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# NUMERICAL ANALYSIS OF AN UNBOUNDED OPERATOR ARISING FROM AN ELECTROMAGNETIC INTERIOR SCATTERING PROBLEM

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ABSTRACT. In this paper a 1-D singular integral equation motivated by the well-known singular volume integral equation associated with electromagnetic interior scattering is considered. In the 3-D case the kernel (the dyad Green's function) is $O(R^{-3})$ and in the present 1-D case the kernel is $O(R^{-1})$. The numerical solution is obtained by using a simple Nyström method. The mapping properties of the integral operator and the numerical integral operators are studied in various (Hölder) subspaces of $C([a,b])$. Convergence theorems for the numerical integral operators as well as for the numerical solutions are proved.

**1. Introduction.** For safety and health reasons, it is of considerable interest to assess the short- and long-term effects of electromagnetic (EM) radiation on people working near radars and other similar EM-wave-generating devices. Research to understand this can be classified as epidemiological, experimental and numerical. In numerical electromagnetic dosimetry one is led naturally to the problem of solving the Maxwell's equations inside a highly inhomogeneous and highly dispersive body. One of the solution approaches is to solve an equivalent problem in the frequency domain using a volume integral equation formulation.

Mathematically, in the time-harmonic case, if the body ($V$) is incident by an electric field $\mathbf{E}^{\mathbf{i}}(\mathbf{r})$ and if $\mathbf{E}(\mathbf{r})$ is the total electric field inside the body ($\mathbf{r} \in V$), then the scattered field $\mathbf{E}^{\mathbf{s}}(\mathbf{r})$, defined through the relation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathbf{s}}(\mathbf{r}) + \mathbf{E}^{\mathbf{i}}(\mathbf{r}), \tag{1}$$

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has the form [13]

$$\mathbf{E}^{\mathbf{s}}(\mathbf{r}) = \left(\mathbf{I} + \frac{1}{k_o^2}\,\nabla\nabla\cdot\right)\int_V g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(\mathbf{r}')\,dV', \tag{2}$$

in which $\mathbf{F}(\mathbf{r}) := \tau(\mathbf{r})\mathbf{E}(\mathbf{r})$, $\tau(\mathbf{r}) := k^2(\mathbf{r}) - k_o^2$, $g_3(\mathbf{r},\mathbf{r}') := (e^{jk_o r}/(4\pi r))$ and $r := |\mathbf{r}-\mathbf{r}'|$. Here $j = \sqrt{-1}$ and $k_o$ and $k(\mathbf{r})$ are the wave numbers associated with free space and the body, respectively. One can manipulate equation (2) to avoid the differentiations outside the integral and rewrite equation (1) as a vector integral equation of the form (see Appendix A):

$$\begin{aligned}\overline{\mathbf{A}}(\mathbf{r})\mathbf{F}(\mathbf{r}) &- \int_V g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(\mathbf{r}')\,dV' \\ &- \frac{1}{k_o^2}\int_V \nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')(\mathbf{F}(\mathbf{r}') - \mathbf{F}(\mathbf{r}))\,dV' = \mathbf{E}^{\mathbf{i}}(\mathbf{r})\end{aligned} \tag{3}$$

where

$$\overline{\mathbf{A}}(\mathbf{r}) = \frac{1}{\tau(\mathbf{r})}\,\overline{\mathbf{I}} - \frac{1}{k_o^2}\int_{\partial V}\nabla' g_3(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}'\,dS'.$$

For $\mathbf{r}$ on the boundary, the integral in the dyad $\overline{\mathbf{A}}(\mathbf{r})$ is singular but exists as an improper integral [**7**]. The integrand in the second volume integral in equation (3), $\nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')$, has a singularity of type $O(|\mathbf{r}-\mathbf{r}'|^{-3})$. Previous attempts [**9**], [**10**] to solve equation (3) using either a Moment method or a Nyström method have been successful only for a restricted class of parameters. This motivated us to analyze a 1-D integral equation analogous to (3). Starting with a scalar version of equation (2) and again eliminating the differentiations outside the integral, we arrive (Appendix B) at an integral equation analogous to equation (3):

$$\lambda(t)\phi(t) - K_g\phi(t) - K_b\phi(t) = \chi_1(t) \tag{4}$$

for $t \in (a,b)$ and $\chi_1 \in C((a,b))$. Here

$$\lambda(t) = \frac{1}{\tau(t)} - \frac{1}{k_o^2}\,[g_t(t,a) - g_t(t,b)] \tag{5}$$

$$K_g\phi(t) = \int_a^b g(t,s)\phi(s)\,ds \tag{6}$$

$$K_b\phi(t) = \frac{1}{k_o^2}\int_a^b g_{ss}(t,s)\,[\phi(s) - \phi(t)]\,ds. \tag{7}$$

