JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 13, Number 4, Winter 2001

RADON TRANSFORM OVER CONES AND RELATED DECONVOLUTION PROBLEMS

JEAN-FRANÇOIS CROUZET

ABSTRACT. We introduce a new kind of radon transform, consisting in integrating a function (to be recovered) over a special family of cones. It is in fact a formal generalization of the "Coded-aperture gammagraphy" imaging method, encountered in medicine and astronomy. We show that it is a natural geometric operation, but which does not have the fine properties of similar integral transform. Nevertheless, several inverse problems (like classical radon transform, deconvolution) are related to it, and also new kinds on integral transforms : essentially the "Quasi-convolution". After this study, where we show that the problem is severely ill-posed (essentially because of insufficient data), an inversion is performed in the case of complete data.

1. Introduction.

1.1 Notations and tools. A point $x \in \mathbf{R}^n$ is written $x = (x', x_n)$. The Euclidean scalar product of x and y is $x.y$, and the associated norm of x is $|x|$. We denote closed balls $\mathbf{B}(a,r) = \{x \in \mathbf{R}^n,\ |x-a| \le r\}$, spheres $\mathbf{S(a,r)} = \{\mathbf{x} \in \mathbf{R^n},\ |\mathbf{x}-\mathbf{a}| = \mathbf{r}\}$. Let $\mathbf{B}(0,1) = \mathbf{B}$, $\mathbf{S}(0,1) = \mathbf{S}$, and call $\mathbf{S}_+$, the half-unit sphere of $\mathbf{R}^n$ for positive x_n.

We introduce $\mathbf{\Pi}(\lambda) = \{x \in \mathbf{R}^n,\ x_n = \lambda\}$, and let $\mathbf{\Pi}(0) = \mathbf{\Pi}$, the hyperplane delimiting the two open half-spaces $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n,\ x_n > 0\}$ and $\mathbf{R}^n_- = \{x \in \mathbf{R}^n,\ x_n < 0\}$. The set $\mathbf{\Pi}$ will be called the "plane of the code," and the previous half-spaces will respectively be the "region of the source" and the "region of the detector."

The characteristic function of a compact set $\mathbf{K}$ of $\mathbf{R}^n$ is denoted

$$\chi_{\mathbf{K}}(x) = \begin{cases} 1 & \text{if } x \in \mathbf{K}, \\ 0 & \text{if } x \notin \mathbf{K}. \end{cases}$$

For $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$, we write $f_\lambda(x) = \lambda^n f(\lambda x)$.

1.1.1 *The Fourier transform and the convolution.* In the context of the spaces $L^p = \{f : \mathbf{R}^n \mapsto \mathbf{R}\ ;\ \left(\int_{\mathbf{R}^n} |f(x)|^p\, dx\right)^{1/p} = \|f\|_p < +\infty\}$, for $p \geq 1$, the Fourier transform of an L^1 function and its adjoint transform are written, as well as their extensions to the L^2 space of square integrable functions,

$$\mathcal{F} f = \hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix.\xi}\, dx\ , \qquad \mathcal{F}^* f(\xi) = \int_{\mathbf{R}} f(x) e^{ix.\xi}\, dx\ .$$

($*$ will also denote the complex conjugation). We also write $\mathcal{F}_X f$ the Fourier transform along a set of variables X. (For a brief review of the properties of the Fourier transform that we will need, see Appendix A and also [**23**] or [**25**]).

The usual convolution operator of two functions f and g is denoted $f \star g$ and represents the integral transform

$$f \star g\ (y) = \int_{\mathbf{R}^n} f(x)\, g(y-x)\, dx\ .$$

(If $f \in L^p$ and $g \in L^q$, with $1/p + 1/q \geq 1$, then $f \star g \in L^r$, with $1/r = 1/p + 1/q - 1$.) If f is in L^2 (typically the "source-function" corresponding to the information on the body, that one aims to recover in gammagraphy), and $g \in L^1$ (typically $g = \chi_{\mathbf{K}}$, the characteristic function of a compact "code" $\mathbf{K} \subset \mathbf{\Pi}$), the product $f \star g$ remains, so that we can write $\widehat{f \star g} = \widehat{f}\, \widehat{g}$, and justify the existence of most of the integrals encountered later.

1.1.2. *The classical radon transforms over hyperplanes (RT).* Let $\theta \in \mathbf{S}$, $s \in \mathbf{R}$, and f be, for sake of simplicity, a rapidly decreasing Schwartz function (see Appendix A for a review of the definition). The radon transform Rf represents the integral of f over the hyperplane $\Pi(\theta, s) = \{x \in \mathbf{R}^n\ , x.\theta = s\}$ and reads

$$Rf(\theta, s) = R_\theta f(s) = \int_{\mathbf{\Pi}(\theta,s)} f(x)\, d\mu(x)\ ,$$

where $d\mu(x)$ is the Lebesgue measure on $\mathbf{\Pi}(\theta, s)$.

We denote $\mathbf{Z} = \mathbf{S} \times \mathbf{R}$ the unit cylinder, and $\mathbf{Z}_+ = \mathbf{S}_+ \times \mathbf{R}$ the half unit cylinder on which Rf is completely defined (as soon as $Rf(-\theta, -s) = Rf(\theta, s)$).

Note that if the support of f, $\operatorname{supp} f$, is inside $\mathbf{B}(O,r)$, then $Rf(\theta,s)=0$, for $|s|>r$. The properties of the radon transform needed later can be found in Appendix B, and also in [**16**] or [**23**].

1.2 The radon transforms over cones (RTC).

1.2.1 *A general definition of the RTC.*

Definition 1.1. Consider as a "source" a closed subset $\mathbf{I}\subset\mathbf{R}^n_-$, and as a "detector" a closed subset $\mathbf{D}\subset\mathbf{R}^n_+$. Let $f\in L^2$, supported in $\mathbf{I}$, be the "source function", and $\mathcal{C}\in L^1(\mathbf{\Pi})$ (typically $\mathcal{C}=\chi_{\mathbf{K}}$, where $\mathbf{K}$ is the compact "code") the "code function". The radon transform over cones of f, defined on $\mathbf{D}$, is denoted $V_{\mathcal{C}}f$, and has the following integral expression:

$$\begin{aligned} V_{\mathcal{C}}f(y) &= V_{\mathcal{C}}f(y',y_n)\\ &= \int_{\mathbf{R}^n} f(x',x_n)\,\mathcal{C}\left(\frac{-x_n}{-x_n+y_n}\,y' + \frac{y_n}{-x_n+y_n}\,x'\right)\\ &\qquad\qquad \cdot\frac{dx'dx_n}{(-x_n+y_n)^{n-1}}\ . \end{aligned} \tag{1.1}$$

Here is its geometrical meaning: Take $\mathcal{C}=\chi_{\mathbf{K}}$, where $\mathbf{K}$ is a compact subset of $\mathbf{\Pi}$, and, for sake of simplicity, replace $\mathbf{D}$ by $\mathbf{R}^n_+$. From a point $y\in\mathbf{D}$ the source function f is integrated over the cone emerging from y and generated by $\mathbf{K}$ (playing the role of an aperture, as we will see in next subsection). This is the reason why occurs the convex combination $\eta=(-x_n/(-x_n+y_n))y'+(y_n/(-x_n+y_n))x'$, which determines the intersection of the plane $\mathbf{\Pi}$ with the half-line emerging from $x\in\mathbf{I}$ and getting through $y\in\mathbf{D}$:

Remark. The presence of the attenuation term $1/(-x_n+y_n)^{n-1}$ is explained in the next subsection, but one can already notice that, as $\mathbf{I}$ is closed, there exists a positive real number γ such that $-x_n\geq\gamma>0$, so that it remains bounded.

1.2.2 *The original "coded aperture transform" in medical imaging.* The origins of the RTC are to be found in the "coded aperture

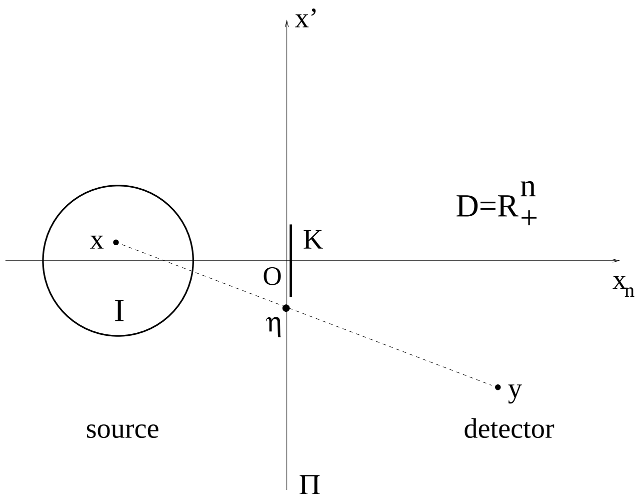

FIGURE 1.

transform" introduced in medical imaging in the 70's ([**4**], [**12**], [**14**], [**15**], [**24**]) and used also in astronomical measurements ([**1**], [**17**], [**21**]).

