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FACTORIZATION OF SINGULAR INTEGRAL OPERATORS WITH A CARLEMAN SHIFT AND SPECTRAL PROBLEMS

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Dedicated to Professor G.S. Litvinchuk
on the occasion of his seventieth birthday

ABSTRACT. In this paper we study singular integral operators with a linear fractional Carleman shift preserving the orientation on the unit circle. The main goal is to characterize the spectrum of some of these operators. To this end a special factorization of the operator is derived with the help of a factorization of a matrix function in a suitable algebra. After developing methods which permit us to obtain a factorization of a matrix function for some classes of interest, the spectral analysis of some types of singular integral operators with a Carleman shift is done.

1. Introduction. In this paper we study singular integral operators with a Carleman shift on the unit circle $\mathbf{T}$ of a special nature, the so-called linear fractional Carleman shift preserving the orientation of $\mathbf{T}$ (see (1.1)). The main objective is to develop tools which permit us to give a complete description of the spectrum for some classes of these operators acting in the Lebesgue space $L_p(\mathbf{T})$, $p \in (1, \infty)$. This includes the identification of the Fredholm and non-Fredholm parts of the spectrum of such an operator T (the latter, in general, is known for the type of operators which we are going to consider here), as well as the calculation of the defect numbers of the operator $T - \lambda I$, for all λ belonging to the Fredholm part of the spectrum of T, and a representation for the resolvent operator. In this respect, this paper may be seen as a continuation of [**7**], where the spectral analysis was made for a very special type of operators.

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Regarding the tools we are going to use, the following must be said. The idea of using the concept of factorization of functions to obtain a factorization of singular integral operators is a very old one and quite well understood. In the case of singular integral operators with a linear fractional Carleman shift, this idea was used in [**9**]. Unfortunately, at the beginning of that paper some lapses were made which, while they do not change the final results much, completely change the contents. This was brought to light several months ago when Prof. G. Litvinchuk posed to the first author of the present paper some interesting questions concerning the referenced paper. The authors are very grateful to Prof. Litvinchuk for these useful questions. So a part of this paper is devoted to the characterization of singular integral operators with a linear fractional Carleman shift preserving the orientation of $\mathbf{T}$ via the factorization of matrix-valued functions of a particular type. Afterwards methods to achieve an explicit factorization of these matrix-valued functions are developed.

Before going any further, let us make precise the notation and terminology that we are going to use.

Let $\mathbf{T}$ denote the unit circle with the usual anti-clockwise orientation, and denote by $\mathbf{T}_+$ and $\mathbf{T}_-$ the interior and exterior of $\mathbf{T}$, respectively. It is well known that the singular integral with Cauchy kernel $S\varphi$, given almost everywhere on $\mathbf{T}$ by

$$S\varphi(t) = \frac{1}{\pi i} \int_{\mathbf{T}} \frac{1}{\tau - t} \varphi(\tau)\, d\tau,$$

where the integral is understood in the sense of its principal value, defines a bounded linear operator in $L_p(\mathbf{T})$ for all $p \in (1, \infty)$. A very useful property of this operator is that $S^2 = I$, I denoting the identity operator in $L_p(\mathbf{T})$, $p \in (1, \infty)$. This permits us to introduce in $L_p(\mathbf{T})$ a pair of complementary projection operators by $P_\pm = (I \pm S)/2$ and decompose $L_p(\mathbf{T}) = L_p^+(\mathbf{T}) \oplus \mathring{L}_p^-(\mathbf{T})$ with $L_p^+(\mathbf{T}) = \operatorname{im} P_+$ and $\mathring{L}_p^-(\mathbf{T}) = \operatorname{im} P_-$. We also set $L_p^-(\mathbf{T}) = \mathring{L}_p^-(\mathbf{T}) \oplus \mathbf{C}$.

On the unit circle we consider the so-called linear fractional Carleman shift, i.e., the function α defined by

$$\alpha(t) = \frac{t - \beta}{\bar{\beta} t - 1}, \quad |\beta| < 1. \tag{1.1}$$

This choice of the constant β implies that the shift α preserves the orientation of $\mathbf{T}$. Let us mention that if $\beta \neq 0$, which we suppose to hold from now on, then α does not have fixed points on $\mathbf{T}$, but if we consider it defined in $\mathbf{C} \setminus \{1/\bar{\beta}\}$ there are two fixed points $\xi_\pm \in \mathbf{T}_\pm$, given by $\xi_\pm = (1 \mp \delta)/\bar{\beta}$, with $\delta = \sqrt{1 - |\beta|^2}$. Associated with the shift function α we consider the weighted shift operator $U : L_p(\mathbf{T}) \to L_p(\mathbf{T})$, $p \in (1, \infty)$, defined by

$$(Uf)(t) = -\alpha_+(t) f(\alpha(t))$$

where α_+ is the left factor in the following factorization of α:

$$\alpha = \alpha_+ \, t \, \alpha_- \tag{1.2}$$

with

$$\alpha_+(t) = \frac{\delta}{\bar{\beta} t - 1}, \qquad \alpha_-(t) = \frac{t - \beta}{\delta t}, \quad t \in \mathbf{T}. \tag{1.3}$$

With this definition the operator U satisfies the following properties:

(i) $U^2 = I$,

(ii) $US = SU$.

Note that $\alpha_+(\alpha(t)) = \alpha_+^{-1}(t)$, and so property (i) is a consequence of the Carleman condition $\alpha(\alpha(t)) = t$, $t \in \mathbf{T}$. Also this rule for the composition plays a role in property (ii), which can be verified by a straightforward computation. It should be emphasized that the main reason to include the weight α_+ in the definition of the shift operator U is precisely to have this property (it is well known that, for an unweighted shift operator, i.e., $V\varphi(t) = \varphi(\alpha(t))$, where α preserves the orientation of $\mathbf{T}$, $VS = SV + K$, where K is a compact operator). Since we would like to consider spectral problems, the fulfillment of property (ii) is very important.

An operator acting in $L_p(\mathbf{T})$ of the form

$$A = a\,I + b\,U, \tag{1.4}$$

where a, b are functions defined on $\mathbf{T}$, is called a functional operator and the functions a and b are referred to as coefficients of this operator.

The purpose of this paper is to study singular integral operators of the form

$$T(A,B) = AP_+ + BP_-, \tag{1.5}$$

where A and B are functional operators. For simplicity, in what follows we shall deal most of the time with the case of functional operators having continuous coefficients and afterwards we consider the extension of the results to the case of coefficients belonging to $L_\infty(\mathbf{T})$.

The paper is organized as follows. In Section 2 we complete the results of [**9**]. The Fredholm properties of a singular integral operator with a linear fractional Carleman shift are expressed in terms of a factorization of a 2×2 matrix function. This connection was also used in [**7**] to obtain the spectrum of some singular integral operators with a Carleman shift, in the case of multiplication operators by the characteristic function of some arc of the unit circle. As was mentioned at the beginning, one of our objectives is to characterize the spectrum of other singular integral operators with a Carleman shift. Therefore, in Section 3 we consider the factorization problem for a class of 2×2 matrix functions, which is a subclass of the one known as Daniele-Khrapkov class, the elements of which can be interpreted as continuous functions of a multiplication operator by a rational matrix function. The factorization problem for another class of matrix functions related to singular integral operators with a Carleman shift is also considered. Finally, in Section 4 we use the results obtained so far to characterize the spectrum of some singular integral operators with a linear fractional Carleman shift of the form (1.1).

2. Factorization of singular integral operators with a Carleman shift. First we need to introduce the notation and recall some basic results. Let $\mathfrak{B}$ denote a decomposing algebra of continuous functions on the unit circle, $\mathfrak{B} = \mathfrak{B}_+ \oplus \overset{\circ}{\mathfrak{B}}_-$, with $\mathfrak{B}_+ = P_+\mathfrak{B}$, $\overset{\circ}{\mathfrak{B}}_- = P_-\mathfrak{B}$ and set $\mathfrak{B}_- = \overset{\circ}{\mathfrak{B}}_- \oplus \mathbf{C}$ (see, for instance, [**5**] or [**14**]). By $\mathbf{A}$ we denote the algebra of all functional operators with coefficients in $\mathfrak{B}$. Analogously, $\mathbf{A}_+$ and $\overset{\circ}{\mathbf{A}}_-$ are the sets of functional operators with coefficients, respectively, in $\mathfrak{B}_\pm$ and $\overset{\circ}{\mathfrak{B}}_-$. From the commutation relation between U and S, it follows that

$$\mathbf{A} = \mathbf{A}_+ \oplus \overset{\circ}{\mathbf{A}}_-, \qquad P_\mp \mathbf{A}_\pm P_\pm = \{0\},$$

and so the operator algebra $\mathbf{A}$ is decomposing. Therefore it makes sense to consider the factorization problem for the elements of $\mathbf{A}$.

Definition 2.1. Let α be the Carleman shift (1.1). In the sequel we use the symbol $\mathfrak{B}_\alpha^{2\times 2}$ for the algebra of all 2×2 matrix functions of the form

$$\mathcal{A} = \begin{pmatrix} a & b \\ b(\alpha) & a(\alpha) \end{pmatrix} \tag{2.1}$$

where $a, b \in \mathfrak{B}$.

Here and in what follows by $x(\alpha)$ we represent the composition $x \circ \alpha$. In the case where x is a matrix function, its elements are defined componentwise.

We shall use frequently the following simple characterization of the elements of $\mathfrak{B}_\alpha^{2\times 2}$: $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$ if and only if $\mathcal{A} \in \mathfrak{B}^{2\times 2}$ and

$$\mathcal{A} = e\,\mathcal{A}(\alpha)e, \quad e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

It is easy to verify that the subalgebra $\mathfrak{B}_\alpha^{2\times 2}$ of $\mathfrak{B}^{2\times 2}$ is also a decomposing algebra

$$\mathfrak{B}_\alpha^{2\times 2} = \mathfrak{B}_{+,\alpha}^{2\times 2} \oplus \overset{\circ}{\mathfrak{B}}{}_{-,\alpha}^{2\times 2}$$

where $\mathfrak{B}_{+,\alpha}^{2\times 2} = P_+\mathfrak{B}_\alpha^{2\times 2}$ and $\overset{\circ}{\mathfrak{B}}{}_{-,\alpha}^{2\times 2} = P_-\mathfrak{B}_\alpha^{2\times 2}$. We put $\mathfrak{B}_{-,\alpha}^{2\times 2} = \overset{\circ}{\mathfrak{B}}{}_{-,\alpha}^{2\times 2} \oplus \mathbf{C}^{2\times 2}$.

It is well known that the map

$$\pi : A = a\,I + b\,U \longmapsto \mathcal{A} = \begin{pmatrix} a & b \\ b(\alpha) & a(\alpha) \end{pmatrix} \tag{2.3}$$

is an algebraic isomorphism from the algebra $\mathbf{A}$ onto the algebra $\mathfrak{B}_\alpha^{2\times 2}$, for every Carleman shift α (see, for instance, [**10**]). Moreover, π is a homomorphism which in the present case, where α is the linear fractional Carleman shift (1.1), satisfies the following invariance properties

$$\pi(\mathbf{A}_\pm) = \mathfrak{B}^{2\times 2}_{\pm,\alpha}.$$

This property means that if we want to factorize a singular integral operator with coefficients belonging to the algebra $\mathbf{A}$, we must look for a factorization of the matrix function $\mathcal{A} \in \mathfrak{B}^{2\times 2}_\alpha$ in (within) the algebra $\mathfrak{B}^{2\times 2}_\alpha$.

2.1 Factorization in the algebra $\mathfrak{B}^{2\times 2}_\alpha$. Now we proceed in order to define a concept of factorization for the elements of the algebra $\mathfrak{B}^{2\times 2}_\alpha$ which will be convenient for our purposes.

Let us recall that a factorization of a matrix function $\mathcal{A} \in \mathfrak{B}^{2\times 2}$ in the algebra $\mathfrak{B}^{2\times 2}$ is a representation of it in the form

$$\mathcal{A} = \mathcal{A}_+ \, \Lambda \, \mathcal{A}_-,$$

where $\mathcal{A}_+^{\pm 1} \in \mathfrak{B}^{2\times 2}_+$, $\mathcal{A}_-^{\pm 1} \in \mathfrak{B}^{2\times 2}_-$, and

$$\Lambda(t) = \operatorname{diag}\{t^{\kappa_1}, t^{\kappa_2}\}, \tag{2.4}$$

where κ_1, κ_2 are integers, which we suppose to satisfy $\kappa_1 \geq \kappa_2$ and are called partial indices of $\mathcal{A}$ (they are uniquely determined by $\mathcal{A}$). The number $\kappa = \kappa_1 + \kappa_2$ is referred to as the total index of $\mathcal{A}$ and equals $\operatorname{ind}_{\mathbf{T}} \det \mathcal{A}$. If $\kappa_1 = \kappa_2 = 0$ the factorization of $\mathcal{A}$ is said to be canonical.

Now note that if $\mathcal{A} \in \mathfrak{B}^{2\times 2}_\alpha$ admits a noncanonical factorization, then the central factor in a factorization of $\mathcal{A}$ does not belong to $\mathfrak{B}^{2\times 2}_\alpha$, i.e., $\Lambda \notin \mathfrak{B}^{2\times 2}_\alpha$.

However, the situation is clear if $\mathcal{A} \in \mathfrak{B}^{2\times 2}_\alpha$ admits a canonical factorization. In fact, if this is the case then, as is well known, $\mathcal{A}$ can be uniquely represented in the form

$$\mathcal{A} = \mathcal{A}_+ \, \mathcal{A}_-, \quad \text{with } \mathcal{A}_+(\xi_+) = e_0,$$

where ξ_+ is the fixed point of the shift α in the interior of $\mathbf{T}$ and e_0 is the 2×2 identity matrix. Since $\mathcal{A} = e\mathcal{A}(\alpha)e$ and

$$\mathcal{A} = e\, \mathcal{A}_+(\alpha) e\, e\, \mathcal{A}_-(\alpha) e, \quad \text{with } e\, \mathcal{A}_+(\alpha(\xi_+)) e = e_0,$$

it follows that $\mathcal{A}_+ = e\mathcal{A}_+(\alpha)e$ and $\mathcal{A}_- = e\mathcal{A}_-(\alpha)e$ or, what is the same, $\mathcal{A}_+ \in \mathfrak{B}^{2\times 2}_{+,\alpha}$ and $\mathcal{A}_- \in \mathfrak{B}^{2\times 2}_{-,\alpha}$.

In the next proposition we draw some conclusions about the factors of an arbitrary factorization of $\mathcal{A}$.

