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SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS ON THE HALF-AXIS WITH RAPIDLY DECREASING NON-DIFFERENCE KERNELS

ANNA MITINA

ABSTRACT. The purpose of this paper is to investigate the set of all solutions of the integro-differential equation (1) and to obtain a convenient algorithm for calculation of any solution. Both objectives are obtained in the case when the integral kernel $R_1(x)$ is even and both kernels $R_1(x)$ and $R_2(x)$ in the equation rapidly decrease as x approaches infinity, although the integrals

$$\phi_i(0) \equiv \int_{-\infty}^{\infty} R_i(x)\, dx, \quad i = 1, 2$$

are not assumed to be small. To be sure that integrals in the equation converge, the sought for solutions are supposed to satisfy a condition of the type:

$$|y(x)| < \text{const} \cdot e^{\lambda x}.$$

The asymptotic behavior of solutions as $x \to \infty$ is defined by the number $\phi_1(0)$. If $\phi_1(0) < 1$, then there is a positive number $p^\star$ such that all solutions grow proportionally to $e^{p^\star x}$ except specific ones which tend to zero as $e^{-p^\star x}$. If $\phi_1(0) = 1$, then all solutions grow as linear functions except the specific ones which tend to a constant as $x \to \infty$. If $\phi_1(0) > 1$, there exists a purely imaginary number $p^\star$ such that the asymptotic behavior of solutions as $x \to \infty$ is described by an oscillating function which is a linear combination of two specific solutions which behave as $e^{i|p^\star|x}$ and $e^{-i|p^\star|x}$, respectively.

In all these cases the condition

$$y'(0) = \mu y(0)$$

is found which ensures a solution to be specific.

In many physical applications involving the considered problem it is the coefficient μ which is important. To evaluate the parameter μ an asymptotic series convergent to μ is found.

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1. Formulation of the problem. This paper deals with integro-differential equations of the following type:

$$-\frac{d^2y}{dx^2}+y=\int_0^\infty R_1(x-t)y(t)\,dt + \int_0^\infty R_2(x+t)y(t)\,dt,\quad x>0. \tag{1}$$

Equations of this kind arise in various fields of physics. As such we may mention radiative equilibrium of stars [**5**], anomalous skin- effect in metals [**4**], [**6**] and [**12**], stationary neutron density in multiplying media [**1**], [**7**] and [**8**], and wave propagation in acoustic and electrodynamic waveguides [**11**], [**13**] and [**14**]. In all these fields of research there are many particular problems which lead to the equation (1) with $R_2\equiv 0$. These cases have been exhaustively treated with the standard Wiener-Hopf technique. However, there are many problems which cannot be simplified in this way. That is why equation (1) in its general form deserves an independent investigation.

An existence and uniqueness theorem for the Cauchy problem was proved in [**10**]. An integro-differential equation where the integrand is the product of a differential operator on an unknown function and a kernel of the same type as in (1), as well as systems of such equations were considered in [**2**] and [**3**].

Boundary problems and asymptotic behavior of solutions are treated in the present paper as well as in [**7**], [**9**] and [**10**].

This paper deals with both Cauchy and boundary problems for a special class of equations of type (1) with a large parameter characterizing the rate of decrease of the two kernels $R_1(x)$ and $R_2(x)$ as x approaches infinity. These kernels are assumed to be rapidly decreasing although the integrals

$$\int_0^\infty |R_1(t)|\,dt \quad\text{and}\quad \int_0^\infty |R_2(t)|\,dt, \tag{2}$$

are not assumed to be small. As a convenient model of such functions we consider

$$R_1(x)=\nu R_1^{(0)}(\nu x)\quad\text{and}\quad R_2(x)=\nu R_2^{(0)}(\nu x) \tag{3}$$

where ν is a large parameter. The functions $R^{(0)}$ are chosen in such a way that, for some positive constants s_0, c and C, the estimates

$$|R_1^{(0)}(x)| < ce^{-s_0|x|}, \qquad |R_2^{(0)}(x)| < Ce^{-s_0 x}, \quad x > 0$$

are true, so that

$$|R_1(x)| < c\nu e^{-s|x|}, \qquad |R_2(x)| < C\nu e^{-sx}, \quad s = s_0\nu,\ x > 0. \tag{4}$$

Obviously the integrals (2) do not depend on the choice of ν.

We assume that the functions $R_1(x)$ and $R_2(x)$ are piecewise continuous and R_1 is even. Solutions $y(x)$ are sought in the class of twice differentiable functions which satisfy the following condition:

$$|y(x)| \le ce^{\lambda x}, \quad \lambda = \nu\lambda_0 \tag{5}$$

where λ_0 is some fixed real number

$$\lambda_0 < s_0. \tag{6}$$

This inequality takes care of convergence of the integrals in (1).

We make the following assumption

Assumption 1.1. *The subspace of all solutions of equation* (1) *is at least two-dimensional.*

In what follows we assume that the parameter ν is large enough unless the opposite is stated.

Introduction of a large parameter in the present paper allows us to find properly defined Cauchy and boundary problems and obtain a convenient algorithm for calculating approximate solutions of these two problems.

In the next three Sections (2 to 4) the main results of the paper are formulated in the form of five theorems. Their proofs are given in Sections 5 to 8. The explicit formulae for approximate solutions both of Cauchy- and boundary problems with error proportional to ν^{-1} are given in the last section of the paper.

2. Main results – General. To formulate the main results the so-called characteristic function $G(p)$ is needed. If $\phi_j,\ j=1,2$, are Fourier transforms of the two kernels (we put $R_2(-x)=R_2(x)$):

$$\phi_j(\alpha)=\int_{-\infty}^{\infty}e^{ix\alpha}R_j(x)\,dx \quad j=1,2,$$

then by the definition

$$G(p)=1-p^2-\phi_1(ip). \tag{7}$$

There is a simple relation between the function $\phi_j(\alpha)$ and the Fourier transform $\phi_j^{(0)}(\alpha)$ of the kernels $R_j^{(0)}(x)$:

$$\phi_j(\alpha)=\phi_j^{(0)}\left(\frac{\alpha}{\nu}\right),\quad j=1,2. \tag{8}$$

The characteristic function as well as the Fourier transforms $\phi_j(p)$ are analytic in any strip

$$\Pi=\{p:|\Re p|\le\beta,\ \beta=\beta_0\nu\} \tag{9}$$

where β_0 is an arbitrary positive number $\beta_0<s_0$. The Fourier transforms $\phi_j^{(0)}(p)$ are analytic in the strip

$$\Pi_0=\{p:|\Re p|\le\beta_0\}. \tag{10}$$

Theorem 2.1. *The strip* Π *contains exactly two zeros,* p_1 *and* p_2 *of the characteristic function* $G(p)$*. Moreover,* $p_2=-p_1$*;* p_1 *is positive if* $\phi_1(0)<1$*,* $p_1=0$ *if* $\phi_1(0)=1$*, and* p_1 *is purely imaginary if* $\phi_1(0)>1$*.*