Let us first describe the medical imaging coded-aperture method from a geometrical point of view and then make more physical considerations (for more details see the original work in [**4**] and also [**8**]).

For $n = 3$ consider a three-dimensional compact source $\mathbf{I} \subset \mathbf{R}^3_-$ (for instance, $\mathbf{I} = \mathbf{B}(a, 1)$, with $a_3 < -1$), that will be the body of interest, supporting the unknown function f.

A detector is modeled by the whole plane $\mathbf{\Pi}(\lambda)$, with $\lambda > 0$. Between those two objects, a third one is interposed on $\mathbf{\Pi}$: an opaque mask, with an aperture (a sort of window, which is called, in fact, the "code") pierced in it, and corresponding to the compact $\mathbf{K}$. Thus we have $\mathcal{C} = \chi_{\mathbf{K}}$, and the following RTC expression:

(1.2)

$$V_{\mathcal{C}} f(y', \lambda) = \int_{\mathbf{R^3}} f(x', x_3)\, \chi_{\mathbf{K}}\left(\frac{-x_3}{-x_3 + \lambda} y' + \frac{\lambda}{-x_3 + \lambda} x' \right) \frac{dx'\, dx_3}{(-x_3 + \lambda)^2}\ .$$

Technically, the body of interest (for example one myocardium of a human heart) has previously received an injection of radioactive serum, and now emits, in the whole space, gamma-rays corresponding to the

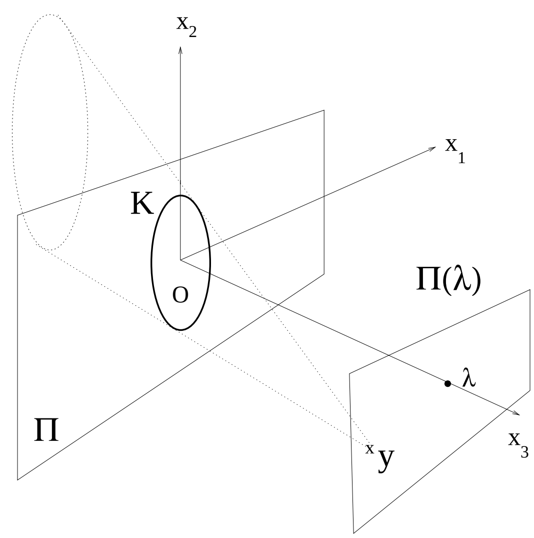

FIGURE 2.

serum fixation f in the tissue (lower if the tissue is sick). In order to reconstruct this information f, this "coding" (of "filtering") dispositive thus was imaged in nuclear medicine ([**4**]). Figure 2 illustrates the coding process and the conical nature of the measurements (here **K** is a thin annular aperture).

The code can be an annular thin aperture ([**5**], [**6**], [**7**], [**8**]), or concentric annular thin apertures (called a Fresnel-zone, [**14**]). Sometimes the code function $\mathcal{C}$ is approximated by distributional measures, so that the scope of the RTC is enlarged: In [**1**] or [**24**] authors use regular grids of pinholes (modeled by ponctual Dirac δ measures). Systems of rectilign slits, as in [**11**] and [**15**], are also studied (and even stochastic apertures in [**22**]). Pinholes lead to the "divergent-beam transform" ([**23**]), and rectilign slits lead to the usual radon transform over hyperplanes. In both cases the data will of course be very incomplete (see Appendix B).

Remark. The attenuation term $1/(-x_n+y_n)^{n-1}$ now reads $1/(-x_3+\lambda)^2$. It can be interpreted as a loss of energy of the gamma-rays, being modeled by the classical "$1/r^2$" law. It is in fact a good approximation of the physical reality, which is much more distorted; see the thesis again [**4**].

2. Special cases and related deconvolution problems. The integral transform $V_{\mathcal{C}}$, called RTC, does not have the usual linear properties of the radon transform RT. Nevertheless, there are many particular cases that lead to problems of interest. One of them is well known (and now revisited in another context), and others are quite new.

2.1. The case of $(n-1)$-dimensional sources. In this section we consider sources that are supported by a hypersurface (that can be typically written $x_n = s(x'))$. The detector should, at least in this paper, also be chosen as an $(n–1)$-dimensional hypersurface in order to economize on onerous image registrations or numerical computations. In the medical context of gammagraphy, the detector should then classically be a hyperplane.

2.1.1. *The convolution case.* In this simple case, both source and detector are parallel to the hyperplane $\mathbf{\Pi}$ of the code. Let us take for instance $\mathbf{I} = \mathbf{\Pi}(\mu)$ and $\mathbf{D} = \mathbf{\Pi}(\lambda)$, with $\mu < 0 < \lambda$. Thus, the integral expression of $V_{\mathcal{C}} f$ becomes

$$V_{\mathcal{C}} f\,(y', \lambda) = \int_{\mathbf{R}^{n-1}} f(x', \mu)\, \mathcal{C}\left(\frac{-\mu}{-\mu+\lambda} y' + \frac{\lambda}{-\mu+\lambda} x'\right) \frac{dx'}{(-\mu+\lambda)^{n-1}}\ ,$$

which is the expression of a convolution. Indeed, after denoting $V_{\mathcal{C}} f\,(y', \lambda) = V_{\mathcal{C}} f(y')$ and $f(x', \mu) = f(x')$, we obtain

$$V_{\mathcal{C}} f = (-\mu)^{1-n} f_{\mu/\lambda} \star \mathcal{C}_{-\mu/(-\mu+\lambda)}\ . \tag{2.1}$$

This convolution becomes natural when considering the following 2D illustration of two measurements (1) and (2) and applying the Thales theorem.

Let us now follow the classical ideas of deconvolution, found in [**2**] and [**3**]. Take a compact code, $\mathcal{C} = \chi_{\mathbf{K}}$, and apply the Fourier transform to the left and righthand side of 2.1. This yields

$$\widehat{V_{\mathcal{C}} f} = (-\mu)^{1-n} \widehat{f_{\mu/\lambda}}\, \widehat{\mathcal{C}}_{-\mu/(-\mu+\lambda)}\ . \tag{2.2}$$

As soon as the analytic function $\widehat{\mathcal{C}}_{-\mu/(-\mu+\lambda)}$ vanishes infinitely many, it is impossible to reconstruct analytically the function $\hat{f}$ (and f). We

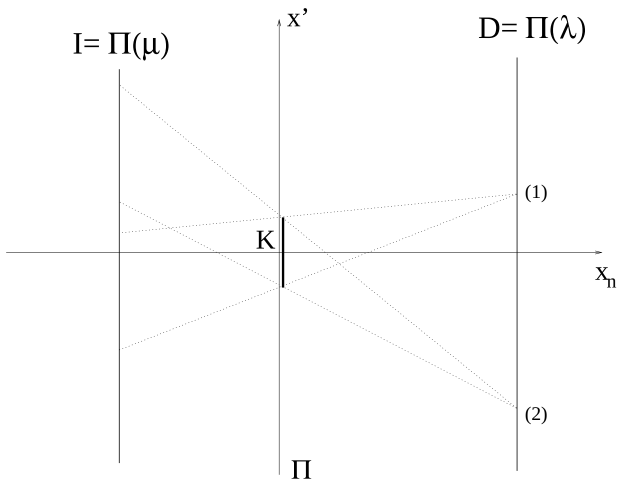

FIGURE 3.

are thus led to take two coded images in order to avoid the zeros of the "transfer function": One possible physical issue is, as soon as supp f is compact, to construct a compact "double-code," $\mathbf{K} = \mathbf{K}_1 \cup \mathbf{K}_2$ with $\mathcal{C} = \chi_{\mathbf{K}} = \mathcal{C}_1 + \mathcal{C}_2$, such that the two corresponding images are disjoint:

$$\operatorname{supp}(V_{\mathcal{C}_1} f) \cap \operatorname{supp}(V_{\mathcal{C}_2} f) = \varnothing \ .$$

(It should always be possible as soon as $\mathbf{K}_2$ is "far enough" from $\mathbf{K}_1$).