Since $\nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')$ has a singularity of type $O(|\mathbf{r}-\underline{\mathbf{r}}'|^{-3})$, we seek a function $g(t,s)$ whose second derivatives have a singularity of type $O(|s-t|^{-1})$. We picked $g_{ss}(t,s)$ to be

$$g_{ss}(t,s) = \frac{1}{|t-s|}. \tag{8}$$

(As pointed out by one of the reviewers, a more appropriate choice would be $g_{ss}(t,s) = 1/(t-s)$, as its 3-D counterpart is Cauchy integrable. The singularity resulting from the present choice is more severe.) A function $g(t,s)$ that satisfies equation (8) is (see Appendix C)

$$g(t,s) = |t-s|\,(\ln|t-s| + A_g) + B_g, \tag{9}$$

where $A_g$ and $B_g$ are arbitrary constants. It has properties analogous to those of $g_3(\mathbf{r},\mathbf{r}')$:

$$g(t,s) = g(s,t)$$
$$g_t(t,s) = -\,g_s(t,s).$$

In this paper we will assume

$$A_g \leq -\ln(b-a) - 1/2 \quad \text{and} \quad B_g \leq (b-a)/8. \tag{10}$$

These will guarantee the nonpositivity of some quantities we will use later. It should be mentioned that the problem considered here is not equivalent to the 1-D Maxwell's equations wherein $\mathbf{E}$ is dependent on only one spatial dimension. It is well known that the Green's function for the 1-D Maxwell's equations is much better behaved.

As a preliminary analysis of equation (4), we numerically solved it using several variants of the Nyström method:

1. Product method with extrapolation at the end intervals [**1**],

2. Gauss-Legendre method [**1**], and

3. A Simple Nyström method (or Simple method, for short) in which uniformly spaced integration points and uniform weights are used. This method will be explained in detail in Section 4.

In each case, apparent convergence was obtained. The main result of this paper is a rigorous mathematical proof of the convergence of the Simple method for this problem.

In Section 2 we will consider a special case of equation (4) and put it into perspective. In Section 3 we will investigate the properties of an operator $K$ that arises from this special case. The numerical method (the Simple method) used to solve the problem will be defined in Section 4, and some preliminary properties of the associated numerical integral operators $K_n$ will be explored. Several convergence theorems for the numerical integral operators $K_n$ will be proven in Section 5. In Section 6, a convergence theorem for the numerical solution of the special case is proved. A generalization of the special case to the general case given by equation (4) is described in Section 7. The results of some numerical experiments to ascertain the convergent rates of the Simple method and other similar methods will be given in Section 8. Finally, we will conclude with some closing remarks in Section 9.

**2. A special case.** Since $g(t,s)$ is continuous in $[a,b]\times[a,b]$, the operator $K_g$ in equation (4) is compact. If the operator $K_b$ were absent, this would be a classical and well-studied problem. The special feature of the problem here is the presence of the operator $\mathrm{K}_b$ (which we will show is noncompact). This motivated us to study a problem involving only the operator $K_b$ first. The knowledge gained is then used to study the full equation. Hence, with $K_g = 0$ and $g(t,s)$ defined by equation (9), equation (4) becomes

$$\lambda(t)\phi(t) - \gamma_1 \int_a^b |t-s|^{-1}[\phi(s)-\phi(t)]\,ds = \chi_1(t), \qquad t\in(a,b). \tag{11}$$

Here the constant $\gamma_1 := (1/k_o^2)$ is positive and $\lambda(t)$ becomes (see Appendix B)

$$\lambda(t) = \frac{1}{\tau(t)} - \gamma_1\{\ln[(t-a)(b-t)] + 2(A_g+1)\}.$$

The quantity within the braces is negative for all $t\in(a,b)$ due to the assumption on $A_g$ in equation (10). As the "contrast" $\tau(t)$, which gives a measure of the difference between the electromagnetic properties of the body and that of free space, is normally bounded above zero, we have

$$\lambda(t) \geq c > 0 \tag{12}$$

for all $t \in (a,b)$. We further assume the contrast $\tau(t)$ is continuous in $(a,b)$. It follows that $\lambda(t) \in C((a,b))$ and approaches $\infty$ as $t \to a$ and as $t \to b$. Under these assumptions, equation (11) can be transformed to an integral equation of the second kind:

$$\phi(t) - \gamma(t) \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\,ds = \chi(t), \quad t \in (a,b). \tag{13}$$