Proposition 2.2. *If $\mathcal{A} \in \mathfrak{B}^{2\times 2}_{\alpha}$ admits factorization in $\mathfrak{B}^{2\times 2}$, say $\mathcal{A} = \mathcal{A}_+\Lambda\mathcal{A}_-$, then the outer factors satisfy the following identities*

$$\begin{aligned}\mathcal{A}_+ &= e\,\mathcal{A}_+(\alpha)\Lambda_+\mathcal{H}_{\varepsilon,p}\\ \mathcal{A}_- &= \Lambda^{-1}\mathcal{H}_{\varepsilon,p}\,\Lambda\,\Lambda_-\mathcal{A}_-(\alpha)e\end{aligned}$$

where e is the constant matrix given in (2.2), $\Lambda_\pm = \operatorname{diag}\{\alpha_\pm^{\kappa_1}, \alpha_\pm^{\kappa_2}\}$ *and $\mathcal{H}_{\varepsilon,p}$ is a triangular matrix function: $\mathcal{H}_{\varepsilon,p} = \left(\begin{smallmatrix}\varepsilon & p\\ 0 & -\varepsilon\end{smallmatrix}\right)$ where $\varepsilon \in \{1,-1\}$ and p is a polynomial of degree at most $\kappa_1-\kappa_2$ which is an even number, such that*

$$p(\alpha) = \alpha_+^{\kappa_1-\kappa_2}\,p. \tag{2.5}$$

Proof. For $\mathcal{A} \in \mathfrak{B}^{2\times 2}_{\alpha}$ we have $\mathcal{A} = e\mathcal{A}(\alpha)e$ and then

$$\mathcal{A} = (e\,\mathcal{A}_+(\alpha)\Lambda_+)\,\Lambda\,(\Lambda_-\mathcal{A}_-(\alpha)e)$$

is another factorization of $\mathcal{A}$. It is well known that there exists a triangular matrix function $\mathcal{H} = \left(\begin{smallmatrix}\mu & p\\ 0 & \nu\end{smallmatrix}\right)$, where $\mu,\nu \in \mathbf{C}$ and p is a polynomial of degree not greater than $\kappa_1 - \kappa_2$ (see [**14**]), such that the outer factors in the two factorizations of $\mathcal{A}$ are related by

$$\mathcal{A}_+ = e\,\mathcal{A}_+(\alpha)\Lambda_+\mathcal{H}, \tag{2.6}$$

$$\mathcal{A}_- = \Lambda^{-1}\mathcal{H}^{-1}\Lambda\,\Lambda_-\mathcal{A}_-(\alpha)e. \tag{2.7}$$

The shift (1.1) has two fixed points, $\xi_\pm \in \mathbf{T}_\pm$. Calculating the determinant of both sides of (2.6) and (2.7) and evaluating them at the fixed points ξ_+ and ξ_-, respectively, we obtain

$$\mu\,\nu(-1)^{\kappa_1+\kappa_2} = -1 \quad\text{and}\quad \mu\,\nu = -1,$$

where we have used the equalities $\alpha_+(\xi_+) = -1$ and $\alpha_-(\xi_-) = 1$. Thus, $\kappa_1 + \kappa_2$, and also $\kappa_1 - \kappa_2$, is necessarily an even number. Note that this conclusion could actually be obtained from the fact that $\det \mathbf{A}$ is invariant under the shift α. On the other hand, from (2.6), we get $e_0 = \Lambda_+(\alpha)\mathcal{H}(\alpha)\Lambda_+\mathcal{H}$. Then we have $\mu^2 = \nu^2 = 1$, and, therefore,

$$\mu = -\nu = \varepsilon, \quad \varepsilon \in \{1, -1\},$$

and we also conclude that p must satisfy the equality (2.5). Setting $\mathcal{H}_{\varepsilon,p} = \mathcal{H}$, we also have $\mathcal{H}_{\varepsilon,p}^{-1} = \mathcal{H}_{\varepsilon,p}$. $\square$

It should be noted that with a little extra effort, it can be shown that the number ε appearing in this proposition is uniquely determined by the matrix function $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$, thus not depending on the factorization under consideration.

We are now prepared to introduce the concept of factorization in the algebra $\mathfrak{B}_\alpha^{2\times 2}$.

Definition 2.3. Let $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$ be nonsingular. By a *factorization of $\mathcal{A}$ in the algebra $\mathfrak{B}_\alpha^{2\times 2}$*, we mean its representation in the form

$$\mathcal{A} = \mathcal{A}_{+,\alpha}\, \mathcal{R}\, \mathcal{A}_{-,\alpha},$$

where

(i) $\mathcal{A}_{+,\alpha}^{\pm 1} \in \mathfrak{B}_{+,\alpha}^{2\times 2}$, $\mathcal{A}_{-,\alpha}^{\pm 1} \in \mathfrak{B}_{-,\alpha}^{2\times 2}$,

(ii) $\mathcal{R} \in \mathfrak{B}_\alpha^{2\times 2}$ is of the form

$$\mathcal{R} = \mathcal{K}\, \mathcal{S}^{-1} \mathrm{diag}\,\{\chi^{\kappa_1}, \chi^{\kappa_2}\}\, \mathcal{S},$$

where κ_1, κ_2 are integers such that $\kappa_1 \geq \kappa_2$, $\mathcal{K} = \mathrm{diag}\,\{(1, (-1)^{\kappa_1}\}$, $\mathcal{S}$ is a constant nonsingular matrix, and $\chi \in \mathfrak{B}$ satisfies the following properties:

$$\chi(\alpha) = -\chi \quad \text{and} \quad \mathrm{ind}_{\mathbf{T}}\, \chi = 1. \tag{2.8}$$

Here $\mathrm{ind}_{\mathbf{T}}\, \chi$ denotes the winding number of the function χ on $\mathbf{T}$.

Let us make some remarks concerning this definition.

For some time it was not clear to us what we should demand of the middle factor $\mathcal{R}$ in the above definition. However, it should be noted that, since $\mathfrak{B}_\alpha$ is a subalgebra of $\mathfrak{B}$, any factorization of $\mathcal{A}$ in $\mathfrak{B}_\alpha^{2\times 2}$ should give rise to a factorization of $\mathcal{A}$ in $\mathfrak{B}^{2\times 2}$. This is ensured by the condition $\mathrm{ind}_{\mathbf{T}}\,\chi = 1$ since this implies that χ can be factorized in $\mathfrak{B}$ as $\chi = \chi_+\, t\, \chi_-$ and, consequently, we obtain a factorization of $\mathcal{R}$ in $\mathfrak{B}^{2\times 2}$ as follows

$$\mathcal{R} = (\mathcal{K}\,\mathcal{S}^{-1}\mathrm{diag}\,\{\chi_+^{\kappa_1}, \chi_+^{\kappa_2}\})\mathrm{diag}\,\{t^{\kappa_1}, t^{\kappa_2}\}(\mathrm{diag}\,\{\chi_-^{\kappa_1}, \chi_-^{\kappa_2}\}\,\mathcal{S}).$$

Substituting this factorization in the factorization $\mathcal{A} = \mathcal{A}_{+,\alpha}\mathcal{R}\mathcal{A}_{-,\alpha}$ gives the required factorization of $\mathcal{A}$ in $\mathfrak{B}^{2\times 2}$. This procedure shows that the numbers κ_1 and κ_2 are nothing else but the partial indices of $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$ which according to the previous proposition are both even or both odd numbers.

Let us now explain the role of the other condition imposed on the function χ, namely, $\chi(\alpha) = -\chi$. First of all, this condition is compatible with the first one, since every α anti-invariant function has odd index. Further, for instance, in the case of κ_1 (and also κ_2) being an even number, when $\mathcal{K} = e_0$, it implies that $\mathcal{R}(\alpha) = \mathcal{R}$. So, in order that $\mathcal{R} \in \mathfrak{B}_\alpha^{2\times 2}$, we must have $e\mathcal{R}e = \mathcal{R}$. This permits us to determine explicitly the constant matrix $\mathcal{S}$ in order to fulfill that requirement: $\mathcal{S}$ is a diagonalizing matrix for e. Analogously, in the case of κ_1 (and also κ_2) being an odd number, when $\mathcal{K} = \mathrm{diag}\,\{1, -1\}$, that condition on χ implies that $\mathcal{R}(\alpha) = -\mathcal{R}$. Since $\mathcal{K}$ anti-commutes with e, in order that $\mathcal{R} \in \mathfrak{B}_\alpha^{2\times 2}$, the same conclusion must hold for $\mathcal{S}$. Among the diagonalizing matrices for e, we may choose those being unitary matrices, yielding

$$\mathcal{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & v \\ -v & 1 \end{pmatrix}, \quad \text{with } v = \pm 1. \tag{2.9}$$

From now on, we write $\mathcal{S}_v$ instead of $\mathcal{S}$ for this choice.

As in a usual factorization, we may choose the function χ as simple as possible. We give two examples of possibly good choices. We can take for χ the function defined by $\chi(t) = \alpha(t) - t$, $t \in \mathbf{T}$, which can be factorized as $\chi = \chi_+\, t\, \chi_-$, with $\chi_+(t) = \bar{\beta}((t - \xi_-)/(\bar{\beta}t - 1))$ and $\chi_-(t) = 1 - (\xi_+/t)$. In this case the matrix function $\mathcal{R}$ is rational. Another example is provided by a convenient branch of the multi-valued function $\sqrt{t\,\alpha(t)}$, namely $\chi(t) = \sqrt{\alpha_+}\, t\, \sqrt{\alpha_-}$, where $\sqrt{\alpha_\pm}$ are branches preserving analyticity in $\mathbf{T}_\pm$. In the present context, this choice can be made if we can guarantee that $\sqrt{\alpha_\pm} \in \mathfrak{B}$.

For further reference, let us mention the following properties of the factors of any factorization of an arbitrary function χ satisfying the two conditions (2.8):

$$\chi_+(\alpha) = -\alpha_+^{-1}\,\chi_+, \qquad \chi_-(\alpha) = \alpha_-^{-1}\,\chi_-. \tag{2.10}$$

Now we are ready to state the main result about the factorization in $\mathfrak{B}_\alpha^{2\times 2}$.

Theorem 2.4. *Let $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$. Then the following assertions are equivalent*:

(i) $\mathcal{A}$ *admits a factorization in* $\mathfrak{B}_\alpha^{2\times 2}$,

(ii) $\mathcal{A}$ *admits a factorization in* $\mathfrak{B}^{2\times 2}$.

Moreover, from any of the above factorizations of $\mathcal{A}$ one can obtain explicitly the other.

Proof. From the remarks made before this theorem, it is clear that (i) $\Rightarrow$ (ii). So it remains to prove that (ii) $\Rightarrow$ (i). Suppose that

$$\mathcal{A} = \mathcal{A}_+\,\Lambda\,\mathcal{A}_- \tag{2.11}$$

is a factorization of $\mathcal{A}$ in $\mathfrak{B}^{2\times 2}$ with $\Lambda = \operatorname{diag}\{t^{\kappa_1}, t^{\kappa_2}\}$. If this is a canonical factorization ($\Lambda = e_0$), then as we have shown before, it is also a factorization of $\mathcal{A}$ in $\mathfrak{B}_\alpha^{2\times 2}$. We suppose that $\Lambda \neq e_0$.

Since $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$, by Proposition 2.2, we know that the outer factors in the factorization (2.11) satisfy

$$\mathcal{A}_+ = e\,\mathcal{A}_+(\alpha)\Lambda_+\mathcal{H}_{\varepsilon,p}, \qquad \mathcal{A}_- = \Lambda^{-1}\mathcal{H}_{\varepsilon,p}\,\Lambda\,\Lambda_-\mathcal{A}_-(\alpha)e$$

for some ε, p satisfying the conditions given in that proposition.

Let $v \in \{1,-1\}, \mathcal{S}_v$ be the matrix defined in (2.9) and $\mathcal{K} = \operatorname{diag}\{1, (-1)^{\kappa_1}\}$. Further, let $\chi \in \mathfrak{B}$ be a function satisfying the conditions mentioned in Definition 2.3, and set

$$\mathcal{R}_v = \mathcal{K}\,\mathcal{S}_v^{-1}\Lambda_\chi\,\mathcal{S}_v, \qquad \Lambda_\chi = \operatorname{diag}\{\chi^{\kappa_1}, \chi^{\kappa_2}\}. \tag{2.12}$$

We have $\mathcal{R}_v \in \mathfrak{B}_\alpha^{2\times 2}$ (see the notes after Definition 2.3). Consider a factorization of $\chi = \chi_+\,t\,\chi_-$ in $\mathfrak{B}$ and the corresponding factorization of $\mathcal{R}_v$ in $\mathfrak{B}^{2\times 2}$:

$$\mathcal{R}_v = \tilde{\mathcal{R}}_{v+}\Lambda\,\tilde{\mathcal{R}}_{v-} = (\mathcal{K}\,\mathcal{S}_v^{-1}\Lambda_{\chi_+})\,\Lambda\,(\Lambda_{\chi_-}\mathcal{S}_v),$$

where $\Lambda_{\chi_\pm} = \operatorname{diag}\{\chi_\pm^{\kappa_1}, \chi_\pm^{\kappa_2}\}$. Using the first one of the properties given in (2.10), it is straightforward to verify that $\tilde{\mathcal{R}}_{v+} = e\,\tilde{\mathcal{R}}_{v+}(\alpha)\Lambda_+\mathcal{H}_{v,0}$, where we have used the notation in Proposition 2.2.

We set $v = \varepsilon$ and $\mathcal{R} = \mathcal{R}_\varepsilon$. Since $\mathcal{R}$ also belongs to $\mathfrak{B}_\alpha^{2\times 2}$ and has the same partial indices as $\mathcal{A}$, if we consider an arbitrary factorization of it, say

$$\mathcal{R} = \mathcal{R}_+\,\Lambda\,\mathcal{R}_-, \tag{2.13}$$

using again Proposition 2.2, we have

$$\mathcal{R}_+ = e\,\mathcal{R}_+(\alpha)\Lambda_+\mathcal{H}_{\varepsilon,q}, \qquad \mathcal{R}_- = \Lambda^{-1}\mathcal{H}_{\varepsilon,q}\,\Lambda\,\Lambda_-\mathcal{R}_-(\alpha)e \tag{2.14}$$

for some polynomial q satisfying the same requirements as p.

Substituting (2.13) in (2.11), we obtain

$$\mathcal{A} = (\mathcal{A}_+\mathcal{R}_+^{-1})\,\mathcal{R}\,(\mathcal{R}_-^{-1}\mathcal{A}_-).$$

Now we claim that from the given factorization of $\mathcal{A}$, which is characterized by the polynomial p, we can choose a factorization of $\mathcal{R}$, which is characterized by the polynomial q, such that the above representation is a factorization of $\mathcal{A}$ in the algebra $\mathfrak{B}_\alpha^{2\times 2}$. In fact, if we take $\mathcal{R}_+ = \tilde{\mathcal{R}}_+\mathcal{H}_{\varepsilon,p/2}$ and $\mathcal{R}_- = \Lambda^{-1}\mathcal{H}_{\varepsilon,p/2}\,\Lambda\tilde{\mathcal{R}}_-$, then (2.14) is fulfilled with $q = p$ and, since $\mathcal{H}_{\varepsilon,p} = \mathcal{H}_{\varepsilon,p}^{-1}$, it follows that

$$\begin{aligned}\mathcal{A}_{+,\alpha} &= \mathcal{A}_+\mathcal{R}_+^{-1} = e\,\mathcal{A}_+(\alpha)\Lambda_+\mathcal{H}_{\varepsilon,p}\mathcal{H}_{\varepsilon,p}^{-1}\,\Lambda_+^{-1}\mathcal{R}_+^{-1}(\alpha)e\\ &= e(\mathcal{A}_+\mathcal{R}_+^{-1})(\alpha)e = e\,\mathcal{A}_{+,\alpha}(\alpha)e\end{aligned}$$

and

$$\mathcal{A}_{+,\alpha}^{-1} = e\,\mathcal{A}_{+,\alpha}^{-1}(\alpha)e.$$

The verification of the corresponding properties for the factor $\mathcal{A}_{-,\alpha} = \mathcal{R}_-^{-1}\mathcal{A}_-$ is straightforward, and so the proof is complete. □

It should be emphasized that the proof of this theorem furnishes a constructive procedure to obtain a factorization of a matrix function $\mathcal{A} \in \mathfrak{B}_\alpha^{2\times 2}$ in the algebra $\mathfrak{B}_\alpha^{2\times 2}$ from a factorization of it in $\mathfrak{B}^{2\times 2}$. In fact, if $\mathcal{A} = \mathcal{A}_+\,\Lambda\,\mathcal{A}_-$ is a factorization of it in $\mathfrak{B}^{2\times 2}$, with partial indices κ_1, κ_2, its outer factors satisfy the relations established in

Proposition 2.2 (that is, the number $\varepsilon \in \{-1, 1\}$ and the polynomial p are fixed). Then a factorization of $\mathcal{A}$ in $\mathfrak{B}_\alpha^{2\times 2}$ is given by

$$\mathcal{A} = \mathcal{A}_{+,\alpha}\, \mathcal{R}\, \mathcal{A}_{-,\alpha},$$

where $\mathcal{R} \in \mathfrak{B}_\alpha^{2\times 2}$ is given by (2.12), with $v = \varepsilon$, where $\chi \in \mathfrak{B}$ is any function satisfying (2.8); for instance, those given before this theorem. A simple computation shows that $\mathcal{R}$ can be written as

$$\mathcal{R} = \begin{pmatrix} u & \varepsilon v \\ \varepsilon v_\alpha & u_\alpha \end{pmatrix} = \begin{cases} \begin{pmatrix} u & \varepsilon v \\ \varepsilon v & u \end{pmatrix} & \text{if } \kappa_1 \text{ and } \kappa_2 \text{ are even,} \\ \begin{pmatrix} u & \varepsilon v \\ -\varepsilon v & -u \end{pmatrix} & \text{if } \kappa_1 \text{ and } \kappa_2 \text{ are odd,} \end{cases} \tag{2.15}$$

with

$$u = \frac{1}{2}\,(\chi^{\kappa_1} + \chi^{\kappa_2}), \qquad v = \frac{1}{2}\,(\chi^{\kappa_1} - \chi^{\kappa_2}). \tag{2.16}$$

The outer factors $\mathcal{A}_{\pm,\alpha}$ of the factorization of $\mathcal{A}$ in $\mathfrak{B}_\alpha^{2\times 2}$ can be obtained by considering a convenient factorization of $\mathcal{R}$ in $\mathfrak{B}^{2\times 2}$.