To find $y(x)$ and its Laplace transform $Y(p)$, $\Re p=\beta,\ \lambda<\beta<s$, we use the following key theorem

Theorem 2.2. *If* $\phi_1(0)\ne 1$*, any solution of equation* (1) *in the class of functions* (5) *satisfies the following equation:*

$$y(x)=\sum_{k=1,2}A_k\,e^{p_kx}-\Psi(x) \tag{11}$$

where

$$A_k = -\frac{1}{G'(p_k)}\left\{y'(0) + p_k y(0) - \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\left[\frac{\phi_1(i\zeta)}{p_k-\zeta} + \frac{\phi_2(i\zeta)}{p_k+\zeta}\right] d\zeta\right\}, \quad k = 1, 2; \tag{12}$$

$$\Psi(x) = \frac{1}{\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{e^{x\eta}\, d\eta}{G(\eta)} \cdot \left[y'(0) - \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)[\phi_1(i\zeta)-\phi_2(i\zeta)]\,\frac{\zeta d\zeta}{\eta^2-\zeta^2}\right], \tag{13}$$

and

$$Y(p) = \int_0^\infty y(x)e^{-px}\, dx, \quad \Re\, p = \beta,\ 0 < \sigma < \beta,\ \lambda < \beta < s. \tag{14}$$

We assume that σ, λ and β are proportional to ν:

$$\sigma = \nu\sigma_0, \quad \lambda = \nu\lambda_0, \quad \beta = \nu\beta_0,$$

and $\lambda_0 < \beta_0 < s_0$.

In the case $\phi_1(0) = 1$, relation (10) has to be replaced with

$$y(x) = B_1 x + B_2 - \Psi(x) \tag{15}$$

where

$$B_1 = -\frac{2}{G''(0)}\left\{y'(0) + \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_1(i\zeta)-\phi_2(i\zeta)}{\zeta}\, d\zeta\right\}, \tag{16}$$

$$B_2 = -\frac{2}{G''(0)}\left[y(0) + \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_1(i\zeta)+\phi_2(i\zeta)}{\zeta^2}\, d\zeta\right]. \tag{17}$$

3. Main results – Initial value problem.

Theorem 3.1. *For any fixed pair of numbers $y(0)$ and $y'(0)$ equations (11) and (15) have exactly one solution each.*

Thus, for sufficiently large ν, we obtain a properly defined initial value problem for equation (11) and for equation (15) by fixing two values $y(0)$ and $y'(0)$. The same is true for equation (1) under Assumption 1.1. This is an immediate consequence of Theorems 2.2 and 3.1. The subspace of all solutions of equation (1) coincides with that of equation (11) or (15) if $\phi(0) \neq 1$ or $\phi(0) = 1$, respectively. Thus, under Assumption 1.1, equation (1) is equivalent either to equation (11) or to equation (15).

To solve equation (11) we use it in a natural way to construct the following iterative algorithm:

$$Y_0(p) \equiv 0, \tag{18}$$

$$y_{n+1}(x) = \sum_{k=1,2} A_n^k \, e^{p_k x} - \Psi_n(x), \tag{19}$$

$$Y_n(p) = \int_0^\infty y_n(x) e^{-px} \, dx, \quad \Re\, p = \beta, \tag{20}$$

where A_n^k and $\Psi_n(x)$ are obtained from (12) and (13), respectively, by replacing $Y(\zeta)$ with $Y_n(\zeta)$:

$$\begin{aligned} A_n^{(k)} = -\frac{1}{G'(p_k)} \Big\{ & y'(0) + p_k y(0) \\ & - \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} Y_n(\zeta) \left[\frac{\phi_1(i\zeta)}{p_k - \zeta} + \frac{\phi_2(i\zeta)}{p_k + \zeta} \right] d\zeta \Big\}, \quad k = 1, 2, \end{aligned} \tag{21}$$

$$\begin{aligned} \Psi_n(x) = \frac{1}{\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} & \frac{e^{x\eta} d\eta}{G(\eta)} \\ & \cdot \left[y'(0) + \frac{1}{2} \int_{\beta - i\infty}^{\beta + i\infty} Y_n(\zeta) [\phi_1(i\zeta) - \phi_2(i\zeta)] \, \frac{\zeta d\zeta}{\eta^2 - \zeta^2} \right]. \end{aligned} \tag{22}$$

For equation (15) we construct a similar iterative process replacing (19) with

$$y_{n+1}(x) = B_n^{(1)} x + B_n^{(2)} - \Psi_n(x) \tag{23}$$

where $B_n^{(k)}$, $k = 1, 2$, are obtained from (16) and (17):

(24)
$$B_n^{(1)} = -\frac{2}{G''(0)} \left\{ y'(0) + \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} Y_n(\zeta) \, \frac{\phi_1(i\zeta) - \phi_2(i\zeta)}{\zeta} \, d\zeta \right\},$$

(25)
$$B_n^{(2)} = -\frac{2}{G''(0)} \left[y(0) + \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} Y_n(\zeta) \, \frac{\phi_1(i\zeta) + \phi_2(i\zeta)}{\zeta^2} \, d\zeta \right].$$

The effectiveness of these algorithms is described by the following

Theorem 3.2. *The sequence* $\{y_n(x)\}_1^\infty$ *converges to the solution* $y(x)$ *of the corresponding system in the following sense:*

$$\lim_{n\to\infty} ||y_n(x) - y(x)|| \equiv \lim_{n\to\infty} \max\{|y_n(x) - y(x)| e^{-\beta x}\} = 0, \quad \phi_1(0) \neq 1,$$

$$\lim_{n\to\infty} \max\{|y_n(x) - y(x)|(1+x)^{-1}\} = 0, \quad \phi_1(0) = 1,$$

$$\lim_{n\to\infty} A_n^k = A_k, \quad k = 1, 2,$$

$$\lim_{n\to\infty} B_n^k = B_k, \quad k = 1, 2,$$

$$\lim_{n\to\infty} |\Psi_n(x) - \Psi(x)| e^{\sigma x} = 0.$$

The sequence $\{Y_n(p)\}_0^\infty$ *converges uniformly on the line* $\Re p = \beta$. *The rate of convergence is as that of a geometrical sequence with ratio proportional to* ν.

Theorems 3.1 and 3.2 give the complete solution to the initial value problem for integro-differential equation (1) and supply a practical algorithm for evaluating its solution.

