JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 12, Number 4, Winter 2000

BOUNDEDNESS OF THE GLOBAL ERROR OF SOME LINEAR AND NONLINEAR METHODS FOR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. The boundedness of the global error of Runge-Kutta and direct quadrature methods for nonconvolution linear systems of Volterra integral equations of the second kind is analyzed.

1. Introduction. In this paper we are interested in the boundedness of the global error of Volterra Runge-Kutta (VRK) and direct quadrature (DQ) methods for linear Volterra integral equations (VIEs) of the type

$$y(t) = g(t) + \int_0^t k(t,s)y(s)\,ds, \quad t \in [0,T] \qquad (1.1)$$
$$y,\, g \in R^d, \quad k(t,s) \in R^{d\times d}.$$

Because of the hereditary character of the problem (1.1), any numerical methods applied to it give rise to a Volterra Discrete Equation (VDE) which is a difference equation with unbounded order. This makes the analysis of the behavior of the global error of numerical methods for Volterra problems a very involved task.

The behavior of the global error is strictly connected with the stability of numerical methods. In numerical analysis the necessity of studying stability arises anytime one is faced with a general step by step method for computing a sequence of values (for example numerical methods for ODEs or VIEs) and the name numerical stability is used with several different meanings. We could summarize the definitions of stability, both for numerical methods for ODEs and VIEs, in two concepts which, in some cases, coincide. The first concept is the basis of the weak stability theory for numerical methods for ODEs (see [**11**,

Key words and phrases. Global error, Runge-Kutta, direct quadrature, Volterra integral equations.
AMS *Mathematics Subject Classification.* 65R20, 65Q05.
Received by the editors in November 1998, and in revised form on May 29, 2000.

p. 64]) and according to it a numerical method is stable if the error introduced by the discretization does not grow catastrophically as the computation proceeds (see, for example, [**1**, p. 791, ff]). The second one is a more general concept requiring that the numerical solution possess certain properties which are analogous to the stability properties of the analytical solution (see, for example, [**2**, p. 409]).

As we have already specified, the two concepts often coincide. For example, in the case of linear problems, the numerical solution and the global error satisfy difference equations of the same type; therefore, the two concepts are equivalent provided that the analytical solution is bounded (see, for example, [**12**]).

The most complete result on the stability of VRK and DQ methods of (ρ, σ) type has been given in [**10**], [**12**] for convolution linear equations (i.e., $k(t,s) = k(t-s)$). More recently, in [**7**], [**8**], [**9**], some of these results have been extended to the case of nonlinear VIEs of the type

$$y(t) \, = \, g(t) + \int_0^t k\,(t-s)\phi(y(s))\,ds, \quad t \in [0,T]$$
$$y,\, g,\, \phi \in R^d, \quad k\,(t) \in R^{d\times d}.$$

Moreover, by using some recent results on the theory of VDEs [**3**], [**4**], [**5**] the author and others proved in [**3**], [**4**] some sufficient conditions for the boundedness of the global error of certain DQ methods for nonconvolution linear equations.

In this paper we continue our analysis and we give some conditions which assure the boundedness of the global error of a larger class of numerical methods for VIEs. This class includes any DQ and VRK methods.

In particular, in Section 2 we give a result on the boundedness of a general VDE which in Section 3 will be applied to the VDEs arising by the application of the VRK and DQ methods to equation (1.1). This leads to our main theorems which assure the stability (in the sense of boundedness of the global error) of the VRK and DQ methods for nonconvolution equations. Such theorems require a restriction on the stepsize of the method and some conditions on the kernel of the considered VIE. These conditions are not very restrictive, and they are very common in the stability analysis of Volterra methods. Finally, a sharper bound for the global error of DQ methods applied to a VIE whose kernel has nonpositive logarithmic norm is proved.

