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GLOBAL STABILITY OF A FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

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Dedicated to Professor S.-O. Londen on the occasion of his 60th birthday.

1. Introduction. We consider the equation

$$u_{tt}(t,x) = \int_0^t b(t-s)u_{txx}(s,x)\,ds + (g(u_x(t,x)))_x, \quad t>0, \quad x\in(0,1) \tag{1.1}$$

with boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t>0, \tag{1.2}$$

and initial values

$$u(0,x) = u^0(x), \qquad u_t(0,x) = u^1(x). \tag{1.3}$$

This problem is motivated by the theory of viscoelastic materials, cf., Pego [**10**], Ball et al. [**1**]. The nonlinearity g behaves typically like a power at infinity,

$$g(\xi) \sim \operatorname{sign}(\xi)|\xi|^m, \quad |\xi|\to\infty,$$

and we assume that it vanishes solely at zero so that zero is the unique stationary state of (1.1).

The convolution term represents a fractional derivative with respect to t; explicitly, let

$$b(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0<\alpha<1.$$

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Then the equation can be rewritten in terms of fractional derivatives,

$$u_{tt} = \frac{\partial^\alpha}{\partial t^\alpha} u_{xx} + g(u_x)_x.$$

In case $b = \delta$ (the Dirac delta function) the strong dissipative character of the instantaneous viscous damping term u_{txx} forces all solutions of the problem to tend to zero; see, e.g., Greenberg [**6**]. The energy of the dynamical system generated by the equation decreases along solutions and can serve as a Lyapunov function.

Due to the presence of memory, our equation does not generate a semi-flow in a natural energy space, the energy is in general not decreasing along solutions, so the theory of Lyapunov functions is not applicable.

The aim of this paper is to prove that, even in the case of fractional derivatives where the damping effect is not so strong, the system still dissipates enough energy so that all solutions tend to zero, the unique stationary state, in energy norm. Our main result reads as follows.

Theorem 1.1. *Let $g \in C^1(\mathbf{R}, \mathbf{R})$, $g(0) = 0$, $g'(0) > 0$, $g(s)s > 0$ for all $s \neq 0$, assume there are $p \geq 1$ and a positive constant K such that*

$$|g(s)|^p \leq K\bigg(\int_0^s g(\xi)\,d\xi + 1\bigg), \quad s \in \mathbf{R}, \tag{1.4}$$

and let

$$b(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \text{where } 0 < \beta < \frac{2p-1}{2p+1}. \tag{1.5}$$

Suppose $u^0 \in \mathring{W}^{1,\infty}(0,1)$ and $u^1 \in L^q(0,1)$ where $q \geq \max\{2, (1+\beta)/(1-\beta)\}$.

Then problem (1.1)–(1.3) *admits a unique global solution $u(t)$ of class*

$$u \in \mathrm{BUC}\,(\mathbf{R}^+; \mathring{W}^{1,\infty}(0,1)) \cap \mathrm{BUC}^1(\mathbf{R}^+; L^q(0,1)).$$

As $t \to \infty$, $u(t)$ converges to zero in $W^{1,\infty}(0,1)$ and $u_t(t)$ converges to zero in $L^q(0,1)$.

If the initial values satisfy in addition, for some $r \in [1,\infty)$, $u^0 \in W^{2,r}(0,1) \cap \overset{\circ}{W}{}^{1,\infty}(0,1)$, $u^1 \in \overset{\circ}{W}{}^{1,r}(0,1)$, *then*

$$u \in \mathrm{BUC}(\mathbf{R}^+; W^{2,r}(0,1)) \cap \mathrm{BUC}^1(\mathbf{R}^+; \overset{\circ}{W}{}^{1,r}(0,1)),$$

and $u(t) \to 0$ *in* $W^{2,r}(0,1)$, $u_t(t) \to 0$ *in* $W^{1,r}(0,1)$ *as* $t \to \infty$.

Our work builds on results obtained in [**7**] where existence of global weak solutions of the equation

$$u_t = a * u_{xx} + b * g(u_x)_x + f$$

on the whole real line is proved for fairly general initial conditions and forcing functions, and a larger class of nonlinearities.

The existence results are based on abstract formulations of the equation. Results from harmonic analysis of vector-valued functions derived in Prüss [**11**] are applied to find estimates of resolvent operators for the problem. The energy inequality together with methods from nonlinear Volterra equations enables us to find a priori estimates which allow to prove global existence and to examine the asymptotic behavior of solutions. We use properties of Triebel-Lizorkin spaces and spectra of vector-valued bounded functions to prove for some $r \in [2,\infty)$ that $u_x \in \mathrm{BUC}^\mu(\mathbf{R}^+; H) \cap L^r(\mathbf{R}^+; H)$ and $\dot{u}(t) \to 0$ in H, hence also $u(t) \to 0$ in $W^{1,2}(0,1)$; here we have put $H = L^2(0,1)$.

The kernel b could be taken more general; the same method applies for log-convex kernels whose Laplace transforms can be estimated with the help of $|\lambda|^{-\beta}$ but the notation would be more complicated. More precisely, as in Prüss [**11**] it would be sufficient for b to be of class $L^1_{\mathrm{loc}}(\mathbf{R}^+)$, positive, nonincreasing, log-convex, and

$$c|\lambda|^{-\beta} \le |\widehat{b}(\lambda)| \le c^{-1}|\lambda|^{-\beta}, \quad \mathrm{Re}\,\lambda > R,$$

for some constants $c > 0$ and $R \ge 0$.

The assumption $g(0) = 0$ is a normalization, since only g' enters the equation. $g'(0) > 0$ means that the linearized problem is asymptotically stable, which will be shown in Lemma 4.4 below. It should be noted that asymptotic stability of the linearized problem is valid for all kernels

b such that $b(t) > 0$ and log-convex for $t > 0$; this has nothing to do with the choice of β. The condition $g(s)s > 0$ for $s \neq 0$ ensures that zero is the only equilibrium state. Even in case $b = \delta_0$ the asymptotic behavior of (1.1) seems to be unclear if the latter condition is violated because then the set of stationary solutions is not discrete, in particular, zero is nonisolated.

Observe that in the case where $sg(s) \sim G(s) \sim |s|^{m+1}$ as $|s| \to \infty$, for some $m \geq 0$, (1.4) holds with $p = 1 + 1/m$, hence the restriction on β in (1.5) becomes $\beta < (m+2)/(3m+2)$. This is certainly not optimal since the linear case $g(s) = \eta s$, $\eta > 0$ corresponds to $m = 1$, hence $\beta < 3/5$. In fact, global existence of weak solutions has been shown in [**7**] provided $\beta < (m+3)/(3m+1)$, which means $\beta < 1$ in the linear case. We need the stronger restriction on β here to obtain a uniform bound on $\|u_x(t;\cdot)\|_{L^\infty}$. Actually, it is possible to prove local asymptotic stability of the trivial steady state in the case $\beta < (m+3)/(3m+1)$, but we shall not do it here.

The paper is organized as follows. In Section 2 several representations for the solutions are derived, estimates on resolvent operators are given and a local existence result is proved. A priori estimates based on the energy inequality are deduced and used to prove the global existence result in Section 3. Finally, convergence of solutions is shown in Section 4.