4. Main results – Boundary problems. To define a boundary problem properly one has to know the asymptotic behavior of the general solution at $x = \infty$. The representation of $y(x)$ in Theorem 2.2 allows us to see in great detail the asymptotic behavior of the solutions of the problem as $x \to \infty$. Asymptotic behavior of each of the three terms in (11) or (15) is affected by the value of parameter ν in quite a different way. As $x \to \infty$ asymptotical behavior of the two terms in the sum $\sum_{k=1,2} A_k e^{p_k x}$ in (11) is determined by the numbers p_1 and p_2 which approach $\pm\sqrt{1-\phi_1(0)}$ respectively as $\nu \to \infty$, so these terms change rather smoothly as ν increases. The asymptotic behavior, $x \to \infty$, of the sum of the first two terms in (15) is linear. The third term $\Psi(x)$ in relations (11) and (15) decreases as fast or faster than $e^{-\nu\sigma_0 x}$, $x \to \infty$. As a consequence the asymptotic behavior of each particular solution, $y(x)$ of equation (1) is determined by the coefficients A_1 and A_2 (or B_1 and B_2).

Now it is easy to define a boundary problem properly. The boundary condition depends on the value of $\phi_1(0)$.

In the case $\phi_1(0) < 1$, the number p_1 is positive and the number p_2 is negative. Therefore, in this case, all solutions grow as $e^{p_1 x}$ unless $A_1 = 0$. If $A_1 = 0$ the solution approaches zero as $e^{-p_1 x}$ as $x \to \infty$. It means that the condition,

(26) $\qquad y(x)$ is bounded on the entire positive half-axis,

makes the coefficient A_1 equal to zero and leaves only a one-dimensional set of solutions. In other words, adding this condition to equation (11) we obtain a properly defined boundary problem. Of course, if we substitute the last condition with the condition

$$|y(x)|e^{qx} < \infty$$

where q is any number such that $p_2 < q < p_1$, the boundary problem would still be properly defined and have the same set of solutions.

In the case $\phi_1(0) = 1$, condition (26) makes the coefficient B_1 equal to zero and leaves us with a one-dimensional set of solutions. Thus, it is an appropriate boundary condition.

In the case $\phi_1(0) > 1$, the numbers p_1 and p_2 are imaginary, and, according to Theorem 2.1, $p_2 = -p_1$. So, the asymptotical behavior of any solution is described by the relation

$$|y(x) - A_1 e^{p_1 x} - A_2 e^{p_2 x}| \longrightarrow 0, \quad x \to \infty. \tag{27}$$

This relation with any pair of fixed coefficients A_1 and A_2 can be regarded as an appropriate boundary condition. We see that in this case there are infinitely many different boundary problems.

Two solutions whose asymptotic behavior is described by $e^{i|p_1|x}$ and $e^{-i|p_1|x}$, respectively, describe in many physical problems two waves going in opposite directions. They are complex conjugate to each other and, therefore, it is sufficient to find only one of them. We will consider the solution that behaves asymptotically as $e^{-i|p_1|x}$, that is, the one with $A_1 = 0$. For this solution the boundary condition has the form

$$|y(x) - A_2 e^{p_2 x}| \longrightarrow 0, \quad x \to \infty. \tag{28}$$

At first glance, a natural way to find an approximate solution of any of the above described boundary value problems is to use approximate solutions of the initial value problem with initial conditions which make A_1, or B_1, equal to zero. Unfortunately, it could be done only approximately. It means that the corresponding approximate solution $y(x)$ contains an unbounded component $A_1 e^{p_1 x}$ (or $B_1 x$) and, therefore, solves only the initial value problem but not the boundary value problem with the boundary condition (26). Nevertheless, the ratio $y'(0)/y(0)$ for the solution $y(x)$ of the boundary value problem can be evaluated in this way with any degree of accuracy. To find the approximate solution $y(x)$ of the boundary problem with uniformly small error we have to find another way.

For that purpose we derive a system of equations with the following property: each solution of this system is a solution of the boundary problem. This system can be solved by an iterative method since ν is large. To obtain the system in the case $\phi_1(0) \neq 1$, we set $A_1 = 0$ in (11) and (12) and get:

$$y(x) = A_2 e^{p_2 x} - \Psi(x), \tag{29}$$

$$y'(0) = - p_1 y(0) - \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} Y(\zeta) \left[\frac{\phi_1(i\zeta)}{p_1 - \zeta} + \frac{\phi_2(i\zeta)}{p_1 + \zeta} \right] d\zeta, \tag{30}$$

and

$$Y(p) = \int_0^\infty y(x)e^{-px}\,dx, \quad \Re p = \beta, \tag{31}$$

where A_2 and $\Psi(x)$ are given by (12) and (13), respectively. In the case $\phi_1(0) = 1$, equations (29) and (30) of the system should be replaced with

$$y(x) = B_2 - \Psi(x) \tag{32}$$

where B_2 is given by (17), and

$$y'(0) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_1(i\zeta) - \phi_2(i\zeta)}{\zeta}\,d\zeta. \tag{33}$$

To find the solutions we use recurrent algorithms as in the case of the initial value problem. In the case $\phi_1(0) \neq 1$, the algorithm is given by:

$$y_{n+1}(x) = A_n^{(2)}e^{p_2 x} - \Psi_n(x), \tag{34}$$

and

$$y_n'(0) = -p_1 y(0) - \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y_n(\zeta)\left[\frac{\phi_1(i\zeta)}{p_1-\zeta} + \frac{\phi_2(i\zeta)}{p_1+\zeta}\right]d\zeta \tag{35}$$

together with (18), (20) and (21) for $k = 2$ and (22).

In the case $\phi_1(0) = 1$, relation (34) should be replaced with

$$y_{n+1}(x) = B_n^{(2)} - \Psi_n(x) \tag{36}$$

and relation (35) with

$$y_n'(0) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y_n(\zeta)\,\frac{\phi_1(i\zeta) - \phi_2(i\zeta)}{\zeta}\,d\zeta. \tag{37}$$

These recurrent relations define three sequences

$$\{y_n(x)\}_1^\infty,\ \{Y_n(p)\}_0^\infty,\ \{y_n'(0)\}_0^\infty. \tag{38}$$

The following theorem justifies the introduction of these recurrent relations.

Theorem 4.1. *For any fixed initial value $y(0)$, each of the two systems* (29)–(31) *and* (31)–(33) *has a unique solution. These solutions solve the respective boundary problems and could be found as limits of the convergent sequences* (38)*. The rate of convergence is at least proportional to ν.*

5. Zeros of the characteristic function. In this section we prove Theorem 2.1, which reveals a simple structure of the set of zeros of the function $G(p)$.

We begin with some preliminary remarks. First, we recall that

$$\phi_1^{(0)}(\alpha) = \int_{-\infty}^{\infty} e^{ix\alpha} R_1^{(0)}(x)\, dx$$

and then we introduce the following notation:

$$m \equiv \sup |\phi_1^{(0)}(q)|, \quad q \in \Pi_0,$$

(see (10)). Due to (8) we have

$$|\phi_1(p)| < m \tag{39}$$

for $p \in \Pi$ (see (9)), and each zero p in the strip Π belongs to the circle $Q = \{p|\ |p| \le r = \sqrt{1+m}\,\}$.

