the order interval $[c^{1/d}, \varrho]$.

Proof. Without loss of generality, we can assume $\omega_0 \neq 0$. Now, using the notation of Theorem 2, put

$$\begin{cases} h(t) := t^d & \forall t \in \mathbf{R}_0^+, \\ f(t,u) := u^r & \text{if } (t,u) \in \Omega \times \mathbf{R}_0^+, \\ g(t,s,u) := k(t,s)\omega_0(s)u^q, & \text{if } (t,s,u) \in \Omega \times \Omega \times \mathbf{R}_0^+, \end{cases} .$$

We claim that all the assumptions of Theorem 2 are satisfied. Indeed B1), B2) and B3) are obviously true. Write, for almost every $t \in \Omega$,

$$a(t) := c^{1/d} \quad \text{as well as} \quad b(t) := \rho.$$

Due to C2) one has

$$\int_\Omega g(t,s,b(s))\,ds \le \varrho^q \int_\Omega k(t,s)\omega_0(s)\,ds \le \varrho^q \alpha \|\omega_0\|_p < +\infty.$$

Therefore, B4) holds. Moreover,

$$h(a(t)) = c \le c + c^{(r+q)/d} \int_\Omega k(t,s)\omega_0(s)\,ds,$$

results, while bearing in mind C3), we have

$$c + \varrho^{r+q} \int_\Omega k(t,s)\omega_0(s)\,ds \le c + \varrho^{r+q}\alpha\|\omega_0\|_p \le \varrho^d = h(b(t))$$

for every $t \in \Omega$. So also B5) is verified. At this point, the conclusion follows from Theorem 2. □

Remark 4. It is worthwhile to note that assumption C3) of Theorem 3 is satisfied by every nonnegative function ω_0 belonging to $L^p(\Omega)$ whenever one has

C_3') $d > r + q$.

Arguing as in Theorem 3 it is possible to prove the following result regarding (3), which is an immediate consequence of Theorem 2.

Corollary 4. *Let* Ω *be a nonempty Lebesgue measurable subset of* $\mathbf{R}^n$*; let* $k : \Omega \times \Omega \to \mathbf{R}_0^+$ *and* $\varphi_0 \in L^p(\Omega)$ *be two functions such that* $k \neq 0$ *and* $\varphi_0 \geq 0$*. Assume that* C1) *and* C2) *hold and, moreover,*

C_3^*)
$$\alpha \|\varphi_0\|_p \leq 1/4.$$

Then equation (3) *admits the minimal solution* v_* *and the maximal solution* v^* *belonging to order interval* $[\varphi_0, \varrho\varphi_0]$ *where*

$$\frac{1 - \sqrt{1 - 4\alpha\|\varphi_0\|_p}}{2\alpha\|\varphi_0\|_p} \leq \varrho \leq \frac{1 + \sqrt{1 - 4\alpha\|\varphi_0\|_p}}{2\alpha\|\varphi_0\|_p}. \tag{11}$$

Proof. We first note that C_3^*) allows us to write (11). Since ϱ satisfies (11), Theorem 2 can be applied to equation (3) by choosing $a(t) = \varphi_0(t)$, $b(t) = \varrho\varphi_0(t)$, $t \in \Omega$ and $h \equiv id$. □

Remark 5. We explicitly observe that in Corollary 4 it is neither assumed that Ω is of finite measure nor that the solution given by the same result is bounded or unbounded according to whether φ_0 is.

Remark 6. As pointed out in [**9**], equation (3) admits a unique solution in a certain sphere of $L^1(0, 1)$ whenever the kernel $k(t, s)$ and φ_0 satisfy the assumptions:

i) $0 < k(t, s) < 1$,

ii) $k(t + s) + k(s, t) = 1$, for all $(t, s) \in \Omega \times \Omega$,

iii) $\|\varphi_0\|_1 \leq 1/2$.

Instead, here, the same conclusion is achieved by requiring that the kernel be non-negative and iii) replaced with C_3^*. Moreover, it is a simple matter to prove that if $1 < \varrho < 2$, then the operator B is defined by putting

$$B(u)(t) := \varphi_0(t) + u(t) \int_\Omega k(t,s)u(s)\,ds \quad \forall u \in [\varphi_0, \varrho\varphi_0],$$

is a contraction on the complete metric space $[\varphi_0, \varrho\varphi_0]$. Thus, due to the Banach-Caccioppoli contraction principle, there exists at most one solution in the order interval

$$\left[\varphi_0, \frac{1 - \sqrt{1 - 4\alpha\|\varphi_0\|_1}}{2\alpha\|\varphi_0\|_1}\varphi_0\right]$$

provided $\alpha\|\varphi_0\|_1 < 1/4$.

Acknowledgments. The author wishes to thank Professors G. Bonanno and S.A. Marano for introducing him to the topics treated in this paper and for many stimulating conversations.

REFERENCES

1. J. Appell, *The superposition operator in function spaces–A survey*, Expo. Math. **6** (1988), 209–270.

2. I.K. Argyros, *Quadratic equations and application to Chandrasekhar's and related equations*, Bull. Austral. Math. Soc. **32** (1985), 275–292.

3. ———, *On some quadratic equations*, Funct. Approx. Comment. Math. **19** (1990), 159–165.

4. J. Banás, J.R. Rodriguez and K. Sadarangani, *On a class of Urysohn-Stieltjes quadratic integral equations and their applications*, J. Comput. Appl. Math. **113** (2000), 35–50.

5. J. Banás, M. Lecko and W.G. El-Sayed, *Existence theorems for some quadratic integral equations*, J. Math. Anal. Appl. **222** (1998), 276–285.

6. G. Bonanno and S.A. Marano, *Fixed points in ordered Banach spaces and applications to elliptic boundary-value problems*, in *Equilibrium problems and variational models*, Kluwer Academic Publishers, F. Giannessi-A. Maugeri-P. Pardalos Eds.

7. Y. Du, *Fixed points of increasing operators in ordered Banach space and applications*, Appl. Anal. **38** (1990), 1–20.

8. J. Dugundji and A. Granas, *Fixed points theory*, Mono. Matem. (1982),

9. G.A. Hively, *On a class of nonlinear integral equations arising in transport theory*, SIAM J. Math. Anal. **9** (1978), 787–792.

10. S. Hu, M. Khavanin and W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal. **34** (1989), 261–266.

11. V. Hutson and J.S. Pym, *Applications of functional analysis and operator theory*, Academic Press, 1980.

12. L. Kantorovitch, *The method of successive approximations for functional equations*, Acta Math. **71** (1939), 63–97.

13. M.A. Krasnoselskij, *Positive solutions of operator equations*, Noordhoff, 1964.

14. R. Leggett, *On certain nonlinear integral equations*, J. Math. Anal. Appl. **57** (1977), 462–468.

15. A. Majorana and S.A. Marano, *Continuous solutions of a non linear integral equation on an unbounded domain*, J. Integral Equations Appl. **1** Vol. 6 (1994), 119–128.

16. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, Singapore, 1987.

17. D.Z. Vulikh, *Introduction to the theory of partially ordered space*, Wolters-Noordoff Scientific Publications, 1967.

18. E. Zeidler, *Applied functional analysis*: *Main principles and their applications*, Springer, New York, 1996.

Dipartimento di Patrimonio Architettonico e Urbanistico, Univerità di Reggio Calabria, Salita Melissari, Feo di Vito, 89100 Reggio Calabria, Italy and Dipartimento di Matematica, Università di Messina, 98166 Sant' Agata (ME), Italy
Email address: candito@ing.unirc.it