Indeed, rewriting the former relation 2.1, $V_{\mathcal{C}} f = \tilde{f} \star \mathcal{H}$, we have

$$V_{\mathcal{C}_1} f = \tilde{f} \star \mathcal{H}_1 \ , \ \ V_{\mathcal{C}_2} f = \tilde{f} \star \mathcal{H}_2 \ .$$

Then, the corresponding Fourier relation 2.2 $\widehat{V_{\mathcal{C}} f} = \widehat{\tilde{f}}\, \widehat{\mathcal{H}}$ splits into two parts: $\widehat{V_{\mathcal{C}_1} f} = \widehat{\tilde{f}}\, \widehat{\mathcal{H}_1}$ and $\widehat{V_{\mathcal{C}_2} f} = \widehat{\tilde{f}}\, \widehat{\mathcal{H}_2}$. Choose two "deconvolutors" G_1 and G_2 such that the Bezout relation

$$\widehat{\mathcal{H}_1} G_1 + \widehat{\mathcal{H}_2} G_2 = 1 \tag{2.3}$$

is satisfied everywhere. For instance, compute the analytic functions

$$G_1 = \frac{\widehat{\mathcal{H}_1}^*}{|\widehat{\mathcal{H}_1}|^2 + |\widehat{\mathcal{H}_2}|^2} \quad \text{and} \quad G_2 = \frac{\widehat{\mathcal{H}_2}^*}{|\widehat{\mathcal{H}_1}|^2 + |\widehat{\mathcal{H}_2}|^2} \ .$$

Then, the combination $\widehat{V_{\mathcal{C}_1}f}\ G_1 + \widehat{V_{\mathcal{C}_2}f}\ G_2 = \widehat{\tilde{f}}$, makes the computation of $\tilde{f}$ and f possible.

Of course, one will aim for G_1 and G_2 to be the Fourier transform of two compactly supported distributions, in order to perform good numerical inversions and to construct in a practical way such deconvolutors. In [**2**] and [**3**], the authors analytically compute, in terms of simple mathematical operations, the analytic Fourier transforms G_1 and G_2 (band-limited Paley-Wiener functions) of two compactly supported distributions $\mathcal{D}_1$ and $\mathcal{D}_1$. Note that the corresponding Bezout relation reads, in the space of compactly supported distributions,

$$\mathcal{H}_1 \star \mathcal{D}_1 + \mathcal{H}_2 \star \mathcal{D}_2 = \delta\ . \tag{2.4}$$

Remark. A remarkable necessary condition for obtaining 2.3 is that $\widehat{\mathcal{H}_1}$ and $\widehat{\mathcal{H}_2}$ have no common zero. This could be, for example, physically performed (with a good approximation) when using two suitable concentric thin annular codes ([**2**], [**3**], [**8**]).

2.1.2. *The mixed convolutions case.* Let us complicate the coding process and choose again a detector $\mathbf{D} = \mathbf{\Pi}(\lambda)$ with $\lambda > 0$, but a source with equation $x_1 = \alpha x_n + \beta$. The previous Thales property has clearly been lost: see Figure 3.

There, the integral expression of $V_{\mathcal{C}}f$ reads

$$V_{\mathcal{C}}f\,(y',\lambda) = \int_{\mathbf{R}^{n-1}} f(\alpha x_n + \beta, x'', x_n)\ \mathcal{C}(Z) \frac{dx''dx_n}{(-x_n+\lambda)^{n-1}}\ ,$$

where we have denoted $\xi'' = (\xi_2, \dots, \xi_{n-1})$, and

$$Z = \mathcal{C}\left(\frac{-x_n}{-x_n+\lambda} y_1 + \frac{\lambda}{-x_n+\lambda}(\alpha x_n + \beta)\,,\ \frac{-x_n}{-x_n+\lambda} y'' + \frac{\lambda}{-x_n+\lambda} x''\right)\ .$$

This is no more the expression of a convolution, but it is a combination of additive and multiplicative convolutions. Indeed, if we again simplify the expression and write $V_{\mathcal{C}}f\,(y',\lambda) = V_{\mathcal{C}}f(y') = V_{\mathcal{C}}f\,(y_1, y'')$ and

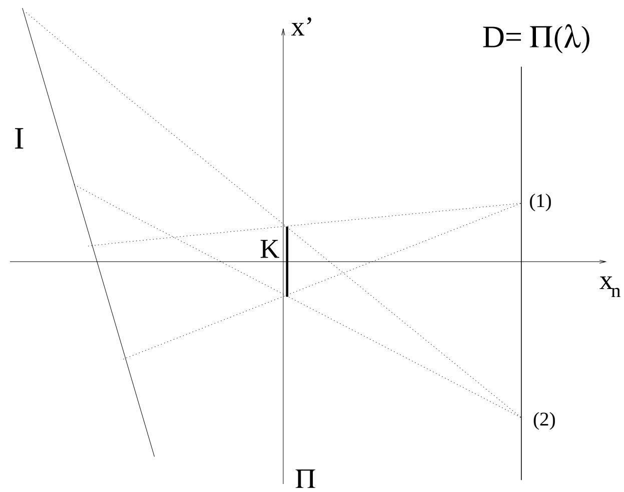

FIGURE 4.

$f(\alpha x_n+\beta, x'', x_n) = f(x'', x_n)$, we obtain

$$\begin{aligned} &V_{\mathcal{C}} f\,(y_1-\lambda\alpha+\beta, y'') \\ &\quad = \int_{\mathbf{R}^{n-1}} f(x'', x_n)\, \mathcal{C}\left(\frac{-x_n}{-x_n+\lambda}\, y_1+\beta,\, \frac{-x_n}{-x_n+\lambda}\left(y''-\frac{\lambda}{x_n}x''\right)\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \frac{dx''dx_n}{(-x_n+\lambda)^{n-1}}\ . \end{aligned}$$

Skipping all details, let us make (as in [**8**]) the change of variables $v''=\lambda x''/x_n$, $u=(x_n-\lambda)/x_n$, to derive an integral expression with the following kernel:

$$\mathcal{C}\left(\frac{1}{u}\, y_1+\beta,\ \frac{1}{u}\,(y''-v'')\right)\ .$$

It is thus possible to view the integral operation as a combination of classical convolution (the term $y''-v''$) and Mellin multiplicative convolution (the term $1/u$). Then, taking a suitable family of codes $\mathcal{C}(x') = \Phi(x_1)\Psi(x'')$, a consecutive Fourier transform along y'' and a Mellin transform along y_1 (as made in [**8**]) leads to reconstruction of the function f.

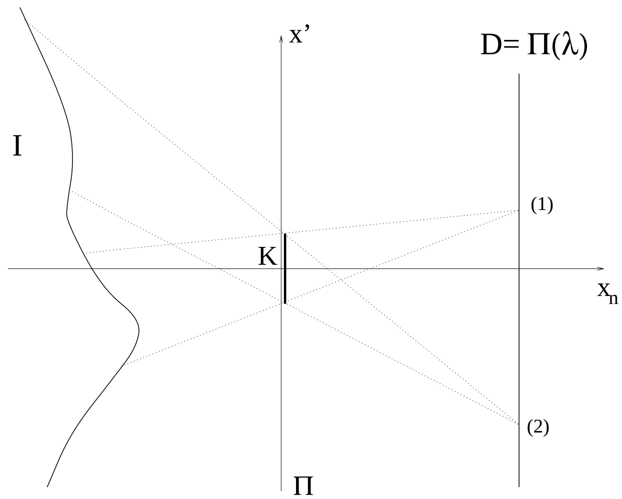

FIGURE 5.

2.1.3. *The "quasi-convolution" case.* Let us now choose a general $(n-1)$-dimensional source with equation $x_n = s(x')$. The only restriction should be a natural geometrical simplification: each ray emerging from a point (y', λ) of the detector and getting through the code should meet the body (the support of the function f) not more than once, as in Figure 4.