Since the radius of this circle does not depend on the parameter ν, we can assume that

$$\beta \equiv \beta_0 \nu \ge r,$$

so the circle Q lies in the strip Π and

$$|p| \le \beta \quad \text{if } G(p) = 0.$$

The last inequality guarantees the convergence of the integral

$$\int_{-\infty}^{\infty} R(x) \cosh(|p|\,|x|)\, dx$$

for any $p \in Q$ and, in particular, for any zero of $G(p)$.

Theorem 5.1. *For any two numbers z_k, $z_k \in Q$, $k = 1, 2$, the absolute value of the fraction*

$$\frac{\phi_1(z_2) - \phi_1(z_1)}{z_2^2 - z_1^2}, \quad z_1^2 \neq z_2^2, \tag{40}$$

is less than the integral

$$K \equiv \int_0^\infty R_1(x) x^2 \cosh(|z|x)\, dx, \quad z = \max\{|z_1|, |z_2|\}.$$

Proof. $R_1(x)$ being an even function, we can write

$$\phi_1(iz) = 2 \int_0^\infty R_1(x) \cosh(xz)\, dx.$$

Therefore,

$$\phi_1(iz_2) - \phi_1(iz_1) = 2 \int_0^\infty R_1(x) \{\cosh(xz_2) - \cosh(xz_1)\}\, dx.$$

Using an elementary inequality

$$|\cosh z_2 - \cosh z_1| \leq \frac{1}{2} |z_2^2 - z_1^2| \cosh |z|, \quad |z| \equiv \max\{|z_1|, |z_2|\},$$

the result follows.

Notice that, due to (4),

$$K \leq \frac{2c}{(\nu\beta_0 - \sqrt{1+m})^3}.$$

Therefore, if

$$\nu > \frac{(1+m)^{1/2}}{\beta_0} + (4c)^{1/3}, \tag{41}$$

then fraction (40) is less than a number d which is less than $(1/2)$. Summarizing, we have

$$\frac{\phi_1(z_2) - \phi_1(z_1)}{z_2^2 - z_1^2} < d < \frac{1}{2}, \quad z_1^2 \neq z_2^2 \tag{42}$$

for ν such as in (41). In what follows we always assume that condition (41) is satisfied.

Now, let z_1 and z_2 be any two zeros of the characteristic function. Then

$$\phi_1(iz_2) - \phi_1(iz_1) = z_1^2 - z_2^2.$$

According to (42) this is possible only if $z_1^2 = z_2^2$.

This proves that there are no more than two zeros of $G(p)$ in Π.

Theorem 5.2. *The characteristic function $G(p)$ has at least one zero in Π.*

Proof. Consider the following numerical sequence z_n, $n = 0, 1, 2, \dots$:

$$z_0 = 0, \quad 1 - z_{n+1}^2 = \phi_1(z_n), \quad 0 \le \arg z_{n+1} < \pi, \quad n = 1, 2, \dots .$$

Obviously,

$$z_n \in Q \quad \text{and} \quad z_{n+2}^2 - z_{n+1}^2 = \phi_1(z_n) - \phi_1(z_{n+1}).$$

Theorem 5.1 yields inequality

$$|z_{n+2}^2 - z_{n+1}^2| \le d|z_{n+1}^2 - z_n^2|, \quad d < \frac{1}{2}.$$

Thus, the sequence $\{z_n\}_0^\infty$ has a limit z which obviously is a zero of the characteristic function $G(z)$ and that completes the proof.

Notice that $z_1^2 = 1 - \phi_1(0)$ and

$$|z^2 - z_1^2| < |z_1^2|. \tag{43}$$

Taking into consideration that the function $R_1(x)$ is even and real, we conclude that $G(p)$ has the following property: if p is a zero of the characteristic function so are the numbers $-p, \bar{p}$ and $-\bar{p}$. That shows that each zero, $p \in \Pi$ of $G(p)$, is either real or purely imaginary. If $p = 0$ is a zero of $G(p)$ then $\phi_1(0) = 1$ and $G(p) = p^2$, so $p = 0$ is the double root of $G(p)$ in Π.

Let $\phi_1(0) > 1$. Then, obviously, z_1^2 is negative. Inequality (43) in that case means that $z^2 = \lim_{n\to\infty} z_n^2$ is also negative. Thus, if $\phi_1(0) > 1$, the two zeros of $G(p)$ in Π are purely imaginary. In the same way we conclude that in the case $\phi_1(0) < 1$, the two zeros of $G(p)$ in Π are real. That completes the proof of Theorem 2.1.

6. Proof of Theorem 2.2. Our first step in proving the theorem is to obtain a relation which is valid in the strip $\lambda^+ < \operatorname{Re} p < \beta$, where $\lambda^+ = \max\{\lambda, 0\}$.

We begin with the Laplace transform of equation (1) and obtain the following relation:

(44)
$$\begin{aligned} Y(p)(1-p^2) + y'(0) + py(0) &= \int_0^\infty e^{-px}\,dx \int_0^\infty R_1(x-t)y(t)\,dt \\ &\quad + \int_0^\infty e^{-px}\,dx \int_0^\infty R_2(x+t)y(t)\,dt, \\ &\operatorname{Re} p > \max\{\lambda, -s\}. \end{aligned}$$

We also use two inversion formulae:

$$R_j(x) = \frac{1}{2\pi}\int_{i\beta_j-\infty}^{i\beta_j+\infty} \phi_j(\alpha)e^{-i\alpha x}\,d\alpha, \quad j = 1, 2, \tag{45}$$

valid if

$$-s < \beta_j < s, \quad j = 1, 2.$$

We chose β_1 and β_2 so that

$$\beta_1 = -\,\beta_2 = \beta, \quad \lambda_+ < \beta < s.$$

which makes the relations

(46)
$$\begin{aligned} &\int_0^\infty e^{-px}\,dx \int_0^\infty R_1(x-t)y(t)\,dt \\ &\qquad = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_1(i\zeta)}{p-\zeta}\,d\zeta, \quad \Re p > s, \end{aligned}$$

$$\int_0^\infty e^{-px}\,dx \int_0^\infty R_2(x+t)y(t)\,dt = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_2(i\zeta)}{p+\zeta}\,d\zeta, \quad \Re p > s, \tag{47}$$

valid. Therefore, (44) could be rewritten in the following form:

$$Y(p)(1-p^2)+y'(0)+py(0) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\Phi(p,\zeta)\,d\zeta, \quad \Re p > \beta, \tag{48}$$

where

$$\Phi(p,\zeta) = \frac{\phi_1(i\zeta)}{p-\zeta} + \frac{\phi_2(i\zeta)}{p+\zeta}. \tag{49}$$

Integral (47) is analytic in the half-plane $\Re p > -\beta$; integral (46) is analytic in the half-plane $\Re p > \beta$ but can be analytically continued on the strip $\lambda_+ < \Re p < \beta$ where it can be written as:

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_1(i\zeta)}{p-\zeta}\,d\zeta + \phi_1(ip)Y(p), \quad \lambda_+ < \Re p < \beta. \tag{50}$$

Thus, in the strip $\lambda_+ < \Re p < \beta$ relations (48), (49) and (50) yield:

$$Y(p)(1-p^2-\phi_1(ip))+y'(0)+py(0) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\Phi(p,\zeta)\,d\zeta, \quad \lambda_+ < \Re p < \beta, \tag{51}$$

or

$$Y(p) = \frac{1}{G(p)}\left\{\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\Phi(p,\zeta)\,d\zeta - y'(0) - py(0)\right\}, \quad \lambda_+ < \Re p < \beta, \tag{52}$$

where $G(p)$ is the function defined by (7). Now we have a relation with the righthand side analytic in the strip $\lambda_+ < \Re p < \beta$ with the possible exception of some poles at the zeros of the function $G(p)$. Therefore, the same is true for the function $Y(p)$. Thus, (52) is valid in the strip

$\lambda_+ \Re p < \beta$ and shows that the Laplace transform $Y(p)$ is an analytic function on this strip with no other singularities than possible poles at the zeros of $G(p)$.

Now it is possible to derive a modified version of equation (52) using the Wiener-Hopf technique.

Let $\sigma_1 > 0$ and $\sigma_2 > \sigma_1$ be any two numbers in the strip $\lambda_+ < \mathrm{Re}\, p < \beta$. In the strip $\sigma_1 < \Re p < \sigma_2$ the function $(\Phi(p,\zeta)/G(p))$ could be represented in the form:

(53)

$$\frac{\Phi(p,\zeta)}{G(p)} = \frac{1}{2\pi i}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)}\,\frac{d\eta}{\eta-p} - \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)}\,\frac{d\eta}{\eta-p}$$

$$-\sum \frac{\Phi(p_k,\zeta)}{G'(p_k)}\,\frac{1}{p_k-p}, \quad \sigma_1 < \Re p < \sigma_2,$$

where summation is over all zeros p_k of $G(p)$ in the strip $\sigma_1 \le \Re p \le \sigma_2$. All zeros p_k in (53) are not singular points of the function $Y(p)$ since they are in the half-plane $\Re p > \lambda$ where $Y(p)$ is analytic. It follows, therefore, from equation (52) that for $p = p_k$ the following identity holds:

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\Phi(p_k,\zeta)\,d\zeta - y'(0) - p_k y(0) = 0. \tag{54}$$

Similarly, we represent function $(y'(0)+py(0))/G(p)$ in the form:

$$\frac{y'(0)+py(0)}{G(p)} = \frac{1}{2\pi i}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{y'(0)+\eta y(0)}{G(\eta)}\,\frac{d\eta}{\eta-p}$$

$$- \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{y'(0)+\eta y(0)}{G(\eta)}\,\frac{d\eta}{\eta-p} \tag{55}$$

$$-\sum \frac{y'(0)+p_k y(0)}{G'(p_k)}\,\frac{1}{p_k-p}, \quad \sigma_1 < \Re p < \sigma_2.$$

As we substitute (53) and (55) in (52) and combine terms which depend on p_k, we see that due to (54) these terms cancel out. Thus, we get:

$$
\begin{aligned}
&Y(p) + \frac{1}{(2\pi)^2}\int_{\beta-i\infty}^{\beta+i\infty} d\zeta Y(\zeta)\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)}\,\frac{d\eta}{\eta-p} \\
&\quad + \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{y'(0)+\eta y(0)}{G(\eta)}\,\frac{d\eta}{\eta-p} \\
&= \frac{1}{(2\pi)^2}\int_{\beta-i\infty}^{\beta+i\infty} d\zeta Y(\zeta)\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)}\,\frac{d\eta}{\eta-p} \\
&\quad + \frac{1}{2\pi i}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{y'(0)+\eta y(0)}{G(\eta)}\,\frac{d\eta}{\eta-p}, \quad \sigma_1 < \Re p < \sigma_2.
\end{aligned} \tag{56}
$$

The lefthand side of this equation is analytic in the half-plane $\Re p > \sigma_1$ while the righthand side is analytic in the half-plane $\Re p < \sigma_2$. These half-planes have a nonempty common strip

$$\sigma_1 < \Re p < \sigma_2.$$

It follows from Liouville's theorem that the lefthand side of this equation identically equals zero in the half-plane $\Re p > \sigma_1$, and the righthand side equals zero in $\Re p < \sigma_2$. If we replace p with $-p$ in the righthand side of (56) we get a function which is identically equal to zero in $\Re p > -\sigma_2$:

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\beta-i\infty}^{\beta+i\infty} d\zeta Y(\zeta)\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)}\,\frac{d\eta}{\eta+p} \\
&\quad + \frac{1}{2\pi i}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{y'(0)+\eta y(0)}{G(\eta)}\,\frac{d\eta}{\eta+p} = 0, \quad \Re p > -\sigma_2.
\end{aligned} \tag{57}
$$

Restricting p to the half-plane $\Re p > \sigma_1$ we can add the last equation to the lefthand side of (56) and obtain the desired modified equation:

(58)

$$Y(p) = \frac{1}{(2\pi)^2}\int_{\beta-i\infty}^{\beta+i\infty} d\zeta$$
$$\cdot Y(\zeta)\left\{\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{\Phi(\eta,\zeta)}{G(\eta)}\frac{d\eta}{\eta-p} - \int_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{\Phi(\eta,\zeta)}{G(\eta)}\frac{d\eta}{\eta+p}\right\}$$
$$+\frac{1}{2\pi i}\left\{\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{y'(0)+\eta y(0)}{G(\eta)}\frac{d\eta}{\eta-p}\right.$$
$$\left.-\int_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{y'(0)+\eta y(0)}{G(\eta)}\frac{d\eta}{\eta+p}\right\}, \quad \Re p > \sigma_1.$$

We will use this equation for $\Re p = \beta$ only. Notice that parameter $\sigma_2 > \lambda$ in (58) could be chosen arbitrarily close to s.