2. Solution formulae. Let $\|\cdot\|_H$, $\langle\cdot,\cdot\rangle$ denote the norm and the scalar product in $H = L^2(0,1)$, $D = d/dx$, $\dot{v} = v_t$, $k * v = \int_0^t k(t-s)v(s)\,ds$; C stands for a generic positive constant. The Laplace transform of a function f is denoted by $\widehat{f}$ or $\mathcal{L}(f)$. We introduce (1.1) in an appropriate functional analytic setting.

Let $1 \leq p < \infty$, and define

$$\mathcal{D}(A_p) = W^{2,p}(0,1) \cap \overset{\circ}{W}{}^{1,p}(0,1),$$
$$A_p\phi = -\frac{d^2}{dx^2}\phi = -D^2\phi.$$

For simplicity we drop the subscript p when there is no danger of confusion. It is well known that A_p is an invertible, sectorial operator on $L^p(0,1)$, A_2 is self-adjoint, positive definite in H, $\sigma(A_p) \subset [\pi^2, +\infty)$

consists only of simple eigenvalues and there is $M(\theta) > 0$ such that

$$\|A^{\alpha}(z+A)^{-1}\|_{\mathcal{B}(L^p)} \le \frac{M(\theta)}{|\pi^2+z|^{1-\alpha}}, \quad 0 \le \alpha \le 1, \qquad |\arg(z+\pi^2)| \le \pi - \theta, \quad \theta > 0. \tag{2.1}$$

We reformulate (1.1) as

$$\ddot{u} + b * A\dot{u} = Dg(Du) \tag{2.2}$$

and apply results of [**12**] to solve it. The Laplace transform of (2.2) yields

$$\lambda(\lambda + \widehat{b}(\lambda)A)\widehat{u}(\lambda) = (\lambda + \widehat{b}(\lambda)A)u^0 + u^1 + \mathcal{L}(Dg(Du)).$$

Let S be the solution of the operator equation

$$\dot{S} + b * AS = 0, \quad S(0) = I,$$

so that

$$\widehat{S}(\lambda) = (\lambda + \widehat{b}(\lambda)A)^{-1}, \quad \lambda \notin (-\infty, 0]$$

and denote

$$S_1(t) = \int_0^t S(s)\,ds.$$

Then the solution u can be expressed by means of the variation of parameters formula

$$u(t) = u^0 + S_1(t)u^1 + S_1 * Dg(Du)(t). \tag{2.3}$$

Using estimates collected in the forthcoming lemma together with the relations

$$Au(t) = Au^0 + AS_1(t)u^1 - AS_1 * g'(Du)Au(t), \tag{2.4}$$

$$\dot{u}(t) = S(t)u^1 - S * g'(Du)Au(t), \tag{2.5}$$

and the embedding $W^{2,p}(0,1) \hookrightarrow C^1[0,1]$, we may apply the contraction mapping principle to obtain local existence and uniqueness of a strong solution

$$u \in C([0,t_0);\ \mathcal{D}(A_p)) \cap C^1([0,t_0);\ \overset{\circ}{W}{}^{1,p}(0,1))$$

provided

$$u^0 \in \mathcal{D}(A_p) = W^{2,p}(0,1) \cap \mathring{W}^{1,p}(0,1)$$
$$u^1 \in \mathcal{D}(A_p^{1/2}) = \mathring{W}^{1,p}(0,1),$$

and in addition $g \in C^2(\mathbf{R})$. This relies on the boundedness of $S(t)$ and on the estimate $\|AS_1(t)\|_{\mathcal{B}(L^p(0,1))} \leq ct^{-\beta}$, $t > 0$; see (2.18) below. In fact, by means of these estimates we can even get more regularity of the solution $u(t)$, but this is not the goal here.

However, via this approach, strong solutions will only be local; they only exist as long as $Au(t)$ stays bounded in $L^p(0,1)$, and this cannot be guaranteed a priori. But we can do better by means of weak solutions. In fact, since (2.4) is linear in Au, one obtains global existence of strong solutions as soon as weak solutions exist such that $|Du(t)|_\infty$ stays bounded.

To prove existence of weak solutions we need to get rid of the outer x-derivative of the term $g(u_x)_x$. To do so we integrate the equation with respect to x and get a similar equation for a function v where $v_x = u$:

$$\frac{\partial}{\partial x}(v_{tt} - b * v_{txx} - g(u_x)) = 0.$$

We choose v with mean zero and solve the Neumann boundary value problem for v, treating the term $g(u_x)$ as given.

For $\phi \in L^p(0,1)$, denote

$$P\phi = \int_0^1 \phi(x)\,dx, \qquad L_0^p(0,1) = \{\phi \in L^p(0,1);\ P\phi = 0\},$$

and define an operator $Q : L^p(0,1) \to W^{1,p}(0,1)$ by

$$Q\phi(x) = \int_0^x y\phi(y)\,dy - \int_x^1 (1-y)\phi(y)\,dy.$$

Then

$$PQ\phi = 0, \qquad DQ\phi = \phi, \qquad QD\phi = \phi - P\phi.$$

The function $v = Qu$ satisfies

$$v_{tt} - b * v_{txx} - g(u_x) = K, \qquad Pv = 0.$$

Applying the operator P we determine the constant $K : PK = K = -Pg(u_x)$. Thus we have to solve the problem

$$\begin{gathered} v_{tt} - b * v_{txx} - g(u_x) = -Pg(u_x), \\ v(0,x) = v_0(x) := Qu^0(x), \qquad \dot{v}(0,x) = v_1(x) := Qu^1(x), \\ v_x(t,0) = v_x(t,1) = 0. \end{gathered}$$

The last equation can be reformulated as

$$\ddot{v} + b * A^N \dot{v} = g(Du) - Pg(Du)$$

where A^N denotes the Laplace operator with Neumann boundary conditions, i.e.,

$$\begin{aligned} &\mathcal{D}(A_p^N) = \{\phi \in L_0^p : \phi \in W^{2,p}(0,1),\ \phi_x(0) = \phi_x(1) = 0\}, \\ &A_p^N \phi = -\frac{d^2}{dx^2}\phi = -D^2\phi. \end{aligned}$$

Everything that has been said about the operator A, in particular formula (2.1), remains valid when replacing A_p by A_p^N and L^p by L_0^p, and we let $A^N = A_2^N$.

As before, the solution v is given by the formula

$$v(t) = v_0 + S_1^N(t)v_1 + S_1^N * (g(Du) - Pg(Du))(t),$$

S^N being the solution operator corresponding to A^N, $\widehat{S}^N(\lambda) = (\lambda + \widehat{b}(\lambda)A^N)^{-1}$. For the solution of (1.1), $u = Dv$, we have

$$u(t) = u^0 + DS_1^N(t)Qu_1 + DS_1^N * (g(Du) - Pg(Du))(t).$$

Remark. We can derive this formula directly from (2.3) as soon as we realize that

$$A = DA^NQ$$

which implies

$$S(t) = DS^N(t)Q, S_1(t) = DS_1^N(t)Q.$$

We denote $P_1 = I - P$ and express u and its derivatives as follows.

$$u = u^0 + S_1u^1 + DS_1^N * P_1g(Du), \tag{2.6}$$

$$Du = Du^0 + DS_1u^1 - A^NS_1^N * P_1g(Du), \tag{2.7}$$

$$\dot{u} = Su^1 + DS^N * P_1g(Du), \tag{2.8}$$

$$D\dot{u} = DSu^1 - A^NS^N * P_1g(Du), \tag{2.9}$$

$$\ddot{u} = \dot{S}u^1 + DS^NP_1g(Du^0) + DS^N * P_1(g'(Du)D\dot{u}). \tag{2.10}$$

As all formulae, estimates and proofs for A, S and A^N, S^N are identical, we omit the superscript N throughout the further text.