2. A boundedness result on a general VDE. Let us consider the following explicit VDE

$$
\begin{aligned}
x_n &= \sum_{l=n_0}^{n-1} B(n,l)x_l + p_n, \quad n \geq n_0, \\
x_{n_0} &= x_0,\, x_n,\, p_n \in R^d, \quad B(n,l) \in R^{d\times d}.
\end{aligned} \tag{2.1}
$$

In order to prove the main theorem of this section, we need the following comparison result, which, in the case of finite order difference equations, is well known. The proof is omitted and can be found in [**3**], [**6**].

Lemma 2.1. *Consider* (2.1) *and scalar* VDE

$$
\begin{aligned}
\tilde{x}_n &= \sum_{l=n_0}^{n-1} \tilde{B}(n,l)\,\tilde{x}_l + \tilde{p}_n, \quad n \geq n_0, \\
\tilde{x}_{n_0} &= \tilde{x}_0 = \|x_0\|, \quad \tilde{x}_n,\, \tilde{p}_n,\, \tilde{B}(n,l) \in R,
\end{aligned}
$$

such that

$$
\|p_n\| \leq \tilde{p}_n, \quad \|B(n,l)\| \leq \tilde{B}(n,l), \quad l \geq n_0, \quad n \geq n_0.
$$

Then there results

$$
\|x_n\| \leq \tilde{x}_n, \quad n \geq n_0. \tag{2.2}
$$

Now we consider the VDE

$$
\begin{aligned}
x_n &= \sum_{l=n_0}^{n} B(n,l)\, x_l + p_n, \quad n \geq n_0 + 1, \\
x_{n_0} &= x_0,\, x_n,\, p_n \in R^d, \quad B(n,l) \in R^{d\times d}.
\end{aligned} \tag{2.3}
$$

The following result on the boundedness of the solution of the implicit VDE (2.3) can be proved.

Theorem 2.1. *Assume that there exists a norm in R^d and a subordinate one in $R^{d\times d}$ such that*

(i) $\sup_{n\geq n_0+1}\|B(n,n)\| \leq \beta < 1$,
(ii) $\|B(n,l)\| \leq \beta_l$, $n \geq n_0+1$, $l = n_0,\dots,n-1$,
(iii) $\sum_{l=n_0+1}^{\infty}\beta_l \leq \beta^*$,
(iv) $\|p_n\| \leq p$, $n \geq n_0+1$.

Then the solution x_n *of* (2.3) *satisfies*

$$\|x_n\| \leq \frac{1}{1-\beta}\,(p+\beta_{n_0}\|x_0\|)\,e^{(1/(1-\beta))\beta^*}. \tag{2.4}$$

Proof. From (i) we have that the matrix $[I_d - B(n,n)]$ is invertible and the equation (2.3) can be written in the explicit form

$$x_n = [I_d - B(n,n)]^{-1}p_n + \sum_{l=n_0}^{n-1}[I_d - B(n,n)]^{-1}B(n,l)x_l, \quad n > n_0.$$

Once again, from (i) we obtain

$$\|[I_d - B(n,n)]^{-1}\| \leq \frac{1}{1-\|B(n,n)\|} < \frac{1}{1-\beta}$$

and, in view of (ii) and (iv) there results

$$\|x_n\| \leq \frac{1}{1-\beta}\,p + \frac{1}{1-\beta}\sum_{l=n_0}^{n-1}\beta_l\|x_l\|.$$

Now let us consider the related scalar VDE

$$\tilde{x}_n = \frac{1}{1-\beta}\,p + \frac{1}{1-\beta}\sum_{l=n_0}^{n-1}\beta_l\tilde{x}_l, \quad \tilde{x}_0 = \tilde{x}_{n_0} = \|x_0\|$$

whose solution can be expressed as follows

$$\tilde{x}_n = \frac{1}{1-\beta}\,(p+\beta_{n_0}\tilde{x}_0)\prod_{l=n_0+1}^{n-1}\left(1+\frac{\beta_l}{1-\beta}\right).$$

Finally Lemma 1 yields

$$\|x_n\| \le \tilde{x}_n \le \frac{1}{1-\beta}\left(p + \beta_{n_0}\|x_0\|\right) e^{1/(1-\beta)\sum_{l=n_0+1}^{n-1}\beta_l},$$

and the thesis follows by taking into account hypothesis (iii). □

3. Boundedness of the global error. In spite of its simplicity the result of the previous section is very useful for studying the behavior of the global error of a large class of numerical methods applied to VIEs characterized by kernels $k(t,s)$ whose norm is bounded with respect to t and summable with respect to s.