Here both $\gamma(t) := \gamma_1/\lambda(t)$ and $\chi(t) := \chi_1(t)/\lambda(t) \in C((a,b))$. Moreover, $\gamma(t) > 0$ in $(a,b)$ and approaches 0 as $t \to a$ and as $t \to b$. Hence, we may assume $\gamma(t) \in C([a,b])$. If we formally define the operator $K$ by

$$K\phi(t) := \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\,ds, \tag{14}$$

then equation (13) can be written in the familiar operator notation as

$$(I - \gamma K)\phi = \chi. \tag{15}$$

It will be seen later that $K\phi(t) \in C([a,b])$ if $\phi(t) \in C([a,b])$ and satisfies an additional mild condition. Hence we will assume $\chi(t) \in C([a,b])$ also. This in turn places some obvious limitations on $\chi_1(t)$.

Our problem is then to analyze the numerical solution of this equation when, in particular, a simple Nyström method (to be described in Section 4 below) is used.

The problem being addressed here differs from conventional weakly singular integral equations in at least two fundamental ways. First, while equation (13) contains the difference term used in the well-known Singularity Subtraction method, namely,

$$K\phi(t) = \int_a^b k(t,s)[\phi(s) - \phi(t)]\,ds,$$

the subtracted term

$$\int_a^b k(t,s)\phi(t)\,ds = \phi(t) \int_a^b k(t,s)\,ds$$

in our case is divergent. This is in stark contrast to the conventional case where the subtracted term is and must be finite.

Second, for the weakly singular integral of the second kind

$$\phi(t) - \int_a^b k_w(t,s)\phi(s)\,ds = \chi(t)$$

or

$$(I - K_w)\phi = \chi$$

where $|k_w(t,s)| \leq C|s-t|^{\alpha-1}$, $0 < \alpha \leq 1$, $K_w$ is compact from $C([a,b]) \to C([a,b])$. Consequently, the analysis of a typical numerical method taking the form

$$(I - K_n)\phi_n = \chi \tag{16}$$

can be based on Anselone's Collectively Compact operators wherein the operators $K_n$ are each compact from $C([a,b]) \to C([a,b])$. (See, for example, [**1**]). Unfortunately, in our problem the operators are not compact, as we shall see below.

**3. Mapping properties of $K$.** We first investigate the mapping properties of the operator in equation (14):

$$K\phi(t) := \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\,ds.$$

We recall a function $f$ is *uniformly Hölder continuous of order* $\alpha$, $0 < \alpha \leq 1$, on an interval $[a,b]$ if there exists a constant $C$ such that

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all $x$ and $y$ in $[a,b]$. Define

$$C^{(0,\alpha)}([a,b]) : \begin{cases} \text{The space of all uniformly Hölder continuous} \\ \text{functions of order } \alpha \text{ on an interval } [a,b]. \end{cases}$$

The following properties of $C^{(0,\alpha)}([a,b])$ are well known:

**Proposition 1.** *For* $0 < \alpha < \beta \leq 1$,

1. $C^{(0,\beta)}([a,b]) \subset C^{(0,\alpha)}([a,b])$
2. $C^{(0,\beta)}([a,b])$ *is a subalgebra of* $C([a,b])$.
3. $C^{(0,\alpha)}([a,b])$ *is a Banach space under the norm*

$$\|f\|_\alpha = \|f\|_\infty + |f|_\alpha$$

*where*

$$|f|_\alpha := \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \,\middle|\, x \neq y \right\} \tag{17}$$

*is a semi-norm.*

4. *Imbedding maps* $I_{\beta,\alpha} : C^{(0,\beta)}([a,b]) \to C^{(0,\alpha)}([a,b])$ *are compact.*

*Proof.* See [**6**], [**3**]. □

**Corollary 2.** *For* $0 < \alpha < \beta \leq 1$, *the mapping* $I^{-1}_{\beta,\alpha}$ *from* $(C^{(0,\beta)}([a,b]), \|\cdot\|_\alpha)$ *onto* $(C^{(0,\beta)}([a,b]), \|\cdot\|_\beta)$ *is unbounded.*

*Proof.* Otherwise, the identity map $I_{\beta,\beta}$ would be compact on the infinite dimension space $C^{(0,\beta)}([a,b])$. □

**Corollary 3.** *For* $0 < \alpha < \beta \leq 1$, $C^{(0,\beta)}([a,b])$ *is not a Banach subspace of* $C^{(0,\alpha)}([a,b])$.

