JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 12, Number 2, Summer 2000

A GALERKIN-WAVELET METHOD FOR A SINGULAR CONVOLUTION EQUATION ON THE REAL LINE

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ABSTRACT. A Galerkin method based on bandlimited wavelet bases is proposed for solving the singular convolution equation on the real line of the form

$$\int_{-\infty}^{\infty} (H(t-s) + |t-s|^{-\alpha}) f(s) ds = g(t), \quad 0 < \alpha < 1, t \in R.$$

The proposed method allows the discretized equation to be well posed due to the exceptional property of bandlimited wavelets in Fourier domain. Under certain smoothness conditions on righthand side function g, we prove that the approximate solution is convergent of higher order.

1. Introduction. There has been considerable interest in solving differential and integral equations using techniques which involve wavelet bases. In most of the cases, compactly supported wavelets, such as Daubechies wavelets or coiflets, are used as bases [**4**]. These lead to sparse matrix representation which are useful for the efficient numerical implementation. They have been used with much success in second kind integral equations, but are little used in first kind equations, since their advantages (vanishing moments, finite number of terms in dilation equations) are offset by the ill posed nature of such equations. In this case the compact support in the frequency domain seems to serve us better.

For example, consider the following convolution:

$$\int_{-\infty}^{\infty} K(t-s) f(s)\, ds = g(t), \quad t \in R. \tag{1.1}$$

Received by the editors on February 5, 1998, and in revised form on July 7, 1998.
1991 AMS *Mathematics Subject Classification.* Primary 42A10, 42A15, Secondary 41A05.
Key words and phrases. Galerkin method, Meyer wavelet, raised-cosine wavelet, singular kernel, convolution equation.

with associated integral operator defined by:

$$\mathbf{K} : f(t) \longrightarrow \int_{-\infty}^{\infty} K(t-s) f(s)\, ds \quad t \in R. \tag{1.2}$$

We assume the kernel $K(z) = H(z) + |z|^{-\alpha}$, $0 < \alpha < 1$, the second part of which carries the singularity of the kernel, while H satisfies the following:

H1.1. its Fourier transform $\widehat{H}(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt > 0$, and

H1.2. $H \in \mathbf{H}^{\beta}(R)$, $\beta \geq 2$, where $\mathbf{H}^{\beta}(R)$ is the Sobolev space with parameter β.

We observe that, under the assumptions H1.1 and H1.2, the kernel $K \notin L^1(R)$; therefore, the associated operator is not necessarily compact on $L^2(R)$. In other words, (1.1) may possibly be well-posed. However if we use the compactly supported wavelet bases we would normally consider both the integral and domain to be over a finite interval. This would usually lead to a finite section equation as its discretized system [**3**], [**6**] and [**7**]. We thus would end up with an ill-posed problem since the discretized integral operator is compact [**2**]. Consequently, the numerical solution may not be stable, even if it exists uniquely in the first place. In other words, a suitable choice of basis for this type of equation is crucial, and should be one, *which does not lead to a finite section equation and then keep the well-posed character.*

In this paper we go to another approach. We consider the Galerkin method with Meyer type wavelets, i.e., band limited wavelets, as bases. Since this set of bases is not compactly supported in the time domain, we avoid the need to consider a finite section equation. On the other hand, we still benefit from the combination of localization property of wavelets and the singularity of the kernel. We expect a relatively stable numerical procedure.

To be specific, we choose a particular Meyer wavelet family, the "raised-cosine wavelets" as bases. These raised-cosine wavelets are based on some pulses frequently used in digital communication and signal processing that are characterized by a raised-cosine spectrum. Like Shannon wavelets, they have simple analytic forms [**14**], but unlike Shannon wavelets, they have a vanishing moment. Other Meyer wavelets with as many vanishing moments as we wish may be used instead. We shall work mostly with the raised cosine in this paper.

We organize this paper as follows. This section follows by Section 2, in which we list some related definitions and properties of raised-cosine wavelets. In Section 3 we introduce the Galerkin method with raised-cosine wavelet basis. By showing the invertibility of the discrete linear system, the existence and uniqueness of the solution in scaling subspaces are derived in Section 4. Also, the method is shown to be of greater than linear approximation order under certain assumptions on the righthand side function and the solution function. In fact, this convergence rate is limited only by the properties of the particular kernel. An error bound for this approximation is also obtained.

2. Preliminary properties of raised-cosine wavelets. The scaling function and 1/2-shifted mother wavelet of raised cosine wavelet are defined as:

$$\phi(t) = \frac{1}{\pi t[1-(4\beta t)^2]}\left[\sin \pi(1-\beta)t - 4\beta t \cos \pi(1+\beta)t\right] \tag{2.1}$$

$$\begin{aligned}\psi\left(t+\frac{1}{2}\right) &= \frac{1}{\pi t[(4\beta t)^2-1]}\left[\sin \pi(1+\beta)t - 4\beta t\cos\pi(1-\beta)t\right] \\ &\quad - \frac{1}{\pi t[(8\beta t)^2-1]}\left[\sin 2\pi(1-\beta)t + 8\beta t\cos 2\pi(1+\beta)t\right]\end{aligned} \tag{2.2}$$

where $0 \le \beta \le 1/3$.