3.1. *The case of VRK methods.* Let us consider the equation (1.1) and the classical m-stage VRK method for its resolution ([**2**, p. 170])

$$\begin{aligned} Y_{nj} &= F_n(t_n + \theta_j h) + h \sum_{i=1}^{m} a_{ji} k\,(t_n + d_{ji}h, t_n + c_i h) Y_{ni}, \\ &\qquad i = 1, \dots, m \\ y_{n+1} &= F_n(t_{n+1}) + h \sum_{i=1}^{m} b_i k\,(t_{n+1} + (e_i - 1)h, t_n + c_i h) Y_{ni}, \\ &\qquad n \ge 0, \quad y_0 = y(0) \end{aligned}$$

with $t_n = nh$, $y_n \approx y(t_n)$. The vectors

$$\theta = (\theta_j), \quad c = (c_j), \quad e = (e_j), \quad b = (b_j), \quad j = 1, \dots, m \tag{3.1}$$

and the square matrices

$$A = (a_{ij}), \quad D = (d_{ij}), \quad i, j = 1, \dots, m \tag{3.2}$$

are given.

As is known, VRK method is called of mixed type if it is

$$F_n(t_n + \theta_j h) = g\,(t_n + \theta_j h) + h \sum_{l=0}^{n} w_{n,l} k\,(t_n + \theta_j h, t_l) y_l; \tag{3.3}$$

here $w_{n,l}$ are the weights of a quadrature formula. Moreover, if it is

(3.4)
$$F_n(t_n+\theta_j h) \;=\; g\,(t_n+\theta_j h)+h \sum_{l=0}^{n-1}\sum_{i=1}^{m} b_i k\,(t_n+(\theta_j+e_i-1)h, t_l+c_i h)Y_{li}$$

then the method is called of extended type.

Now we want to show how the global error $\varepsilon_n = y(t_n) - y_n$ of the VRK methods applied to (1.1) satisfies a VDE of the type (2.3).

Toward this purpose we set

$$\gamma_{n,j} \;=\; Y_{nj} - \overline{Y}_{nj}$$

with

$$\overline{Y}_{nj} \;=\; \overline{F}_n(t_n+\theta_j) + h \sum_{i=1}^{m} a_{ji} k\,(t_n+d_{ji}h, t_n+c_i h)\overline{Y}_{ni}, \quad i = 1, \dots, m.$$

The expression of $\overline{F}_n(t_n+\theta_j)$ for the mixed methods is

$$\overline{F}_n(t_n+\theta_j h) \;=\; g\,(t_n+\theta_j h) + h \sum_{l=0}^{n} w_{n,l} k\,(t_n+\theta_j h, t_l) y(t_l),$$

whereas for the extended methods it is

$$\overline{F}_n(t_n+\theta_j h) \;=\; g\,(t_n+\theta_j h)+h \sum_{l=0}^{n-1}\sum_{i=1}^{m} b_i k\,(t_n+(\theta_j+e_i-1)h, t_l+c_i h)\overline{Y}_{li}.$$