*Proof.* Else $I^{-1}_{\beta,\alpha}$ would be bounded, by the closed graph theorem [**5**], since $I_{\beta,\alpha}$ and therefore $I^{-1}_{\beta,\alpha}$ are closed operators. □

**Proposition 4.** *For* $0 < \alpha < \beta \leq 1$, $K : C^{(0,\beta)}([a,b]) \to C^{(0,\alpha)}([a,b])$ *is compact.*

*Proof.* Mimicking the steps in one of the proofs in [**3**], one can show that $K$ is bounded from $C^{(0,\beta)}([a,b]) \to C^{(0,\delta)}([a,b])$, where $\delta := (\alpha+\beta)/2$. Using the fact that the imbedding from $C^{(0,\delta)}([a,b]) \to C^{(0,\alpha)}([a,b])$ is compact, the proposition follows immediately. □

Unfortunately, classical Fredholm theory does not apply here because of the following observation.

**Proposition 5.** *$C^{(0,\alpha)}([a,b])$ is not invariant under $K$ for any $0 < \alpha \leq 1$.*

*Proof.* We first consider the case where $0 < \alpha < 1$. Without loss of generality, we may assume $[a,b] = [0,1]$. Since $\phi(t) := t^\alpha \in C^{(0,\alpha)}([0,1])$, it suffices to show that $K\phi(t) \notin C^{(0,\alpha)}([0,1])$. By manipulating the (improper) integrals involved, one can readily show that $K\phi(t) = I_1(t) + I_2(t)$, where

$$I_1(t) = \int_0^t (t-s)^{-1}[s^\alpha - t^\alpha]\, ds = [\psi(1) - \psi(1+\alpha)]t^\alpha$$
$$I_2(t) = \int_t^1 (s-t)^{-1}[s^\alpha - t^\alpha]\, ds = \frac{1}{\alpha} + t^\alpha \ln t - \frac{t^\alpha}{\alpha} + t^\alpha I_3(t).$$

Here $\psi(*)$ is the digamma function and

$$\begin{aligned} I_3(t) = \int_1^{1/t} \frac{x^\alpha - 1}{x(x-1)}\, dx &\leq \int_1^2 \frac{x^\alpha - 1}{x(x-1)}\, dx \\ &+ \int_2^\infty \frac{x^\alpha - 1}{x(x-1)}\, dx \leq \ln 2 + \frac{2^\alpha}{1-\alpha} \end{aligned}$$

is bounded for all $t \in [0,1]$. Now $I_1(t)$ clearly $\in C^{(0,\alpha)}([0,1])$. However,

$$\frac{|I_2(s) - I_2(0)|}{|0-s|^\alpha} = \Big| \ln s - \frac{1}{\alpha} + I_3(s) \Big|$$

is not bounded in the neighborhood of 0. Hence, $I_2(t)$ and, consequently, $K\phi(t) \notin C^{(0,\alpha)}([0,1])$.

Due to the term $1-\alpha$ in the denominator in the bound for $I_3(t)$, the above proof does not extend to the case $\alpha = 1$. For this latter case, it is convenient to assume $[a,b] = [0,1/2]$ and consider the function $\phi(t) = t/\sqrt{-\ln(t)}$, $t \in [0,1/2]$. Since $\phi(t)$ is continuously differentiable on $[0,1/2]$, $\phi(t) \in C^{(0,1)}([0,1/2])$. It suffices to show that $(K\phi(t) - K\phi(0))/t$ is unbounded in the neighborhood of 0. We have

$$K\phi(t) - K\phi(0) = I_4(t) + I_5(t)$$

where

$$I_4(t) = \int_0^t \left[\frac{\phi(s)-\phi(t)}{t-s} - \frac{\phi(s)-\phi(0)}{s-0}\right] ds$$