With the same notations as before we now have

$$V_{\mathcal{C}} f\,(y') = \int_{\mathbf{R}^{n-1}} f(x')\, \mathcal{C}\left(\frac{-s(x')}{-s(x')+\lambda}\, y' + \frac{\lambda}{-s(x')+\lambda} x'\right) \frac{dx'}{(-s(x')+\lambda)^{n-1}} \;.$$

Let us authorize change of variables $X' = \lambda x'/s(x') = \phi(x')$, and denote

$$\Psi(X') = (s(x') - \lambda)/s(x'),$$
$$g(X') = f(\phi^{-1}(X'))\, \frac{d\left(\phi^{-1}(X')\right)/dX'}{\left((-s\left(\phi^{-1}(X')\right) + \lambda)\right)^{n-1}}$$

to obtain

$$V_{\mathcal{C}} f\,(y') = \int_{\mathbf{R^{n-1}}} g(X')\, \mathcal{C}\left(\frac{y' - X'}{\Psi(X')}\right) dX' = f \,\widetilde{\star}_{\Psi}\, \mathcal{C}\,(y')\;.$$

The integral transform now reads like a new operation that has been introduced in [**8**] and [**9**], called "quasi-convolution." The kernel $\mathcal{C}\left((y' - X')/\Psi(X')\right)$ indicates that the shape of the intersection between the emerging cones and the surface supporting the source is no more constant, but depends on the function Ψ.

If $s(x')$ is affine, $\Psi(X')$ is a rational function, and the quasi-convolution can be regarded as a mixed multiplicative and additive convolution, as in the previous section. Otherwise, the problem has not yet been solved.

It is also worth studying the dual transform

$$f \star_\Phi \mathcal{C}\,(X') = \int_{\mathbf{R}^{n-1}} f(y')\,\mathcal{C}\left(\frac{y' - X'}{\Phi(X')}\right) dy' ,$$

which may constitute the effective "quasi-convolution." Indeed, Φ plays the geometrical rule of a scaling function, and connections may be made with wavelets, and even with convolutions in spaces with nonconstant curvature (like hyperbolic spaces). Of course, one aims to follow the classical deconvolution line, and try to derive a formula corresponding to

$$\mathcal{F}(g \star \mathcal{C}) = \mathcal{F}(g)\mathcal{F}(\mathcal{C}) .$$

It should suffice here to find a Fourier-like operator $\mathcal{F}_\Phi$ such that

$$\mathcal{F}_\Phi(g \star_\Phi \mathcal{C}) = \mathcal{G}_\Phi(g)\mathcal{H}_\Phi(\mathcal{C}) ,$$

where $\mathcal{G}_\Phi$ is an invertible transform.

In fact, for nonaffine functions s (and rational Φ) only very particular code functions $\mathcal{C}$ have yet provided results (see [**8**], [**9**]).

Remarks. For $n = 3$, this particular case of a 2D surface source can, for instance, be the model of a myocardium in heart imaging. It becomes rather thin when it ends its periodic cycles and can be approximated by a half-ellipsoid. (It also should be correctly placed in front of a sufficiently small code, in order to get the geometrical hypothesis made at the beginning of this subsection).

One can also consider a nonplane detector, and thus enlarge the scope of the integral transform. This yields, if one takes a detector with

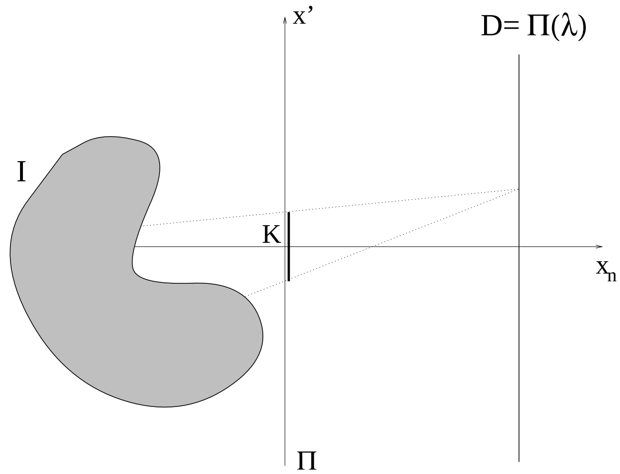

FIGURE 6.

equation $x_n = q(x')$, the expression

$$V_{\mathcal{C}} f\,(y') = \int_{\mathbf{R}^{n-1}} f(x')\,\mathcal{C}\left(\frac{s(x')}{s(x')+q(y')} y' + \frac{q(y')}{s(x')+q(y')} x'\right) \frac{dx'}{(s(x')+q(y'))^{n-1}}\ .$$

2.2. The case of n-dimensional sources.

2.2.1. *The necessity of n-dimensional detectors.* Let us go back to the initial problem: the source $\mathbf{I}$ is now some closed subset of the half-space $\mathbf{R}^n_-$, and the detector some closed subset of the half-space $\mathbf{R}^n_+$. Take a source function $f \in L^2(\mathbf{I})$ and a code function $\mathcal{C} \in L^1(\mathbf{\Pi})$, as in Figure 6.

In the corresponding general expression,

$$V_{\mathcal{C}} f(y) = \int_{\mathbf{R}^n} f(x', x_n)\,\mathcal{C}\left(\frac{-x_n}{-x_n+y_n} y' + \frac{y_n}{-x_n+y_n} x'\right) \frac{dx' dx_n}{(-x_n+y_n)^{n-1}}\ ,$$

one cannot make the economy of the y_n variable any more, and thus cannot decide to use a plane detector as in the previous cases. Indeed, if the unknown function f is an n-variable function (even with compact support), it is clear that the data $V_{\mathcal{C}} f$ of the integral transform must also be one. Although the characterization of the null-space of the transform is certainly difficult, one can, however, consider the following intuitive example.

Let us take $\mathbf{D} = \mathbf{\Pi}(\lambda)$, let Θ be any function in $L^2(\mathbf{R}^{n-1})$, and let f be closed to the distribution (corresponding to two parallel plane sources)

(2.5)
$$\begin{aligned}\tilde{f}(x', x_n) = \delta(x_n - \mu_1)\,(-\mu_1)^{n-1}\left[\Theta_{\lambda/\mu_1} \star \mathcal{C}_{-\mu_2\lambda/((-\mu_2+\lambda)\mu_1)}\right](x')\\ - \delta(x_n - \mu_2)\,(-\mu_2)^{n-1}\left[\Theta_{\lambda/\mu_2} \star \mathcal{C}_{-\mu_1\lambda/((-\mu_1+\lambda)\mu_2)}\right](x')\,.\end{aligned}$$

Applying the relation 2.1 yields

$$\begin{aligned}V_{\mathcal{C}}\tilde{f}(y', \lambda) = \left[\Theta \star \mathcal{C}_{-\mu_2/(-\mu_2+\lambda)}\right] \star \mathcal{C}_{-\mu_1/(-\mu_1+\lambda)}\,(y')\\ - \left[\Theta \star \mathcal{C}_{-\mu_1/(-\mu_1+\lambda)}\right] \star \mathcal{C}_{-\mu_2/(-\mu_2+\lambda)}\,(y') = 0\,.\end{aligned}$$

Thus, the coding process may not detect such sources for a fixed plane detector, and the variable y_n has to vary in order to operate reconstructions.

Guided by this example, we should now write the RTC in terms of sums of convolutions: it was the historical way physicists handled the gammagraphy problem ([**5**], [**24**]).

2.2.2. *An approach by sums of convolutions.* In fact, $V_{\mathcal{C}} f$ can be viewed as a continuous sum of convolutions; denote $f_{x_n,y_n}(x') = (-x_n)^{1-n} f_{x_n/y_n}(x', x_n)$, and rewrite

$$\begin{aligned}V_{\mathcal{C}} f(y', y_n) &= \int_{\mathbf{R}^n} f(x', x_n)\,\mathcal{C}_{-x_n/(-x_n+y_n)}\left(y' - \frac{y_n}{x_n}x'\right)\frac{dx'dx_n}{(-x_n)^{n-1}}\\ &= \int_{\mathbf{R}^n} (-x_n)^{1-n}(x_n/y_n)^{n-1} f((x_n/y_n)x', x_n)\\ &\qquad\qquad \cdot\,\mathcal{C}_{-x_n/(-x_n+y_n)}\,(y' - x')\;dx'\,dx_n\,.\end{aligned}$$

This provides

$$V_{\mathcal{C}} f(y', y_n) = \int_{\mathbf{R}} \left[f_{x_n,y_n} \star \mathcal{C}_{-x_n/(-x_n+y_n)}\right](y')\,dx_n\,.$$

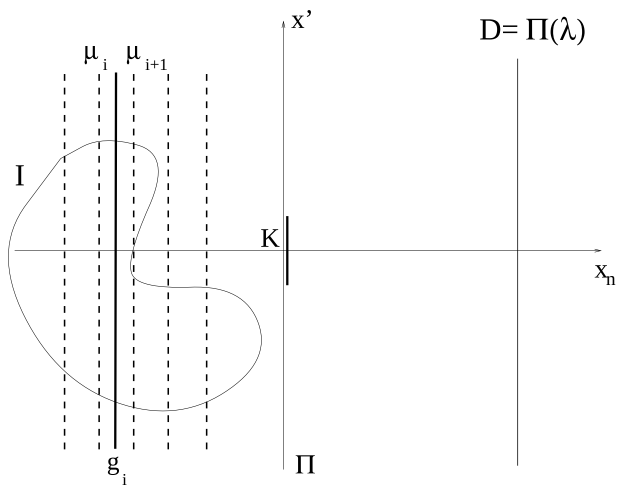

FIGURE 7.