The global error equations of the two methods are, respectively,

(3.5)
$$\begin{aligned}
\varepsilon_{n+1} \;&=\; T_{n+1}(h) + h \sum_{l=0}^{n} w_{n,l} k\,(t_n+\theta_j h, t_l)\varepsilon_l \\
&\quad + h \sum_{i=1}^{m} b_i k\,(t_{n+1}+(e_i-1)h, t_n+c_i h)\gamma_{n,i}, \quad n \ge 0, \\
\gamma_{n,j} \;&=\; h \sum_{l=0}^{n} w_{n,l} k\,(t_n+\theta_j h, t_l)\varepsilon_l \\
&\quad + h \sum_{i=1}^{m} a_{ji} k\,(t_n+d_{ji}h, t_n+c_i h)\gamma_{n,i}, \quad j = 1, \dots, m \\
\varepsilon_0 \;&=\; 0, \quad \gamma_{0,j} \;=\; 0, \quad j = 1, \dots, m
\end{aligned}$$

(3.6)
$$
\begin{aligned}
\varepsilon_{n+1} &= T_{n+1}(h) + h\sum_{l=0}^{n-1}\sum_{i=1}^{m} b_i k\,(t_{n+1}+(e_i-1)h, t_l+c_ih)\gamma_{l,i}\\
&\quad + h\sum_{i=1}^{m} b_i k\,(t_{n+1}+(e_i-1)h, t_n+c_ih)\gamma_{n,i}, \quad n\ge 0\\
\gamma_{n,j} &= h\sum_{l=0}^{n-1}\sum_{i=1}^{m} b_i k\,(t_n+(\theta_j+e_i-1)h, t_l+c_ih)\gamma_{l,i}\\
&\quad + h\sum_{i=1}^{m} a_{ji} k\,(t_n+d_{ji}h, t_n+c_ih)\gamma_{n,i}, \quad j=1,\dots,m\\
\varepsilon_0 &= 0, \quad \gamma_{0,j} = 0, \quad j=1,\dots,m.
\end{aligned}
$$

In both the formulas (3.5) and (3.6), $T_{n+1}(h)$ represents the local truncation error of the VRK methods.

Now it can be easily seen that both the error equations can be put in the form (2.3).

To be more specific (3.5) can be rewritten in the form (2.3) simply by setting

(3.7)
$$
\begin{aligned}
x_n &= [\gamma_{n-1,1},\dots,\gamma_{n-1,m},\varepsilon_n]^T \in R^{d(m+1)},\\
&\quad n_0 = 0, \quad \gamma_{-1,1} =,\dots,= \gamma_{-1,m} = 0\\
p_n &= [0,\dots,0,T_n(h)]^T \in R^{d(m+1)}, \quad n\ge 1
\end{aligned}
$$

(3.8)
$$
\begin{aligned}
B(n,l) &= (B_{ij}(n,l))_{i,j=1,\dots,m+1}\\
&= \begin{cases} 0 & i=1,\dots,m+1,\ j=1,\dots,m,\\ hw_{n-1,l}k\,(t_{n-1}+\theta_ih, t_l) & i=1,\dots,m,\ j=m+1,\\ hw_{n-1,l}k\,(t_n,t_l) & i=m+1,\ j=m+1,\end{cases}\\
&\qquad l=0,\dots,n-1, \quad n\ge 1
\end{aligned}
$$

(3.9)

$$B(n,n) = (B_{ij}(n,n))_{i,j=1,\dots,m}$$

$$= \begin{cases} ha_{ij}k\left(t_{n-1}+d_{ij}h, t_{n-1}+c_jh\right) & i=1,\dots,m1, \\ & j=1,\dots,m \\ 0 & i=1,\dots,m+1, \\ & j=m+1,\ n\geq 1 \\ hb_jk\left(t_n+(e_j-1)h, t_{n-1}+c_jh\right) & i=m+1, \\ & j=1,\dots,m. \end{cases}$$

The same holds for the VDE (3.6) by defining x_n, p_n and $B(n,n)$ as in (3.7) and (3.9), respectively, and