$$I_5(t) = \int_t^{1/2} \frac{t\phi(s)-s\phi(t)}{s(s-t)}\, ds.$$

Since $\phi(t) \in C^{(0,1)}([0,1/2])$, $I_4(t)/t$ is bounded. Define $\eta(t) := \sqrt{-\ln(t)}$, $t \in (0,1/2]$. Then $\phi(t) = t/\eta(t)$ in $(0,1/2]$, and

$$\begin{aligned}|I_5(t)/t| &= \left|\int_t^{1/2} \frac{t\phi(s)-s\phi(t)}{ts(s-t)}\, ds\right| \\ &= \left|\int_t^{1/2} \frac{\eta(t)-\eta(s)}{\eta(s)\eta(t)(s-t)}\, ds\right|.\end{aligned}$$

Since $\eta(s) > 0$ and monotone decreasing on $(0,1/2]$,

$$\begin{aligned}|I_5(t)/t| &\geq \frac{1}{\eta(t)^2}\int_t^{1/2} \frac{\eta(t)-\eta(s)}{(s-t)}\, ds \\ &= \frac{1}{\eta(t)^2}\int_t^{1/2} \frac{\eta(t)^2-\eta(s)^2}{(s-t)[\eta(t)+\eta(s)]}\, ds \\ &\geq \frac{1}{2\eta(t)^3}\int_t^{1/2} \frac{\eta(t)^2-\eta(s)^2}{(s-t)}\, ds \\ &= \frac{1}{2\eta(t)^3}\int_1^{1/2t} \frac{\ln(s)}{s-1}\, ds.\end{aligned}$$

Finally, applying the l'Hospital's rule twice, we obtain

$$\lim_{t\to 0^+} \frac{1}{2\eta(t)^3}\int_1^{1/2t} \frac{\ln(s)}{s-1}\, ds = \lim_{t\to 0^+} \frac{2}{3}\,\eta(t) = \infty.$$

This completes the proof. $\square$

For theoretical as well as numerical reasons, it is desirable to consider operators $L$ whose range is contained in its domain, so that $L^2$, for example, is defined. This leads us to the following spaces. For $0 \leq \alpha < 1$, we define

$$X_\alpha := \bigcup \{C^{(0,\beta)}([a,b]) \mid \alpha < \beta \leq 1\}.$$

In particular, $X_0$ is the set of all functions defined on $[a, b]$ which are uniformly Hölder continuous of some order $\alpha \in (0, 1]$.

**Lemma 6.** *For $0 < \alpha < 1$, $(X_\alpha, \|\cdot\|_\alpha)$ is a normed linear space and is invariant under $K$.*

*Proof.* $(X_\alpha, \|\cdot\|_\alpha)$ is a linear subspace of $(C^{(0,\alpha)}([a, b]), \|\cdot\|_\alpha)$. The invariance follows from Proposition 4. □

While the semi-norm $|\cdot|_\alpha$ in equation (17) and hence the norm $\|\cdot\|_\alpha$ are defined for $\alpha \in (0, 1]$ on $C^{(0,\alpha)}([a, b])$, it is convenient (and also consistent) to define

$$\|f\|_0 := \|f\|_\infty, \quad f \in X_0.$$

**Lemma 7.** *$(X_0, \|\cdot\|_0)$ is a normed linear space and is invariant under $K$.*

*Proof.* $(X_0, \|\cdot\|_0) = (X_0, \|\cdot\|_\infty)$ is a linear subspace of $(C([a, b]), \|\cdot\|_\infty)$, and the invariance follows again from the last proposition. □

While not germane to our discussion here, it can be shown that the closure of $(X_\alpha, \|\cdot\|_\alpha)$ in $C^{(0,\alpha)}([a, b])$ is not $C^{(0,\alpha)}([a, b])$, even though $C^{(0,\alpha)}([a, b])$ contains $C^{(0,\beta)}([a, b])$ for all $\beta > \alpha$.

**Proposition 8.** *$K$ is unbounded on $(X_\alpha, \|\cdot\|_\alpha)$ for any $0 \le \alpha < 1$.*

*Proof.* Assume, without loss of generality, $[a, b] = [0, 1]$. One can then readily show that

$$\phi_n(t) := t^{(\alpha+1)/n}$$

is a bounded sequence in $X_\alpha$, but $\|K\phi_n\|_\alpha \to \infty$ as $n \to \infty$. □

**4. A Simple Nyström method and the associated numerical integral operators, $K_n$.** In this section we will state the Simple

(Nyström) method we used to find numerical solutions of equation (15) and investigate some of the properties of the associated numerical integral operators.