Since $\widehat{b}(\lambda) = \lambda^{-\beta}$ there is a positive number $\theta < (\pi/2)$ such that

$$\arg(\lambda/\widehat{b}(\lambda)) \leq \pi - \theta \quad \text{when } |\arg(\lambda)| \leq \frac{\pi}{2} + \theta, \lambda \neq 0. \tag{2.11}$$

Hence $[(\lambda/\widehat{b}(\lambda)) + A]^{-1}$ exists for these values of λ, (2.1) holds, and we can integrate over the contour Γ_R, $R > 0$, which consists of three parts

$$\begin{cases} \Gamma_1 \ldots & -ire^{-i\theta} \quad R \leq r < \infty, \\ \Gamma_2 \ldots & \mathrm{R}e^{i\phi}, \quad -\theta - \frac{\pi}{2} \leq \phi \leq \theta + \frac{\pi}{2} \\ \Gamma_3 \ldots & ire^{i\theta} \quad R \leq r < \infty, \end{cases}$$

to obtain the following formulae:

$$S(t) = \frac{1}{2\pi i}\int_{\Gamma_R} e^{\lambda t}(\lambda + \widehat{b}(\lambda)A)^{-1}\, d\lambda, \tag{2.12}$$

$$\widehat{S}(\lambda) = (\lambda + \widehat{b}(\lambda)A)^{-1} = \frac{1}{\widehat{b}(\lambda)}\left(\frac{\lambda}{\widehat{b}(\lambda)} + A\right)^{-1},$$

$$S_1(t) = \int_0^t S(s)\, ds, \qquad \widehat{S}_1(\lambda) = \frac{1}{\lambda}(\lambda + \widehat{b}(\lambda)A)^{-1}, \tag{2.13}$$

$$\begin{aligned} \widehat{AS}_1(\lambda) &= \frac{1}{\lambda\widehat{b}(\lambda)}(I - \lambda(\lambda + \widehat{b}(\lambda)A)^{-1}) \\ &= \frac{1}{\lambda\widehat{b}(\lambda)} - \frac{1}{\widehat{b}(\lambda)}(\lambda + \widehat{b}(\lambda)A)^{-1} = \widehat{k}(\lambda) - \widehat{T}(\lambda) \end{aligned} \tag{2.14}$$

so

$$AS_1(t) = k(t) - T(t) \tag{2.15}$$

where

$$k(t) = t^{-\beta}/\Gamma(1-\beta), \quad T(t) = \frac{d}{dt} k * S(t).$$

In the following lemma we collect estimates on S, S_1 and T.

Lemma 2.1. *Let S, S_1 and T be given by the formulae* (2.12), (2.13), (2.15), $1 \le p < \infty$, $0 \le \alpha \le 1$. *Then there is a constant C such that*

$$\|A^\alpha S(t)\phi\|_{L^p} \le C \min\{t^{-\alpha(1+\beta)}, t^{-1-\beta}\}\|\phi\|_{L^p}, \tag{2.16}$$

$$\|\dot{S}(t)\phi\|_{L^p} \le C t^{-1}\|\phi\|_{L^p}, \tag{2.17}$$

$$\|A^\alpha S_1(t)\phi\|_{L^p} \le C \min\{t^{1-\alpha(1+\beta)}, t^{-\beta}\}\|\phi\|_{L^p}, \tag{2.18}$$

$$A^\alpha S \in L^1(\mathbf{R}^+, \mathcal{B}(L^p(0,1))), \quad \text{if } \alpha < \frac{1}{1+\beta}, \tag{2.19}$$

$$\|T(t)\phi\|_{\mathcal{D}(A_p^\alpha)} \le C \min\{t^{-(\alpha+\beta+\alpha\beta)}, t^{-1-2\beta}\}\|\phi\|_{L^p}, \tag{2.20}$$

(2.21)

$$\|(T(t+h) - T(t))\phi\|_{L^p} \le C h^{1-\beta} \frac{1 + h^\beta t^{-\beta}}{t+h}\|\phi\|_{L^p},$$

$$T \in L^1(\mathbf{R}^+, \mathcal{B}(L^p(0,1), C^\mu(0,1)) \tag{2.22}$$

$$\text{when } 0 < \mu < 2\frac{1-\beta}{1+\beta} - \frac{1}{p}.$$

Proof. We shall take advantage of the following result on vector-valued analytic functions in $\mathbf{C}^+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ (see [**11**, Theorem 1]).

Lemma 2.2. *Suppose $h : \mathbf{C}^+ \to X$ is holomorphic and satisfies*

$$\|h(\lambda)\| + \|\lambda h'(\lambda)\| \le c|\lambda|^{-\gamma}, \quad \operatorname{Re}\lambda > 0,$$

for some $\gamma \in (0,1)$. Then there is a continuous function $v : (0,\infty) \to X$ such that $\widehat{v}(\lambda) = h(\lambda)$ for $\operatorname{Re}\lambda > 0$. In addition, the following estimates *hold*

$$\|v(t)\| \le Mt^{\gamma-1}, \quad t > 0, \tag{2.23}$$

$$\|tv(t) - sv(s)\| \le M|t-s|^{\gamma}, \quad 0 < s < t < \infty. \tag{2.24}$$

Here M denotes a constant depending only on γ and c.

Estimate (2.16) is obtained by using (2.12), (2.1) and (2.11).

$$\begin{aligned}\|A^\alpha \widehat{S}(\lambda)\phi\|_{L^p} &= \left\|\frac{1}{\widehat{b}(\lambda)} A^\alpha \left(\frac{\lambda}{\widehat{b}(\lambda)} + A\right)^{-1} \phi\right\|_{L^p} \\ &\le C \frac{|\lambda|^\beta}{|1+\lambda/\widehat{b}(\lambda)|^{1-\alpha}} \|\phi\|_{L^p} \\ &\le C_1 |\lambda|^{-1+\alpha(1+\beta)} \|\phi\|_{L^p} \quad \text{for } \operatorname{Re}\lambda > 0.\end{aligned}$$

The same estimate we get for $\|A^\alpha \lambda (d/d\lambda)\widehat{S}(\lambda)\phi\|_{L^p}$. If $\alpha < 1/(1+\beta)$, we directly apply Lemma 2.2 with $\gamma = 1-\alpha(1+\beta)$ while if $\alpha \ge 1/(1+\beta)$, we use the fact that $\mathcal{L}(tS(t)) = -(d/d\lambda)\widehat{S}(\lambda)$ to get the same result. The latter observation is also used to prove

$$\|A^\alpha S(t)\phi\|_{L^p} \le Ct^{-1-\beta}\|\phi\|_{L^p},$$

estimating

$$\begin{aligned}\left\|A^\alpha \frac{d}{d\lambda}\widehat{S}(\lambda)\phi\right\|_{L^p} &= \left\|A^\alpha (I + \widehat{b}'(\lambda)A)(\lambda + \widehat{b}(\lambda)A)^{-2}\phi\right\|_{L^p} \\ &\le C|\lambda|^{\beta-1}\|\phi\|_{L^p}.\end{aligned}$$

Hence (2.16) and, consequently, (2.19), follow.