Thus far we have not used the assumption that ν is large. But now we will use it to be sure that the strip $|\Re p| < \beta$ contains no more than two zeros of $G(p)$. Keeping in mind that ν is large ($\nu >> 1$) we transform equation (58) in order to obtain an equation with easily identifiable order of terms as $\nu \to \infty$. With the help of the two obvious identities

$$\int_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{\Phi(\eta,\zeta)}{G(\eta)}\frac{d\eta}{\eta+p} = \int_{-\sigma_2-i\infty}^{-\sigma_2+i\infty}\frac{\Phi(-\eta,\zeta)}{G(\eta)}\frac{d\eta}{p-\eta} \tag{59}$$

and

(60)

$$\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{\Phi(\eta,\zeta)}{G(\eta)}\frac{d\eta}{\eta-p} = \int_{-\sigma_2-i\infty}^{-\sigma_2+i\infty}\frac{\Phi(\eta,\zeta)}{G(\eta)}\frac{d\eta}{\eta-p}$$
$$+2\pi i\sum_{p_k}\frac{\Phi(p_k,\zeta)}{G'(p_k)}\frac{1}{p_k-p}, \quad \phi_1(0)\neq 1,$$

where summation is over all zeros p_k of the function $G(p)$ in the strip

$-\sigma_2 < \operatorname{Re} p < \sigma_1$, we get the following identity

$$\begin{aligned}
&\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)} \frac{d\eta}{\eta - p} - \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \frac{\Phi(\eta,\zeta)}{G(\eta)} \frac{d\eta}{\eta + p} \\
&= \int_{-\sigma_2 - i\infty}^{-\sigma_2 + i\infty} [\Phi(\eta,\zeta) + \Phi(-\eta,\zeta)] \frac{d\eta}{G(\eta)(\eta - p)} \\
&\quad + 2\pi i \sum_{p_k} \frac{\Phi(p_k,\zeta)}{G'(p_k)(p_k - p)}.
\end{aligned} \tag{61}$$

In a similar way we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\left\{ \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{y'(0) + \eta y(0)}{G(\eta)} \frac{d\eta}{\eta - p} - \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \frac{y'(0) + \eta y(0)}{G(\eta)} \frac{d\eta}{\eta + p} \right\} \\
&= \frac{1}{\pi i} y'(0) \int_{-\sigma_2 - i\infty}^{-\sigma_2 + i\infty} \frac{d\eta}{G(\eta)(\eta - p)} + \sum_{p_k} \frac{y'(0) + p_k y(0)}{G'(p_k)(p_k - p)}.
\end{aligned} \tag{62}$$

Noticing that

$$\Phi(\eta,\zeta) + \Phi(-\eta,\zeta) = \frac{2\zeta}{\eta^2 - \zeta^2} [\phi_1(i\zeta) - \phi_2(i\zeta)]$$

we rewrite (58) in the following final form

$$\begin{aligned}
Y(p) &= \frac{1}{2\pi^2} \int_{\beta - i\infty}^{\beta + i\infty} \zeta \, d\zeta Y(\zeta) [\phi_1(i\zeta) - \phi_2(i\zeta)] \\
&\quad \cdot \int_{-\sigma_2 - i\infty}^{-\sigma_2 + i\infty} \frac{d\eta}{G(\eta)(\eta - p)(\eta^2 - \zeta^2)} \\
&\quad - \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} d\zeta Y(\zeta) \sum_{p_k} \frac{\Phi(p_k,\zeta)}{G'(p_k)} \frac{1}{p_k - p} \\
&\quad + \frac{y'(0)}{\pi i} \int_{-\sigma_2 - i\infty}^{-\sigma_2 + i\infty} \frac{d\eta}{G(\eta)(\eta - p)} \sum_{p_k} \frac{y'(0) + p_k y(0)}{G'(p_k)(p_k - p)}, \\
&\qquad \Re p = \beta, \ \phi_1(0) \neq 1.
\end{aligned} \tag{63}$$

In the case $\phi_1(0) = 1$, $p_1 = p_2 = 0$ and the last formula has to be replaced with

$$\begin{aligned} Y(p) = &-\frac{1}{2\pi^2}\int_{\beta-i\infty}^{\beta+i\infty} \zeta\, d\zeta Y(\zeta)\,[\phi_1(i\zeta)-\phi_2(i\zeta)] \\ &\cdot \int_{-\sigma_2-i\infty}^{-\sigma_2+i\infty} \frac{d\eta}{G(\eta)(\eta-p)(\eta^2-\zeta^2)} \\ &+ \frac{1}{2\pi i p^2 G''(0)}\int_{\beta-i\infty}^{\beta+i\infty} Y(\zeta)\,\frac{\phi_1(i\zeta)(p+\zeta)+\phi_2(i\zeta)(p-\zeta)}{\zeta^2}\,d\zeta \\ &- \frac{y'(0)}{\pi i}\int_{-\sigma_2-i\infty}^{-\sigma_2+i\infty} \frac{d\eta}{G(\eta)(\eta-p)} + \frac{y'(0)+py(0)}{p^2G''(0)}, \\ &\Re\, p = \beta,\ \phi_1(0)=1. \end{aligned} \tag{64}$$

Now that the value σ_1 is no longer present, there is no reason to keep index "2." So we will write σ instead of σ_2. Application of the inverse Laplace transform operator to relations (63) and (64) yields relations (11) and (15) of Theorem 2.2, respectively, if notations (12), (13), (16) and (17) are used.

In these formulae all terms approach zero as $\nu \to \infty$ except the ones proportional to $y(0)$ and $y'(0)$.

7. Proof of Theorems 3.1 and 3.2. To prove Theorem 3.1 we use the usual scheme of reasoning. Suppose $\{z_1(x), Z_1(p)\}$ and $\{z_2(x), Z_2(p)\}$ are two solutions of equation (11) with the same initial values

$$z_1(0) = z_2(0), \qquad z_1'(0) = z_2'(0).$$

Their difference $\{\Delta z(x), \Delta Z(p)\}$ satisfies the following relations:

$$\Delta z(x) = \sum_{k=1,2} \Delta A_k e^{p_k x} - \Delta\Psi(x) \tag{65}$$

where

$$\Delta A_k = -\frac{1}{G'(p_k)}\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \Delta Z(\zeta)\left[\frac{\phi_1(i\zeta)}{p_k-\zeta}+\frac{\phi_2(i\zeta)}{p_k+\zeta}\right]d\zeta, \quad k=1,2, \tag{66}$$

(67)
$$\Delta\Psi(x) = \frac{1}{2\pi^2}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{e^{x\eta}d\eta}{G(\eta)} \int_{\beta-i\infty}^{\beta+i\infty} \Delta Z(\zeta)\,[\phi_1(i\zeta)-\phi_2(i\zeta)]\,\frac{\zeta d\zeta}{\eta^2-\zeta^2},$$

and

$$\Delta Z(\zeta) = \int_0^\infty \Delta z(x)e^{-\zeta x}\,dx, \quad \Re\,\zeta = \beta. \tag{68}$$