In practice, this property leads one to take y_n constant, the usual plane detectors are $\mathbf{\Pi}(\lambda)$, and to approximate the compactly supported function f by parallel plane sources g^i, considering that it is almost constant on each thin slice $[\mu_i, \mu_{i+1}]$, as in Figure 7.

Thus taking, for N large,

$$f(x', x_n) = \sum_{i=1}^{N} \delta(x_n - \mu_i)\, g^i(x')\ ,$$

with fixed μ_i, this gives

$$V_{\mathcal{C}} f(y', \lambda) = \sum_{i=1}^{N} g^i_{\mu_i, \lambda} \star \mathcal{C}_{-\mu_i/(-\mu_i+\lambda)}\,(y')\ .$$

Suppose λ is then assigned to vary in a discrete set $\Lambda = \{\lambda_1, \dots, \lambda_M\}$, or, in other words, one makes several measurements, as in Figure 8, and thus registers the family of images

$$V_{\mathcal{C}} f(y', \lambda_j) = \sum_{i=1}^{N} g^i_{\mu_i, \lambda_j} \star \mathcal{C}_{-\mu_i/(-\mu_i+\lambda_j)}\,(y'), \quad j = 1, \dots, M\ .$$

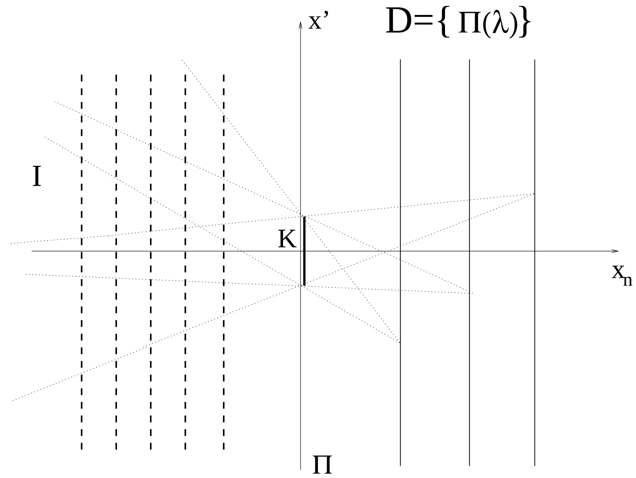

FIGURE 8.

After rewriting this set of equalities,

$$S_j = \sum_{i=1}^{N} f_i \star \mathcal{C}_{ij}, \quad j = 1, \dots, M \ ,$$

the unknown source that we now aim to recover is now the set of plane source functions

$$F = \{f_1, \dots, f_N\}$$

solution of a linear system of convolutions.

Of course, retrieving the set F with a large number of slices N could ensure a good version of f as far as the problem is well posed in the sense of Hadamard (see [**23**]), which is obviously not the case; the inverse of the integral operator $V_{\mathcal{C}}$ is certainly not continuous, and in many cases should not even exist at all.

The previous example 2.5 now shows clearly the noninjectivity of the (approximated) transform: when making the substitutions

$$f_k \hookrightarrow f_k + \Theta \star \mathcal{C}_{lj}, \qquad f_l \hookrightarrow f_l - \Theta \star \mathcal{C}_{kj} \ ,$$

the data S_j remains unchanged:

$$S_j \hookrightarrow S_j + \Theta \star \mathcal{C}_{lj} \star \mathcal{C}_{kj} - \Theta \star \mathcal{C}_{kj} \star \mathcal{C}_{lj} = S_j \ .$$

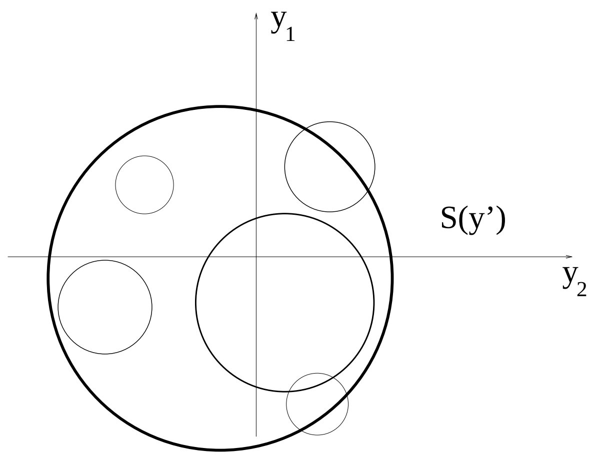

FIGURE 9.

This leads to the necessity of taking a detector **D** not reduced to a single hyperplane, but to choose a large enough set $\{\mathbf{\Pi}(\lambda), \lambda \in \Lambda\}$. We examine this problem in the next subsection.

2.2.3. *The "multi-deconvolution" problem.* In other words, the number M of registrations S_j should be large when the number N of slices used to approximate the source functions is large, which should be the case most of the time ([**5**]). This is actually the core of the problem when one makes this approach. Nevertheless, let us first consider briefly some empirical but cheap approximative methods of reconstruction.

After making severe restrictions on f (say F), one should be allowed to use only one or two plane detectors $\mathbf{\Pi}(\lambda)$. Indeed, let us first consider the case when f is roughly approximated by a discrete set of Dirac δ-measures (several isolated luminous peaks) placed at the points (X_i, μ_i), $i = 1, \ldots, N$. The data $S_j = S$ then reads as the combination of characteristic functions:

$$S(y') = \sum_{i=1}^{N} \alpha_i \, \mathcal{C}_{-\mu_i/(-\mu_i+\lambda)} \; (y' - \lambda/\mu_i X_i) \;,$$

where the α_i are given. Thus the inversion consists in the problem of dissociating the different characteristic functions (in order to calculate

(X_i, μ_i), $i = 1, \dots, N$), as in Figure 9, representing a 2D plane image S when the code is a thin annulus.

If the source function cannot be so roughly approximated, the following empirical deconvolutions methods are nevertheless worth looking at.

Suppose indeed one has two registered images S_1 and S_2 (compactly supported). Then construct (or compute) two deconvolutors $\mathcal{D}_{k1}$ and $\mathcal{D}_{k2}$ such that the Bezout relation (as 2.4)

$$\mathcal{C}_{k1} \star \mathcal{D}_{k1} + \mathcal{C}_{k2} \star \mathcal{D}_{k2} = \delta$$

is satisfied. For example, the deconvolutors are the following tempered distributions

$$\mathcal{D}_{k1} = \mathcal{F}^{-1}\left(\frac{\widehat{\mathcal{C}_{k1}}^{*}}{|\widehat{\mathcal{C}_{k1}}|^2 + |\widehat{\mathcal{C}_{k2}}|^2}\right), \qquad \mathcal{D}_{k2} = \mathcal{F}^{-1}\left(\frac{\widehat{\mathcal{C}_{k2}}^{*}}{|\widehat{\mathcal{C}_{k1}}|^2 + |\widehat{\mathcal{C}_{k2}}|^2}\right).$$

Now compute what one could call a "focalization" on the slice k:

$$\begin{aligned} S_1 \star \mathcal{D}_{k1} + S_2 \star \mathcal{D}_{k2} &= f_k \star \delta + \sum_{i \neq k, i=1}^{N} f_i \star [\mathcal{C}_{i1} \star \mathcal{D}_{k1} + \mathcal{C}_{i2} \star \mathcal{D}_{k2}] \\ &= f_k + \varepsilon_k \ . \end{aligned}$$

The idea is now that the additional term ε_k should be a negligible "blur" in front of the function f_k. In other words, f_k (for example a "peak") should be extractible from the focused image $f_k + \varepsilon_k$. After removing the contributions of the level f_k, one thus continues to focus on another level k' in $S_1 - f_k \star \mathcal{C}_{k1}$ and $S_2 - f_k \star \mathcal{C}_{k2}$.