Estimates (2.17) and (2.18) are obtained in the same way. To see (2.20), we use the expression

$$\widehat{A^\alpha T}(\lambda)\phi = \frac{1}{\widehat{b}^2(\lambda)} A^\alpha \left(\frac{\lambda}{\widehat{b}(\lambda)} + A\right)^{-1} \phi$$

and

$$\begin{aligned}\left\|\frac{1}{\widehat{b}^2(\lambda)} A^\alpha \left(\frac{\lambda}{\widehat{b}(\lambda)} + A\right)^{-1} \phi\right\|_{L^p} &\le C \frac{|\lambda|^{2\beta}}{(1+|\lambda|^{1+\beta})^{1-\alpha}} \|\phi\|_{L^p} \\ &\le C|\lambda|^{-1+\alpha+\beta+\alpha\beta}.\end{aligned}$$

Taking $\alpha = 0$, $\gamma = 1 - \beta$, we get (2.21) from (2.24).

According to [**9**, Theorem 1.6.1],

$$\mathcal{D}(A_p^\alpha) \subset C^\mu(0,1) \quad \text{if } 0 < \mu < 2\alpha - \frac{1}{p}. \tag{2.25}$$

Now it is sufficient to realize that $(\mu/2)+(1/2p) < (1-\beta)/(1+\beta)$ when $0 < \mu < 2(1-\beta)/(1+\beta) - (1/p)$. Choosing $\alpha \in \big((\mu/2) + (1/2p), (1 - \beta)/(1+\beta)\big)$ in (2.20), we get (2.22). □

Now we are in a position to prove local existence and uniqueness of a weak solution of (1.1).

Proposition 2.1. *Let the assumptions of Theorem* 1.1 *be fulfilled.*

Given any data $u^0 \in \overset{\circ}{W}{}^{1,\infty}(0,1)$, $u^1 \in L^q(0,1)$, *a unique time* $t_0 = t_0(u^0, u^1) > 0$ *and a unique weak solution* $u \in C([0,t_0); \overset{\circ}{W}{}^{1,\infty}(0,1)) \cap C^1([0,t_0); L^q(0,1))$ *exist; if* $t_0 < \infty$ *then* $\overline{\lim}_{t\to t_0+} \|Du(t)\|_{L^\infty} = \infty$. *Moreover,*

$$Du \in C^{(1-\beta)/2}([0,t_0), L^q(0,1)), \tag{2.26}$$

and

$$D\dot{u},\ \ddot{u} - \dot{S}u^1 \in L^r((0,t_0); L^q(0,1)), \tag{2.27}$$

provided $1/r > (1+\beta)/2$. *If, in addition,* $u_0 \in \mathcal{D}(A_q)$, $u_1 \in \mathcal{D}(A_q^{1/2})$, *then* u *is a strong solution of* (1.1), $u \in C([0,t_0); \mathcal{D}(A_q)) \cap C^1([0,t_0); \overset{\circ}{W}{}^{1,q}(0,1))$.

Proof. We apply the contraction mapping principle to the operator defined by the righthand side of (2.6) in the class $C([0,t_1]; \overset{\circ}{W}{}^{1,\infty}(0,1))$ with a suitable $t_1 > 0$. For this, estimates from Lemma 2.1, identities (2.7) and (2.8) are employed as well as the fact that

$$A^{1/2}Q,\ DA^{-1/2} \quad \text{are bounded operators in } L^q(0,1).$$

Inclusion (2.25) with $\alpha > 1/(2q)$, $p = q$, allows for the estimate

$$\|DS_1(t)u^1\|_{L^\infty} \le C\|A^\alpha A^{1/2} S_1(t)u^1\|_{L^q} \le Ct^{1-(\alpha+(1/2))(1+\beta)}\|u^1\|_{L^q}.$$

Observe that, by proper choice of α, the exponent on the righthand side of this equation can be chosen positive provided $(1/q) < (1-\beta)/(1+\beta)$. From (2.15) and (2.22) we get $AS_1 \in L^1_{\text{loc}}(\mathbf{R}^+; \mathcal{B}(L^\infty(0,1)))$, so $AS_1 * P_1 g(Du) \in \text{BUC}([0,t_1]; L^\infty(0,1))$, $AS_1 * P_1 g(Du)(0) = 0$. The estimates for $\dot{u}$ are similar, using this time (2.8) and (2.16).

We can continue the solution as long as $\|u(t)\|_{W^{1,\infty}}$ remains bounded; this implies boundedness of $\|\dot{u}(t)\|_{L^q}$ on the same interval. To prove (2.26) we observe that (2.21) yields the same estimate for AS_1, and Hölder continuity follows by a straightforward computation involving Lemma 2.1. It is then used in the proof of (2.27) where we have to solve the following equation for $v = D\dot{u}$ in $L^r((0,t_0); L^q(0,1))$.

$$v(t) = DS(t)u^1 - \int_0^t AS_1(t-s)P_1(g'(Du(s))v(s))\, ds - AS_1(t)P_1 g(Du_0)$$

and use (2.16) and (2.20). The remaining assertions follow from (2.10) and (2.17).

The contraction mapping principle applied in the space

$$C([0,t_1]; \mathcal{D}(A_q)) \cap C^1([0,t_1]; \overset{\circ}{W}{}^{1,q}(0,1))$$

yields existence of a strong solution to (1.1). $\square$

3. A priori estimates and global existence. One of the important features of (1.1) is the fact that there is an energy inequality. Dissipation is not so strong as in the case when the viscous damping term $D^2\dot{u}$ is present where the energy

$$E(u) := \int_0^1 \left(\frac{1}{2}\dot{u}^2 + G(Du)\right) dx, \qquad G(s) = \int_0^s g(r)\, dr$$

decreases along solutions. In our case we have

$$E(u)(t) + \int_0^t \langle b * D\dot{u}(t), D\dot{u}(t)\rangle\, dt = E(u)(0),$$

and consequently,

$$E(u)(t) + \int_0^t \|a * D\dot{u}\|_H^2 \le E(u)(0) \tag{3.1}$$

holds for every kernel a such that b is a-positive, i.e.,

$$\int_0^t (b*\phi)(s)\phi(s)\,ds \ge \int_0^t |a*\phi(s)|^2\,ds \quad \text{for all } t>0,\ \phi\in C(\mathbf{R}^+). \tag{3.2}$$

It follows that the energy stays bounded on $\mathbf{R}^+$. Inequality (3.1) holds for strong solutions of (1.1) provided b satisfies the positivity assumption (3.2). Using density arguments we get (3.1) also for weak solutions. This is the starting point of our discussion of the long time behavior of solutions of (1.1).

An immediate consequence of (3.1), (3.2) and growth assumption (1.4) is

$$\|\dot u(t)\|_H \le C_0, \quad \|g(Du(t))\|_{L^p} \le C_0, \quad 0\le t<t_0 \tag{3.3}$$

where $C_0 = C_0(\|u^0\|_{W^{1,\infty}}, \|u^1\|_H)$. Another important consequence of (3.1) is

$$a * D\dot u \in L^2((0,t_0),H) \tag{3.4}$$

for every kernel a satisfying (3.2). Now we use the energy estimates to obtain global existence.