In frequency domain, we have,

$$\hat{\phi}(\omega) = \begin{cases} 1 & 0 \le |\omega| \le \pi(1-\beta), \\ \cos\left[\dfrac{|\omega|}{4\beta} - \dfrac{\pi(1-\beta)}{4\beta}\right] & \pi(1-\beta) \le |\omega| \le \pi(1+\beta), \\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

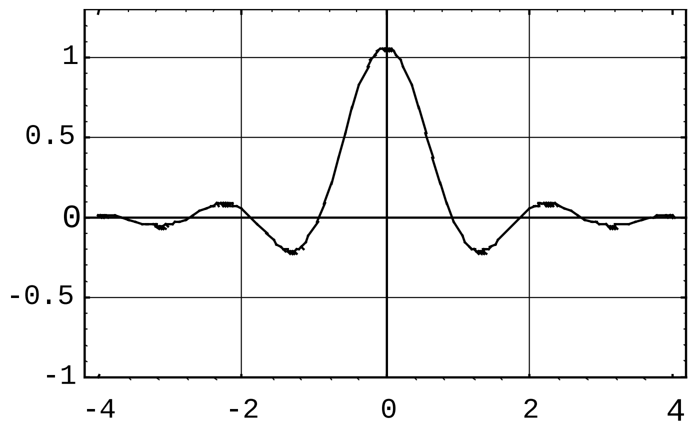

FIGURE 1. The scaling function $\phi(t)$.

(2.4)

$$\begin{aligned}\hat{\psi}(\omega) &= e^{-i(\omega/2)}[\hat{\phi}(\omega+2\pi)+\hat{\phi}(\omega-2\pi)]\hat{\phi}(\omega/2)\\ &= \begin{cases} 0 & 0 \le |\omega| \le \pi(1-\beta),\\ \cos\left[\dfrac{\pi-|\omega|}{4\beta}+\dfrac{\pi}{4}\right] & \pi(1-\beta) \le |\omega| \le \pi(1+\beta),\\ 1 & \pi(1+\beta) \le |\omega| \le 2\pi(1-\beta),\\ \cos\left[\dfrac{|\omega|/2-\pi(1-\beta)}{4\beta}\right] & 2\pi(1-\beta) \le |\omega| \le 2\pi(1+\beta),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Their graphs are shown in Figures 1–4.

From now on we work with raised-cosine wavelets. The properties in this section refer to the raised-cosine wavelets only.

We will state some basic properties for these wavelets. For the sake of simplicity, through this paper, we take $\beta = 1/4$; other cases could be carried out in the same way. Since we are dealing with a particular wavelet with closed form, most of the properties can be derived from direct calculations.

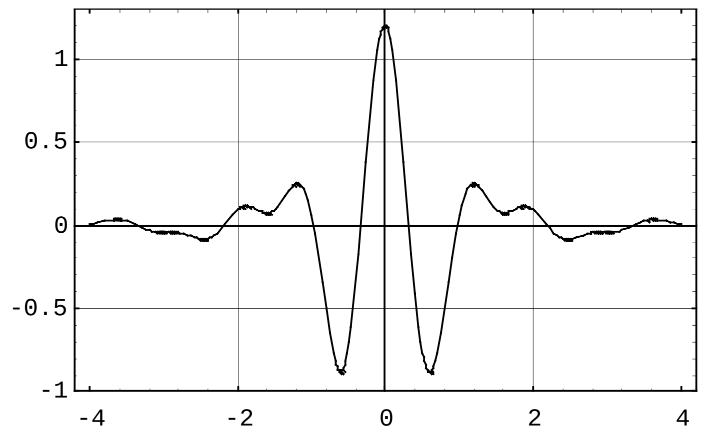

FIGURE 2. The 1/2-shifted mother wavelet $\psi(t+1/2)$.

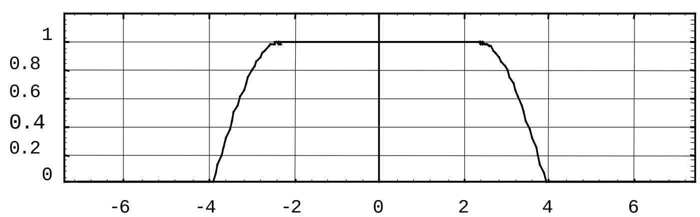

FIGURE 3. The Fourier transform of scaling function $\hat{\phi}(\omega)$.

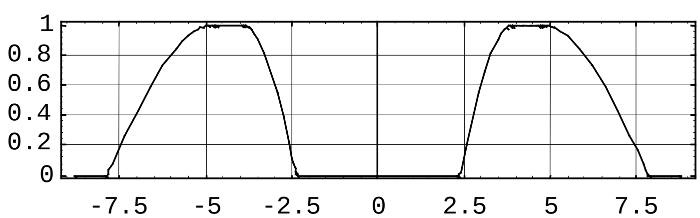

FIGURE 4. The absolute value of $\hat{\psi}(\omega)$.

Property 2.1 [**14**]. *The dilation coefficients in the refinement equation are given by* $h(k) = (1/\sqrt{2})\phi(k/2)$, *with Fourier transform* $m_0(\omega) = \sum_{k=-\infty}^{\infty} \hat{\phi}(2\omega + 4\pi)$. *Moreover* $|h(k)| = O(k^{-2})$, $k \in R$.