This is an empirical scenario which could work in practice, possibly with the help of a priori knowledge, but which is mathematically inconsistent. Indeed, in Fourier space, the terms $\widehat{\mathcal{C}_{i1}}\,\widehat{\mathcal{D}_{k1}} + \widehat{\mathcal{C}_{i2}}\,\widehat{\mathcal{D}_{k2}}$ have no reason to be small.

The same scenario (originating in [**5**]) can be followed with only one image S, multiplying its Fourier transform $\widehat{S}$ by the "Pseudo-Wiener" deconvolutor

$$\widehat{\mathcal{D}_k} = \frac{\widehat{\mathcal{C}_k}^{*}}{|\widehat{\mathcal{C}_k}|^2 + \varepsilon^2} \ ,$$

where ε is small. This yields

$$\widehat{S}\widehat{\mathcal{D}_k} = \frac{|\widehat{\mathcal{C}_k}|^2}{|\widehat{\mathcal{C}_k}|^2 + \varepsilon^2}\,\widehat{f_k} + \sum_{i\neq k, i=1}^{N} \frac{\widehat{\mathcal{C}_i}\,\widehat{\mathcal{C}_k}^*}{|\widehat{\mathcal{C}_k}|^2 + \varepsilon^2}\,\widehat{f_i} \simeq \widehat{f_k} + \eta_k\ .$$

Here again, the "blur" η_k should be small in front of $\widehat{f_k}$, which seems, at least mathematically, impossible.

A third idea is to use, for numerical purposes, a linear least-square inversion: At each point of a regular grid of $\left(\mathbf{R}^{n-1}\right)^N$, we know

$$\widehat{S_j} = \sum_{i=1}^{N} \widehat{f_i}\,\widehat{\mathcal{C}_{ij}} \quad j = 1, \dots, M\ ,$$

and thus we can compute the best approximation in the Euclidean norm (using for example a singular value decomposition [**23**]) of the set $\{\widehat{f_1}, \dots, \widehat{f_N}\}$. Then, we can use the Shannon sampling theorem, see [**25**], on the grid to reconstruct an approximation of the source $F = \{f_1, \dots, f_N\}$.

This approximation will be better if M is large, which was supposed to be an obvious requirement at the beginning of this section. If $M = N$, one could try to approach a Cramer system with square matrix $[\widehat{\mathcal{C}_{ij}};\ i, j = 1, \dots, N]$, and perform a stable numerical method, like the P.O.C.S. iterative method (see [**18**]).

Our idea in the next section is to use a completely different approach. We write the RTC transform $V_{\mathcal{C}}f$ in terms of the classical radon transform Rf, and then perform an inversion when the data is considered as complete.

3. Inversion in the case of complete data. We recall the expression

(3.1)

$$V_{\mathcal{C}}f(y) = \int_{\mathbf{R^n}} f(x', x_n)\,\mathcal{C}\left(\frac{x_n}{x_n + y_n}\,y' + \frac{y_n}{x_n + y_n}\,x'\right)\frac{dx'dx_n}{(x_n + y_n)^{n-1}}\ ,$$

of the RTC, and we suppose now that the data is complete, in the sense that $V_{\mathcal{C}}f(y)$ is known on the whole half-space $\mathbf{R}^n_+$. We have the two following results for $n \geq 3$.

Theorem 3.1. *Let $f \in L^2(\mathbf{I})$, where I is a closed subset of the open half-space $\mathbf{R}^n_-$, $n \geq 3$, and let $\mathcal{C} \in L^1(\mathbf{\Pi}) \cap L^2(\mathbf{\Pi})$. Suppose that the n and $n-1$ dimensional radon transform Rf and $R\mathcal{C}$ exist. Denote $f_0(x) = f(x)/(-x_n)^{n-2}$ and, for sake of simplicity, also denote $V_{\mathcal{C}}f$ the extension of $V_{\mathcal{C}}f$ vanishing for negative values of y_n. For $\rho \in \mathbf{R}$, $s \in \mathbf{R}$ and $\theta \in \mathbf{S}_+$, we have the following relations*:

$$\rho \widehat{V_{\mathcal{C}}f}(\rho\theta) = \widehat{f}_0(\rho\theta) \star_\rho \widehat{\mathcal{C}}(\rho\theta') \tag{3.2}$$

$$\mathcal{F}^*_\rho \left(\rho \widehat{V_{\mathcal{C}}f}(\rho\theta)\right)(s) = Rf_0(\theta, s)\; R\mathcal{C}\left(\theta'/|\theta'|,\, s/|\theta'|\right)/|\theta'|. \tag{3.3}$$

Remarks. The existence of the two radon transforms is the minimal requirement, and demands, as it is the case in practice, localized and regular enough functions f and $\mathcal{C}$.

For $n = 2$ the formulae may also be valid, but in the sense of tempered distributions.

Theorem 3.2. *With the same hypothesis as in Theorem* 3.1*, let us suppose that $\mathcal{C} = \chi_{\mathbf{K}}$, where the code $\mathbf{K}$ is a compact set of Π with the following property:*

There exist real numbers r such that each affine hyperplane intersecting $\mathbf{B}(O, r)$ also intersects $\mathbf{K}$ on a set of positive $n-1$ dimensional Lebesgue measure.

Denote α the supremum of the positive real numbers r. Suppose also that f is compactly supported in $\mathbf{I}$, and denote β the infimum of the real numbers b such that $\mathbf{I} \subset \mathbf{B}(a, b)$, with $a = (0, \dots, b)$.

Then, if $\alpha > \beta$, the source function f can be reconstructed.

Remark. In practice the code should be an $n-1$-dimensional set, somehow centered at the origin, as in the following examples.

The source may also be centered along the x_n-axis, near the origin, and the hypothesis roughly means that, the code $\mathbf{K}$ (with "diameter" α) has to be "larger" than the source support $\mathbf{I}$ (with "diameter" β).

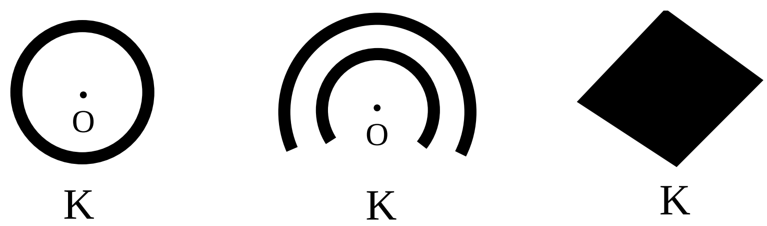

FIGURE 10.

A large code will provide a large α and, in fact, lead to a much better posed problem (which is in general severely ill-posed, as we will see in the proof).

The same kinds of results may, of course, be obtained with a wider class of compact codes and with weaker hypotheses on the support $\mathbf{I}$ of f.

Proof of Theorem 3.1. Let us proceed in four steps. We shall first prove that $V_{\mathcal{C}}f \in L^2(\mathbf{R}^n)$, then calculate its Fourier transform and, finally, establish the two equations.

i) Applying the Cauchy-Scwharz inequality yields

$$|V_{\mathcal{C}}f(y', y_n)|^2 \le \int_{\mathbf{R^n}} f^2(x', x_n)\, \mathcal{C}^2\left(\frac{-x_n}{-x_n+y_n}\, y' + \frac{y_n}{-x_n+y_n}\, x'\right) \frac{dx'dx_n}{(-x_n+y_n)^{2n-2}} \ .$$

Thus, thanks to the Fubini rule,

$$\int_{\mathbf{R}^{n-1}} |V_{\mathcal{C}}f(y', y_n)|^2 \, dy' \le \int_{\mathbf{R}^n} f^2(x', x_n) \left[\int_{\mathbf{R}^{n-1}} \mathcal{C}^2\left(\frac{-x_n}{-x_n+y_n}\, y' + \frac{y_n}{-x_n+y_n}\, x'\right)\right] \frac{dx'dx_n}{(-x_n+y_n)^{2n-2}} \ .$$

Then, after the change of variable

$$Y = \frac{-x_n}{-x_n+y_n}\, y' + \frac{y_n}{-x_n+y_n}\, x' \ ,$$

we get, as there exists a positive real number γ such that $-x_n > \gamma$ in the integral (**I** is closed),

$$\int_{\mathbf{R}^{n-1}} |V_{\mathcal{C}} f(y', y_n)|^2 \, dy' \le \|\mathcal{C}\|_2^2 \int_{\mathbf{R}^n} f^2(x', x_n) \frac{dx' dx_n}{(-x_n)^{n-1}(-x_n + y_n)^{n-1}} \; .$$