The functions $\Delta z(x)$ and $\Delta Z(\zeta)$ belong to the spaces of continuous functions $\{f\}$ and $\{F\}$ with finite norms:

$$\|f\| = \sup_x |f(x)e^{-\lambda x}|, \quad 0 \le x < \infty,$$

and

$$\|F\| = \sup_\zeta |F(\zeta)|, \quad \Re\,\zeta = \beta.$$

Obviously,

$$\|\Delta z(x)\| \le \sum_{k=1,2} |\Delta A_k| + \|\Delta\Psi(x)\|,$$

$$|\Delta A_k| \le \|\Delta Z(\zeta)\|\,\frac{1}{|G'(p_k)|}\,\frac{1}{2\pi}\int_{\beta-i\infty}^{\beta+i\infty}\left[\frac{|\phi_1(i\zeta)|}{|p_k-\zeta|}+\frac{|\phi_2(i\zeta)|}{|p_k+\zeta|}\right]|d\zeta|,$$
$$k=1,2,$$

$$\|\Delta\Psi(x)\| \le \|\Delta Z(\zeta)\|\,\frac{1}{2\pi^2}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{|d\eta|}{|G(\eta)|} \cdot \int_{\beta-i\infty}^{\beta+i\infty} |\phi_1(i\zeta)-\phi_2(i\zeta)|\,\frac{|\zeta d\zeta|}{|\eta^2-\zeta^2|},$$

and

$$\|\Delta Z(\zeta)\| \le \|\Delta z(x)\|\,\frac{1}{\beta-\lambda}.$$

Therefore,

(69)
$$\|\Delta z(x)\| \le \frac{\|\Delta z(x)\|}{\beta-\lambda}\Bigg\{\sum_{k=1,2}\frac{1}{|G'(p_k)|}\,\frac{1}{2\pi}\int_{\beta-i\infty}^{\beta+i\infty}\left[\frac{|\phi_1(i\zeta)|}{|p_k-\zeta|}+\frac{|\phi_2(i\zeta)|}{|p_k+\zeta|}\right]|d\zeta| + \frac{1}{2\pi^2}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{|d\eta|}{|G(\eta)|}\int_{\beta-i\infty}^{\beta+i\infty}|\phi_1(i\zeta)-\phi_2(i\zeta)|\,\frac{|\zeta d\zeta|}{|\eta^2-\zeta^2|}\Bigg\}.$$

The last relation shows that $\|\Delta z(x)\| = 0$ if the coefficient for $\|\Delta z(x)\|$ in the righthand side is less than 1. It is easy to see that this coefficient denoted by $Q(\nu)$ is given by

$$Q(\nu) = \frac{1}{\nu(\beta_0 - \lambda_0)} \left\{ \sum_{k=1,2} \frac{1}{|G'(p_k)|} \frac{1}{2\pi} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \left[\frac{|\phi_1^{(0)}(i\zeta)|}{|p_k - \nu\zeta|} + \frac{|\phi_2^{(0)}(i\zeta)|}{|p_k + \nu\zeta|} \right] |d\zeta| + T(\nu) \right\} \tag{70}$$

where

$$T(\nu) \equiv \frac{\nu}{2\pi^2} \int_{-\sigma_0 - i\infty}^{-\sigma_0 + i\infty} \frac{|d\eta|}{|1 - \nu^2\eta^2 - \phi_1^{(0)}(i\eta)|} \cdot \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} |\phi_1^{(0)}(i\zeta) - \phi_2^{(0)}(i\zeta)| \frac{|\zeta d\zeta|}{|\eta^2 - \zeta^2|}, \quad \phi_1(0) \neq 1.$$

Therefore, the inequality

$$Q(\nu) < 1 \tag{71}$$

is a sufficient condition for the uniqueness of the solution. Clearly, it is satisfied if the parameter ν is large enough because

$$\lim_{\nu \to \infty} Q(\nu) = 0.$$

Theorem 3.1 is proven for the case $\phi_1(0) \neq 1$. The case $\phi_1(0) = 1$ can be considered in much the same way. Equation (70) in this case should be replaced with:

$$Q(\nu) = \frac{1}{\nu(\beta_0 - \lambda_0)} \left\{ \frac{1}{\nu\pi\lambda e|G''(0)|} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{|\phi_1^{(0)}(i\zeta) - \phi_2^{(0)}(i\zeta)|}{|\zeta|} |d\zeta| + \frac{1}{\nu\pi|G''(0)|} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{|\phi_1^{(0)}(i\zeta) + \phi_2^{(0)}(i\zeta)|}{|\zeta^2|} |d\zeta| + T(\nu) \right\}, \quad \phi_1(0) = 1. \tag{72}$$

To prove Theorem 3.2, we introduce differences

$$\Delta y_n(x) = y_{n+1}(x) - y_n(x), \quad n = 1, 2, \dots ,$$

and

$$\Delta Y_n(\zeta) = Y_{n+1}(\zeta) - Y_n(\zeta), \quad n = 0, 1, 2, \dots .$$

According to (19)

$$\Delta y_n(x) = \sum e^{p_k x} \Delta A_{n-1}^{(k)} - \Delta\Psi_{n-1}(x), \quad \phi_1(0) \neq 1, \tag{73}$$

where

$$\Delta A_{n-1}^{(k)} = -\frac{1}{G'(p_k)}\,\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \Delta Y_{n-1}(\zeta)\left[\frac{\phi_1(i\zeta)}{p_k-\zeta} + \frac{\phi_2(i\zeta)}{p_k+\zeta}\right] d\zeta, \qquad k = 1, 2, \tag{74}$$

$$\begin{aligned}\Delta\Psi_{n-1}(x) = \frac{1}{2\pi^2}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{e^{x\eta}d\eta}{G(\eta)} \\ \cdot \int_{\beta-i\infty}^{\beta+i\infty} \Delta Y_{n-1}(\zeta)[\phi_1(i\zeta) - \phi_2(i\zeta)]\,\frac{\zeta d\zeta}{\eta^2 - \zeta^2},\end{aligned} \tag{75}$$

and

$$\Delta Y_{n-1}(\zeta) = \int_0^\infty \Delta y_{n-1}(x) e^{-\zeta x}\, dx, \quad \Re\,\zeta = \beta. \tag{76}$$

These relations define an algorithm of obtaining $\Delta y_n(x)$ if one starts with $\Delta y_{n-1}(x)$, which is precisely the same as that we have used to estimate the righthand side of equation (65) starting with Δz. Therefore, we can write:

$$\|\Delta y_n(x)\| \leq Q(\nu)\|\Delta y_{n-1}(x)\|.$$

Thus, the same condition $Q(\nu) < 1$, which is sufficient for the uniqueness, is also sufficient for the convergence of the sequence $\{y_n(x)\}_{n=1}^\infty$ and, consequently, convergence of the sequences $\{A_n^{(k)}\}_{n=1}^\infty$, $k = 1, 2$, and $\{Y_n(p)\}_{n=1}^{\infty}$. Letting $n \to \infty$ in (19) and (20) we obtain the solution of the initial value problem:

$$y(x) = \lim_{n\to\infty} y_n(x),$$
$$Y(p) = \lim_{n\to\infty} Y_n(p).$$

The case $\phi_1(0) = 1$ can be treated in a similar way.