We say f satisfies a fast decay condition if

$$f \in A_r = \{f \mid x^r f \in L^1(R), \text{ for some } r > 1\}.$$

Property 2.2. *If* $f \in \mathbf{H}^\alpha(R) \cap A_r$, α, $r \geq 2$, *then*

$$(f, \phi_{m,n}) = C_1 2^{(5m/2)} n^{-2}, \qquad (f, \psi_{m,n}) = C_2 2^{(5m/2)} n^{-2},$$

where $\phi_{m,n}(t) = 2^{m/2}\phi(2^m t - n)$, $\psi_{m,n}(t) = \psi(2^m t - n)$, *and* C_i, $i = 1, 2$, *are constants.*

The proofs are straightforward calculations.

Property 2.3. *Let* V_m *be the* MRA *associated with the raised-cosine scaling function and* $\mathbf{P}_m$ *the orthogonal projection onto* V_m. *If* $f \in \mathbf{H}^\beta(R)$, $\beta > (3/2)$, *we have*

$$||P_m f - f||_\infty = C_\beta ||f||_{\mathbf{H}^\beta}(2^{-2m}), \quad m \in N, \tag{2.5}$$

where C_β *depends on* β *but is independent of* m *and* f.

For the proof, we refer the reader to [**10**, pp. 125–128]. Although there is a slight change in the hypothesis, the proof can go through in a similar way. □

The next lemma is important in the setting of discretization.

Lemma 2.1. *Assume that the Fourier transform of* K *exists in the sense of temped distribution. If* $f \in V_m$, *then* $K * f \in V_{m+1}$.

Proof. Consider the expansion

$$K * f(t) = \sum_{n=-\infty}^{\infty} (f, \phi_{mn}) K * \phi_{mn}(t).$$

By taking the Fourier transform, we have

$$\begin{aligned}\widehat{K}(\omega)\hat{f}(\omega) &= \left(\frac{1}{2^m}\sum_{n=-\infty}^{\infty}(f,\phi_{mn})e^{-i(n/2^m)\omega}\widehat{K}(\omega)\hat{\phi}\left(\frac{\omega}{2^m}\right)\right)\\ &= \alpha_m^f(\omega)\widehat{K}(\omega)\hat{\phi}\left(\frac{\omega}{2^m}\right),\end{aligned}$$

where $\alpha_m^f(\omega) = (1/2^m)\sum_{n=-\infty}^{\infty}(f,\phi_{mn})e^{-i(n/2^m)\omega}$ is a periodic function with period $2^{m+1}\pi$.

Since $\hat{\phi}(\omega/2^{m+1}) = 1$ on the support of $\hat{\phi}(\omega/2^m)$, we can rewrite the above expression as

$$\widehat{K}(\omega)\hat{f}(\omega) = \alpha_m^f(\omega)\widehat{K}(\omega)\hat{\phi}\left(\frac{\omega}{2^m}\right)\hat{\phi}\left(\frac{\omega}{2^{m+1}}\right).$$

This shows that the support of $\widehat{K}(\omega)\hat{f}(\omega)$ is contained in $[-2^{m+1}(4\pi/3), 2^{m+1}(4\pi/3)]$. We extend $\widehat{K}(\omega)\hat{f}(\omega)$ periodically $2^{m+2}\pi$, to get

$$y_{m+1}(\omega) = \sum_{k=-\infty}^{\infty}\widehat{K}(\omega + 2^{m+2}\pi)\hat{\phi}\left(\frac{\omega + 2^{m+2}\pi k}{2^m}\right).$$

Since $y_{m+1}(\omega) = \widehat{K}(\omega)\hat{\phi}(\omega/2^m)$ on the support of $\hat{\phi}(\omega/2^m)$, we may write

$$\hat{f}(\omega) = \alpha_{m+1}(\omega)\hat{\phi}\left(\frac{\omega}{2^{m+1}}\right),$$

where $\alpha_{m+1}(\omega) = \alpha_m^f(\omega)y_{m+1}(\omega)$ is a periodic function of period $2^{m+1}\pi$. Therefore $K * f \in V_{m+1}$. This completes the proof of the lemma. $\square$

3. Galerkin method with raised-cosine basis.

3.1 The Galerkin formalism. Define the projection $\mathbf{P}_m : L^2(R) \to V_m = \overline{\text{span}\,\{\phi_{m,k}(t)\}}_{k=-\infty}^{\infty}$ by

$$\mathbf{P}_m(f)(t) := \sum_{k=-\infty}^{\infty} f_{m,k}\phi_{m,k}(t), \quad m \in Z. \tag{3.1}$$

Now we are in position to define the Galerkin-wavelet method. We start with the following projection equation:

$$\mathbf{P}_{m+1}(\mathbf{K}\mathbf{P}_m)f(t) = \mathbf{P}_{m+1}g(t). \tag{3.2}$$

We observe the fact that if $P_m(f)(t) \in V_m$, then $\mathbf{K}\mathbf{P}_m(f)(t) \in V_{m+1}$, see Lemma 2.1. We then have

$$(\mathbf{K}\mathbf{P}_m)f(t) = \mathbf{P}_{m+1}g(t),$$

that is,

$$\sum_{k=-\infty}^{\infty} f_{m,k} \int_{-\infty}^{\infty} K(s-t)\phi_{m,k}(s)\,ds = \sum_{k=-\infty}^{\infty} g_{m+1,k}\phi_{m+1,k}(t). \tag{3.3}$$

Since $V_{m+1} = V_m \oplus W_m$, we can rewrite the righthand side as:

$$\sum_{k=-\infty}^{\infty} (g_{mk}^{(1)}\phi_{mk}(t) + g_{mk}^{(1)}\psi_{mk}(t)) = g_m^{(1)}(t) + g_m^{(2)}(t).$$