For each integer $n > 0$, we define a partition $P_n$ on the interval $[a, b]$ by partitioning the interval into $2^n$ equal subintervals. We associate with the partition $P_n$ the operator $K_n$ defined on $C([a, b])$ as follows.

$$K_n\phi(t) := \sum_{j=1}^{k_n} w_{n,j}\, g_{n,j}(t)\Delta_{n,j}\phi(t), \quad \phi \in C([a, b])$$

where

$$\begin{aligned} k_n &= 2^n \\ w_{n,j} &= (b-a)/k_n =: h_n \\ g_{n,j}(t) &= \begin{cases} |t - t^*_{n,j}|^{-1} & t \notin [t_{n,j-1}, t_{n,j}] \\ 2/h_n & t \in [t_{n,j-1}, t_{n,j}] \end{cases} \\ \Delta_{n,j}\phi(t) &= \phi(t^*_{n,j}) - \phi(t) \\ t_{n,j} &= a + jh_n, \quad j = 0, \dots, k_n \\ t^*_{n,j} &= (t_{n,j-1} + t_{n,j})/2, \quad j = 1, \dots, k_n. \end{aligned}$$

For the Simple method, we have chosen each weight $w_{n,j}$ associated with the $j$th subinterval in $P_n$ to be dependent only on the integer $n$ and not on $j$. More sophisticated choice for the weights is of course possible, but the resulting analysis would be more complicated. Also $K_n\phi$ is actually defined for any function $\phi$ that is merely defined on the interval $[a, b]$. However, here we are only interested in those functions that are at least continuous. It is obvious that if $\phi \in C([a, b])$, then so is $K_n\phi$. That is, $C([a, b])$ is invariant under $K_n$.

In the Simple method, the numerical solution $\phi_n$ to equation (15) is obtained by solving the approximating equation

$$(I - \gamma K_n)\phi_n = \chi. \tag{18}$$

By collocation at the $k_n$ midpoints $\{t^*_{n,j}\}$, the following system of linear equations is obtained

$$(I - \gamma K_n)\phi_n(t^*_{n,j}) = \chi(t^*_{n,j}), \quad j = 1, \dots, k_n \tag{19}$$

from which $\{\phi_n(t^*_{n,j})\}$ can be solved.

For comparison with the operator $K$, we will look at some properties of $K_n$. Unless specified otherwise, we will always assume $n$ is a positive integer in the following. Also, for later convenience, we introduce the following functions, each of which depends only on $n$:

$$\psi_n(t) := \sum_{j=1}^{k_n} w_{n,j}\, g_{n,j}(t).$$

We will first look at some properties of $K_n$ on $C([a,b])$ and then its properties on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$. As we have already noted, we have

**Proposition 9.** *$C([a,b])$ is invariant under $K_n$.*

Moreover, we have

**Proposition 10.** *$K_n$ is bounded on $C([a,b])$ with $\|K_n\|_\infty = 8\sum_{j=1}^{k_n-1} 1/(2j-1)$.*

*Proof.* From the definition of $K_n$, it follows immediately that

$$K_n\phi(t) = K_{n,1}\phi(t) + K_{n,2}\phi(t)$$

where

$$\begin{aligned} K_{n,1}\phi(t) &= \sum_{j=1}^{k_n} w_{n,j}\, g_{n,j}(t)\phi(t^*_{n,j}),\\ K_{n,2}\phi(t) &= -\,\psi_n(t)\phi(t). \end{aligned}$$

Now $K_{n,1}$ is compact and hence bounded on $C([a,b])$, because it has finite dimensional range. Since $g_{n,j}(t) \in C([a,b])$, so does $\psi_n(t)$. Hence $K_{n,2}$ is also bounded on $C([a,b])$. It follows that $K_n$ must be bounded on $C([a,b])$. To find the norm of $K_n$ on $C([a,b])$, let $t^* = (a+b)/2$. One can verify directly that

$$\|\psi_n\|_\infty = \psi_n(t^*).$$

For any $\phi \in C([a,b])$,

$$\begin{aligned}|K_n\phi(t)| &\le 2\|\phi\|_\infty \psi_n(t)\\ &\le 2\|\phi\|_\infty \psi_n(t^*).\end{aligned}$$