Proposition 3.1. *Let the assumption of Theorem* 1.1 *be fulfilled. Given any data* $u^0 \in \overset{\circ}{W}{}^{1,\infty}(0,1)$, $u^1 \in L^q(0,1)$, *there exists a unique weak solution of the problem* (1.1)–(1.3) *and a constant* $K = K(\|u_0\|_{W^{1,\infty}}, \|u_1\|_{L^q})$ *such that*

$$u \in \mathrm{BUC}\,(\mathbf{R}^+; \overset{\circ}{W}{}^{1,\infty}(0,1)) \cap \mathrm{BUC}^1(\mathbf{R}^+; L^q(0,1)), \tag{3.5}$$

and

$$\sup_{t\in\mathbf{R}^+} (\|u(t)\|_{W^{1,\infty}} + \|\dot u(t)\|_{L^q}) \le K.$$

Moreover,

$$u \in C^1((0,\infty); \overset{\circ}{W}{}^{1,q}(0,1)),$$

and there is a nondecreasing function $C(T)$ such that

$$\|Du(t+h)-Du(t)\|_{L^q} \le C(T)ht^{-(1+\beta)/2}, \quad \text{for } t \le T, \tag{3.6}$$

$$\|D\dot{u}(t)\|_{L^q} \le C(T)t^{-(1+\beta)/2} \quad \text{for } t \le T. \tag{3.7}$$

If, in addition, $u^0 \in \mathcal{D}(A_q)$, $u^1 \in \mathcal{D}(A_q^{1/2})$, then u is a strong solution of (1.1),

$$u \in C(\mathbf{R}^+; \mathcal{D}(A_q)) \cap C^1(\mathbf{R}^+; \mathring{W}^{1,q}(0,1)).$$

Proof. We can continue the solution as long as $\|Du(t)\|_{L^\infty}$ remains bounded. It follows from (3.3) that $\dot{u}$ is bounded in H and $g(Du) \in L^\infty((0,t_0); L^p(0,1))$. The convolution $T * P_1 g(Du)$ belongs to $\mathrm{BUC}([0,t_0), L^\infty(0,1))$ according to (2.22). This, together with (2.15), allows us to continue the solution to $\mathbf{R}^+$ and rewrite (2.7) in the form

$$v + k * g(v) = h_1 + k * h_2,$$

with $v = Du$, $h_1 = Du^0 + DS_1u^1 + T * P_1 g(Du)$, $h_2 = Pg(Du)$ and log-convex kernel $k(t) = t^{-\beta}/\Gamma(1-\beta)$; observe that h_1 and h_2 are bounded. The theory of scalar integral equations yields boundedness of v, hence of $\|Du(t)\|_{L^\infty}$; see [**8**, Chapter 20, Theorem 3.1] or the discussion in [**7**]. $\dot{u} \in \mathrm{BUC}(\mathbf{R}^+, L^q(0,1))$ is due to (3.3) and (2.8).

To prove (3.6) we use the mean value theorem together with (2.16) and (2.18).

$$\begin{aligned}
&\|Du(t+h) - Du(t)\|_{L^q} \\
&\quad \le \|[DS_1(t+h) - DS_1(t)]u^1\|_{L^q} \\
&\qquad + \left\| \int_0^t AS_1(s)P_1(g(Du(t+h-s)) - g(Du(t-s)))\, ds \right\|_{L^q} \\
&\qquad + \left\| \int_t^{t+h} AS_1(s)g(Du(t+h-s))\, ds \right\|_{L^q} \\
&\quad \le C\left(ht^{-(1+\beta)/2} + \int_0^t (t-s)^{-\beta} \|Du(s+h) - Du(s)\|_{L^q} + \int_t^{t+h} s^{-\beta} \right) \\
&\quad \le C(T)ht^{-(1+\beta)/2} + C\int_0^t (t-s)^{-\beta} \|Du(s+h) - Du(s)\|_{L^q},
\end{aligned}$$

and the generalized Gronwall lemma applies. We employed (3.6) when proving (3.7).

$$\begin{aligned}\|D\dot{u}(t)\|_{L^q} &= \|DS(t)u^1 - AS * P_1 g(Du)(t)\|_{L^q}\\ &\le C\bigg(t^{-(1+\beta)/2} + \int_0^{t/2}(t-s)^{-1-\beta}\,ds\\ &\qquad + \bigg\|A\int_{t/2}^t S(t-s)P_1 g(Du(t))\,ds\bigg\|_{L^q}\\ &\qquad + \int_{t/2}^t (t-s)^{-1-\beta}\|Du(t)-Du(s)\|_{L^q}\,ds\bigg)\\ &\le C\bigg(t^{-(1+\beta)/2} + t^{-\beta} + \int_{t/2}^t (t-s)^{-\beta}s^{-(1+\beta)/2}\,ds\bigg)\\ &\le C(T)(t^{-(1+\beta)/2} + t^{(1-3\beta)/2}) \le C(T)t^{-(1+\beta)/2}.\end{aligned}$$

As for strong solutions, having

$$u^0 \in \mathcal{D}(A_q), \qquad u^1 \in \mathcal{D}(A_q^{1/2}),$$

we get boundedness of $A_q u$, $A_q^{1/2}\dot{u}$ using expressions (2.4) and (2.5) and estimates (2.18) and (2.16). For $\|Au\|_{L^q}$ we have

$$\|Au(t)\|_{L^q} \le C_1 + C_2\int_0^t (t-s)^{-\beta}\|Au(s)\|_{L^q}\,ds$$

and use the Gronwall lemma, while

$$\begin{aligned}\|A^{1/2}\dot{u}(t)\|_{L^q} &\le \|S(t)A^{1/2}u^1\|_{L^q} + \|A^{1/2}S * g'(Du)Au(t)\|_{L^q}\\ &\le C + \|A^{1/2}S\|_{L^1(\mathbf{R}^+;B(L^q))}\sup_{s\le t}\|g'(Du(s))\|_{L^\infty}\|Au(s)\|_{L^q}.\end{aligned}$$

□

To prove convergence of solutions we employ the energy inequality once more and derive estimates we will need in the following section.

Let $a_1(t) = t^{(\beta/2)-1}(\sqrt{\cos(\beta\pi/2)}/\Gamma(\beta/2))$. Then

$$\operatorname{Re}\widehat{b}(i\rho) \ge |\widehat{a}_1(i\rho)|^2, \quad \rho \in \mathbf{R}.$$

Moreover, according to Lemma 2 in [**4**], there exists a function a_2 such that

$$a_2 \in L^1(\mathbf{R}^+), \quad \widehat{a}_2(i\rho) \neq 0 \quad \text{on } \mathbf{R}, \qquad \operatorname{Re}\widehat{b}(i\rho) \geq |\widehat{a}_2(i\rho)|^2, \quad \rho \in \mathbf{R}. \tag{3.8}$$

It follows that b is both a_1 and a_2-positive and, from (3.4), we have

$$a_1 * D\dot{u} \in L^2(\mathbf{R}^+, H), \quad a_2 * D\dot{u} \in L^2(\mathbf{R}^+, H). \tag{3.9}$$

Important consequences of (3.9) are collected in the following lemma.