By Mallat's decomposition algorithm, see, for example, [**10**, p. 44], we have

$$g_{mk}^{(1)} = \sum_{k=-\infty}^{\infty} g_{m+1,k}c_{k-2n}, \tag{3.4}$$

$$g_{mk}^{(2)} = \sum_{k=-\infty}^{\infty} g_{m+1,k}(-1)^k c_{1-k+2n}. \tag{3.5}$$

Scaling function solution. We multiply both sides of (3.3) by $\phi_{mn}(t)$ and integrate with respect to t on the real line. By using the orthogonality, we have the following linear system:

$$\sum_{k=-\infty}^{\infty} \sigma_{nk}^m f_{mk} = g_{mn}^{(1)}, \quad n \in Z, \tag{3.6}$$

where

$$\sigma^m_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(s-t)\phi_{mn}(s)\phi_{mk}(t)\,ds\,dt, \quad n,k \in Z. \tag{3.7}$$

$$g^{(1)}_{mn} = \sum_{k=-\infty}^{\infty} g_{m+1,k}c_{k-2n}, \quad n \in Z. \tag{3.8}$$

In this case, the dilation coefficients are simply the function values, that is,

$$c_{k-2n} = \frac{1}{\sqrt{2}}\phi\biggl(\frac{k-2n}{2}\biggr), \tag{3.9}$$

see Property 2.1. A proof of the general case for Meyer wavelets can be found in [**11**, p. 503].

To calculate the elements of the coefficient matrix, we rewrite σ^m_{nk} formally as follows:

$$\begin{aligned}\sigma^m_{nk} &= 2^m \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(s-t)\phi(2^m s-n)\phi(2^m t-k)\,ds\,dt \\ &= 2^{-m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K\biggl(\frac{x-y}{2^m}\biggr)\phi(x)\phi(y-(n-k))\,dx\,dy,\end{aligned}$$

which enables us to write

$$\sigma^m_{nk} = \sigma^m_{n-k}. \tag{3.10}$$

By symmetry we only need to calculate σ^m_k, $k = 0, 1, \ldots$.

By using Parseval's identity, we have

$$\begin{aligned}\sigma^m_k &= 2^{-m}\int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty} K\biggl(\frac{x-y}{2^m}\biggr)\phi(x)\,dx\phi(y-k)\biggr)\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty}\widehat{K}(2^m\omega)\overline{\hat{\phi}(\omega)}e^{-iy\omega}d\omega)\phi(y-k)\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty}\phi(y-k)e^{-iy\omega}\,dy\biggr)\widehat{K}(2^m\omega)\overline{\hat{\phi}(\omega)}\,d\omega \\ &= \frac{1}{2\pi}\int_{-(5\pi/4)}^{(5\pi/4)} e^{-ik\omega}\widehat{K}(2^m\omega)\hat{\phi}^2(\omega)\,d\omega.\end{aligned} \tag{3.11}$$

The interchange of integrals is allowed by Fubini's theorem since $\phi \in L^1(R)$ and $\hat{\phi}$ has compact support.

In particular, for the raised-cosine wavelets, the coefficients are given by

(3.12)
$$\sigma_k^m = \frac{1}{2\pi}\bigg[\int_{|\omega|\le(3\pi/4)} e^{-ik\omega}\widehat{K}(2^m\omega)\,d\omega + \int_{(3\pi/4)\le|\omega|\le(5\pi/4)} e^{-ik\omega}\widehat{K}(2^m\omega)\cos^2\left(|\omega| - \frac{3\pi}{4}\right)d\omega\bigg].$$

They will be discretized by using a quadrature formula and then used in numerical implementations.

We rewrite (3.6) as the following matrix equation:

$$\mathbf{S}_m\mathbf{X}_m = \mathbf{G}_m^{(1)}, \tag{3.13}$$

where $\mathbf{S}_m = \{\sigma_{n-k}^m\}$, $\mathbf{X}_m = \{f_{mk}\}$, and $\mathbf{G}_m^{(1)} = \{g_{mk}^{(1)}\}$. Notice that (3.13) is an infinite discrete convolution equation.

Mother wavelet solution. We multiply both sides of (3.3) by $\psi_{mn}(t)$ and go through the exactly same calculation procedures. We obtain another discrete convolution equation:

$$\sum_{k=-\infty}^{\infty} b_{n-k}^m f_{mk} = g_{mn}^{(2)}, \quad n \in Z, \tag{3.14}$$

where

$$b_{n-k}^m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(s-t)\phi_{mn}(s)\psi_{mk}(t)\,ds\,dt, \quad n,k \in Z. \tag{3.15}$$

$$g_{mn}^{(2)} = \sum_{k=-\infty}^{\infty} (-1)^k g_{m+1,k}c_{1-k+2n}, \quad n \in Z. \tag{3.16}$$

By using (2.4), we have,

$$b_k^m = \frac{1}{2\pi}\int_{(3\pi/4)\le|\omega|\le(5\pi/4)} e^{-i(k+(1/2))\omega}\widehat{K}(2^m\omega)\hat{\phi}(\omega)\hat{\phi}\left(\frac{\omega}{2}\right) \cdot [\hat{\phi}(\omega+2\pi) + \hat{\phi}(\omega-2\pi)]\,d\omega. \tag{3.17}$$

In particular, for raised-cosine wavelets, the coefficients are given by

$$b_k^m = \frac{1}{4\pi} \int_{(3\pi/4)\le|\omega|\le(5\pi/4)} e^{-i(k+(1/2))\omega} \widehat{K}(2^m\omega) \cos 2\omega \, d\omega. \tag{3.18}$$

In this case, we have the following matrix equation:

$$\mathbf{B}_m \mathbf{X}_m = \mathbf{G}_m^{(2)}, \tag{3.19}$$

where $\mathbf{B}_m = \{b_{n-k}^m\}$, $\mathbf{X}_m = \{f_{mk}\}$ and $\mathbf{G}_m^{(2)} = \{g_{mn}^{(2)}\}$.