2.2 Operator factorization. Now we apply the results of the previous subsection to analyze singular integral operators with a Carleman shift or, to be more precise, singular integral operators with coefficients belonging to the algebra $\mathbf{A}$.

Without loss of generality, we restrict ourselves to the study of operators of the form:

$$T(A) = T(I, A) = P_+ + AP_-, \tag{2.17}$$

where $A \in \mathbf{A}$ is a functional operator. In fact, it is well known that a necessary condition for an operator of the form (1.5) to be Fredholm is that its coefficients, the functional operators A and B, are invertible operators (see, e.g., [**8**]).

The first result that we can deduce is that the operator $T(A)$ admits a factorization whenever the matrix function $\pi(A)$ admits a factorization in the algebra $\mathfrak{B}_\alpha^{2\times 2}$ (see 2.3 for the definition of the map π). In fact, we have

Theorem 2.5. *Let A be a functional operator of the form* (1.4) *with coefficients from a decomposing algebra $\mathfrak{B}$, $A \in \mathbf{A}$, Suppose that $\mathcal{A} = \pi(A)$ admits a factorization in the algebra $\mathfrak{B}_\alpha^{2\times 2}$, say $\mathcal{A} = \mathcal{A}_{+,\alpha}\mathcal{R}\mathcal{A}_{-,\alpha}$, where $\mathcal{R}$ is the matrix function* (2.15). *Then the operator $T(A)$ admits the factorization*

$$T(A) = A_+\, T(R)\, (A_+^{-1}P_+ + A_-P_-), \tag{2.18}$$

where $A_\pm = \pi^{-1}(\mathcal{A}_{\pm,\alpha})$ and $R = \pi^{-1}(\mathcal{R})$.

Proof. This representation is a simple consequence of the properties of the map π. In fact, if $\mathcal{A} = \mathcal{A}_{+,\alpha}\mathcal{R}\mathcal{A}_{-,\alpha}$ is a factorization of $\mathcal{A} = \pi(A)$ in $\mathfrak{B}_\alpha^{2\times 2}$, then for the functional operator $A \in \mathbf{A}$ we have $A = A_+RA_-$, where $A_\pm = \pi^{-1}(\mathcal{A}_{\pm,\alpha})$ are invertible functional operators, with inverses given by $A_\pm^{-1} = \pi^{-1}(\mathcal{A}_{\pm,\alpha}^{-1})$, such that $P_\mp A_\pm P_\pm = 0$ and $P_\mp A_\pm^{-1} P_\pm = 0$. Therefore,

$$\begin{aligned} T(A) &= P_+ + A\,P_- = A_+(P_+ + RP_-)\,(A_+^{-1}P_+ + A_-P_-) \\ &= A_+\,T(R)\,(A_+^{-1}P_+ + A_-P_-). \qquad \square \end{aligned}$$

In the next theorem, we derive some results on the generalized invertibility of $T(A)$, $A \in \mathbf{A}$. In order to have a useful representation for a generalized inverse, we fix the matrix function $\mathcal{R}$ to be the one given by (2.15)–(2.16) with $\chi(t) = \alpha(t) - t$, $t \in \mathbf{T}$.

Theorem 2.6. *Let $A \in \mathbf{A}$ and suppose that the conditions of the previous theorem are fulfilled, and let κ_1, κ_2 be the partial indices of $\mathcal{A}$. Let $\mathcal{R}$ be the matrix function given above and $R = \pi^{-1}(\mathcal{R})$. Then $T(A)$ is generalized invertible and a generalized inverse of it is given by*

$$T(A)^{(-1)} = (A_+P_+ + A_-^{-1}P_-)\,RT\,(R^{-1})R^{-1}A_+^{-1}.$$

Moreover, we have

(i) *$T(A)$ is left invertible if $\kappa_1 \le 0$,*

(ii) *$T(A)$ is right invertible if $\kappa_2 \ge 0$*

and, therefore, $T(A)$ is an invertible operator if $\kappa_1 = \kappa_2 = 0$. In each of these cases, the one-sided or two-sided inverse of $T(A)$ is given by the expression above.

Proof. First of all, note that the outer factors A_+ and $A_+^{-1}P_+ + A_-P_-$ in (2.18) are invertible operators with inverses given by A_+^{-1} and $A_+P_+ + A_-^{-1}P_-$, respectively, where $A_\pm = \pi^{-1}(\mathcal{A}_{\pm,\alpha})$. Therefore, we have to prove that a generalized inverse of the operator $T(R)$ is the operator

$$\begin{aligned} T(R)^{(-1)} &= R\,T(R^{-1})R^{-1} = R(P_+ + R^{-1}P_-)R^{-1} \\ &= (R\,P_+ + P_-)R^{-1}, \end{aligned} \tag{2.19}$$

which is the right inverse of $T(R)$ if $\kappa_2 \geq 0$, and is the left inverse of $T(R)$ if $\kappa_1 \leq 0$.

To this end, we introduce some notation and useful representations for the operators under consideration. For the sake of simplicity in writing formulas, let us rename the lowest partial index of $\mathcal{A}$: $\kappa_{-1} =: \kappa_2$. Recall that we already know that the partial indices of $\mathcal{A}$ are both even or both odd numbers. Since $U^2 = I$, we introduce the pair of complementary projection operators in $L_p(\mathbf{T})$, $p \in (1, \infty)$, by

$$Q^\pm = \frac{1}{2}(I \pm U). \tag{2.20}$$

Thus

$$\begin{aligned} R &= uI + \varepsilon vU = (u + \varepsilon v)Q^+ + (u - \varepsilon v)Q^- \\ &= \chi^{\kappa_\varepsilon}Q^+ + \chi^{\kappa_{-\varepsilon}}Q^-, \end{aligned}$$

where we have used (2.16). Now, for $\varepsilon = \pm 1$, introduce the singular integral operators (without shift) in $L_p(\mathbf{T})$

$$\begin{aligned} V_\varepsilon &= P_+ + \chi^{\kappa_\varepsilon}P_-, \qquad & V_\varepsilon^{(-1)} &= P_+ + \chi^{-\kappa_\varepsilon}P_-, \\ \Lambda_\varepsilon &= \chi^{\kappa_\varepsilon}P_+ + P_-, \qquad & \Lambda_\varepsilon^{(-1)} &= \chi^{-\kappa_\varepsilon}P_+ + P_-. \end{aligned}$$

Note that with the choice made for the function χ in the definition of R, we have $P_-\,\chi\,P_+ = 0$. Most of the proof is based on the fact that, as is well known from the theory of singular integral operators (without shift), the operator Λ_ε is either left or right invertible according to whether κ_ε is nonnegative or nonpositive, respectively, and moreover, its one-sided inverse is the operator $\Lambda_\varepsilon^{(-1)}$. As a consequence, the following identity holds: $V_\varepsilon\Lambda_\varepsilon\Lambda_\varepsilon^{(-1)} = V_\varepsilon$ for $\varepsilon = \pm 1$. Since each of the projections $P_\pm$ commute with each of the projections $Q^\pm$, we have

$$\begin{aligned} T(R) &= P_+ + R\,P_- = P_+ + (\chi^{\kappa_\varepsilon}Q^+ + \chi^{\kappa_{-\varepsilon}}Q^-)P_- \\ &= V_\varepsilon\,Q^+ + V_{-\varepsilon}\,Q^-. \end{aligned} \tag{2.21}$$

Also it is easy to see that $R^{-1} = (u^2 - v^2)^{-1}(u\,I - (-1)^{\kappa_1}\varepsilon v U)$. Similar to what we have done above, we get

$$R^{-1} = \begin{cases} \chi^{-\kappa_\varepsilon}\,Q^+ + \chi^{-\kappa_{-\varepsilon}}Q^- & \text{if } \kappa_{\pm1} \text{ are even,} \\ \chi^{-\kappa_{-\varepsilon}}\,Q^+ + \chi^{-\kappa_\varepsilon}Q^- & \text{if } \kappa_{\pm1} \text{ are odd.} \end{cases}$$

Note that, by (2.8), the following commutation relations hold:

$$\begin{aligned} Q^\pm\,\chi^{\kappa_\varepsilon} &= \chi^{\kappa_\varepsilon}Q^\pm, \quad \text{if } \kappa_{\pm1} \text{ are even;} \\ Q^\pm\,\chi^{\kappa_\varepsilon} &= \chi^{\kappa_\varepsilon}Q^\mp, \quad \text{if } \kappa_{\pm1} \text{ are odd.} \end{aligned} \tag{2.22}$$

This implies that $V_\varepsilon Q^\pm = Q^\pm V_\varepsilon$ as well as $\Lambda_\varepsilon Q^\pm = Q^\pm \Lambda_\varepsilon$, if κ_ε is even, and $V_\varepsilon Q^\pm = Q^\mp V_\varepsilon \pm UP_+$ as well as $\Lambda_\varepsilon Q^\pm = Q^\mp \Lambda_\varepsilon \pm UP_-$, if κ_ε is odd.

As a consequence of $P_-\,\chi\,P_+ = 0$, we have

$$P_- R P_+ = 0, \quad \text{if } \kappa_2 \geq 0 \quad \text{and} \quad P_- R^{-1} P_+ = 0, \quad \text{if } \kappa_{-1} \leq 0.$$

A straightforward computation permits us to show that the operator $T(R)^{(-1)}$ is the left inverse of $T(R)$, if $\kappa_1 \leq 0$, and the right inverse of $T(R)$, if $\kappa_{-1} \geq 0$. Therefore, it remains to prove that $T(R)^{(-1)}$, defined in (2.19), is a generalized inverse of $T(R)$ if $\kappa_1 > 0$ and $\kappa_{-1} < 0$.

Suppose that $\kappa_1 > 0$ and $\kappa_{-1} < 0$. Using the relations established above, we first obtain

$$\begin{aligned} &(R\,P_+ + P_-)R^{-1}T(R) \\ &\qquad = \begin{cases} \Lambda_\varepsilon\,\Lambda_\varepsilon^{(-1)}Q^+ + \Lambda_{-\varepsilon}\,\Lambda_{-\varepsilon}^{(-1)}Q^- & \text{if } \kappa_{\pm1} \text{ are even,} \\ \Lambda_\varepsilon\,\Lambda_\varepsilon^{(-1)}Q^- + \Lambda_{-\varepsilon}\,\Lambda_{-\varepsilon}^{(-1)}Q^+ & \text{if } \kappa_{\pm1} \text{ are odd.} \end{cases} \end{aligned}$$

Suppose that $\kappa_{\pm1}$ are even numbers. Then we immediately obtain

$$\begin{aligned} Z &\stackrel{\text{def}}{=} T(R)(R\,P_+ + P_-)R^{-1}T(R) \\ &= Q^+V_\varepsilon\,\Lambda_\varepsilon\,\Lambda_\varepsilon^{(-1)}Q^+ + Q^-V_{-\varepsilon}\,\Lambda_{-\varepsilon}\,\Lambda_{-\varepsilon}^{(-1)}Q^- \\ &= Q^+V_\varepsilon + Q^-V_{-\varepsilon} = T(R). \end{aligned}$$

Now suppose that $\kappa_{\pm1}$ are odd numbers. Then using the commutation relation (2.22), instead of (2.21), we may write

$$T(R) = Q^-V_\varepsilon + Q^+V_{-\varepsilon}. \tag{2.23}$$

Consequently, in this case,

$$\begin{aligned} Z = {} & Q^- V_\varepsilon \, \Lambda_\varepsilon \, \Lambda_\varepsilon^{(-1)} Q^- + Q^- V_\varepsilon \, \Lambda_{-\varepsilon} \, \Lambda_{-\varepsilon}^{(-1)} Q^+ \\ & + Q^+ V_{-\varepsilon} \, \Lambda_\varepsilon \, \Lambda_\varepsilon^{(-1)} Q^- + Q^+ V_{-\varepsilon} \, \Lambda_{-\varepsilon} \, \Lambda_{-\varepsilon}^{(-1)} Q^+ \end{aligned}$$

or, equivalently,

$$\begin{aligned} Z & = Q^- V_\varepsilon Q^- + Q^- V_\varepsilon \, \Lambda_{-\varepsilon} \, \Lambda_{-\varepsilon}^{(-1)} Q^+ + Q^+ V_{-\varepsilon} \, \Lambda_\varepsilon \, \Lambda_\varepsilon^{(-1)} Q^- \\ & \quad + Q^+ V_{-\varepsilon} Q^+ \\ & = T(R) + Q^- V_\varepsilon (\Lambda_{-\varepsilon} \, \Lambda_{-\varepsilon}^{(-1)} - I) Q^+ + Q^+ V_{-\varepsilon} (\Lambda_\varepsilon \, \Lambda_\varepsilon^{(-1)} - I) Q^-. \end{aligned}$$

For $\kappa_1 > 0$ and $\kappa_{-1} < 0$, we have $\Lambda_{-1} \Lambda_{-1}^{(-1)} = I$ and

$$Q^\pm V_{-1} (\Lambda_1 \, \Lambda_1^{-1} - I) Q^\mp = Q^\pm (\chi^{\kappa_{-1}} - P_+ \chi^{\kappa_1}) P_- \chi^{-\kappa_1} P_+ Q^\mp = 0,$$

as a consequence of the commutation relation (2.22). It follows that, for any value of $\varepsilon \in \{-1, 1\}$, the last two terms in the righthand side of the above representation of Z vanish, and thus we arrive at the desired result. $\square$

It is necessary to remark here that the previous theorem only gives sufficient conditions for the (one-sided or two-sided) invertibility of the operator $T(A)$. For instance, as is well known, for the invertibility of a usual singular integral operator $P_+ + \mathcal{A} P_-$, where we denote by the same symbol the matrix function $\mathcal{A} \in \mathfrak{B}^{2\times 2}$ and the multiplication operator by that function, it is necessary and sufficient that $\mathcal{A}$ admits a canonical factorization in $\mathfrak{B}^{2\times 2}$. However, in the case of our singular integral operators with shift, the condition of $\mathcal{A} = \pi(A)$ to admit a canonical factorization is no longer necessary for the invertibility of $T(A)$ (see Example 2.11 below).