Lemma 3.1. *Let* $a_1(t) = t^{(\beta/2)-1}(\sqrt{\cos(\beta\pi/2)}/\Gamma(\beta/2))$, *u a solution of* (1.1) *satisfying* (3.5) *and the energy inequality* (3.1). *Then*

$$Du \in \mathrm{BUC}^{(1-\beta)/2}(\mathbf{R}^+; H), \tag{3.10}$$

$$a_1 * \frac{d}{dt} g(Du) \in L^2(\mathbf{R}^+; H), \tag{3.11}$$

$$\dot{u}(t) \to 0 \quad \text{in } H \quad \text{as } t \to \infty. \tag{3.12}$$

Proof. To prove Lemma 3.1 we use properties of the homogeneous Triebel-Lizorkin spaces $\dot{F}^\sigma_{2,2}(\mathbf{R})$ endowed with norm

$$\|v\|_{\dot{F}^\sigma_{2,2}} = \int_{\mathbf{R}} |\xi|^{2\sigma} |\tilde{v}(\xi)|^2 \, d\xi$$

where $\tilde{v} = \mathcal{F}(v)$ denotes the Fourier transform of v. This space is equivalent to the homogeneous Besov space $\dot{B}^\sigma_{2,2}(\mathbf{R})$ with norm

$$\|v\|_{\dot{B}^\sigma_{2,2}} = \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{|v(t) - v(s)|^2}{|t-s|^{1+2\sigma}} \, ds \, dt.$$

The operator

$$\dot{I}_\eta f = \mathcal{F}^{-1}[|\xi|^\eta \mathcal{F} f]$$

maps $\dot{F}^\sigma_{2,2}(\mathbf{R})$ isomorphically onto $\dot{F}^{\sigma-\eta}_{2,2}(\mathbf{R})$ and $\dot{B}^\sigma_{2,2}(\mathbf{R})$ onto $\dot{B}^{\sigma-\eta}_{2,2}(\mathbf{R})$ respectively, and

$$\dot{F}^0_{2,2}(\mathbf{R}) = L^2(\mathbf{R}).$$

The space $\dot{B}^{\sigma}_{2,2}(\mathbf{R})$ is continuously embedded into $\mathrm{BUC}^{\sigma-(1/2)}(\mathbf{R})$, $\sigma > (1/2)$; see [**13**] for the theory of these spaces. The definitions and properties just mentioned extend to functions with values in a Hilbert space.

We continue Du by Du^0, $a_1(t) = 0$ for t negative to obtain

$$a_1 * D\dot{u} \in L^2(\mathbf{R}, H), \quad \mathcal{F}(a_1 * D\dot{u})(\xi) = (i\xi)^{1-(\beta/2)}\mathcal{F}(Du)(\xi).$$

Hence Du belongs to the space $\dot{F}^{1-(\beta/2)}_{2,2}(\mathbf{R}; H) \sim \dot{B}^{1-(\beta/2)}_{2,2}(\mathbf{R}; H) \hookrightarrow \mathrm{BUC}^{(1-\beta)/2}(\mathbf{R}; H)$. (This assertion can also be obtained directly as in the proof of Proposition 2.1.)

Now it is sufficient to realize that $Du \in L^\infty(\mathbf{R}^+; H)$ and that g is locally Lipschitz continuous and use the Hilbert space-valued version of the foregoing results to see that

$$a_1 * D\dot{u} \in L^2(\mathbf{R}^+; H) \Longrightarrow a_1 * \frac{d}{dt} g(Du) \in L^2(\mathbf{R}^+; H).$$

To prove (3.12) we employ the following notion of the spectrum of bounded functions which, roughly speaking, says that $\rho \in \mathrm{sp}\,(u)$ if and only if its Laplace transform cannot be extended to a neighborhood of $i\rho$ on the imaginary axis as an L^1_{loc}-function.

Definition. Let X be a Banach space and $u \in L^\infty(\mathbf{R}^+; X)$. A number $\rho \in \mathbf{R}$ does not belong to $\mathrm{sp}\,(u)$ if there is an $r > 0$ and a function $h \in L^1(B_r(\rho); X)$ such that

$$\int_{\rho-r}^{\rho+r} \phi(s)\widehat{u}(\sigma + is)\,ds \to \int_{\rho-r}^{\rho+r} \phi(s)h(s)\,ds \quad \text{as } \sigma \to 0_+$$

for all $\phi \in C^\infty(\mathbf{R})$ with $\mathrm{supp}\,\phi \subset B_r(\rho) \subset \mathbf{R}$.

It is shown, e.g., in Chill [**2**] that BUC functions with void spectrum belong to $C_0(\mathbf{R}^+; X)$. So it is sufficient to show that $sp(\dot{u}) = \varnothing$ to obtain (3.12). Let a_2 be such that (3.8) and, consequently, (3.9) are satisfied. This together with the Poincaré inequality yields

$$a_2 * \dot{u} \in L^2(\mathbf{R}^+; H).$$

Since H is a Hilbert space, the Laplace transform $\psi(\lambda) = \widehat{a}_2(\lambda) \cdot \widehat{\dot{u}}(\lambda) \in \mathcal{H}^2(\mathbf{C}^+; H)$, the Hardy space of exponent 2, and has boundary values on the imaginary axis in $L^2(\mathbf{R}; H)$. Hence,

$$\widehat{\dot{u}}(i\rho) = \lim_{\lambda \to i\rho} \widehat{\dot{u}}(\lambda) = \frac{\psi(i\rho)}{\widehat{a}_2(i\rho)}$$

exists in $L^2_{\mathrm{loc}}(\mathbf{R}, H)$ since $\widehat{a}_2(i\rho) \neq 0$. Thus $\operatorname{sp}(\dot{u}) = \varnothing$ and $\dot{u}(t) \to 0$ in H as $t \to \infty$. □

4. Global asymptotic stability. In this section we prove $Du = u_x \in L^r(\mathbf{R}^+, H)$ for some $r \in [2, \infty)$ which, together with (3.10), implies $Du \to 0$ in H, i.e., $u(t) \to 0$ as $t \to \infty$ in $W^{1,2}(0,1)$. To do so we will need the following lemma, which expresses the fact that the operator $(d/dt)b*$ is accretive in $L^p(\mathbf{R}^+; X)$ for each $p \in [1, \infty]$ and each Banach space X, see Clément and Prüss [**3**].

Lemma 4.1. *Let X be a Banach space, $1 \leq p < \infty$, $b \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$ nonnegative, nonincreasing, and let B_p be defined in $L^p(\mathbf{R}^+; X)$ by*

$$(B_p u)(t) = \frac{d}{dt} b * u(t), \quad t \geq 0, \quad u \in D(B_p),$$

with domain

$$D(B_p) = \{u \in L^p(\mathbf{R}^+; X) : b * u \in \overset{\circ}{W}{}^{1,p}(\mathbf{R}^+; X)\}.$$

Then B_p is m-accretive. In particular, if $X = H$ is a Hilbert space, then

$$\int_0^T \langle Bu(t), u(t)\rangle \|u(t)\|_H^{p-2}\, dt \geq 0, \quad T > 0,$$

for each $u \in D(B_p)$.