3.2 About the coefficient matrices $\mathbf{S}_m$, $\mathbf{B}_m$ and their associated operators. From the definitions, it is clear that both two coefficient matrices are Toeplitz matrices. We summarize their properties as in the following:

Lemma 3.1. *The elements of the coefficient matrix $\mathbf{S}_m$ satisfy the following properties*:

(i) $\sigma_0^m > 0$,

(ii) S_m *is symmetric if the Fourier transform of* H *is*,

(iii) $|\sigma_k^m| \le (C_\alpha/2^{(1-\alpha)m})(1/k^\alpha))$, *where* C_α *is a constant independent of* m *and* k.

Since we assume that the Fourier transform of the kernel is nonnegative, (i) holds. (ii) is derived from the definition of σ_k^m. To prove (iii), we write

$$\begin{aligned}
\sigma_k^m &= \frac{1}{2\pi} \int_{-(5\pi/4)}^{(5\pi/4)} e^{-ik\omega} \hat{\phi}^2(\omega) (\widehat{H}(2^m\omega) + |2^m\omega|^{\alpha-1}) \, d\omega \\
&= \frac{1}{2\pi} \int_{-(5\pi/4)}^{(5\pi/4)} e^{-ik\omega} \hat{\phi}^2(\omega) (\hat{H}(2^m\omega) \, d\omega \\
&\quad + C(m,\alpha) \bigg[\int_0^{(3\pi/4)} \frac{\cos k\omega}{\omega^{1-\alpha}} \, d\omega + \frac{1}{2} \int_{(3\pi/4)}^{(5\pi/4)} \frac{\cos k\omega}{\omega^{1-\alpha}} (1 - \sin 2\omega) \, d\omega \bigg] \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $C(m,\alpha) = (2/\pi) 2^{(\alpha-1)m} \Gamma(1-\alpha) \cos(\pi/2)(1-\alpha)$.

Since the condition H2 implies $\widehat{H}(\omega)$ is differentiable and in $L^2(R)$, and because the same is true for $\hat{\phi}^2(\omega)$, $I_1 = O(k^{-2})$. For I_2, a similar argument can be used to I_3, we can write,

$$C(m,\alpha)\int_0^{(3\pi/4)} \frac{\cos k\omega}{\omega^{1-\alpha}}\, d\omega = \frac{C(m,\alpha)}{k^\alpha}\int_0^{(3\pi k/4)} \frac{\cos k\xi}{\xi^{1-\alpha}}\, d\xi.$$

Since $\int_0^{(3\pi k/4)}(\cos k\xi/\xi^{1-\alpha})\, d\xi$ converges, we have our result.

As an example, we choose $H \equiv 0$, then (3.12) is given by:

(3.20)

$$\sigma_k^m = C(m,\alpha)\Big[\int_0^{(3\pi/4)} \frac{\cos k\omega}{\omega^{1-\alpha}}\, d\omega + \frac{1}{2}\int_{(3\pi/4)}^{(5\pi/4)} \frac{\cos k\omega}{\omega^{1-\alpha}}(1-\sin 2\omega)\, d\omega\Big].$$

We define the associate linear operator $\mathfrak{S}_m$ with respect to $\mathbf{S}_m$ as follows

$$\mathfrak{S}_m(\xi) = \sum_{k=-\infty}^{\infty} \sigma_k^m e^{i\xi k}. \tag{3.21}$$

This series may not converge pointwise but always converges in the sense of tempered distributions. That is,

$$\langle \mathfrak{S}_m, \theta\rangle = \lim_{N\to\infty}\Big\langle \sum_{k=-N}^{N} \sigma_k^m e^{i\xi k}, \theta(\xi)\Big\rangle$$

exists, where θ is the testing function.

Similar to Lemma 3.1, we have

Lemma 3.2. The elements of the coefficient matrix satisfy the following properties:

(i) $\mathbf{B}_m$ *is not symmetric, even if the Fourier transform of kernel of* (1.1) *is symmetric.*

(ii) $\{b_k^m\} \in l^2 \cap l^1$, *in fact,* $|b_k^m| = O(\frac{1}{k^2})$.

(iii) $\lim_{m\to\infty} \mathbf{B}_m = 0$, *uniformly.*

3.3 Solvability. In this subsection, we discuss the solvability of the infinite discrete convolution equations (3.13) and (3.19).

We adopt the following definition.

Definition 3.1 [**5**]. *The Galerkin method is applicable to the operator K related to the system $(\alpha_j, \mathbf{P}_j)_j$ if for any g, the system* (3.2), *or equivalently,* (3.3), *beginning with some N, has a unique solution.*

The following theorem will lead to the result that Galerkin method, based on our choice of the basis and the definition of projections, is applicable to the operator $\mathbf{K}$.