Now our purpose is to characterize the kernel of $T(A)$. To this end, the representations for the operator $T(R)$ used in the preceding proof are important. However, as we have seen in the proof of the above theorem, only in the case of even partial indices the operator V_ε commutes with the projections $Q^\pm$. To distinguish both cases, let us introduce a parameter $\gamma \in \{1, -1\}$ as follows:

$$\gamma = \begin{cases} 1 & \text{if } \kappa_1 \text{ and } \kappa_2 \text{ are even} \\ -1 & \text{if } \kappa_1 \text{ and } \kappa_2 \text{ are odd.} \end{cases} \tag{2.24}$$

In the case $\gamma = -1$, together with (2.21)–(2.23), we also have

$$T(R) = V_\varepsilon\, L^- + V_{-\varepsilon}\, L^+, \tag{2.25}$$

where $L^\pm$ are the pair of complementary projection operators in $L_p(\mathbf{T})$ defined by

$$L^\pm = P_+\, Q^\pm + P_-\, Q^\mp = Q^\pm\, P_+ + Q^\mp\, P_-. \tag{2.26}$$

In fact,

$$\begin{aligned} Q^\pm\, V_{\mp\varepsilon} &= Q^\pm(P_+ + (u \mp \varepsilon v)P_-) = P_+\, Q^\pm + (u \mp \varepsilon v)P_-\, Q^\mp \\ &= (P_+ + (u \mp \varepsilon v)P_-)(P_+\, Q^\pm + P_-\, Q^\mp) = V_{\mp\varepsilon}\, L^\pm. \end{aligned}$$

Let us make a slightly different convention about the notation of operators. Since for $\varepsilon = 1$ the operator V_ε only depends on the partial index κ_1 and for $\varepsilon = -1$ it only depends on the partial index κ_2 (see (2.21)–(2.16)), we rename these operators and put $T_{\kappa_1} = V_1$ and $T_{\kappa_2} = V_{-1}$. More generally, for further reference, we set

$$T_n = P_+ + \chi^n\, P_-, \quad n \in \mathbf{Z}. \tag{2.27}$$

We have

Proposition 2.7. *Let $A \in \mathbf{A}$, suppose that the conditions of Theorem 2.5 are fulfilled, let κ_1, κ_2 denote the partial indices of $\mathcal{A}$ and let $\varepsilon = \pm 1$ be the number which appears in Proposition 2.2. Then $\ker T(A) = \ker T(R)$ and*

(2.28)

$$\ker T(R) = \begin{cases} [\ker T_{\kappa_1} \cap \ker Q^-] \oplus [\ker T_{\kappa_2} \cap \ker Q^+] & \text{if } \varepsilon = 1 \text{ and } \gamma = 1 \\ [\ker T_{\kappa_1} \cap \ker L^+] \oplus [\ker T_{\kappa_2} \cap \ker L^-] & \text{if } \varepsilon = 1 \text{ and } \gamma = -1 \\ [\ker T_{\kappa_1} \cap \ker Q^+] \oplus [\ker T_{\kappa_2} \cap \ker Q^-] & \text{if } \varepsilon = -1 \text{ and } \gamma = 1 \\ [\ker T_{\kappa_1} \cap \ker L^-] \oplus [\ker T_{\kappa_2} \cap \ker L^+] & \text{if } \varepsilon = -1 \text{ and } \gamma = -1, \end{cases}$$

where $Q^\pm$ and $L^\pm$ are the projection operators defined in (2.20) and (2.26), respectively.

Proof. From representation (2.18), we deduce that $\dim\ker T(A) = \dim\ker T(R)$. Moreover, from (2.21) and (2.22) it follows that

$$\ker T(R) = [\ker V_\varepsilon \cap \ker Q^-] \oplus [\ker V_{-\varepsilon} \cap \ker Q^+]$$

if $\gamma = 1$ and, from (2.23) and (2.25), we get

$$\ker T(R) = [\ker V_\varepsilon \cap \ker L^+] \oplus [\ker V_{-\varepsilon} \cap \ker L^-]$$

if $\gamma = -1$. Using the convention of notation established before this proposition, we arrive at (2.28). □

Note that $\ker T_n = \{0\}$ for $n < 0$. Our next aim is to identify the subsets in the righthand side of (2.28) for $\kappa_1, \kappa_2 > 0$. To this end, it is convenient to introduce another pair of complementary projection operators in $L_p(\mathbf{T})$ by

$$Q^{\pm,n} = \frac{1}{2}\,(I \pm \alpha_+^{-n}\, U), \quad n \in \mathbf{N}. \tag{2.29}$$

For $n \in \mathbf{N}$, let P^{n-1} denote the vector space of all polynomials in $\mathbf{T}$ of degree less than n. It is easily seen that the subspace P^{n-1} of $L_p(\mathbf{T})$ is invariant under $Q^{\pm,n}$. We shall denote by $Q_r^{\pm,n}$ the restriction operator of $Q^{\pm,n}$ to P^{n-1}.

Proposition 2.8. *Let $n \in \mathbf{N}$. Then*

$$\dim(\ker T_n \cap \ker Q^{\pm}) = \dim\ker Q_r^{\pm,n} \quad \textit{if } n \textit{ is even},$$

and

$$\dim(\ker T_n \cap \ker L^{\pm}) = \dim\ker Q_r^{\mp,n} \quad \textit{if } n \textit{ is odd}.$$

Proof. It is well known from the theory of singular integral operators that, for $n \in \mathbf{N}$, $\dim\ker T_n = n$ and that the elements in $\ker T_n$ can be characterized in terms of a factorization of χ, say $\chi = \chi_+\, t\, \chi_-$, as follows

$$\varphi = p_{n-1}\,\nu_n, \quad \text{with } \nu_n = \chi_+^n - t^{-n}\,\chi_-^{-n} = \chi_+^n(1 - \chi^{-n}), \tag{2.30}$$

with p_{n-1} being a polynomial of degree less than n, $p_{n-1} \in P^{n-1}$.

Suppose that $n \in \mathbf{N}$ is an even number. The condition $\varphi \in \ker T_n \cap \ker Q^{\pm}$ means that

$$\varphi \pm U\varphi = p_{n-1}\,\nu_n \pm \nu_n(\alpha)(U\,p_{n-1}) = 0. \tag{2.31}$$

Since $\chi(\alpha) = -\chi$ and $\chi_+(\alpha) = -\alpha_+^{-1}\,\chi_+$ (see (2.8) and (2.10)), we have $\chi^n(\alpha) = \chi^n$ and $\chi_+^n(\alpha) = \alpha_+^{-n}\,\chi_+^{-n}$. Consequently, $\nu_n(\alpha) = \alpha_+^{-n}\nu_n$ and, therefore, (2.31) implies that

$$Q^{\pm,n}\,p_{n-1} = p_{n-1} \pm \alpha_+^{-n}\,U\,p_{n-1} = 0.$$

So $\varphi \in \ker T_n \cap \ker Q^{\pm}$ implies that φ is of the form given in (2.30), with $p_{n-1} \in \ker Q_r^{\pm,n}$. It is now straightforward to verify that, reciprocally, if $p_{n-1} \in \ker Q^{\pm,n}$, then the function φ given by (2.30) belongs to $\ker T_n \cap \ker Q^{\pm}$. Consequently, the result about the dimension follows.

Now suppose that $n \in \mathbf{N}$ is an odd number. First note that the condition $\varphi \in \ker L^{\pm}$ is equivalent to

(i) $Q^{\pm}P_+\varphi = 0$

and

(ii) $Q^{\mp}P_-\varphi = 0$.

We show that, for $\varphi \in \ker T_n$, (i) and (ii) are equivalent. In fact, if $\varphi \in \ker T_n \cap \ker Q^{\pm}\,P_+$, then

$$\begin{aligned} Q^{\pm}P_+\,\varphi &= p_{n-1}\,\chi_+^n \pm \chi_+^n(\alpha)U\,p_{n-1} \\ &= \chi_+^n(p_{n-1} \mp \alpha_+^{-n}\,U\,p_{n-1}) = \chi_+^n\,Q^{\mp,n}\,p_{n-1} = 0, \end{aligned}$$

where we have used again the properties of χ mentioned above. Analogously, if $\varphi \in \ker T_n \cap \ker Q^{\mp}P_-$, then

$$\begin{aligned} Q^{\mp}P_-\,\varphi &= -\chi_+^n\,\chi^{-n}\,p_{n-1} \mp \chi_+^n(\alpha)\chi^{-n}\,U\,p_{n-1} \\ &= -\chi_+^n\,\chi^{-n}(p_{n-1} \pm \alpha_+^{-n}\,U\,p_{n-1}) = -\chi_+^n\,\chi^{-n}Q^{\mp,n}\,p_{n-1} = 0. \end{aligned}$$

Therefore, $\varphi \in \ker T_n \cap \ker L^{\pm}$ if and only if φ is of the form (2.30) with $p_{n-1} \in \ker Q_r^{\mp,n}$. Consequently the result about the dimension follows. □

The characterization of the finite dimensional projection operators $Q_r^{\pm,n}$ is easy.

Lemma 2.9. *Let* $n \in \mathbf{N}$ *and* $Q_r^{\pm,n} : P^{n-1} \to P^{n-1}$ *be the projection operators defined in* (2.29). *Then* $\dim\ker Q_r^{+,n} = [(n+1)/2]$, $\dim\ker Q_r^{-n} = [n/2]$ *and a basis for* $\ker Q_r^{\pm,n}$ *is formed by the functions*

$$q_m^{\pm} = \frac{1}{2}\, t^{n-1}(\alpha_-^{m-1} \pm \alpha_-^{n-m}), \quad m = 1, \dots, \dim\ker Q_r^{\pm,n}. \tag{2.32}$$

Proof. All the results are obtained straightforwardly if we consider the following basis of P^{n-1}:

$$t^{n-1}\,\alpha_-^{m-1}, \quad m = 1, \dots, n. \qquad \square$$

We can now state the main result about the subspace $\ker T(A)$.

Theorem 2.10. *Let* $A \in \mathbf{A}$, *suppose that the conditions of Theorem* 2.5 *are fulfilled, let* κ_1, κ_2 *be the partial indices of* $\mathcal{A}$ *and let* $\varepsilon = \pm 1$ *be the number which appears in Proposition* 2.2. *Then*

$$\dim\ker T(A) = \begin{cases} 0 & \text{if } \kappa_1 \le 0 \\ \left[\dfrac{\kappa_1}{2} + \dfrac{1-\varepsilon}{4}\right] & \text{if } \kappa_1 > 0,\ \kappa_2 \le 0, \\ \dfrac{\kappa_1 + \kappa_2}{2} & \text{if } \kappa_2 > 0. \end{cases}$$

Moreover, for $n = \kappa_1, \kappa_2$, *let* ν_n *be the functions given in* (2.30), *let* $q_m^{\pm}$ *be the elements in a basis of* $\ker Q_r^{\pm,n}$ *as given in* (2.32), *and let* $B = A_+P_+ + A_-^{-1}P_-$. *Then*

$$\ker T(A) = \mathit{span}\,\{B(q_m^+\,\nu_{\kappa_1})\}, \quad m = 1, \dots, \left[\frac{\kappa_1}{2} + \frac{1-\varepsilon}{4}\right]$$

if $\kappa_1 > 0$, $\kappa_2 \le 0$, *and*

$$\ker T(A) = \mathit{span}\,\{B(q_m^+\,\nu_{\kappa 1}) \cup B(q_l^-\,\nu_{\kappa 2})\},$$
$$m = 1, \dots, \left[\frac{\kappa_1}{2} + \frac{1-\varepsilon}{4}\right]$$
$$l = 1, \dots, \left[\frac{\kappa_2}{2} + \frac{1+\varepsilon}{4}\right]$$

if $\kappa_1 > 0,\ \kappa_2 > 0.$

Proof. Using Propositions 2.7, 2.8 and Lemma 2.9, we have, independently of the value of $\gamma \in \{1, -1\}$,

$$\dim\ker T(A) = \begin{cases} \dim\ker Q_r^{-,\kappa_1} = \left[\dfrac{\kappa_1}{2}\right] & \text{if } \varepsilon = 1, \\ \dim\ker Q_r^{+,\kappa_1} = \left[\dfrac{\kappa_1+1}{2}\right] & \text{if } \varepsilon = -1, \end{cases}$$

if $\kappa_1 > 0$, $\kappa_2 \le 0$, and

$$\dim\ker T(A) = \begin{cases} \dim\ker Q_r^{-\kappa_1} + \dim\ker Q_r^{+,\kappa_2} = \left[\dfrac{\kappa_1}{2}\right] + \left[\dfrac{\kappa_2+1}{2}\right] \\ \qquad \text{if } \varepsilon = 1 \\ \dim\ker Q_r^{+,\kappa_1} + \dim\ker Q_r^{-,\kappa_2} = \left[\dfrac{\kappa_1+1}{2}\right] + \left[\dfrac{\kappa_2}{2}\right] \\ \qquad \text{if } \varepsilon = -1 \end{cases}$$

if $\kappa_1 > 0$, $\kappa_2 > 0$. Hence, for any $\varepsilon \in \{-1, 1\}$, if $\kappa_1 > 0$, $\kappa_2 \le 0$, we have

$$\dim\ker T(A) = \left[\frac{\kappa_1}{2} + \frac{1-\varepsilon}{4}\right]$$

and if $\kappa_1 > 0$, $\kappa_2 > 0$, the following holds

$$\dim\ker T(A) = \left[\frac{\kappa_1}{2} + \frac{1-\varepsilon}{4}\right] + \left[\frac{\kappa_2}{2} + \frac{1+\varepsilon}{4}\right] = \frac{1}{2}\,(\kappa_1 + \kappa_2).$$

The rest of the proof is a consequence of the representation (2.18) and the description of $\ker T_n \cap \ker Q^{\pm}$ for even $n \in \mathbf{N}$, and of $\ker T_n \cap \ker L^{\pm}$ for odd $n \in \mathbf{N}$, given in the proof of Proposition 2.8. $\square$

Let us add some comments to the results in the previous theorem. It is well known that the index of $T(A)$, $A \in \mathbf{A}$, is related to the index of the singular integral operator (without shift) $P_+ + \mathcal{A}P_-$, $\mathcal{A} = \pi(A)$, by

$$\operatorname{ind} T(A) = \frac{1}{2}\operatorname{ind}(P_+ + \mathcal{A}\,P_-) = \frac{1}{2}\,(\kappa_1 + \kappa_2), \tag{2.33}$$

where $\kappa_1 \ge \kappa_2$ are the partial indices of $\mathcal{A}$. By the previous theorem, we have

$$\operatorname{ind} T(A) = \dim\ker T(A) = (\kappa_1 + \kappa_2)/2 \quad \text{if } \kappa_2 > 0,$$
$$\operatorname{ind} T(A) = -\operatorname{codim}\operatorname{im} T(A) = (\kappa_1 + \kappa_2)/2 \quad \text{if } \kappa_1 < 0,$$

which means that, in case the partial indices have the same sign, the (one- or two-sided) invertibility of $T(A)$ follows the same rule as that of the operator $P_+ + \mathcal{A}P_-$.

However, if the partial indices have different signs, namely, if the partial indices of $\mathcal{A}$ are equal to $+1$ and -1, and contrary to what happens for the operator $P_+ + \mathcal{A}P_-$, it is still possible that $T(A)$ is invertible. This can happen because, in such a case, although the index of $T(A)$ does not depend on the parameter ε, the dimension of its kernel (and, therefore, also the dimension of its cokernel), depends on ε. This dependence is such that if $\varepsilon = -1$, the operator $T(A)$ is noninvertible, whereas if $\varepsilon = 1$ the operator $T(A)$ is invertible. The following example illustrates the latter situation.