8. Proof of Theorem 4.1. The proof of Theorem 4.1 basically repeats and combines the proofs of Theorems 3.1 and 3.2. The uniqueness condition (71) obtained in the proof of Theorem 3.1 should be replaced with the condition $\hat{Q}(\nu) < 1$ where

$$\hat{Q}(\nu) = Q(\nu) + \frac{1}{\beta-\lambda}\frac{1}{2\pi^2}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{|d\eta|}{|G(\eta)|} \cdot \int_{\beta-i\infty}^{\beta+i\infty}\left|\frac{\phi_1(i\zeta)}{p_1-\zeta}+\frac{\phi_2(i\zeta)}{p_1+\zeta}\right||d\zeta| < 1 \tag{77}$$

when $\phi_1(0) \neq 1$, or with

$$\hat{Q}(\nu) = Q(\nu) - \frac{1}{\pi(\beta_0-\lambda_0)}\int_{\beta_0-i\infty}^{\beta_0+i\infty}\left|\frac{\phi_1^{(0)}(i\zeta)-\phi_2^{(0)}(i\zeta)}{\zeta}\,d\zeta\right| \cdot\left\{\frac{1}{\nu^2\lambda e|G''(0)|}+\frac{1}{2}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left|\frac{d\eta}{1-\eta^2-\phi_1^{(0)}(i\eta)}\right|\right\} \tag{78}$$

if $\phi_1(0) = 1$.

The same inequality implies the convergence of sequences (38) to their respective limits $y(x)$, $Y(p)$, and $y'(x)$ and guarantees that these functions satisfy (11)–(14), or (15)–(17) in the case $\phi_1(0) = 1$.

9. Approximate solutions. The solutions obtained for problem (1), (4), and (5)are difficult to apply to practical problems. We can obtain useful approximate formulae as follows.

For the case $\phi_1(0) \neq 1$ we start with the exact relation (11) and find expressions for A_k and $\Psi(x)$ in the form:

$$A_k = \tilde{A}_k + \mathcal{O}\left(\frac{1}{\nu^3}\right), \quad k = 1, 2, \tag{79}$$

$$\Psi(x) = \tilde{\Psi}(x) + \mathcal{O}\left(\frac{1}{\nu^3}\right) e^{-\sigma x}, \tag{80}$$

so that

$$y(x) \approx \sum_{k=1,2} \tilde{A}_k e^{p_k x} - \tilde{\Psi}(x). \tag{81}$$

Using the iterative algorithm and ignoring terms of the order ν^{-3} we get:

(82)
$$\tilde{A}_k = -\frac{1}{G'(p_k)} \Bigg[y'(0) + p_k y(0) + \frac{2y(0)p_1}{\nu G'(p_1)} \int_{-\infty}^{0} t \big[R_1^{(0)}(t) - R_2^{(0)}(t) \big] \, dt - \frac{y'(0) + p_k(0)}{G'(p_k)} \frac{p_k}{\nu^2} \int_{-\infty}^{0} t^2 \big[R_1^{(0)}(t) + R_2^{(0)}(t) \big] dt \Bigg],$$

$$\tilde{\Psi}(x) = -\frac{y(0)p_1}{G'(p_1)} \int_{-\infty}^{0} t^2 \big[R_1^{(0)}(t - \nu x) - R_2^{(0)}(t - \nu x) \big] \, dt. \tag{83}$$

Notice that the kernel $R_2(x)$ contributes only to the terms which decay with the growth of ν.

If $\phi_1(0) = 1$ the exact formula (15) gives the approximation:

$$y(x) \approx \tilde{B}_1 x + \tilde{B}_2 - \tilde{\Psi}(x) \tag{84}$$

where

$$\begin{aligned} \tilde{B}_1 = -\frac{2}{G''(0)} \Bigg\{ & y'(0) - \frac{1}{\nu} \frac{y(0)}{G''(0)} \int_{-\infty}^{0} t \big[R_1^{(0)}(t) - R_2^{(0)}(t) \big] \, dt \\ & + \frac{y(0)}{\nu^2} \left[\frac{2}{G''(0)} \int_{-\infty}^{0} t \big[R_1^{(0)}(t) - R_2^{(0)}(t) \big] \, dt \right]^2 \Bigg\}, \end{aligned} \tag{85}$$

$$\tilde{B}_2 = -\frac{1}{G''(0)}\Big[2y(0) + \frac{y'(0)}{\nu^2 G''(0)} \int_{-\infty}^{0} t^2 \big[R_1^{(0)}(t) - R_2^{(0)}(t)\big]\,dt - \frac{2y(0)}{\nu^2 G''(0)} \int_{-\infty}^{0} t^2 \big[R_1^{(0)}(t) + R_2^{(0)}(t)\big]\,dt\Big], \tag{86}$$

and

$$\tilde{\Psi}(x) = \frac{y(0)}{\nu^2 G''(0)} \int_{-\infty}^{0} t^2 \big[R_1^{(0)}(t-\nu x) - R_2^{(0)}(t-\nu x)\big]\,dt. \tag{87}$$

To obtain similar approximations for the solutions of the boundary value problem we set $\tilde{A}_1 = 0$ and $\tilde{B}_1 = 0$. As a result we get relationships between $y'(0)$ and $y(0)$ for both cases:

(88)
$$y'(0) = -\, p_1 y(0)\Big\{1 + \frac{2}{\nu G'(p_1)} \int_{-\infty}^{0} t\big[R_1^{(0)}(t) - R_2^{(0)}(t)\big]\,dt - \frac{y'(0)+p_k(0)}{G'(p_k)}\,\frac{p_k}{\nu^2} \int_{-\infty}^{0} t^2\big[R_1^{(0)}(t)+R_2^{(0)}(t)\big]\,dt + \mathcal{O}\Big(\frac{1}{\nu^3}\Big)\Big\},$$
$$\phi_1(0) \neq 1,$$

(89)
$$y'(0) = y(0)\Big\{\frac{1}{\nu G''(0)} \int_{-\infty}^{0} t\big[R_1^{(0)}(t) - R_2^{(0)}(t)\big]\,dt - \frac{1}{\nu^2}\Big[\frac{2}{G''(0)} \int_{-\infty}^{0} t\big[R_1^{(0)}(t)-R_2^{(0)}(t)\big]\,dt\Big]^2 + \mathcal{O}\Big(\frac{1}{\nu^3}\Big)\Big\},$$
$$\phi_1(0) = 1.$$

Thus, in the case $\phi_1(0) \neq 1$, an approximate solution of the boundary value problem is given by (81) where $\tilde{A}_1 = 0$, $\tilde{A}_2$ is defined by (82) with $y'(0)$ replaced by (88), and $\tilde{\Psi}(x)$ is defined by (83). In the case $\phi_1(0) = 1$ we should use (84) where $\tilde{B}_1 = 0$, $\tilde{B}_2$ is defined by (86) and (89), $\tilde{\Psi}(x)$ is given by (87).

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Department of Mathematics, Northeastern Illinois University, 5500 N. St. Louis Avenue, Chicago, IL 60625-4699
E-mail address: A-Mitina@neiu.edu