Next we decompose Du given by (2.7) into two parts, corresponding to the decomposition (2.15).

$$\begin{aligned} Du(t) &= Du^0 + DS_1(t)u^1 + T * P_1 g(Du)(t) - k * P_1 g(Du)(t) \\ &= w(t) - k * P_1 g(Du)(t), \end{aligned} \tag{4.1}$$

where

$$w(t) = Du^0 + DS_1(t)u^1 + T * P_1 g(Du)(t). \tag{4.2}$$

Let B denote the inverse operator to $k*$, i.e.,

$$Bu(t) = \frac{d}{dt} b * u(t), \quad t \geq 0.$$

Then

$$\widehat{Bv}(\lambda) = \lambda \widehat{b}(\lambda)\widehat{v}(\lambda), \qquad Bv = \frac{d}{dt}(b * v),$$

and from (4.1) we have

$$B(Du - w) + P_1 g(Du) = 0.$$

The scalar product with $Du\|Du\|_H^{r-2}$ gives

$$\langle B(Du-w)(t), Du(t)\rangle \|Du(t)\|_H^{r-2} + \langle g(Du(t)), Du(t)\rangle \|Du(t)\|_H^{r-2} = 0,$$

and, according to Lemma 4.1 assumptions on g and boundedness of $\|Du\|_{L^\infty}$, there is a constant $c > 0$ such that

$$\int_0^T \langle B(Du)(t), Du(t)\rangle \|Du(t)\|_H^{r-2}\, dt \geq 0,$$
$$\langle g(Du)(t), Du(t)\rangle \|Du(t)\|_H^{r-2} \geq c\|Du(t)\|_H^r.$$

Hence

$$c\int_0^T \|Du(t)\|_H^r\, dt \leq \int_0^T \langle Bw(t), Du(t)\rangle \|Du(t)\|_H^{r-2}\, dt \quad \text{for all } T > 0. \tag{4.3}$$

To estimate further we need

Lemma 4.2. *Let u be a weak solution of* (1.1) *and w be given by* (4.2). *Then*

$$Bw \in L^r((1,\infty), H) \cap L^1((0,1), H), \tag{4.4}$$

for each $r \geq 1$ with $(1/r) < (1-\beta)/2$.

Proof. Having in mind $b * T = S$, we express Bw as follows

(4.5)
$$\begin{aligned} Bw(t) = \frac{d}{dt} b * w(t) &= b(t)Du^0 + b * DS(t)u^1 + \frac{d}{dt} S * P_1 g(Du)(t) \\ &= b(t)Du^0 - b * DS(t)u^1 + S(t)P_1 g(Du^0) + S * \frac{d}{dt} P_1 g(Du)(t). \end{aligned}$$

The first term belongs to $L^r((1,\infty);H) \cap L^1((0,1);H)$ for $(1/r) < 1-\beta$, $SP_1 g(Du^0) \in L^1(\mathbf{R}^+;H) \cap L^\infty(\mathbf{R}^+;H)$, according to (2.16). The remaining terms are handled once more with the help of Laplace transforms.

We use the following generalization of the Paley-Wiener theorem to the Laplace transform of Hilbert space valued L^2 functions.

Lemma 4.3. *Let $v \in L^2(\mathbf{R}^+, H)$. Then $\widehat{v}$ belongs to the Hardy space $\mathcal{H}_2(\mathbf{C}^+;H)$, i.e., is analytic for $\operatorname{Re}\lambda > 0$ and*
$$\sup_{\sigma>0} \int_{\mathbf{R}} \|\widehat{v}(\sigma + i\omega)\|_H^2 \, d\omega = 2\pi \int_{\mathbf{R}} \|v(t)\|_H^2 \, dt < \infty.$$
Conversely, to every H-valued function $\phi \in \mathcal{H}_2(\mathbf{C}^+;H)$, there is a unique function $v \in L^2(\mathbf{R}^+, H)$ satisfying $\widehat{v}(\lambda) = \phi(\lambda)$ for $\operatorname{Re}\lambda > 0$.

The Laplace transform of the second term in (4.5) is essentially given by
$$\mathcal{L}(b * DS)(\lambda)u^1 \sim \widehat{b}(\lambda) A^{1/2} \widehat{S}(\lambda) u^1,$$
which is bounded by $C(1/|1 + \lambda^{1+\beta}|^{1/2})\|u^1\|_H$. According to the preceding lemma, this implies $b * A^{1/2} S u^1 \in F^\gamma_{2,2}(\mathbf{R}^+;H)$ for $\gamma < \beta/2$. Since
$$F^\gamma_{2,2}(\mathbf{R}^+;H) \hookrightarrow L^r(\mathbf{R}^+;H) \quad \text{for } \frac{1}{r} \leq \frac{1}{2} - \gamma,$$
we obtain $b * A^{1/2} S u^1 \in L^r(\mathbf{R}^+;H)$ for $(1/r) < (1-\beta)/2$. Because of boundedness of $DA^{-1/2}$ we then also obtain $b * DSu^1 \in L^r(\mathbf{R}^+;H)$.

As for the last term, it is sufficient to show that $S * (d/dt) g(Du) \in L^r(\mathbf{R}^+;H)$. We have
$$\mathcal{L}\Big(S * \frac{d}{dt} g(Du)\Big) = \mathcal{L}\Big(T_1 * a_1 * \frac{d}{dt} g(Du)\Big),$$

where

$$\widehat{T}_1(\lambda) = \frac{1}{\widehat{a}_1(\lambda)}\widehat{S}(\lambda), \qquad a_1(t) = t^{(\beta/2)-1}/\Gamma\left(\frac{\beta}{2}\right).$$

$(1 + \lambda^{1-(\beta/2)})\widehat{T}_1(\lambda)$ is bounded in $\operatorname{Re}\lambda > 0$ and $a_1 * (d/dt)g(Du) \in L^2(\mathbf{R}^+, H)$ according to Lemma 3.1. Hence Lemma 4.2 applies and

$$S * \frac{d}{dt}g(Du) \in F_{2,2}^{1-(\beta/2)}(\mathbf{R}^+; H) \hookrightarrow \mathrm{BUC}^{(1-\beta)/2}(\mathbf{R}^+; H).$$

Combining these results we get $Bw \in L^r((1,\infty); H)$. □

From the relation (4.3) we have

$$\begin{aligned} c\int_0^T \|Du(t)\|_H^r\, dt &\le \int_0^T \langle Bw(t), Du(t)\rangle \|Du(t)\|_H^{r-2}\, dt \\ &\le \|Bw(t)\|_{L^1((0,1);H)} \sup_{t\in[0,1]} \|Du(t)\|_{L^\infty}^{r-1} \\ &\quad + \|Bw\|_{L^r((1,T),H)} \|Du\|_{L^r((1,T),H)}^{r-1} \\ &\le C + \|Bw\|_{L^r((1,\infty),H)} \|Du\|_{L^r((0,T),H)}^{r-1} \end{aligned}$$

for all $T > 0$. This implies $Du \in L^r(\mathbf{R}^+, H)$. By $Du \in \mathrm{BUC}^{(1-\beta)/2}(\mathbf{R}^+; H)$ we obtain $Du(t) \to 0$ in H as $t \to \infty$, and then also $Du(t) \to 0$ in every $L^q(0,1)$ because of uniform boundedness of Du in $L^\infty(0,1)$. By means of the integral equation for $\dot{u}$ and the properties of S and DS, this in turn yields $\dot{u}(t) \to 0$ in $L^q(0,1)$ as $t \to \infty$.