Example 2.11. Let $\mathfrak{B}$ be either the Wiener algebra or the algebra of Hölder continuous functions on $\mathbf{T}$, and take for χ (see Definition 2.3 and the remarks after it) the function $\chi(t) = \sqrt{\alpha_+}\, t\, \sqrt{\alpha_-}$, $t \in \mathbf{T}$. According to (2.10), in this case we have $\chi_+ = \sqrt{\alpha_+}$, $\chi_+(\alpha) = -\chi_+^{-1}$ and $\chi_- = \sqrt{\alpha_-}$, $\chi_-(\alpha) = \chi_-^{-1}$. Consider the functional operator

$$R = \frac{1}{2}\,(\chi + \chi^{-1})I + \frac{1}{2}\,(\chi - \chi^{-1})U = \chi\, Q^+ + \chi^{-1}Q^-,$$

to which corresponds the matrix function

$$\mathcal{R} = \pi(R) = \frac{1}{2}\begin{pmatrix} \chi + \chi^{-1} & \chi - \chi^{-1} \\ \chi - \chi^{-1} & \chi + \chi^{-1} \end{pmatrix}.$$

In this case $\mathcal{R}$ admits the noncanonical factorization

$$\mathcal{R} = (\mathcal{K}\,\mathcal{S}^{-1}\Lambda_+^{1/2})\,\Lambda\,(\Lambda_-^{1/2}\mathcal{S})$$

with $\Lambda_{\chi_\pm} = \operatorname{diag}\{\chi_\pm, \chi_\pm^{-1}\}$, $\Lambda = \operatorname{diag}\{t, t^{-1}\}$, $\mathcal{K} = \operatorname{diag}\{1, -1\}$ and $\mathcal{S} = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. So, according to our notation $\kappa_1 = 1$, $\kappa_2 = -1$ and $\varepsilon = 1$.

From Theorem 2.10 and (2.33), we have for the operator $T(R) = P_+ + R\,P_-$,

$$\dim\ker T(R) = \operatorname{ind} T(R) = 0$$

and, therefore, $T(R)$ is invertible in $L_p(\mathbf{T})$, $p \in (1, \infty)$.

It is possible to give an expression for the inverse of $T(R)$. Introduce the operators

$$N_\pm = \chi_+^{-1} P_\pm \chi_+, \qquad K = N_+ \chi Q^+ P_-,$$

(note that $N_\pm^2 = N_\pm$ and $N_+ + N_- = I$). Then the inverse of $T(R)$ is given by

$$T(R)^{-1} = N_+ + (R + (\chi - 1)K \chi^{-1} Q^-)N_-.$$

In the rest of this section, we consider a generalization of the results obtained so far for the case of more general, not even continuous but essentially bounded, coefficients a and b of the functional operator A, given in (1.4). This can be done using the concept of generalized factorization of 2×2 matrix functions (see [**14**], where the term Φ-factorization was used, and also [**5**], for the case of a right factorization).

For $p \in (1, \infty)$, let $q = p/(p-1)$. Given a function $\mathcal{G} \in L_\infty(\mathbf{T})^{n \times n}$, $n \in \mathbf{N}$, we say that $\mathcal{G}$ admits a (left) factorization in (or relative to) $L_p(\mathbf{T})$ if

(i) $\mathcal{G}$ can be represented as

$$\mathcal{G} = \mathcal{G}_+ \operatorname{diag}\{t^{\kappa_1}, \dots, t^{\kappa_n}\} \mathcal{G}_-,$$

where $\mathcal{G}_+ \in L_p^+(\mathbf{T})^{n \times n}$, $\mathcal{G}_+^{-1} \in L_q^+(\mathbf{T})^{n \times n}$, $\mathcal{G}_- \in L_q^-(\mathbf{T})^{n \times n}$, $\mathcal{G}_-^{-1} \in L_p^-(\mathbf{T})^{n \times n}$ and $\kappa_1 \geq \dots \geq \kappa_n$ are integers (they are called the partial indices of $\mathcal{G}$),

(ii) The operator B acting according to the rule $B\varphi = \mathcal{G}_+ P_- \mathcal{G}_+^{-1} \varphi$ is a bounded linear operator on $L_p^n(\mathbf{T})$.

It should be noted that a continuous matrix function admits a generalized factorization in $L_p(\mathbf{T})$, $p \in (1, \infty)$, if and only if it is nonsingular on $\mathbf{T}$ and, in this case, the outer factors of any factorization of it belong to the space $L_r(\mathbf{T})$, for every $r \in (1, \infty)$, although they are not necessarily continuous (see, for instance, [**14**]). For this reason, if $\mathcal{G} \in C(\mathbf{T})^{n \times n}$, we simply talk about a factorization of $\mathcal{G}$ instead of a factorization of $\mathcal{G}$ in $L_p(\mathbf{T})$.

Continuing to associate to the functional operator A the matrix function $\mathcal{A}$ given by (2.1) (which of course satisfies (2.2)), following the lines of the proof of Theorem 2.4, it is not difficult to see that the following two statements are equivalent: for $p \in (1, \infty)$

(1) $\mathcal{A}$ admits a generalized factorization in $L_p(\mathbf{T})$,

(2) $\mathcal{A}$ admits a generalized factorization in $L_p(\mathbf{T})$ of the form $\mathcal{A} = \mathcal{A}_{+,\alpha}\mathcal{R}\mathcal{A}_{-,\alpha}$, where the outer factors in addition to the conditions in the previous definition also satisfy the invariance relations $\mathcal{A}_{+,\alpha}^{\pm 1} = e\mathcal{A}_{+,\alpha}^{\pm 1}(\alpha)e$, $\mathcal{A}_{-,\alpha}^{\pm 1} = e\mathcal{A}_{-,\alpha}^{\pm 1}(\alpha)e$ and $\mathcal{R}$ is given by (2.15)–(2.16).

Using this equivalence, all the results about the factorization of the functional operator A and the singular integral operator with shift $T(A)$ remain valid in this new context if we interpret them in a generalized sense, that is, all the operators are bounded in a dense subset of $L_p^n(\mathbf{T})$ and admit continuous extensions to $L_p^n(\mathbf{T})$. In particular, the results about the kernel of the operator $T(A)$ are the same.

3. Factorization of 2×2 matrix functions. In the previous section we saw that the characterization of a singular integral operator with the linear fractional Carleman shift (1.1) in what concerns the Fredholm properties, such as the determination of the dimensions and basis for the kernel and cokernel of such an operator, is strictly related to the factorization problem for a 2×2 matrix function of the form (2.1).

One of the purposes of this work is to identify the spectrum of some integral operators with a Carleman shift. To this end it is essential to deal with the factorization problem for some 2×2 matrix functions. As is well known, the factorization problem for matrix functions of any order $n \geq 2$ is a very difficult one and, in particular, for the order 2, only for some special classes is the question treatable. Among these classes we refer to the algebra of the so-called Daniele-Khrapkov matrix functions, i.e., those 2×2 matrix functions $\mathcal{G}$ which can be given the form

$$\mathcal{G} = c\mathcal{I} + d\mathcal{D}, \qquad \mathcal{D} = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}, \tag{3.1}$$

where c, d are scalar coefficients belonging to an algebra of continuous functions on $\mathbf{T}$, r is a fixed rational function without zeros or poles in $\mathbf{T}$ and I is the identity matrix of order 2.

In recent years a series of papers appeared that were devoted to the factorization problem for Daniele-Khrapkov matrix functions on the real line (but the results can be transferred to the unit circle). As a rule, the methods used to touch on that question highly depend on the number (and the location) of zeros or poles of the function r. The simplest case corresponds to the situation where r is itself the square of a rational function (see [**11**] and [**15**]). The first rigorous study of the class of Daniele-Khrapkov for the case where r is the quotient of two polynomials of the first degree was done in [**18**], where the problem of obtaining a canonical factorization was considered. For the same case in [**12**] the factorization problem was solved, including the determination of the partial indices as well as formulas for the factors in a factorization (although the Lebesgue space L_2 was considered in [**12**], the results are also valid for the space L_p, $p \in (1, \infty)$). In [**1**] these results were generalized for a class of matrix functions of order greater than 2 by a different method. In [**3**] and [**4**] (see also [**2**]) the case where r is a quotient of polynomials of degree two was considered. From these results we cite a criterion for the existence of a canonical factorization (both partial indices are zero), formulas for the factors in a canonical factorization and, with some other restrictions, the determination of the partial indices. We also refer to [**17**], which was written already for the unit circle and the class of matrix functions considered can be reduced by a rational transformation to the form (3.1). In this paper an algorithm is present in order to achieve a factorization.

In the first two subsections of the present section we shall consider a subalgebra of the algebra of Daniele-Khrapkov matrix functions which is not characterized by any restriction concerning the number of zeros (equal to the number of poles) of r, but with another kind of restriction which is more convenient for our purposes. The basic idea is to use the method developed in [**6**] (see also [**16**]), which permits us to interpret the elements of our subalgebra as continuous functions of a multiplication operator by a rational matrix function.

The last part of this section is devoted to the factorization problem for matrix functions belonging to a special class of the subalgebra $\mathfrak{B}_\alpha^{2\times 2}$. This constitutes a new class of matrix functions which can be factorized explicitly.

3.1 The algebra $\mathfrak{A}$ $(\mathcal{V})$. Let $\mathcal{V}$ be a 2×2 matrix function of the form

$$\mathcal{V} = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}, \tag{3.2}$$

where θ_1 and θ_2 are the finite Blaschke products

$$\theta_i(t) = c_i\, t^{m_i} \prod_{j=1}^{N} \frac{t-\beta_{i,j}}{\bar{\beta}_{i,j}t-1}, \tag{3.3}$$

where $|c_i| = 1$, $m_i \in \mathbf{N} \cup \{0\}$, $|\beta_{i,j}| < 1$, $i = 1, 2$, $j = 1, \dots, N$.

Now consider in $L_p^2(\mathbf{T})$, $p \in (1, \infty)$, the operator of multiplication on the left by the matrix function $\mathcal{V}$, which we shall denote by the same symbol $\mathcal{V}$. It is clear that the following properties for the invertible operator $\mathcal{V}$ hold:

(i) The spectral radius of both $\mathcal{V}$ and $\mathcal{V}^{-1}$ is equal to one, in symbols:

$$r_\sigma(\mathcal{V}) = r_\sigma(\mathcal{V}^{-1}) = 1,$$

and

(3.4)

(ii) $\quad \mathcal{V}P_+ = P_+\mathcal{V}P_+, \quad \mathcal{V}^{-1}P_- = P_-\mathcal{V}^{-1}P_- \quad \text{and} \quad \mathcal{V}P_+ \neq P_+\mathcal{V}.$

Let $\mathfrak{A} = \mathfrak{A}(\mathcal{V})$ stand for the closed subalgebra of $\mathcal{L}(L_p^2(\mathbf{T}))$ containing the identity operator which is generated by $\mathcal{V}$ and $\mathcal{V}^{-1}$. Naturally the elements of $\mathfrak{A}$ can be seen as 2×2 matrix functions. Thus a matrix function $\mathcal{A} \in \mathfrak{A}$ if and only if it has the form

$$\mathcal{A} = \sum_{j \in \mathbf{Z}} a_j \mathcal{V}^j,$$

the series being convergent in the operator norm. It was proved in [**16**] that, by virtue of (i) and (ii), there exists a symbol map, i.e., a continuous homomorphism, $\varphi : \mathfrak{A} \to \mathfrak{S}$, where $\mathfrak{S}$ is a subalgebra of $C(\mathbf{T})$ (with the supremum norm $\|\cdot\|_\infty$), that associates to a matrix function $\mathcal{A}$ the scalar function defined on $\mathbf{T}$ by

$$a(t) = \sum_{j \in \mathbf{Z}} a_j\, t^j,$$

which is called the symbol of $\mathcal{A}$ and satisfies $\|a\|_\infty \le \|\mathcal{A}\|$. We shall use the notation $\mathcal{A} = a(\mathcal{V})$ to express the connection between an operator in $\mathfrak{A}$ and the corresponding symbol. This symbol map is such that $\mathcal{A} = a(\mathcal{V}) \in \mathfrak{A}$ is invertible if and only if $a(t) \neq 0$, $t \in \mathbf{T}$.

Moreover, in the present case ($\mathcal{V}$ is the multiplication operator), this map is even a Banach algebra isomorphism and, since $\|\mathcal{V}\| = \|\mathcal{V}^{-1}\| = 1$, $\mathfrak{S}$ contains the Wiener algebra $\mathcal{W}$ (the Banach algebra of functions on $\mathbf{T}$ which can be expanded into an absolutely convergent Fourier series, the norm of an element being given by the norm in $\ell_1(\mathbf{Z})$ of its Fourier coefficients). It is also known that, in the case $p = 2$, there holds $\mathfrak{S} = C(\mathbf{T})$ since $\mathcal{V}$ is a unitary operator in $L_2^2(\mathbf{T})$ (see [**16**]).

There are some more special features in the present case. Set

$$\theta = \theta_1 \, \theta_2.$$

In the next lemma we derive a useful representation for the elements of $\mathfrak{A}$. From now on, we shall use the following notation for the composition of functions: $f_h = f \circ h$.

Lemma 3.1. *The following assertions are equivalent*:

(i) $\mathcal{A} \in \mathfrak{A}$.

(ii) *There exist* $b, c \in C(\mathbf{T})$ *such that*:

$$\mathcal{A} = b_\theta \, I + c_\theta \, \mathcal{V}. \tag{3.5}$$

Moreover, if $\mathcal{A} = a(\mathcal{V}) \in \mathfrak{A}$, *then*

$$a(t) = b(t^2) + t \, c(t^2). \tag{3.6}$$

Proof. Introduce in $C(\mathbf{T})$ the pair P_e, P_o of complementary projection operators defined by

$$(P_e a)(t) = \frac{1}{2}\,(a(t) + a(-t)), \qquad (P_o a)(t) = \frac{1}{2}\,(a(t) - a(-t)), \quad t \in \mathbf{T}.$$

For any $a \in C(\mathbf{T})$ we have

$$a(t) = \tilde{b}(t) + t\,\tilde{c}(t), \quad t \in \mathbf{T}$$

with $\tilde{b} = P_e a$ and $\tilde{c}(t) = (1/t)(P_o a)(t)$, $t \in \mathbf{T}$. Note that in this representation both $\tilde{b}$ and $\tilde{c}$ are even functions on $\mathbf{T}$ since, for $a(t) = \sum_{j\in\mathbf{Z}} a_j t^j$, we get

$$\tilde{b}(t) = \sum_{j\in\mathbf{Z}} a_{2j}\, t^{2j}, \qquad \tilde{c}(t) = \sum_{j\in\mathbf{Z}} a_{2j+1}\, t^{2j}, \quad t \in \mathbf{T}. \tag{3.7}$$

Therefore, we may define $b, c \in C(\mathbf{T})$ by

$$b(t^2) = \tilde{b}(t), \qquad c(t^2) = \tilde{c}(t), \quad t \in \mathbf{T},$$

and so

$$a(t) = b(t^2) + t\, c(t^2), \quad t \in \mathbf{T}.$$

Let $\mathcal{A} = a(\mathcal{V})$. Since $a \in \mathfrak{S} \subset C(\mathbf{T})$, we can represent it as above. On the other hand, if we make use of the identity $\mathcal{V}^2 = \theta I$, we have

$$\begin{aligned}\mathcal{A} = \sum_{j\in\mathbf{Z}} a_j\, \mathcal{V}^j &= \Big(\sum_{j\in\mathbf{Z}} a_{2j}\, \theta^j\Big) I + \Big(\sum_{j\in\mathbf{Z}} a_{2j+1}\, \theta^j\Big)\mathcal{V} \\ &= b_\theta\, I + c_\theta\, \mathcal{V}.\end{aligned}$$