(3.10)

$$B(n,l) = (B_{ij}(n,l))_{i,j=1,\dots,m+1}$$

$$= \begin{cases} hb_jk\left(t_{n-1}+(\theta_i+e_j-1)h, t_l+c_jh\right) & i=1,\dots,m, \\ & j=1,\dots,m, \\ hb_jk\left(t_n+(e_j-1)h, t_l+c_jh\right) & i=m+1, \\ & j=1,\dots,m, \\ 0 & i=m+1, \\ & j=m+1, \end{cases}$$

$$l=0,\dots,n-1, \quad n\geq 1.$$

Remark 3.1. Equations (3.5) and (3.6) represent the global error of a nonlinear method applied to a linear VIE. They are very similar to the equation of the VRK method itself. This is due to the linearity of the kernel $k(t,s)$ and, of course, it is not true anymore if a nonlinear continuous problem is considered.

Moreover, it is stressed that the vector x_n given in (3.7) does not contain the only component ε_n but it contains also some spurious component $\gamma_{n-1,1},\dots,\gamma_{n-1,m}$.

Now we assume the usual hypothesis of boundedness of the local truncation error, i.e.

$$T_n(h) < T^*, \quad n \geq 0$$

and we give a result on the boundedness of the global error of VRK methods.

Set

$$\alpha = \max\{\|b\|_\infty, \|A\|_\infty\}, \quad w^* = \max_{\substack{n \geq 1 \\ l=0,\dots,n}} \{|w_{n,l}|\},$$

where A and b are given in (3.1) and (3.2), respectively.

Theorem 3.1. *Assume that*

(i) $\|k\,(t,s)\|_\infty \leq f(s) \leq k^*$, $t \geq 0$, $s \geq 0$,

(ii) $h < 1/(\alpha\, k^*)$,

(iii) $f(s)$ *ultimately decreasing,*

(iv) $\int_0^\infty f(s)\, ds < \infty$.

Then the global error of the VRK *method is bounded.*

Proof. Let us prove the theorem in the case of VRK methods of mixed type. The proof for the extended case is analogous.

Consider the VDE

$$x_n = p_n + \sum_{l=0}^{n} B(n,l)x_l,$$

with x_n, p_n and $B(n,l)$ given by (3.7) and (3.8)–(3.9), respectively. Let us prove that it satisfies the hypotheses of Theorem 2.1.

From (i) and (ii) we have

$$\|B(n,n)\|_\infty \leq h\alpha k^* < 1.$$

The first hypothesis of Theorem 2.1 is satisfied by setting $h\alpha k^* = \beta$. Assumption (i) also yields

$$\|B(n,l)\|_\infty \leq hw^* f(t_l), \quad n \geq 1, \quad l \geq 0 \tag{3.11}$$

and from (iii) we have

$$\sum_{l=0}^{\infty} \|B(n,l)\|_\infty \le hw^* \sum_{l=0}^{q} f(t_l) + hw^* \sum_{l=q+1}^{\infty} f(t_l), \tag{3.12}$$

where q is such that $f(s)$ is decreasing for $s \ge hq$. In view of (iii) and (iv) there results

$$\sum_{l=0}^{\infty} \|B(n,l)\|_\infty \le hw^* \sum_{l=0}^{q} f(t_l) + w^* \int_{hq}^{\infty} f(s)\, ds = \beta^*. \tag{3.13}$$

The inequalities (3.11) and (3.13) prove the second and the third hypothesis of Theorem 2.1. The last hypothesis is obviously assured by taking into account the boundedness of T_n. Hence, Theorem 2.1 can be applied to the VDE considered, and there exists a constant $M(x_0)$ such that

$$\|\varepsilon_n\|_\infty \le \|x_n\|_\infty \le M(x_0), \quad n \ge 0.$$

Finally, by recalling that $x_0 = 0$, we get the following expression for $M(x_0)$,

$$M(x_0) = \frac{T^*}{1 - h\alpha k^*} e^{(1/(1-h\alpha k^*))\beta^*}. \qquad \square$$