To prove the remaining assertions of Theorem 1.1 we need another lemma which deals with the linearization of (1.1), i.e., with the linear problem

$$\ddot{u} + b * A\dot{u} + \eta A u = f, \quad t > 0, \quad u(0) = u^0,\ \dot{u}(0) = u^1. \tag{4.6}$$

The solution of this problem is given by the variation of parameters formula

$$u(t) = C(t)u^0 + R(t)u^1 + \int_0^t R(t-s)f(s)\, ds, \quad t \ge 0. \tag{4.7}$$

By means of results from Prüss [**12**] (see also the discussions in Fašangová and Prüss [**4**], [**5**]), we have the following properties of the operator families $C(t)$ and $R(t)$, which resemble the operator-cosine and -sine families for the case $b \equiv 0$. Their Laplace transforms are given by

$$\widehat{C}(\lambda) = (\lambda + \widehat{b}(\lambda)A)\widehat{R}(\lambda),$$

and

$$\widehat{R}(\lambda) = (\lambda^2 + \lambda\widehat{b}(\lambda)A + \eta A)^{-1},$$

for $\operatorname{Re}\lambda > 0$, and we let

$$\widehat{T}(\lambda) = \widehat{R}(\lambda)/\lambda.$$

Lemma 4.4. *Let $b(t) = t^{\beta-1}/\Gamma(\beta)$ for $t > 0$, and let $\eta > 0$. Then the operator families $C(t)$, $A^{1/2}R(t)$, $AT(t)$, $S(t) = \dot{R}(t)$, $A^{-1/2}\dot{C}(t)$ exist, are strongly continuous on $\mathbf{R}^+$ in $X = L^q(0,1)$, $1 \le q < \infty$, and in $X = C_0[0,1]$ and are uniformly bounded. There is a function $\phi \in L^1(\mathbf{R}^+) \cap C_0(\mathbf{R}^+)$ such that*

$$\|S(t)\|_{\mathcal{B}(X)} + \|A^{1/2}R(t)\|_{\mathcal{B}(X)} + \|A^{-1/2}\dot{C}(t)\|_{\mathcal{B}(X)} \le \phi(t), \quad t \ge 0,$$

i.e., $S(t)$, $A^{1/2}R(t)$, $A^{-1/2}\dot{C}(t)$ are integrable and converge to zero in $\mathcal{B}(X)$. Also we have $\lim_{t\to\infty} C(t) = 0$ as well as $\lim_{t\to\infty} AT(t) = 1/\eta$ strongly in $\mathcal{B}(X)$. In addition, AR, $A^{1/2}S \in L^1(\mathbf{R}^+;\mathcal{B}(X))$. These assertions are also valid in $X = L^\infty(0,1)$, except that $S(t)$ is not continuous at zero.

Proof. The first assertions are contained in Propositions 1 and 2 of [**4**] for the Hilbert space case. However, these claims remain valid in the Banach case, as soon as A generates a uniformly bounded cosine family which is the case here. In fact, it is even sufficient that A is sectorial of sufficiently small spectral angle and invertible. We refer to Prüss [**12**, Sections 4 and 10.3] for $S(t)$. $A^{1/2}R(t)$ can be handled in the same way, and we have the relations

$$AT(t) = 1 * r * r(t) - r * r * S(t), \quad t \ge 0,$$
$$C(t) = 1 - \eta AT(t), \quad \dot{C}(t) = -\eta AR(t), \quad t \ge 0,$$

where $r(t)$ is defined by $\widehat{r}(\lambda) = 1/\sqrt{\lambda\widehat{b}(\lambda)+\eta}$. Observe that $r \in L^1(\mathbf{R}^+)$ is nonnegative and $\widehat{r}(0) = \int_0^\infty r(t)\,dt = 1/\sqrt{\eta}$. So we have to prove only the very last statement. Applying Lemma 2.2 directly, we obtain

$$\|AR(t)\|_{\mathcal{B}(X)} \le ct^{-\beta}, \qquad \|A^{1/2}S(t)\|_{\mathcal{B}(X)} \le ct^{-(1+\beta)/2}, \quad t>0,$$

which shows that both quantities are well-defined and locally integrable. On the other hand, for $tAR(t)$ and $tA^{1/2}S(t)$, Lemma 2.2. yields

$$\|AR(t)\|_{\mathcal{B}(X)} \le ct^{\beta-2}, \qquad \|A^{1/2}S(t)\|_{\mathcal{B}(X)} \le ct^{\beta-2}, \quad t>0,$$

which then yields the asserted integrability. □

Having the information of this lemma available with $\eta > 0$ small enough, we may proceed as before to obtain the following equation for $Du(t)$.

$$Du(t) = C(t)Du^0 + DR(t)u^1 - AR * P_1[g(Du) - \eta Du](t),$$

hence

$$Du + k_\eta * [g(Du) - \eta Du] = w,$$

where

$$w(t) = C(t)Du^0 + DR(t)u^1 + T_\eta * P_1[g(Du) - \eta Du](t) + k_\eta * Pg(Du)(t),$$
$$\widehat{k}_\eta(\lambda) = \widehat{r}^2(\lambda) = 1/(\lambda^{1-\beta} + \eta),$$

and

$$\widehat{T}_\eta(\lambda) = \lambda^2 \widehat{k}_\eta(\lambda)\widehat{R}(\lambda).$$

Since we know $\|Du(t)\|_{L^\infty} \le M$ on $\mathbf{R}^+$ for some constant $M > 0$, by the assumptions on g there is a constant $\eta > 0$ such that $sg(s) - \eta s^2 \ge 0$ for all $|s| \le M$. Therefore we may apply [**8**, Chapter 20, Theorem 3.1] once more to obtain $\|Du(t)\|_{L^\infty} \to 0$ as $t \to \infty$, once we have $\|w(t)\|_{L^\infty} \to 0$. This will follow from the facts that $\|Du(t)\|_{L^r} \to 0$ as $t \to \infty$, for any $r \in [1, \infty)$, and properties of the operator families introduced in Lemma 4.6. In fact, we have $C(t)Du^0 \to 0$ by the properties of $C(t)$; $r_\eta \in L^1(\mathbf{R}^+)$ implies $k_\eta * Pg(Du)(t) \to 0$, and the estimates on $R(t)$ given in the proof of Lemma 4.6 show

$$\|DR(t)u^1\|_{L^\infty} \leq \|DA^{-1}\|_{\mathcal{B}(L^q,L^\infty)}\|AR(t)u^1\|_{L^q} \leq Ct^{\beta-2}\|u^1\|_{L^q} \to 0.$$

To conclude, observe that, by Lemma 2.2, we obtain

$$T_\eta \in L^1(\mathbf{R}^+;\mathcal{B}(L^r(0,1),C^\mu[0,1])), \quad \text{for } 0<\mu<2\frac{1-\beta}{1+\beta}-\frac{1}{r}.$$

This then implies convergence to zero in $L^\infty(0,1)$ of the third term in the definition of $w(t)$, hence $\|Du(t)\|_{L^\infty} \to 0$ as $t \to \infty$.

As for the convergence of strong solutions, with $\eta = g'(0) > 0$, we may rewrite problem (1.1) $\sim$ (1.3) as

$$u(t) = C(t)u^0 + R(t)u^1 + \int_0^t R(t-s)[g'(0)-g'(Du(s))]Au(s)\,ds, \quad t \geq 0,$$

which yields the following inequality for $\phi(t) = \|Au(t)\|_X$

$$\phi(t) \leq \phi_0(t) + \int_0^t \gamma(t-s)\|g'(0)-g'(Du(s))\|_{L^\infty}\phi(s)\,ds, \quad t \geq 0,$$

where

$$\phi_0(t) = \|C(t)Au^0 + A^{1/2}R(t)A^{1/2}u^1\|_X \quad \text{and} \quad \gamma(t) = \|AR(t)\|_{\mathcal{B}(X)}.$$

Because of $\|Du(t)\|_{L^\infty} \to 0$ as $t \to \infty$ and $\phi_0(t) \to 0$, this implies $\phi(t) = \|Au(t)\|_X \to 0$. Here X may be any of the spaces $X = L^p(0,1)$, $1 \leq p < \infty$ or $X = C_0[0,1]$. In a similar way one also gets $\|A^{1/2}\dot{u}(t)\|_X \to 0$ as $t \to \infty$. This proves the last statement of Theorem 1.1. $\square$

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