Theorem 3.3. *The coefficient matrix of the infinite discrete convolution equation* (3.13) *is invertible.*

Proof. Consider

$$\begin{aligned}
\mathfrak{S}_m(\omega) &= \sum_{k=-\infty}^{\infty} \sigma_k^m e^{ik\omega} \\
&= 2^{-m} \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K\left(\frac{x-y}{2^m}\right) \phi(x)\phi(y-k)\, dx\, dy\, e^{ik\omega} \right. \\
&= 2^{-m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K\left(\frac{x-y}{2^m}\right) \left(\sum_{k=-\infty}^{\infty} \phi(y-k) e^{ik\omega} \right) \phi(x)\, dx\, dy \\
&= 2^{-m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K\left(\frac{x-y}{2^m}\right) \left(\sum_{k=-\infty}^{\infty} \hat{\phi}(2n\pi+\omega) e^{iy(2n\pi+\omega)} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \phi(x)\, dx\, dy \\
&= 2^{-m} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K\left(\frac{x-y}{2^m}\right) \phi(x) \right) dx\, e^{iy(2n\pi+\omega)})\, dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \hat{\phi}(2n\pi+\omega) \\
&= 2^{-m} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(K\left(\frac{\cdot}{2^m}\right) * \phi \right)(y) e^{iy(2n\pi+\omega)} dy \hat{\phi}(2n\pi+\omega)
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \widehat{K}(2^m(2n\pi+\omega))\hat{\phi}^2(2n\pi+\omega).$$

These formal series are to be taken in the sense of distributions for which they converge. They may also converge in a stronger sense and, in fact the last series is locally finite. Notice that $\mathfrak{S}_m(\omega) = \mathfrak{S}_m(\omega+2\pi)$, we only need consider the case $|\omega| < \pi$. We then have

$$\begin{aligned}\mathfrak{S}_m(\omega) &= \widehat{K}(2^m\omega)\hat{\phi}^2(\omega)\\ &\quad + \sum_{n=1}^{\infty} \widehat{K}(2^m(2n\pi-\omega))\hat{\phi}^2(2n\pi-\omega)\\ &\quad + \sum_{n=1}^{\infty} \widehat{K}(2^m(2n\pi+\omega))\hat{\phi}^2(2n\pi+\omega)\\ &= \widehat{K}(2^m\omega)\hat{\phi}^2(\omega) + \widehat{K}(2^m(\omega-2\pi))\hat{\phi}^2(\omega-2\pi)\\ &\quad + \widehat{K}(2^m(\omega+2\pi))\hat{\phi}^2(\omega+2\pi) > 0.\end{aligned}$$

Since the eigenvalue of the matrix $\mathbf{S}_m$ is $\mathfrak{S}_m(\omega)$, the result follows. (See [**9**, p. 304] for more discussion on eigenvalues of a Toeplitz matrix.)

Corollary 3.4. *The truncated linear system*:

$$\mathbf{S}_{mn}\mathbf{X}_{mn} = \mathbf{G}_{mn}^{(1)}, \quad n \in N, \tag{3.22}$$

where $\mathbf{S}_{mn} = \{\sigma_{j-k}^m\}_{j,k=-n}^n$, $\mathbf{X}_{mn} = \{f_{mk}^{(n)}\}_{k=-n}^n$ *and has a unique solution for* m *sufficiently large.*

Obviously, from Definition 3.1, we have

Corollary 3.5. *The Galerkin method related to the system* $(\phi_{m,j}, \mathbf{P}_{m,j})_{j=-\infty}^{\infty}$, *is applicable to the operator* K.

As a by-product, we also have the following corollary:

Corollary 3.6. S_m *is a positive matrix. It is positive definite if in addition* $\widehat{H}(\omega)$ *is even.*

In the next section, except subsection 4.1, we will assume, for simplicity of presentation, that $\widehat{H}(\omega)$ is even; therefore, the matrix $\mathbf{S}_m$ is positive definite. Consequently, the coefficient matrix of the truncated linear system is also positive definite for m large. However, most of the results can be extend to the case when $\widehat{H}(\omega)$ is not symmetric.

4. Convergence and error analysis. In this section we make a study of the convergence properties of the scaling solution $f_m = \mathbf{P}_m f$ of (3.13).

Denote the exact solution of (1.1) by f^*; the error function by Galerkin-wavelet approximation in scaling space is given by:

$$E_m(t) := (f_m - f^*)(t). \tag{3.23}$$

4.1 Convergence of the scaling solution. We will start with some lemmas.

Lemma 4.1. *The restriction of integral operator K to V_m, $\mathbf{K}|_{V_m} : V_m \to KV_m$ is continuous with bounded inverse. In fact, we have*

$$||(\mathbf{K}|_{V_m})^{-1}|| \leq \left(\frac{5\pi}{4}\right)^{1-\alpha} 2^{(m+1)(1-\alpha)},$$

where the operator norm is taken in $L^2(R)$.