Remark 3.2. As we already mentioned in the introduction, there are some results in the literature on stability of nonlinear methods for VIEs [**10**], but they are valid only for linear convolution VIEs. To the best of our knowledge, Theorem 3.1 furnishes the first sufficient conditions for the boundedness of the global error of a nonlinear method applied to linear nonconvolution equations. One of these conditions requires a bound on the stepsize and the other is expressed directly in terms of the characteristics of the kernel of the VIE considered. The hypotheses we require on the kernel of the integral equation are not very restrictive, and they occur very often in the stability analysis of numerical methods for VIEs (see, for example, [**3**], [**4**]).

Remark 3.3. Theorem 3.1 cannot be applied in the case of the convolution kernel. Namely, the hypotheses of the theorem imply

$$\inf f(s) = 0. \tag{3.14}$$

Moreover, the hypothesis (iii) in the case of convolution kernel becomes

$$\|k(\tau)\| \le f(t-\tau), \quad \forall\, t,$$

which, together with (3.14), implies $\|k(t)\| \equiv 0$.

Examples. Let us consider the 4-stage VRK method of extended type. It is characterized by the coefficients ([**2**, p. 172])

$$d_{ji} = c_j, \quad \theta_j = c_j, \quad e_j = 1, \quad i,j = 1,\dots,4,$$
$$c = \left[0, \frac{5-\sqrt{5}}{10}, \frac{5+\sqrt{5}}{10}, 1\right]^T,$$
$$b = \left[\frac{1}{12}, \frac{5}{12}, \frac{5}{12}, \frac{1}{12}\right]^T,$$
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{11+\sqrt{5}}{120} & \frac{25-\sqrt{5}}{120} & \frac{25-13\sqrt{5}}{120} & \frac{-1+\sqrt{5}}{120} \\ \frac{11-\sqrt{5}}{120} & \frac{25+13\sqrt{5}}{120} & \frac{25+\sqrt{5}}{120} & \frac{-1-\sqrt{5}}{120} \\ \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{pmatrix}$$

with $\|A\|_\infty = \|b\|_\infty = 1$.

Now, in order to show the reliability of Theorem 3.1, we consider some kernels which satisfy the hypotheses of such a theorem and we compute the restriction on the stepsize required by the application of the theorem itself.

In Table 1 we report the kernels, the relative restriction $h < h_0$ and the bound of the global error $M(x_0)$. Such a value is computed by taking into account that, for the extended methods, the inequality (3.12) corresponds to

$$\sum_{l=0}^{n} \|B(n,l)\| \le h\|b\|_\infty \sum_{l=0}^{q} f(t_l) + h\|b\|_\infty \sum_{l=q+1}^{\infty} f(t_l).$$

TABLE 1.

kernel	h_0	$M(x_0)$
A) $-\frac{1}{(1+t+s)^2}$	1	$\frac{T^*}{1-h}\, e^{1/(1-h)}$
B) $\frac{1}{1+t+s^2}$	1	$\frac{T^*}{1-h}\, e^{(\pi/2)[1/(1-h)]}$
C) $\sin(t)\, e^{-s}$	1	$\frac{T^*}{1-h}\, e^{1/(1-h)}$
A) bis $-\frac{10}{(1+t+s)^2}$	$\frac{1}{10}$	$\frac{T^*}{1-10h}\, e^{10/(1-10h)}$

Observe that in all the cases the restrictions on the stepsize are not severe, whereas the bound on the global error may have a practical interest only in the first three cases. The value of $M(x_0)$ in the last case is too large (note that the kernel **A bis** is slightly different from the kernel **A**). We report the example **A bis** only to show how the value of the integral of $f(s)$ influences the bound on the global error.

3.2. *The case of DQ methods.* In this section we consider the DQ methods (see [**2**, p. 96]) for the solution of (1.1)

$$y_n = g(t_n) + h \sum_{l=0}^{n} w_{n,l} k(t_n, t_l) y_l, \quad n \geq \kappa, \tag{3.15}$$

where $y_0 = g(0)$ and $y_1, \ldots, y_{\kappa-1}$ are given starting values.