Reciprocally, suppose that $\mathcal{A}$ admits the representation (3.5) where $b, c \in C(\mathbf{T})$. Then $b = \lim_{n\to\infty} p_n$ and $c = \lim_{n\to\infty} q_n$ (the limits being taken in the supremum norm) for some polynomials p_n, q_n in the variables t and t^{-1}. For each $n \in \mathbf{N}$ consider the operator

$$\mathcal{A}_n = (p_n)_\theta\, I + (q_n)_\theta\, \mathcal{V} = p_n(\mathcal{V}^2) + q_n(\mathcal{V}^2)\mathcal{V},$$

where again we have used the relation $\mathcal{V}^2 = \theta I$. We have

$$\begin{aligned}\|\mathcal{A} - \mathcal{A}_n\| &= \|(b - p_n)_\theta\, I + (c - q_n)_\theta\, \mathcal{V}\| \\ &\le \|(b - p_n)_\theta\|_\infty + \|(c - q_n)_\theta\|_\infty,\end{aligned}$$

and, consequently, $\lim_{n\to\infty} \|\mathcal{A} - \mathcal{A}_n\| = 0$. Therefore, $\mathcal{A} \in \mathfrak{A}$. Moreover, the symbol a of $\mathcal{A}$ is given by

$$a(t) = \lim_{n\to\infty} [p_n(t^2) + t\, q_n(t^2)] = b(t^2) + t\, c(t^2), \quad t \in \mathbf{T},$$

which completes the proof. $\square$

As a result of the previous lemma, we can interpret the elements of $\mathfrak{A}$ as matrix functions belonging to $C(\mathbf{T})^{2\times 2}$. In what follows we shall use the following notation

$$\mathfrak{S}_\theta^2 = \{(f_\theta, g_\theta) \in C(\mathbf{T})^2 : h(t) = f(t^2) + t\, g(t^2),\, t \in \mathbf{T},\, h \in \mathfrak{S}\}.$$

Suppose that $\mathfrak{R} \subset \mathfrak{S}$ is a decomposing R-algebra of continuous functions on $\mathbf{T}$, i.e., the rational functions without poles on $\mathbf{T}$ are dense in $\mathfrak{R}$ and $\mathfrak{R} = \mathfrak{R}^+ \oplus \mathfrak{R}_0^-$ with $\mathfrak{R} = \mathfrak{R} \cap C^+(\mathbf{T})$ and $\mathfrak{R}_0^- = \mathfrak{R} \cap C_0^-(\mathbf{T})$, where $C^+(\mathbf{T})(C_0^-(\mathbf{T}))$ is the subalgebra of $C(\mathbf{T})$ consisting of functions which are holomorphically extendable to the interior of $\mathbf{T}$ (exterior of $\mathbf{T}$ and vanishing at infinity). Set $\mathfrak{R}^- = \mathfrak{R}_0^- \oplus \mathbf{C}$.

We derive a very useful property of the representation (3.5) concerning the subalgebras $\mathfrak{R}^\pm$.

Lemma 3.2. *Let $a_\pm \in \mathfrak{R}^\pm$, and let $\mathcal{A}_\pm = a_\pm(\mathcal{V}) \in \mathfrak{A}$. Then $\mathcal{A}_\pm$ can be analytically extended to $\mathbf{T}_\pm$ and, consequently, $\mathcal{A}_\pm \in C^\pm(\mathbf{T})^{2\times 2}$. Here $\mathbf{T}_+ = \{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{T}_- = \{z \in \mathbf{C} : |z| > 1\}$.*

Proof. Let $a_+ \in \mathfrak{R}^+$, and let $\mathcal{A}_+ = a_+(\mathcal{V}) \in \mathfrak{A}$. Then, by the previous lemma, $a_+(t) = b_+(t^2) + t\, c_+(t^2)$, $t \in \mathbf{T}$, and it is straightforward to verify that $b_+, c_+ \in C^+(\mathbf{T})$. Since $\theta \in C^+(\mathbf{T})$ and $\theta(\mathbf{T}_+) \subset \mathbf{T}_+$, using the representation of $\mathcal{A}_+$ given in the previous lemma, it follows that $\mathcal{A}_+ \in C^+(\mathbf{T})^{2\times 2}$.

Now let $a_- \in \mathfrak{R}^-$, and let $\mathcal{A}_- = a_-(\mathcal{V}) \in \mathfrak{A}$. Proceeding similarly, first we get $a_-(t) = b_-(t^2) + t\, c_-(t^2) = b_-(t^2) + t^{-1} d_-(t^2)$, $t \in \mathbf{T}$, where $b_- \in C^-(\mathbf{T})$, $c_- \in C_0^-(\mathbf{T})$ and $d_-(t) = t\, c_-(t)$, $d_- \in C^-(\mathbf{T})$. Alternatively, we can write $a_-(t) = b_+(t^{-2}) + t^{-1} d_+(t^{-2})$, $t \in \mathbf{T}$, with $b_+(t) = b_-(t^{-1}) \in C^+(\mathbf{T})$ and $d_+(t) = d_-(t^{-1}) \in C^+(\mathbf{T})$. Using these representations, we get $\mathcal{A}_- = (b_-)_\theta\, I + (d_-)_\theta\, \mathcal{V}^{-1} = (b_+)_{\theta^{-1}} I + (d_+)_{\theta^{-1}} \mathcal{V}^{-1}$. Since $\theta^{-1} = 1/\theta$ is analytic in $\mathbf{T}_-$ and $\theta^{-1}(\mathbf{T}_-) \subset \mathbf{T}_+$, it follows that $\mathcal{A}_- \in C^-(\mathbf{T})^{2\times 2}$, which completes the proof. $\square$

3.2 Factorization for a class of Daniele-Khrapkov matrix functions. As was done at the end of the previous subsection, suppose that $\mathfrak{R} \subset \mathfrak{S}$ is a decomposing R-algebra of continuous functions on $\mathbf{T}$.

At this point we are concerned with the explicit factorization for matrix functions of the form

$$\mathcal{A} = a_1\,\mathcal{I} + a_2\,\mathcal{V}, \tag{3.8}$$

where $(a_1, a_2) \in \mathfrak{R}^2_\theta$ (see the previous subsection for the definition of the set $\mathfrak{R}^2_\theta$) and $\mathcal{V}$ is the rational function defined in (3.2). Matrix functions of this type belong to the Daniele-Khrapkov class (take $c = a_1$, $d = a_2\theta_1$ and $r = \theta_2/\theta_1$ in (3.1)) and, moreover, according to Lemma 3.1, they also belong to the algebra $\mathfrak{A}$, introduced in the previous subsection.

The following result is the key to obtaining a factorization of Daniele-Khrapkov matrix functions of the form (3.8).

Theorem 3.3. *Let $a_1 = f_\theta$, $a_2 = g_\theta$, and suppose that $(a_1, a_2) \in \mathfrak{R}^2_\theta$. If the function $a \in \mathfrak{R}$ defined by*

$$a(t) = f(t^2) + t\,g(t^2), \quad t \in \mathbf{T}, \tag{3.9}$$

does not vanish on $\mathbf{T}$, then the matrix function $\mathcal{A}$ in (3.8) *admits the representation*

$$\mathcal{A} = \mathcal{A}_+\,\mathcal{V}^{\varkappa}\mathcal{A}_-$$

where $\mathcal{A}_+^{\pm 1} \in C^+(\mathbf{T})^{2\times 2}$ and $\mathcal{A}_-^{\pm 1} \in C^-(\mathbf{T})^{2\times 2}$ and $\varkappa = \mathrm{ind}_{\mathbf{T}} a$.

Proof. As was previously mentioned, the matrix function $\mathcal{A}$ belongs to the algebra $\mathfrak{A}$. According to Lemma 3.1, its symbol is the function a given in (3.9). It is well known (see [**16**]) that the condition $a(t) \neq 0$, $t \in \mathbf{T}$, is necessary and sufficient for the existence of a $\mathfrak{R}$-factorization of a, i.e., a representation of a as

$$a = a_-\,t^{\varkappa}\,a_+, \tag{3.10}$$

with $a_-^{\pm 1} \in \mathfrak{R}^-$, $a_+^{\pm 1} \in \mathfrak{R}^+$ and $\varkappa = \mathrm{ind}_{\mathbf{T}} a$.

Since, by hypothesis, $a(t) \neq 0$, $t \in \mathbf{T}$, a admits an $\mathfrak{R}$-factorization like in (3.10). Then this factorization of a induces a factorization of the matrix function $\mathcal{A} = a(\mathcal{V})$ in the form

$$\mathcal{A} = \mathcal{A}_+\,\mathcal{V}^{\kappa}\,\mathcal{A}_-, \qquad \mathcal{A}_\pm = a_\pm(\mathcal{V}). \tag{3.11}$$

Further, from Lemma 3.2, it follows that

$$P_+\,\mathcal{A}_+^{\pm 1}\,P_+ = \mathcal{A}_+^{\pm 1}\,P_+, \qquad P_-\,\mathcal{A}_-^{\pm 1}\,P_- = \mathcal{A}_-^{\pm 1}\,P_-,$$

which means that the outer factors in (3.11) enjoy the properties of the outer factors in a factorization of $\mathcal{A}$. $\square$

As a consequence, we have

Corollary 3.4. *Let $a \in \mathfrak{R}$ be such that $a(t) \neq 0$, $t \in \mathbf{T}$. Then the partial indices of the matrix function $A = a(\mathcal{V})$ coincide with the partial indices of $\mathcal{V}^\kappa$, where $\kappa = \mathrm{ind}_{\mathbf{T}} a$.*

But, actually, much more can be said. In fact, since $\mathcal{V}$ is a rational function, one can obtain not only the partial indices but as well a factorization of A. In the next proposition we characterize the partial indices of $\mathcal{A}$.

Proposition 3.5. *Suppose that the conditions of Theorem 3.3 are fulfilled. Then the partial indices of the matrix function $\mathcal{A}$ in (3.8) are given by*

$$\varkappa_1 = \max\left(\left[\frac{\varkappa+1}{2}\right]\kappa_1 + \left[\frac{\varkappa}{2}\right]\kappa_2, \left[\frac{\varkappa}{2}\right]\kappa_1 + \left[\frac{\varkappa+1}{2}\right]\kappa_2\right)$$
$$\varkappa_2 = \min\left(\left[\frac{\varkappa+1}{2}\right]\kappa_1 + \left[\frac{\varkappa}{2}\right]\kappa_2, \left[\frac{\varkappa}{2}\right]\kappa_1 + \left[\frac{\varkappa+1}{2}\right]\kappa_2\right)$$

where $\varkappa = \mathrm{ind}_{\mathbf{T}} a$ and $\kappa_{1,2} = \mathrm{ind}_{\mathbf{T}} \theta_{1,2}$.

Proof. Taking into account that

$$\mathcal{V}^\varkappa = \begin{cases} \theta^{\varkappa/2}\,\mathcal{I} & \text{if } \varkappa \text{ is even,} \\ \theta^{(\varkappa-1)/2}\,\mathcal{V} & \text{if } \varkappa \text{ is odd,} \end{cases}$$

we obtain the desired formulas. $\square$

We can now describe a procedure which permits the construction of an explicit factorization for a Daniele matrix function of the form (3.8):

1. Construction of an explicit factorization of the symbol a of $\mathcal{A}$, given in (3.9):

$$a(t) = a_+\, t^\varkappa\, a_-.$$

The factors $a_\pm$ can be calculated by the formula $a_\pm = \exp(P_\pm \log(t^{-\varkappa} a))$.

2. Finding of the matrix functions $\mathcal{A}_\pm = a_\pm(\mathcal{V})$, using Lemma 3.1.

3. Construction of a factorization of $\mathcal{V}^\varkappa$:

$$\mathcal{V}^\varkappa = \mathcal{V}_+ \begin{pmatrix} t^{\varkappa_1} & 0 \\ 0 & t^{\varkappa_2} \end{pmatrix} \mathcal{V}_-.$$

This step can easily be performed since $\mathcal{V}$ is a simple rational matrix function (its partial indices were given in Proposition 3.5).

4. Finally a factorization of $\mathcal{A}$ is given by

$$\mathcal{A}(t) = (\mathcal{A}_+\mathcal{V}_+) \begin{pmatrix} t^{\varkappa_1} & 0 \\ 0 & t^{\varkappa_2} \end{pmatrix} (\mathcal{V}_-\mathcal{A}_-).$$

3.3 A class of matrix functions in $\mathfrak{B}_\alpha^{2\times 2}$. In this part we consider the factorization problem for 2×2 matrix functions belonging to the algebra $\mathfrak{B}_\alpha^{2\times 2}$ of a special nature. The matrix functions we want to deal with are those elements in $\mathfrak{B}_\alpha^{2\times 2}$ for which the coefficients a and b in (2.1) are such that its difference is invariant for the shift α, i.e., $a - b = (a - b)_\alpha$. After a simple normalization, and up to a scalar function, to this type of matrix function can be given the form

$$\mathcal{A} = \begin{pmatrix} 1-a & -a \\ -a_\alpha & 1-a_\alpha \end{pmatrix}, \quad a \in \mathfrak{B}. \tag{3.12}$$

In fact, suppose $c, d \in \mathfrak{B}$ are such that $c - d = (c - d)_\alpha$ and consider the matrix function $\mathcal{B}$ given by (2.1) with a and b replaced by c and d, respectively. Then we have $\mathcal{B} = (c - d)\mathcal{A}$, where $\mathcal{A}$ is given by (3.12) with $a = -c(c - d)^{-1}$.

Matrix functions of the form (3.12) appear in one of the spectral problems that we shall consider in the next section. Note that the function $d = \det \mathcal{A}$ being invariant under the shift α has an even index.

The main result concerning the factorization of matrix functions of the form (3.12) reads as follows.

Theorem 3.6. *Let $\mathcal{A}$ be a nonsingular matrix function of the form (3.12). Also let $d = d_+ t^\kappa d_-$ be a factorization of $d = \det \mathcal{A}$, let*

$b = (a + a_\alpha)/2$, $c = (a - a_\alpha)/2$, decompose $c = c_+ + c_-$ with $c_\pm = P_\pm c$, let $e_\pm = 2P_\pm(c_+ d_-^{-1})$ and, for $\kappa > 0$, denote by p the polynomial which contains the first κ terms in the Taylor expansion of e_+ in a neighborhood of the origin. Further, set

$$\begin{aligned} \mathcal{F}_+ &= \begin{pmatrix} d_+ - e_+ \, t^{-\kappa} & -1 \\ d_+ + e_+ \, t^{-\kappa} & 1 \end{pmatrix}, \\ \mathcal{F}_- &= \frac{1}{2} \begin{pmatrix} d_- & d_- \\ 2c_- - 1 + e_- d_- & 2c_- + 1 + e_- d_- \end{pmatrix}. \end{aligned}$$

Then a factorization of $\mathcal{A}$ in the form

$$\mathcal{A} = \mathcal{A}_+ \, \Lambda \, \mathcal{A}_-$$

is obtained as follows:

(i) *If $\kappa \le 0$, then*

$$\mathcal{A}_+ = F_+, \qquad \Lambda = \operatorname{diag}\{t^\kappa, 1\}, \qquad \mathcal{A}_- = \mathcal{F}_-.$$

(ii) *If $\kappa > 1$ and all the derivatives $e_+^{(j)}$, $j = 1, \ldots, \kappa - 1$, vanish at the origin, then*

$$\mathcal{A}_+ = \mathcal{F}_+ \begin{pmatrix} 1 & 0 \\ -p(0) t^{-\kappa} & 1 \end{pmatrix}, \qquad \Lambda = \operatorname{diag}\{t^\kappa, 1\},$$
$$\mathcal{A}_- = \begin{pmatrix} 1 & 0 \\ p(0) & 1 \end{pmatrix} \mathcal{F}_-.$$

(iii) *If $\kappa > 1$ and at least one of the derivatives $e_+^{(j)}$, $j = 1, \ldots, \kappa - 1$, does not vanish at the origin, then*

$$\mathcal{A}_+ = \mathcal{F}_+ \begin{pmatrix} 1 & 0 \\ -p t^{-\kappa} & 1 \end{pmatrix} \mathcal{P}, \qquad \Lambda = \operatorname{diag}\{t^{\kappa-\nu}, t^\nu\}, \qquad \mathcal{A}_- = \mathcal{R}\mathcal{F}_-,$$

where $\nu \in \{1, \ldots, \kappa - 1\}$, the polynomial matrix function $\mathcal{P}$ and the rational matrix function $\mathcal{R}$ can be explicitly determined by the polynomial p.