Proof. Clearly, $\mathbf{K}$ is a one-to-one operator since the kernel satisfies $\widehat{K}(\omega) \neq 0$. Consequently, $\mathbf{K}$ is one-to-one and onto $\mathbf{K}V_m$, therefore its inverse exists. It is well known that $\mathbf{K}^{-1}$ is continuous if $\mathbf{K}V_m$ is closed. To show $\mathbf{K}V_m$ is closed, let $h_n \in \mathbf{K}V_m \subseteq V_{m+1}$, and let

$$h_n \longrightarrow h, \quad \text{as} \quad n \longrightarrow \infty, \text{ in } L^2(R).$$

Then $h \in V_{m+1}$, since V_{m+1} is closed in $L^2(R)$. Hence since $\mathbf{K}^{-1}$ is continuous on V_{m+1},

$$\mathbf{K}^{-1} h_n \longrightarrow \mathbf{K}^{-1} h, \quad \text{in} \quad L^2(R).$$

But $\mathbf{K}^{-1}(\mathbf{K}|_{V_m})^{-1} h_n \in V_m$ and V_m is closed; therefore,

$$(\mathbf{K}|_{V_m})^{-1} h_n = \mathbf{K}^{-1} h_n \longrightarrow f_0 \in V_m.$$

Since $\mathbf{K}|_{V_m}$ is continuous on V_m,

$$(\mathbf{K}|_{V_m})(\mathbf{K}|_{V_m})^{-1}h_n = h_n \longrightarrow (\mathbf{K}|_{V_m})f_0 \in \mathbf{K}V_m.$$

For any $g \in \mathbf{K}V_m$, we have,

$$\begin{aligned}
||(\mathbf{K}|_{V_m})^{-1}g||_{L^2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\hat{g}(\omega)}{\widehat{K}(\omega)}\right)^2 d\omega \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\hat{g}(\omega)}{\widehat{H}(\omega)+C(0,\alpha)/|\omega|^{1-\alpha}}\right)^2 d\omega \\
&\le \frac{1}{2\pi C^2(0,\alpha)}\int_{-\infty}^{\infty}\hat{g}^2(\omega)|\omega|^{2(1-\alpha)}\,d\omega \\
&\le \frac{1}{2\pi C^2(0,\alpha)}\int_{-(5\pi/4)}^{(5\pi/4)}\hat{g}^2(\omega)|\omega|^{2(1-\alpha)}\,d\omega \\
&\le \frac{1}{C^2(0,\alpha)}\left(\frac{5\pi}{4}\right)^{2(1-\alpha)}2^{(m+1)(1-\alpha)}||g||_{L^2}^2.
\end{aligned}$$

Hence

$$\begin{aligned}
||(\mathbf{K}|_{V_m})^{-1}|| &= \sup_{||g||_{L^2}^2=1}\{||(\mathbf{K}|_{V_m})^{-1}g||_{L^2}^2\} \\
&\le C^{-2}(0,\alpha)\left(\frac{5\pi}{4}\right)^{2(1-\alpha)}2^{(m+1)(1-\alpha)}.
\end{aligned}$$

Lemma 4.2. *Let $f \in \mathbf{H}^\gamma(R)$. Then $||\mathbf{P}_m f - f||_{L^2} \le C_\gamma 2^{-\gamma m}||f||_{\mathbf{H}^\gamma}$, where C_γ is a constant independent of m and f.*

Proof. We calculate

$$\begin{aligned}
||\mathbf{P}_m f - f||_{L^2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}_m(\omega)-\hat{f}(\omega)|^2\,d\omega \\
&= \frac{1}{2\pi}\int_{(|\omega|/2^m)\ge(5\pi/4)}^{\infty}|\hat{f}_m(\omega)-\hat{f}(\omega)|^2\,d\omega \\
&\quad + \frac{1}{2\pi}\int_{(3\pi/4)\le(|\omega|/2^m)\le(5\pi/4)}^{\infty}|\hat{f}_m(\omega)-\hat{f}(\omega)|^2\,d\omega \\
&\quad + \frac{1}{2\pi}\int_{(|\omega|/2^m)\le(3\pi/4)}^{\infty}|\hat{f}_m(\omega)-\hat{f}(\omega)|^2\,d\omega \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Then we have $I_3 = 0$, and

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{(|\omega|/2^m)\geq(5\pi/4)}^{\infty} |\hat{f}(\omega)|^2\, d\omega \\ &= \frac{1}{2\pi}\int_{(|\omega|/2^m)\geq(5\pi/4)}^{\infty} |\hat{f}(\omega)|^2 (1+\omega^2)^{\gamma}\frac{1}{(1+\omega^2)^{\gamma}}\, d\omega \\ &\leq \frac{1}{2\pi}\left(\frac{4}{5\pi}\right)^{2\gamma} 2^{-2m\gamma}||f||_{\mathbf{H}^{\gamma}}. \end{aligned}$$

A similar calculation will lead to the bound:

$$I_2 \leq \frac{1}{\pi}\left(\frac{4}{3\pi}\right)^{2\gamma} 2^{-2m\gamma}||f||_{\mathbf{H}^{\gamma}}.$$

Therefore we have

$$||\mathbf{P}_m f - f||_{L^2} \leq C_{\gamma} 2^{-m\gamma}||f||_{\mathbf{H}^{\gamma}}$$

where C_{γ} is a constant only dependent on γ.

Remark. Notice that in the proof of Lemma 4.2, as in the proof of Theorem 3.3, an important property of the basis, that is, every basis function has compact support in the frequency domain, is used. This property is not shared by some other bases, for instance, Hermite functions. This lemma will be used in the proof of the main theorem, Theorem 4.4, of this section.

Lemma 4.3. *Let* $g \in \mathbf{H}^{\beta}$, $\beta > (3/2)$. *Then if* f^* *is the exact solution of* (1.1)*, then the following estimations hold,*

(i) $||(\mathbf{P}_m\mathbf{K} - \mathbf{K})(f^*)||_{L^2} = O(2^{-\beta m})$,

(ii) $||(\mathbf{K}\mathbf{P}_m - \mathbf{K})(f^*)||_{L^2} = O(2^{-\beta m})$.