Also for this method it can be easily seen that the equation of the global error can be put into the form (2.3). To be more specific, the global error satisfies

$$\begin{aligned} \varepsilon_n &= T_n + h \sum_{l=0}^{n} w_{n,l} k(t_n, t_l) \varepsilon_l, \quad n \geq \kappa, \\ \varepsilon_0 &= 0, \quad \varepsilon_1, \ldots, \varepsilon_{k-1} \quad \text{given}, \end{aligned} \tag{3.16}$$

which assumes the form (2.3) simply by setting

$$x_n = \varepsilon_n \in R^d, \quad p_n = T_n \in R^d, \quad n_0 = \kappa, \tag{3.17}$$

$$B(n,l) = h w_{n,l} k(t_n, t_l) \in R^{d \times d}, \quad n \geq k+1, \quad l \geq 0. \tag{3.18}$$

Now set

$$w = \sup_{n\geq\kappa} |w_{n,n}|, \qquad \tilde{w} = \sup_{\substack{n\geq\kappa \\ l=0,\dots,n-1}} |w_{n,l}|,$$

$$\tilde{\beta} = h\tilde{w} \sum_{l=0}^{q} f(t_l) + \tilde{w} \int_{hq}^{\infty} f(s)\, ds, \tag{3.19}$$

$$\tilde{T} = T^* + \max_{n=0,\dots,\kappa-1} \sum_{l=0}^{\kappa-1} |w_{n,l}|\, \|k\,(t_n, t_l)\|\, \|\varepsilon_l\|.$$

In case of DQ methods, Theorem 3.1 reads

Theorem 3.2. *Assume that*

(i) $\|k\,(t,s)\|_\infty \leq f(s) \leq k^*$, $t \geq 0$, $s \geq 0$,

(ii) $h < (1/(wk^*))$,

(iii) $f(s)$ *ultimately decreasing*,

(iv) $\int_0^\infty f(s)\, ds < \infty$.

Then the global error of the DQ method is bounded by

$$\|\varepsilon_n\|_\infty \leq \frac{\tilde{T}}{1 - hwk^*}\, e^{\tilde{\beta}/(1-hwk^*)}.$$

Due to the simplicity of the expression (3.18), a sharper bound on the global error of DQ methods applied to kernels with nonpositive logarithmic norm can be given.

In order to prove this result we need the following lemma which consists in a slight improvement of Lemma 2.1 in [**13**].

Lemma 3.1. *Let* $\Gamma \in R^{d\times d}$ *be a square matrix such that* $\det(I_d - \Gamma) \neq 0$ *and* $\mu[\Gamma] \leq 0$ *where* $\mu[\cdot]$ *is the logarithmic norm related to any natural norm* $\|\cdot\|$. *Then there results*

$$\|(I_d - \Gamma)^{-1}\| \leq 1.$$

Proof. Set $z := z(x) = (I_d - \Gamma)^{-1}x$, for all $x \neq 0$. Then

$$z = \Gamma z + x$$

and

$$\|z\| \le \mu[\Gamma]\,\|z\| + \|x\|, \quad \forall\, x \ne 0.$$

Hence,

$$\frac{\|z\|}{\|x\|} \le \frac{1}{1-\mu[\Gamma]} \le 1, \quad \forall\, x \ne 0. \qquad \square$$

Corollary 3.1. *Assume that all the hypotheses of Theorem* 3.2 *are fulfilled and moreover*

(v) $w_{n,n} \ge 0$, $n \ge \kappa$, $\mu[k\,(t,t)]_\infty \le 0$.