This provides the inequality

$$|V_{\mathcal{C}} f|_2^2 \le \|\mathcal{C}\|_2^2 \int_{\mathbf{R}^n} f^2(x', x_n) \left[\int_{\mathbf{R}_+} \frac{dy_n}{(-x_n + y_n)^{n-1}} \right] \frac{dx' dx_n}{(-x_n)^{n-1}} \; ,$$

which is finite, as soon as $-x_n > \gamma > 0$, $n \ge 3$, and $f \in L^2$.

ii) Let us calculate at the point ξ the Fourier transform of $V_{\mathcal{C}f}$ along y. We first take the Fourier transform along y'. This yields

$$\begin{aligned} &\mathcal{F}_{y'} \, (V_{\mathcal{C}} f)(\xi', y_n) \\ &\qquad = \int_{\mathbf{R}^n} f(x', x_n) \, e^{-i(y_n/x_n)\xi'.x'} \, \widehat{\mathcal{C}}\left(\left(1 - \frac{y_n}{x_n}\right) \xi' \right) \frac{dx' dx_n}{(-x_n)^{n-1}} \; . \end{aligned}$$

This integral can be written

$$- \int_{\mathbf{R}^n} f_0(x', x_n) \, e^{-i(y_n/x_n)\xi'.x'} \, \widehat{\mathcal{C}}\left(\left(1 - \frac{y_n}{x_n}\right) \xi' \right) \frac{dx' dx_n}{x_n} \; ,$$

and thus

$$\mathcal{F}_{y'} \, (V_{\mathcal{C}} f) \, (\xi', y_n) = - \int_{\mathbf{R}} \mathcal{F}_{x'} \, (f_0) \left(\frac{y_n}{x_n} \xi', x_n \right) \widehat{\mathcal{C}} \left(\left(1 - \frac{y_n}{x_n}\right) \xi' \right) \frac{dx_n}{x_n} \; .$$

Denote $t = y_n/x_n$. As $dx_n/x_n = -dt/t$, we get

$$\mathcal{F}_{y'} \, (V_{\mathcal{C}} f) \, (\xi', y_n) = \int_{\mathbf{R}} \mathcal{F}_{x'} \, (f_0) \left(t\xi', \frac{y_n}{t} \right) \widehat{\mathcal{C}} \left((1 - t) \, \xi' \right) \frac{dt}{t} \; .$$

Finally take the Fourier transform along y_n at the point ξ_n, to obtain

$$\widehat{V_{\mathcal{C}} f}(\xi', \xi_n) = \int_{\mathbf{R}} \widehat{f}_0(t\xi', t\xi_n) \, \widehat{\mathcal{C}} \left((1 - t) \, \xi' \right) \, dt \; ,$$

which reads

$$\widehat{V_{\mathcal{C}} f}(\xi) = \int_{\mathbf{R}} \widehat{f}_0(t\xi) \, \widehat{\mathcal{C}} \left((1 - t) \, \xi' \right) \, dt \; .$$

iii) Write now ξ in polar coordinates, $\xi = \rho\theta$ with $\rho \in \mathbf{R}$ and $\theta \in \mathbf{S}_+$. We have

$$\rho\,\widehat{V_{\mathcal{C}}f}(\xi', \xi_n) = \int_{\mathbf{R}} \widehat{f}_0(t\rho\theta)\,\widehat{\mathcal{C}}\,(\rho\theta' - t\rho\theta')\;\rho\,dt = \int_{\mathbf{R}} \widehat{f}_0(t\theta)\,\widehat{\mathcal{C}}\,((\rho - t)\theta')\;dt\,.$$

This means that the Fourier transform of $V_{\mathcal{C}}f$ along the direction θ reads like the 1D convolution of the Fourier transform of f_0 along the same direction, and of the Fourier transform of $\mathcal{C}$ along the same direction projected on the hyperplane Π.

This is the expected formula 3.2:

$$\rho\widehat{V_{\mathcal{C}}f}(\rho\theta) = \widehat{f}_0(\rho\theta)\;\star_\rho\;\widehat{\mathcal{C}}(\rho\theta')\,.$$

iv) After having recalled the classical results on the radon transform collected in Appendix B, take the adjoint Fourier transform along ρ on each side of equation 3.2. This gives, in the sense of the L^2-space, and for $s \in \mathbf{R}$,

$$\mathcal{F}_\rho^*\left(\rho\widehat{V_{\mathcal{C}}f}(\rho\theta)\right)(s) = \mathcal{F}_\rho^*\left(\widehat{f}_0(\rho\theta)\right)(s)\;\mathcal{F}_\rho^*\left(\widehat{\mathcal{C}}(\rho\theta')\right)(s)\,.$$

In terms of radon transform, we have

$$\mathcal{F}_\rho^*\left(\widehat{f}_0(\rho\theta)\right)(s) = Rf_0(\theta, s)\,,$$

while

$$\begin{aligned}
\mathcal{F}_\rho^*\left(\widehat{\mathcal{C}}(\rho\theta')\right)(s) &= \int_{\mathbf{R}} \widehat{\mathcal{C}}(\rho\theta')\,e^{i\rho s}\,d\rho \\
&= \int_{\mathbf{R}} \widehat{\mathcal{C}}(\rho|\theta'|\phi)\,e^{i\rho s}\,d\rho \\
&= \frac{1}{|\theta'|}\int_{\mathbf{R}} \widehat{\mathcal{C}}(\rho\phi)\,e^{i\rho s/|\theta'|}\,d\rho\,,
\end{aligned}$$

where $\phi = \theta'/|\theta'|$ is an element of the unit sphere of $\mathbf{R}^{n-1}$. (In fact θ' varies in the unit ball of $\mathbf{\Pi}$, which is the projection of $\mathbf{S}_+$, the positive half unit sphere of $\mathbf{R}^n$, while ϕ varies in the corresponding $(n-2)$-dimensional unit sphere).

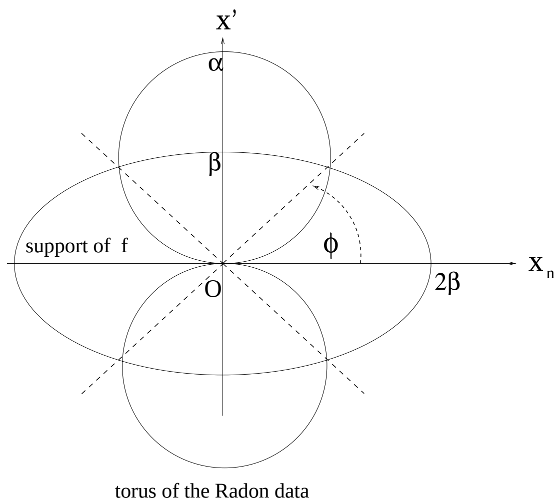

FIGURE 11.

Thus, applying again the Fourier-slice formula, we get $\mathcal{F}_\rho^*\left(\widehat{\mathcal{C}}(\rho\theta')\right)(s) = R\mathcal{C}\left(\theta'/|\theta'|\,,\, s/|\theta'|\right)/|\theta'|$. Finally, formula (3.3) is proved:

$$\mathcal{F}_\rho^*\left(\rho\widehat{V_{\mathcal{C}}f}(\rho\theta)\right)(s) = Rf_0(\theta,s)\; R\mathcal{C}\left(\theta'/|\theta'|\,,\, s/|\theta'|\right)/|\theta'| \tag{3.4}$$

Proof of Theorem 3.2. Because of formula (3.3), $Rf_0(\theta,s)$ is known on the set of points (θ,s) such that $R\mathcal{C}\left(\theta'/|\theta'|\,,\, s/|\theta'|\right) \neq 0$. The hypothesis on $\mathbf{K}$ means that $R\mathcal{C}(\theta,s) \neq 0$ for all $\theta \in \mathbf{S}_+$ and for all $s < \alpha$, so that the previous set of point consists in the torus $\{(\theta,s) \in \mathbf{S}_+ \times \mathbf{R}, -\alpha|\theta'| < s < \alpha|\theta'|\}$. As, by hypothesis, the radon transform of f_0 (and f) has its support included in the ellipsoid $\{(\theta,s) \in \mathbf{S}_+ \times \mathbf{R},\, \beta(-1+|\theta_n|\}) < s < \beta(1+|\theta_n|)\}$, it can be computed in the following cone of $\mathbf{R}^n$: $\{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R},\, |x'| > \tan\phi\, |x_n|\}$, where ϕ is the angle represented in Figure 11.