Proof. By the hypothesis, $\mathbf{K}f^* = g \in \mathbf{H}^{\beta}(R)$. By Lemma 3.2, we have

$$||(\mathbf{P}_m\mathbf{K} - \mathbf{K})(f^*)||_{L^2} = ||\mathbf{P}_m(\mathbf{K}f^*) - \mathbf{K}(f^*)||_{L^2} = O(2^{-\beta m}).$$

To prove (ii), we write

$$(4.1)\qquad ||(\mathbf{KP}_m - \mathbf{K})(f^*)||_{L^2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{K}^2 |\hat{f}_m^* - \hat{f}^*|^2 \, d\omega.$$

Using the same trick as before, we split the integral into three integrals I_1, I_2 and I_3 as in the proof of Lemma 4.2. A routine calculation yields the following bounds

$$I_1 \le \frac{1}{2\pi}\left(\frac{4}{5\pi}\right)^{2\beta} ||g||_{\mathbf{H}^\beta} 2^{-2m\beta},$$
$$I_2 \le \frac{3}{2\pi}\left(\frac{4}{5\pi}\right)^{2\beta} ||g||_{\mathbf{H}^\beta} 2^{-2m\beta},$$

and $I_3 = 0$. (ii) follows immediately.

The next theorem is the main theorem in this section:

Theorem 4.4. *Let the function* $g \in \mathbf{H}^\gamma(R)$, $\gamma > (3/2)$. *If* f^* *is the exact solution of* (1.1) *and satisfies* $f^* \in \mathbf{H}^\mu(R)$, *then the following estimation holds*:

$$||E_m||_{L^2} = C_{\alpha\gamma} 2^{-m(\lambda+\alpha-1)},$$

where $\lambda = \min\{\gamma, \mu\}$ *and* $C_{\alpha\gamma}$ *is a constant only dependent on* α, γ.

Proof. Recall that we start with searching for a solution in scaling subspaces:

$$(4.2)\qquad (\mathbf{KP}_m) f(t) = \mathbf{P}_{m+1} g(t)$$

If f^* is the exact solution of (1.1), we have

$$(4.3)\qquad \mathbf{P}_m \mathbf{K} f^*(t) = \mathbf{P}_m g(t).$$

Subtract (4.3) from (4.2) to get

$$(\mathbf{P}_m \mathbf{K} f^* - \mathbf{KP}_m f)(t) = (g_{m+1} - g_m)(t).$$

This can be rewritten as

$$(\mathbf{P}_m\mathbf{K}f^* - \mathbf{K}\mathbf{P}_m f + \mathbf{K}\mathbf{P}_m f^* - \mathbf{K}\mathbf{P}_m f^*)(t) = (g_{m+1} - g_m)(t).$$

that is,

$$\mathbf{K}(f_m - f_m^*)(t) = (\mathbf{P}_m\mathbf{K} - \mathbf{K}\mathbf{P}_m)f^*(t) + (g_{m+1} - g_m)(t),$$

where $f_m = \mathbf{P}_m f$, $f_m^* = \mathbf{P}_m f^*$.

By Lemma 4.1, we may write

$$(f_m - f_m^*)(t) = ||(\mathbf{K}|_{V_m})^{-1}(\mathbf{P}_m\mathbf{K} - \mathbf{K}\mathbf{P}_m)f^*(t) + (g_{m+1} - g_m)(t).$$

Taking the norm in $L^2(R)$,

$$\begin{aligned}||f_m - f_m^*||_{L^2} &= ||(\mathbf{K}|_{V_m})^{-1}|| ||\mathbf{P}_m\mathbf{K} - \mathbf{K}\mathbf{P}_m)f^*||_{L^2} + ||g_{m+1} - g_m||_{L^2}\\ &\le \left(\frac{5\pi}{4}\right)^{1-\alpha} 2^{(m+1)(1-\alpha)}[O(2^{-\gamma m}) + O(2^{-\gamma(m+1)})]\\ &\le C_{\alpha\gamma}2^{-m(\gamma+\alpha-1)}\end{aligned}$$

where $C_{\alpha\gamma}$ is a constant. We have applied Lemma 4.2 and Lemma 4.3 in the proof. Finally, we have

$$\begin{aligned}||E_m||_{L^2} = ||f_m - f^*||_{L^2} &\le ||f_m - f_m^*||_{L^2} + ||f_m^* - f^*||_{L^2}\\ &\le C_{\alpha\gamma}2^{-m(\gamma+\alpha-1)} + C_\mu ||f||_{\mathbf{H}^\mu} 2^{-\mu m}\\ &\le C_{\alpha\gamma}2^{-m(\lambda+\alpha-1)},\end{aligned}$$

where $\lambda = \min\{\mu, \gamma\}$. This completes the proof of Theorem 4.1. □

Remark. The solvability of this method has been addressed in Section 3.1, see Theorem 3.3 and Corollary 3.4. Discrete Galerkin methods, see [**2**, p. 142], can be derived based on the discussions in this paper. It is remarkable that the Meyer-type wavelets, as opposed to the Daubechies wavelets, have the so-called "oversampling property," see [**11**], [**14**], which allows the calculation of the coefficients by sampling procedures. In the case where the kernel K and the function g are given by the discrete data, this method is therefore likely to be more attractive.

Acknowledgment. The author would like to thank Dr. Gilbert Walter for his help and encouragement.

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