Proof. We start the factorization procedure to obtain a factorization of $\mathcal{A}$ with a very useful observation. Let b and c be as defined in the theorem, and write $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{B}\,\mathcal{C},$$

where

$$\mathcal{B} = \begin{pmatrix} 1-b & -b \\ -b & 1-b \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1-c & -c \\ c & 1+c \end{pmatrix}.$$

Note that $d = \det \mathcal{A} = \det \mathcal{B}$, since $\det \mathcal{C} = 1$.

Now a moment of thought reveals that $\mathcal{C}$ admits a canonical factorization which can be obtained by using the decomposition of $c = c_+ + c_-$. In fact, it is easily verified that

$$\mathcal{C} = \mathcal{C}_+\,\mathcal{C}_- \quad \text{with } \mathcal{C}_\pm = \begin{pmatrix} 1-c_\pm & -c_\pm \\ c_\pm & 1+c_\pm \end{pmatrix}.$$

Further, note that the matrix function $\mathcal{B}$ can be diagonalized by means of a constant matrix. Indeed,

$$\mathcal{B} = \mathcal{S}\,\mathrm{diag}\,\{d,1\}\,\mathcal{S}^{-1} \quad \text{with } \mathcal{S} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore a factorization of $\mathcal{B} = \mathcal{B}_+\,\Lambda\,\mathcal{B}_-$ is obtained by taking

$$\mathcal{B}_+ = \mathcal{S}\,\mathrm{diag}\,\{d_+,1\}, \qquad \Lambda = \mathrm{diag}\,\{t^\kappa,1\}, \qquad \mathfrak{B}_- = \mathrm{diag}\,\{d_-,1\}\mathcal{S}^{-1}.$$

Substituting the factorizations of $\mathcal{B}$ and $\mathcal{C}$ just obtained in the representation (3.12), we get

$$\mathcal{A} = \mathcal{B}_+\,\Lambda\,\mathcal{B}_-\,\mathcal{C}_+\,\mathcal{C}_-.$$

Now, observe that $\mathcal{E} = \mathcal{B}_-\mathcal{C}_+\mathcal{B}_-^{-1}$ is a lower triangular matrix function,

$$\mathcal{E} = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \quad \text{with } e = 2c_+\,d_-^{-1};$$

thus, from the decomposition $e = e_+ + e_-$, $e_\pm = P_\pm e$, we obtain a canonical factorization of $\mathcal{E}$,

$$\mathcal{E} = \mathcal{E}_+\,\mathcal{E}_-, \quad \text{where } \mathcal{E}_\pm = \begin{pmatrix} 1 & 0 \\ e_\pm & 1 \end{pmatrix}.$$

So,

$$\mathcal{A} = \mathcal{B}_+ \, \Lambda \, \mathcal{E}_+ \, \mathcal{E}_- \, \mathcal{B}_- \, \mathcal{C}_-$$

and to obtain a factorization of $\mathcal{A}$ it remains to factorize $\Lambda\mathcal{E}_+$. We have

(i) If $\kappa \leq 0$, then $\Lambda\mathcal{E}_+ = \tilde{\mathcal{E}}_+\Lambda$, where $\tilde{\mathcal{E}}_+ = \left(\begin{smallmatrix} 1 & 0 \\ t^{-\kappa}e_+ & 1 \end{smallmatrix}\right)$. In this case a factorization of $\mathcal{A} = \mathcal{A}_+ \, \Lambda \, \mathcal{A}_-$ is obtained by taking

$$\mathcal{A}_+ = \mathcal{B}_+ \, \tilde{\mathcal{E}}_+ \stackrel{\text{def}}{=} \mathcal{F}_+, \qquad \mathcal{A}_- = \mathcal{E}_- \, \mathcal{B}_- \, \mathcal{C}_- \stackrel{\text{def}}{=} \mathcal{F}_-,$$

and $\Lambda = \operatorname{diag}\{t^\kappa, 1\}$.

Now let p be the polynomial defined in the theorem. We have

(ii) If $\kappa > 1$ and all the derivatives $e_+^{(j)}$, $j = 1, \ldots, \kappa - 1$, vanish at the origin, that is, if $p = p(0) = e_+(0)$, then

$$\Lambda \, \mathcal{E}_+ = \tilde{\mathcal{E}}_+ \begin{pmatrix} 1 & 0 \\ -p(0)t^{-\kappa} & 1 \end{pmatrix} \Lambda \begin{pmatrix} 1 & 0 \\ p(0) & 1 \end{pmatrix}$$

with $\tilde{\mathcal{E}}_+$ defined as before. Therefore a factorization of $\mathcal{A}$ is given by

$$\mathcal{A}_+ = \mathcal{F}_+ \begin{pmatrix} 1 & 0 \\ -p(0)t^{-\kappa} & 1 \end{pmatrix}, \qquad \Lambda = \operatorname{diag}\{t^\kappa, 1\},$$
$$\mathcal{A}_- = \begin{pmatrix} 1 & 0 \\ p(0) & 1 \end{pmatrix} \mathcal{F}_-.$$

(iii) If $\kappa > 1$ and at least one of the derivatives $e_+^{(j)}$, $j = 1, \ldots, \kappa - 1$, does not vanish at the origin (in this case, p is not a constant), we have

$$\Lambda \, \mathcal{E}_+ = \tilde{\mathcal{E}}_+ \begin{pmatrix} 1 & 0 \\ -pt^{-\kappa} & 1 \end{pmatrix} \begin{pmatrix} t^k & 0 \\ p & 1 \end{pmatrix}, \tag{3.13}$$

where the product of the two first left factors on the righthand side as well as its inverse belong to $C^+(\mathbf{T})^{2\times 2}$. So it remains to factorize the polynomial matrix function on the righthand side. This can be done by considering the partial fraction decomposition of the function $r = t^\kappa/p$:

$$r \, = \, p_0 + r_1 \, r^{-1} \, = \, p_0 + \frac{1}{p_1 + r_2 \, r_1^{-1}} \, = \, p_0 + \frac{1}{p_1 + \frac{1}{p_2 + \cdots}},$$

where the p_j are polynomials and the r_j are rational functions. From this representation it is possible to obtain explicitly a factorization of the right factor in the righthand side of (3.13) in the form

$$\begin{pmatrix} t^\kappa & 0 \\ p & 1 \end{pmatrix} = \mathcal{P} \operatorname{diag}\{t^{\kappa-\nu}, t^\nu\} \mathcal{R}$$

for some $\nu \in \{1, \dots, \kappa - 1\}$ and $\mathcal{P}, \mathcal{R}$ with the properties mentioned in the theorem (see [**14**] for the details). Therefore, in this case, we end up with a factorization of $\mathcal{A}$ as follows:

$$\mathcal{A}_+ = \mathcal{F}_+ \begin{pmatrix} 1 & 0 \\ -pt^{-\kappa} & 1 \end{pmatrix} \mathcal{P}, \qquad \Lambda = \operatorname{diag}\{t^{\kappa-\nu}, t^\nu\},$$
$$\mathcal{A}_- = \mathcal{R}\mathcal{F}_-. \qquad \square$$

Let us now talk about the generalization of the results obtained in this section for the case of more general matrix functions.

In principle, the results obtained in subsections 3.1 and 3.2 cannot be further generalized by this method. The main reason for this is that, with the operational calculus considered, in terms of powers of a multiplication operator by rational functions, we can only get continuous matrix functions.

The result in subsection 3.3 (Theorem 3.6) can be generalized directly to the case where the coefficient a in (3.12) belongs to $L_\infty(\mathbf{T})$ with a little change in its formulation. As remarked before, in the case where $a \in C(\mathbf{T})$ the function $d = \det \mathcal{A}$ being nonsingular on $\mathbf{T}$ admits a factorization in every $L_p(\mathbf{T})$ with $p \in (1, \infty)$ and its factors do not depend on the value of p. On the contrary, if we suppose that $a \in L_\infty(\mathbf{T})$, then Theorem 3.6 remains valid for each $p \in (1, \infty)$, that is, it gives a factorization of $\mathcal{A}$ in $L_p(\mathbf{T})$ if we know a factorization of d in $L_p(\mathbf{T})$. Thus, in this case, the existence of a factorization depends on the value of p and, if it exists, the factors can vary with p.

4. Spectrum problems. In this section we make use of the results in the previous sections to describe the spectrum of some singular integral operators with a Carleman linear fractional shift.

We shall use the following notation. If $A : L_p(\mathbf{T}) \to L_p(\mathbf{T})$ is a bounded linear operator, then by $\sigma_{L_p}(A)$ we denote the spectrum of A

in the algebra $\mathcal{L}(L_p(\mathbf{T}))$. By $\sigma_{L_p}^{(e)}(A) \subset \sigma_{L_p}(A)$ we denote the *essential spectrum* of A, i.e., the set of those $\lambda \in \mathbf{C}$ for which $A_\lambda = A - \lambda I$ is not a Fredholm operator in $L_p(\mathbf{T})$.

Let us mention that, apart from the essential spectrum which was characterized in [**13**] (see also [**8**]) very little is known about the spectrum in $L_p(\mathbf{T})$, $p \in (1, \infty)$, of an operator involving the three canonical operators: multiplication, shift and singular integral operator with Cauchy kernel. In fact, to the authors' knowledge, only in the very special case of an operator of the form $\chi U S$, where χ is the characteristic function of an arc of $\mathbf{T}$, the spectrum is known (see [**7**]).

We start the analysis of spectrum problems by considering, in the space $L_p(\mathbf{T})$, $p \in (1, \infty)$, operators of the form

$$T^{(1)} = a\,I + b\,U\,S = (a\,I + b\,U)P_+ + (a\,I - b\,U)P_-.$$

We have

Proposition 4.1. *Let θ_1 be a Blaschke product of the form* (3.3)*, $\theta_2 = (\theta_1)_\alpha$ and $\theta = \theta_1\theta_2$. Suppose that $a, b \in C(\mathbf{T})$ are such that*

$$a = f_\theta, \qquad b = \theta_1\, g_\theta, \tag{4.1}$$

let γ be the function defined on $\mathbf{T}$ by

$$\gamma(t) = f(t^2) + t\,g(t^2) \tag{4.2}$$

and put $\Gamma = \operatorname{im}\gamma = \{\mu \in \mathbf{C} : \mu = \gamma(t), t \in \mathbf{T}\}$. Then

$$\sigma_{L_p}(T^{(1)}) = \sigma_{L_p}^{(e)}(T^{(1)}) = \Gamma.$$

Proof. For any $\lambda \in \mathbf{C}$, we have

$$T_\lambda^{(1)} = A_\lambda^+\, P_+ + A_\lambda^-\, P_-, \quad \text{with } A_\lambda^\pm = (a - \lambda)I \pm b\,U.$$

Put $\Delta_\lambda^\pm = (a - \lambda)(a_\alpha - \lambda) \pm b\, b_\alpha$. It is well known that a necessary and sufficient condition for $T_\lambda^{(1)}$ to be a Fredholm operator in $L_p(\mathbf{T})$ is that Δ_λ^- does not vanish on the unit circle. Supposing that this is the case, we may write

$$T_\lambda^{(1)} = A_\lambda^+(P_+ + C_\lambda P_-), \quad \text{with } C_\lambda = (A_\lambda^+)^{-1}A_\lambda^- = \Delta_\lambda\, I + \nabla_\lambda\, U,$$

where $\Delta_\lambda = \Delta_\lambda^+/\Delta_\lambda^-$ and $\nabla_\lambda = -2(a_\alpha - \lambda)b/\Delta_\lambda^-$. We have (note that $(\Delta_\lambda)_\alpha = \Delta_\lambda$)

$$\mathcal{C}_\lambda = \pi(C_\lambda) = \begin{pmatrix} \Delta_\lambda & \nabla_\lambda \\ (\nabla_\lambda)_\alpha & \Delta_\lambda \end{pmatrix}.$$

If we suppose that the conditions in (4.1) are fulfilled, then $\mathcal{C}_\lambda$ takes the form

$$\mathcal{C}_\lambda = (\tilde{f}_\lambda)_\theta\, I + (\tilde{g}_\lambda)_\theta\, \mathcal{V},$$

where $\mathcal{V}$ is the matrix function given in (3.2) and

$$\begin{aligned} \tilde{f}_\lambda(t) &= \frac{(f(t)-\lambda)^2 + t\, g^2(t)}{(f(t)-\lambda)^2 - t\, g^2(t)}, \\ \tilde{g}_\lambda(t) &= -\frac{2(f(t)-\lambda)g(t)}{(f(t)-\lambda)^2 - t\, g^2(t)}, \quad t \in \mathbf{T}. \end{aligned}$$

Introduce the function c_λ, defined on $\mathbf{T}$ by

$$c_\lambda(t) = \tilde{f}_\lambda(t^2) + t\, \tilde{g}_\lambda(t^2). \tag{4.3}$$

A straightforward computation yields

$$c_\lambda(t) = \frac{\gamma(t)-\lambda}{\gamma(-t)-\lambda}, \quad t \in \mathbf{T}, \tag{4.4}$$

where γ is the function defined by (4.2). Setting $\Gamma = \{\mu \in \mathbf{C} : \mu = \gamma(t), t \in \mathbf{T}\}$, we find that for $\lambda \notin \Gamma$, c_λ is a well-defined continuous function on $\mathbf{T}$ which does not vanish. According to Lemma 3.1, if $\lambda \notin \Gamma$, then $\mathcal{C}_\lambda \in \mathfrak{A}$ and c_λ is its symbol. In the notation introduced in Section 3 we have $(1-\lambda^{-1}f, \lambda^{-1}g) \in \mathfrak{S}_\theta^2$.