In other words, f can be reconstructed by using a method of inversion of the Limited angle radon transform, as soon as $Rf_0(\theta,s)$ is known on a cone, as well as the analytic function $\widehat{f_0}$ (see again the Fourier-slice formula in Appendix B).

The illustration also shows that a larger code provides a larger set of data for the radon transform, so that the problem tends to be well posed.

4. Perspectives. As shown by the previous results, numerical computations should be heavy and unstable in the most general case of 3D bodies. Nevertheless, the RTC is of theoretical interest. Indeed, taking its origin in a simple and natural geometric operation, as a luminous source filtered by clouds, it leads to several integral transform problems, corresponding to many fields of applications.

Many things remain to be cleared up. For instance the range and the null-space of the quasi-convolution, or of the RTC in the cases of incomplete data.

Appendix

A. Review of the Fourier transform. The Schwartz space $\mathcal{S}(\mathbf{R})$ of rapidly decreasing functions is the linear space of infinitely derivable functions for which

$$|f|_{k,l} = \sum_{k' \le k} \sup_{x \in \mathbf{R}} |x^{k'} D^l f(x)|$$

is finite for all positive multi-indices k, l ($D^l f$ is the derivative of order l).

The space $\mathcal{S}'(\mathbf{R})$ is the space of linear functionals T over $\mathcal{S}(\mathbf{R})$ which are continuous in the sense that a positive constant C and two multi-indices k, l exist such that $|Tf| \le C|f|_{k,l}$ for all $f \in \mathcal{S}(\mathbf{R})$.

We also have to recall the simple inversion formula $\mathcal{F}^*\mathcal{F}f = f$, which is valid on $\mathcal{S}(\mathbf{R})$, or, for instance only if both f and $\mathcal{F}f$ belong to L^1.

Moreover, both previous operators can be extended to the L^2 space of square integrable functions, where the Fourier transform becomes an isometry.

Furthermore, the extension can be made on $\mathcal{S}'(\mathbf{R})$, the space of tempered distributions, where it is defined by $\langle \widehat{T}, f\rangle = \langle T, \widehat{f}\rangle$, for all tempered distributions T and all Schwartz test-functions f.

B. Review of the radon transform. Let us denote $\Gamma_a f$ the function $x \mapsto f(x-a)$. Then we have $R\Gamma_a f(\theta, s) = Rf(\theta, s - a.\theta)$. If Rf is known on the unit half cylinder $\mathbf{Z}_+$, the radon transform is said to be with "complete data". In this case f is recoverable by a well-known inversion formula (see [**16**], [**23**]).

Otherwise the transform is "with incomplete data" ([**13**], [**20**], [**23**], [**26**], and many other works). In particular, we are interested in this paper in the "Limited angle radon transform" (LART) ([**10**], [**23**]), where $Rf(\theta, s)$ is known for $(\theta, s) \in \Omega \times \mathbf{R}$, where Ω is a strict subset of the half unit sphere $\mathbf{S}_+$.

In this case, a compactly supported L^2 function f is recoverable, because $\widehat{f}$ is analytic, thanks to the following Fourier-slice formula, indicating that $\widehat{f}$ is known on a cone.

Fourier-slice formula:

$$\widehat{R_\theta f}(\lambda) = \widehat{f}(\lambda\theta), \quad \forall\, \theta \in \mathbf{S}_+, \quad \forall\, \lambda \in \mathbf{R}.$$

We also need the connected equality

$$\mathcal{F}^*_\lambda(\widehat{f}(\lambda\theta))\,(s) = Rf(\theta, s), \quad \forall\, \theta \in \mathbf{S}_+, \quad \forall\, \lambda \in \mathbf{R}, \quad \forall\, s \in \mathbf{R},$$

valid, for example, for compactly supported L^2 functions f.

Remark. The recovering of the function f from its LART is an ill-posed problem that has provided a lot of studies ([**23**], for example). In fact, the Singular Value Decomposition (SVD) gives better numerical results than the extrapolation of band-limited analytic function. Moreover, the singular functions of the LART can be explicitly calculated, as in [**19**].

REFERENCES

1. D. Barret et al., Astrophys. J. Lett. **405** (1993), L59.

2. C.-A. Berenstein, B.-A. Taylor and A. Yger, *Sur quelques formules explicites de déconvolution*, J. Optics **14** (1983), 75–82.

3. C.-A. Berenstein and A. Yger, *Le problème de la déconvolution*, J. Funct. Anal. **54** (1983), 113–160.

4. J. Brunol, *Reconstruction d'images tomographiques en médecine nucléaire*, Thèse d'état de Physique, Paris **11** (1979).

5. J. Brunol, N. de Beaucoudrey, J. Fonroget and S. Lowenthal, *Imagerie tridimensionnelle en gammagraphie*, Optics Comm. **25** (1978), 163–168.

6. J. Brunol and J. Fonroget, *Bruit multiplex en gammagraphie par codage*, Optics Comm. **22** (1977), 301–306.

7. J. Brunol, R. Sauneuf, and J.-P. Gex, *Micro coded aperture imaging applied to laser plasma diagnosis*, Optics Comm. **31** (1979), 129–134.

8. J.-F. Crouzet, *La Gammagraphie par ouverture de codage*, Ph.D. Thesis, Université Bordeaux I, 33400 Talence, France, 1996.

9. J-F. Crouzet and L.-B. Klebanov, *The quasi-convolutions and the applications to the coded images*, Zap. Nauchn. Sem. St. Petersburg Otdel. Math. Inst. Steclov (POMI) **244** (1997), 167–180.

10. M.-E. Davison, *The ill-conditioned nature of the limited angle tomography problem*, SIAM J. Appl. Math. **43** (1983), 428–448.

11. N. de Beaucoudrey and L. Garnero, *Off-axis multi-slit coding for tomographic X-ray imaging of microplasma*, Optics Comm. **49** (1984), 103–107.

12. N. de Beaucoudrey, L. Garnero and J.-P. Hugonin, *Imagerie tomographique par codage et reconstruction*, Traitement Signal **5** (1988), 209–221.

13. A. Faridiani, F. Keinert, F. Natterer, E.L. Ritman, and K.T. Smith, *Local and global tomography*, in *Signal Processing*, Part II, IMA Math. Appl. **23** (1990).

14. J. Fonroget, Y. Belvaux, and S. Lowenthal, *Fonction de transfert de modulation d'un système de Gammagraphie holographique*, Optics Comm. **15** (1975).

15. G.-R. Gindi, R.-G. Paxman and H.-H. Barrett, *Reconstruction of an object from its coded image and object constraints*, Appl. Optics **23** (1984), 851–856.

16. S. Helgason, *The radon transform*, Birkhauser, Boston, 1980.

17. T.-P. Kohman, *Coded-aperture x-ray or gamma-ray telescope with least-squares image reconstruction*, Rev. Sci. Instrum. **60** (1989).

18. R. Lenz, *3-D reconstruction with a projection onto convex sets algorithm*, Optics Comm. **57** (1986), 21–25.

19. A.-K. Louis, *Incomplete data problems in X-Ray computerized tomography: Singular value decomposition of the limited angle transform*, Numer. Math. **48** (1986), 251–262.

20. P. Maass, *The interior radon transform*, SIAM J. Appl. Math. **52** (1992), 710–724.

21. P. Mandrou et al., *title?*, Astronom. and Astrophys. Suppl. **97** (1993).

22. R.-S. May, Z. Akcasu, and G.-F. Knoll, *Gamma-ray imaging with stochastic apertures*, Appl. Optics **13** (1974).

23. F. Natterer, *The mathematics of computerized tomography*, John Wiley & Sons, New York, (1986).

24. N. Ohyama, T. Honda, and J. Tsujiuchi, *Tomogram reconstruction using advanced coded aperture imaging*, Optics Comm. **36** (1981), 434–438.

25. A. Papoulis, *The Fourier integral and its applications*, McGraw-Hill Book Co., Inc., New York, 1962.

26. E. Todd Quinto, *Singular value decompositions and inversion methods for the exterior radon transform and a spherical transform*, J. Math. Anal. Appl. **95** (1983).

Laboratoire A.C.S.I.O.M., Université Montpellier 2, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France
E-mail address: crouzet@math.univ-montp2.fr