Then the global error of the DQ methods is bounded by

$$\|\varepsilon_n\|_\infty \le \tilde{T} e^{\tilde{\beta}}.$$

Proof. The proof is analogous to the one of Theorem 3.2. The only difference is that in this case hypothesis v) and Lemma 3.1 allow us to prove that $\|[I_d - B(n,n)]^{-1}\|_\infty \le 1$. $\square$

Remark 3.4. In [**3, 4**], we have already obtained some results on the boundedness of the global error of certain linear methods applied to (1.1). Compared with those results we note that here we have removed the hypothesis requiring that the kernel decays exponentially (i.e., $k\,(t,s) \le P\nu^{t-s}$, $P > 0$, $\nu \in (0,1)$) and we do not have any restriction on the order of the methods.

Examples. In Tables 2 and 3 we report the bound on the global error (M) obtained by applying Theorem 3.2 and Corollary 3.1 to the kernels **A, B, C, A bis**, respectively. We consider two DQ methods; the first is based on the trapezoidal formula and the latter on the repeated Simpson plus Simpson's 3/8 quadrature rule ([**2**, p. 97]). Moreover, in Table 2 we report the bound on the stepsize (h_0). Such a column is omitted in Table 3 because it coincides with the one in Table 2.

TABLE 2.

Theorem 3.2				
	Trapezoidal		Simpson	
kernel	h_0	M	h_0	M
A)	2	$\frac{\tilde{T}}{1-h/2}\, e^{1/(1-h/2)}$	$\frac{8}{9}$	$\frac{8\tilde{T}}{8-9h}\, e^{8/(8-9h)}$
B)	2	$\frac{\tilde{T}}{1-h/2}\, e^{(\pi/2)[1/(1-h/2)]}$	$\frac{8}{9}$	$\frac{8\tilde{T}}{8-9h}\, e^{4\pi/(8-9h)}$
C)	2	$\frac{\tilde{T}}{1-h/2}\, e^{1/(1-h/2)}$	$\frac{8}{9}$	$\frac{8\tilde{T}}{8-9h}\, e^{8/(8-9h)}$
A) bis	$\frac{1}{5}$	$\frac{\tilde{T}}{1-5h}\, e^{10/(1-5h)}$	$\frac{8}{90}$	$\frac{8\tilde{T}}{8-90h}\, e^{80/(8-90h)}$

TABLE 3.

Corollary 3.1		
	Trapezoidal	Simpson
kernel	M	M
A)	$\tilde{T}e$	$\tilde{T}e^{9/8}$
A) bis	$\tilde{T}e^{10}$	$\tilde{T}e^{90/8}$

Note that, because of hypothesis v), Corollary 3.1 can only be applied to the cases **A** and **A bis**.

Remark 3.5. Consider the VIE (1.1) and assume $\|g(t)\| \leq g^*$. It is easy to prove that, under the hypotheses (i) and (iv) of Theorems 3.1 and 3.2, the analytical solution is bounded. Now observe that, as we already mentioned in the introduction, in the case of linear problems, the global error and the numerical solution satisfy VDEs of the same type. In consequence of this, our results prove that the qualitative behavior of the exact solution is preserved by the numerical solution and Theorems 3.1 and 3.2 assure the stability of the methods also considered according to the second meaning described in the introduction.

Remark 3.6. Observe that the trapezoidal method falls in both classes of VRK and DQ methods. Nevertheless it can be easily seen that the application of Theorem 3.1 to our examples leads to a restriction on h which is more restrictive than the one required by Theorem 3.2. This will happen in general, and it is due to the fact that Theorem 3.1 requires the boundedness of the norm of the whole vector x_n given in (3.7) instead of the boundedness of the only component ε_n as is done in Theorem 3.2.

4. Concluding remarks. Starting from a fairly elementary inequality (2.4) we have proved some results on the boundedness of the global error of numerical methods for linear VIEs. The kernels considered are of nonconvolution type and the methods considered are of linear and nonlinear type with no order restriction. The conditions we require on the kernel are not very restrictive and allow to consider kernels which are not exponentially decreasing.

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