Let $\lambda \notin \Gamma$. Since c_λ does not vanish on $\mathbf{T}$, the matrix function $\mathcal{C}_\lambda$ admits a factorization in $L_p(\mathbf{T})$. Now from (4.4) it follows that $c_\lambda(t)c_\lambda(-t) = 1$, $t \in \mathbf{T}$, and, consequently, $\varkappa_\lambda = 0$. But then $\mathcal{C}_\lambda$ admits a canonical factorization (Proposition 3.5) and therefore, by Theorem 2.6, $T_\lambda^{(1)}$ is invertible. On the other hand, if c_λ vanishes on

$\mathbf{T}$, that is, if $\lambda \in \Gamma$, then $T_\lambda^{(1)}$ is not a Fredholm operator in $L_p(\mathbf{T})$. Therefore, $\sigma_{L_p}(T^{(1)}) = \sigma_{L_p}^{(e)}(T^{(1)}) = \Gamma$. $\square$

Now we consider a general singular integral operator with shift

$$T(A) = P_+ + A\,P_-, \quad A = a\,I + b\,U.$$

We have

Proposition 4.2. *Suppose that the conditions of Proposition* 4.1 *are fulfilled by the coefficients a and b, and let γ and Γ be, respectively, the function and the curve as defined there. In addition, for any $\lambda \in \mathbf{C} \setminus (\Gamma \cup \{1\})$, let $c_\lambda = \lambda - \gamma$, $\varkappa_\lambda = \operatorname{ind}_{\mathbf{T}} c_\lambda$ and put $\Omega = \{\mu \in \mathbf{C} \setminus (\Gamma \cup \{1\}) : \varkappa_\mu \neq 0\}$. Then*

$$\sigma_{L_p}(T(A)) = \Omega \cup \Gamma \cup \{1\}, \quad \sigma_{L_p}^{(e)}(T(A)) = \Gamma \cup \{1\}.$$

Moreover, for any $\lambda \in \Omega$, $T(A)_\lambda$ is a one-sided invertible operator, which is left or right invertible according to $\varkappa_\lambda < 0$ or $\varkappa_\lambda > 0$, respectively, with

$$\operatorname{ind} T(A)_\lambda = \kappa\,\varkappa_\lambda, \qquad \kappa = \operatorname{ind}_{\mathbf{T}} \theta_1.$$

Proof. It is clear that $\lambda = 1$ belongs to $\sigma_{L_p}^{(e)}(T(A))$, since $T(A) - I$ has an infinite dimensional kernel. So for the rest of the proof we suppose that $\lambda \neq 1$. We have

$$T(A)_\lambda = (1-\lambda)[P_+ + (1-\lambda)^{-1} C_\lambda P_-] \quad \text{with } C_\lambda = [(a-\lambda)I + b\,U].$$

According to the results in Section 2 the Fredholm properties of this operator are determined by the factorization properties of the matrix function

$$\mathcal{C}_\lambda = \pi(C_\lambda) = \begin{pmatrix} a-\lambda & b \\ b_\alpha & a_\alpha - \lambda \end{pmatrix}.$$

If we suppose that the conditions in (4.1) are fulfilled, then $\mathcal{C}_\lambda$ takes the form

$$\mathcal{C}_\lambda = (f_\theta - \lambda)\mathcal{I} + g_\theta\,\mathcal{V},$$

where $\mathcal{V}$ is the matrix function given in (3.2) with $\theta_2 = (\theta_1)_\alpha$. Therefore, according to Lemma 3.1, $\mathcal{C}_\lambda \in \mathfrak{A}$ and its symbol is the function c_λ defined on $\mathbf{T}$ by $c_\lambda = \gamma - \lambda$, where γ is defined by (4.2).

If c_λ does not vanish on $\mathbf{T}$, that is, if $\lambda \notin \Gamma$, then $\mathcal{C}_\lambda$ admits a factorization in $L_p(\mathbf{T})$ and, using Proposition 3.5, we find that its partial indices $\kappa_j = \kappa_j(\lambda)$ are equal and are given by

$$\kappa_j = \kappa\, \varkappa_\lambda, \quad \text{where } \kappa = \operatorname{ind}_{\mathbf{T}} \theta_1, \quad \varkappa_\lambda = \operatorname{ind}_{\mathbf{T}} c_\lambda.$$

Since $\kappa > 0$, if $\varkappa_\lambda = 0$, then $\mathcal{C}_\lambda$ admits a canonical factorization and, therefore, by Theorem 2.6, $T(A)_\lambda$ is invertible. If $\varkappa_\lambda \neq 0$, then $\mathcal{C}_\lambda$ admits a noncanonical factorization and, therefore, by Theorem 2.6, $T(A)_\lambda$ is only a one-sided invertible operator. Using Theorem 2.10 we have

$$\operatorname{ind} T(A)_\lambda = \kappa\, \varkappa_\lambda = \begin{cases} \dim \ker T(A)_\lambda & \text{if } \varkappa_\lambda \geq 0, \\ -\operatorname{codim} \operatorname{im} T(A)_\lambda & \text{if } \varkappa_\lambda < 0. \end{cases}$$

If c_λ vanishes on $\mathbf{T}$, that is, if $\lambda \in \Gamma$, then $T(A)_\lambda$ is not a Fredholm operator in $L_p(\mathbf{T})$. So $\sigma^{(e)}_{L_p}(T^{(1)}) = \Gamma \cup \{1\}$. $\square$

In the spectral problems considered above we made some restrictions on the coefficients of the functional operator A which appear in the singular integral operator with shift $T(A)$ under consideration. These restrictions are such that the factorization problem for the corresponding matrix function $\mathcal{A} = \pi(A)$ can be solved. In some cases it is possible to consider general coefficients if there are already methods available to perform the factorization of the matrix function $\mathcal{A}$. The following example fits into this category.

Let $Q^{+,1}$ be the projection operator in $L_p(\mathbf{T})$, $p \in (1, \infty)$, defined by (2.29) with $n = 1$ and consider in $L_p(\mathbf{T})$ operators of the form

$$T^{(2)} = a\, Q^{+,1} P_-$$

with $a \in C(\mathbf{T})$. We have

Proposition 4.3. *Let $b = (a + a_\alpha)/2$, $c = (a - a_\alpha)/2$ and, for $\lambda \in \mathbf{C} \setminus \{0\}$, let $d_\lambda = 1 - (b/\lambda)$. Set $\Gamma = \{\mu \in \mathbf{C} : \mu = b(t), t \in \mathbf{T}\}$ and $\Omega = \{\mu \in \mathbf{C} \setminus (\Gamma \cup \{0\}) :\ \operatorname{ind}_{\mathbf{T}} d_\lambda \neq 0\}$. Then*

$$\sigma_{L_p}(T^{(2)}) = \Omega \cup \Gamma \cup \{0\}, \qquad \sigma^{(e)}_{L_p}(T^{(2)}) = \Gamma \cup \{0\}.$$

Proof. It is clear that $\lambda = 0$ belongs to $\sigma^{(e)}_{L_p}(T^{(2)})$. So in the rest of the proof we suppose that $\lambda \neq 0$. We have

$$T(A)_\lambda = -\lambda[P_+ + C_\lambda P_-] \quad \text{with } C_\lambda = \Big(1 - \frac{a}{2\lambda}\Big)I - \frac{a}{2\lambda}\,\alpha_+^{-1}\,U$$

and

$$\mathcal{C}_\lambda = \operatorname{diag}\{1, \alpha_+\}\,\mathcal{A}_\lambda\ \operatorname{diag}\{1, \alpha_+^{-1}\}, \tag{4.5}$$

where

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 - \dfrac{a}{2\lambda} & -\dfrac{a}{2\lambda} \\ -\dfrac{a_\alpha}{2\lambda} & 1 - \dfrac{a_\alpha}{2\lambda} \end{pmatrix}$$

belongs to the class considered in Theorem 3.6. Here we shall use the conventions about the notation introduced in that theorem and set $x_\lambda = x/\lambda$ if $x \in \{a, b, c, d, e\}$. We have $d_\lambda = \det \mathcal{C}_\lambda = \det \mathcal{A}_\lambda = 1 - (a + a_\alpha)/2\lambda = 1 - (b/\lambda) = 1 - b_\lambda$. Therefore, if d_λ vanishes on $\mathbf{T}$, that is, if $\lambda \in \Gamma$, then $\mathcal{C}_\lambda$ is singular and, consequently, $\lambda \in \sigma^{(e)}_{L_p}(T^{(2)})$.

Suppose that $\lambda \notin \Omega \cup \Gamma \cup \{0\}$ and let $d_\lambda = d_{\lambda-} d_{\lambda+}$ be a canonical factorization of d_λ. According to Theorem 3.6, $\mathcal{A}_\lambda$ also admits a canonical factorization, $\mathcal{A}_\lambda = \mathcal{A}_{\lambda+}\mathcal{A}_{\lambda-}$. Consequently, to obtain a factorization of $\mathcal{C}_\lambda$ it remains for us to consider the influence of the multiplication on the right of the factor $\mathcal{A}_{\lambda-}$ by the matrix function $\operatorname{diag}\{1, \alpha_+^{-1}\}$. From Theorem 3.6, we have

$$\mathcal{A}_{\lambda-} = \frac{1}{2}\begin{pmatrix} d_{\lambda-} & d_{\lambda-} \\ f_{\lambda-} & 2 + f_{\lambda-} \end{pmatrix},$$

where $f_{\lambda-} = 2c_{\lambda-} - 1 + e_{\lambda-}\, d_{\lambda-}$.

Let us note that the function d_λ is invariant for the shift α and this property also holds for the factors of its canonical factorization. From this one easily concludes that $d_-(\infty) = d_-(t_0)$, where $t_0 = 1/\bar{\beta}$ (the point where α_+ has a pole). Now we can consider the factorization of the matrix function $\mathcal{A}_{\lambda-}\operatorname{diag}\{1, \alpha_+^{-1}\}$. The following conclusions hold:

(i) If $f_{\lambda-}(t_0) = 1$, then $\mathcal{A}_{\lambda-}\operatorname{diag}\{1, \alpha_+^{-1}\}$ admits a noncanonical factorization with partial indices $+1$ and -1 given by

$$\mathcal{A}_{\lambda-}\operatorname{diag}\{1, \alpha_+^{-1}\} = \mathcal{M}_+\operatorname{diag}\{t, t^{-1}\}\,\tilde{\mathcal{A}}_{\lambda-}$$

where

$$\mathcal{M}_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ k_1 & \alpha_+^{-1} \end{pmatrix},$$
$$\tilde{\mathcal{A}}_{\lambda-} = \begin{pmatrix} t^{-1}d_{\lambda-} & t^{-1}\alpha_+^{-1}d_{\lambda-} \\ t\,\alpha_+(f_{\lambda-} - k_1 d_{\lambda-}) & t(2 + f_{\lambda-} - k_1 d_{\lambda-}) \end{pmatrix}$$

for $k_1 = d_{\lambda-}^{-1}(t_0)$ (note that $f_{\lambda-}(\infty) = -1$ and so $2 + f_{\lambda-} - k_1 d_{\lambda-} \to 0$ as $t \to \infty$).

(ii) If $f_{\lambda-}(t_0) \neq 1$, then $\mathcal{A}_{\lambda-}\mathrm{diag}\,\{1, \alpha_+^{-1}\} = \mathcal{N}_+\hat{\mathcal{A}}_{\lambda-}$ is a canonical factorization, with

$$\mathcal{N}_+ = \frac{1}{2}\begin{pmatrix} \dfrac{t}{k_2} & -\alpha_+^{-1} \\ \dfrac{k_1}{k_2}\,t + 1 & -k_1\,\alpha_+ \end{pmatrix}$$
$$\hat{\mathcal{A}}_{\lambda-} = \begin{pmatrix} f_{\lambda-} - k_1 d_{\lambda-} & \alpha_+^{-1}(2 + f_{\lambda-} - k_1 d_{\lambda-}) \\ \dfrac{\alpha_+}{k_2}[t\,f_{\lambda-} - (k_1 t + k_2)d_{\lambda-}] & \dfrac{t}{k_2}[(2 + f_{\lambda-} - k_1 d_{\lambda-}) - d_{\lambda-}] \end{pmatrix},$$

where k_1 is defined as before and k_2 is chosen so that $tf_{\lambda-} - (k_1 t + k_2)d_{\lambda-} \to 0$ as $t \to t_0$, yielding $k_2 = k_1 t_0(f_{\lambda-}(t_0) - 1) = t_0[2(c_{\lambda-}(t_0) - 1)d_{\lambda-}^{-1}(t_0) + e_-(t_0)]$.

For the matrix function $\mathcal{C}_\lambda$ we have:

(i) If $f_{\lambda-}(t_0) = 1$, then $\mathcal{C}_\lambda$ admits the noncanonical factorization

$$\mathcal{C}_\lambda = [\mathrm{diag}\,\{1, \alpha_+\}\,\mathcal{A}_{\lambda+}\,\mathcal{M}_+]\,\mathrm{diag}\,\{t, t^{-1}\}\,\tilde{A}_{\lambda-}.$$

(ii) If $f_{\lambda-}(t_0) \neq 1$, then $\mathcal{C}_\lambda$ admits the canonical factorization

$$\mathcal{C}_\lambda = [\mathrm{diag}\,\{1, \alpha_+\}\,\mathcal{A}_{\lambda+}\,\mathcal{N}_+]\,\hat{A}_{\lambda-}.$$

As an immediate consequence, by Theorem 2.6, $T_\lambda^{(2)}$ is invertible if $f_{\lambda-}(t_0) \neq 1$. But $T_\lambda^{(2)}$ is also invertible if $f_{\lambda-}(t_0) = 1$. In fact, from the factors of the factorization of $\mathcal{C}_\lambda$ for case (i), we obtain

$$\mathcal{H} = \Lambda_+^{-1}\,\mathcal{C}_{\lambda+}^{-1}(\alpha)\,e\,\mathcal{C}_{\lambda+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that is, we have $\mathcal{H} = \mathcal{H}_{\varepsilon,p}$ with $\varepsilon = +1$ and $p = 0$ in Proposition 2.2. To establish this result we have used the following easily verifiable relations

$$(c_{\lambda+})_\alpha = -c_{\lambda+} - c_{\lambda-}(t_0), \qquad (c_{\lambda+})_\alpha = -c_{\lambda-} - c_{\lambda-}(t_0),$$
$$(e_{\lambda+})_\alpha = -e_{\lambda_+} - e_{\lambda-}(t_0) + 2c_{\lambda-}(t_0)d_{\lambda-}^{-1}(t_0).$$

Therefore, by Theorem 2.10 and (2.33), $T_\lambda^{(2)}$ is invertible.

Finally, suppose that $\lambda \notin \Gamma \cup \{0\}$. According to Theorem 3.6, if $\lambda \in \Omega$, then one of the partial indices of $\mathcal{A}_\lambda$ is positive (negative) and the other is nonnegative (respectively, nonpositive). Therefore, taking into account Theorem 2.10 and (2.33), as well as the above conclusion on the changes of the partial indices by multiplication by the factor diag $\{1, \alpha_+^{-1}\}$, in this case we have also $\lambda \in \sigma_{L_p}(T^{(2)})$. $\square$

Before ending, let us make some remarks concerning this proposition:

(i) Matrix functions of the form (4.5) were considered in the paper [**2**] from the point of view of the existence of a canonical factorization (they belong to a class which was referred to there as $\mathcal{D} - \mathcal{N}$ class).

(ii) As we can see from the proof of the previous proposition, for $\lambda \notin \Omega \cup \Gamma \cup \{0\}$, $T_\lambda^{(2)}$ provides another example of a singular integral operator with shift $T(A)$ which is invertible in spite of the fact that the matrix function $\mathcal{A} = \pi(A)$ does not admit canonical factorization (see the comments after Theorem 2.10 and Example 2.11).

(iii) With a little extra effort it would be possible to characterize the dimensions of the kernel and cokernel of $T_\lambda^{(2)}$ for $\lambda \in \Omega$. This could be done since we can construct a noncanonical factorization of the matrix function $\mathcal{A}_\lambda$ (see the proof of Theorem 3.6) and perform operations similar to those considered in the above proof, to obtain from it a factorization of the matrix function $\mathcal{C}_\lambda$.

(iv) Proposition 4.3 maintains its validity for an arbitrary coefficient $a \in L_\infty(\mathbf{T})$. In fact, as mentioned at the end of Section 3, the factorization procedure for matrix functions in the class considered in subsection 3.3 is valid in the context of generalized factorization. Therefore, in the proof of Proposition 4.3 we only have to substitute the term "factorization" by the term "factorization in $L_p(\mathbf{T})$." As a consequence, if $a \in L_\infty(\mathbf{T})$ the spectrum of the operator $T^{(2)}$ in

$\mathcal{L}(L_p(\mathbf{T}))$ depends on the number $p \in (1, \infty)$ and so it can be different for different values of p.

As a final comment to the results given in this section, let us emphasize that, whenever it would be needed, the resolvent operator for any of the operators considered can be obtained by the use of the canonical factorization of the corresponding matrix function.

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