Hence $\|K_n\|_\infty \le 2\psi_n(t^*)$. Since $t^* \ne t^*_{n,j}$, $j = 1,\dots,k_n$, there exists $\phi_o \in C([a,b])$ such that $\|\phi_o\|_\infty = 1$ and $\phi_o(t^*) = -\phi_o(t^*_{n,j}) = 1$, $j = 1,\dots,k_n$. Thus, $K_n\phi_o(t^*) = 2\psi_n(t^*)$ and $\|K_n\|_\infty \ge 2\psi_n(t^*)$. It follows that

$$\|K_n\|_\infty = 2\psi_n(t^*).$$

By direct verification, one obtains $\psi_n(t^*) = 4\sum_{j=1}^{k_n-1} 1/(2j-1)$. Hence, $\|K_n\|_\infty = 8\sum_{j=1}^{k_n-1} 1/(2j-1)$. $\square$

**Corollary 11.** *$K_n$ is not a compact operator on $C([a,b])$.*

*Proof.* If $K_n$ were compact, then $K_{n,2} = K_n - K_{n,1}$ would also be compact, since $K_{n,1}$ is compact. Now $1/(\psi_n(t)) \in C([a,b])$ as $\psi_n(t)$ is bounded away from 0. Hence $K_{n,2}/(\psi_n(t)) = -I$ would be compact. This is impossible since the identity operator $I$ is not compact on $C([a,b])$. $\square$

Incidentally, the last proposition implies that if the method of successive approximation is applied to equation (15), it will likely fail as $n$ increases, since $\|K_n\|_\infty$ are not uniformly bounded.

We now turn our attention to the properties of $K_n$ on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$. Unlike the operator $K$, we have

**Proposition 12.** *$C^{(0,\alpha)}([a,b])$ is invariant under $K_n$ for $\alpha \in (0,1]$.*

*Proof.* It suffices to show that $g_{n,j}(\cdot)\Delta_{n,j}\phi(\cdot) \in C^{(0,\alpha)}([a,b])$ for any $\phi \in C^{(0,\alpha)}([a,b])$. If $\phi \in C^{(0,\alpha)}([a,b])$, then clearly $\Delta_{n,j}\phi \in C^{(0,\alpha)}([a,b])$. One can also show that $g_{n,j} \in C^{(0,1)}([a,b])$ and hence it belongs to $C^{(0,\alpha)}([a,b])$ for $\alpha \in (0,1]$. Finally, $g_{n,j}(\cdot)\Delta_{n,j}\phi(\cdot) \in C^{(0,\alpha)}([a,b])$, since the latter is an algebra. $\square$

To investigate the boundedness of $K_n$ on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$, it suffices to consider the individual components of $K_n$, leading us to define the following operators on $C^{(0,\alpha)}([a,b])$:

$$L_{n,j}\phi(t) := g_{n,j}(t)\,\Delta_{n,j}\phi(t), \quad j = 1,\dots,k_n.$$

Clearly, $K_n = \sum_{j=1}^{k_n} w_{n,j}\, L_{n,j}$.

**Lemma 13.** *$L_{n,j}$ is bounded on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$, for $j = 1,\dots,k_n$ and $\|L_{n,j}\|_\alpha \le 2\|g_{n,j}\|_\alpha$.*

*Proof.* For any $\phi \in C^{(0,\alpha)}([a,b])$, we have $|L_{n,j}\phi(t)| = |g_{n,j}(t)\Delta_{n,j}\phi(t)| \le 2\|g_{n,j}\|_\infty\|\phi\|_\infty$, for all $t \in [a,b]$. Hence $\|L_{n,j}\phi\|_\infty \le 2\|g_{n,j}\|_\infty\|\phi\|_\infty$. For any $s$ and $t \in [a,b]$,

$$\begin{aligned}|L_{n,j}\phi(s) - L_{n,j}\phi(t)| &= |g_{n,j}(s)\Delta_{n,j}\phi(s) - g_{n,j}(t)\Delta_{n,j}\phi(t)|\\ &\le |[g_{n,j}(s) - g_{n,j}(t)]\Delta_{n,j}\phi(s)|\\ &\quad + |g_{n,j}(t)[\Delta_{n,j}\phi(s) - \Delta_{n,j}\phi(t)]|\\ &\le 2\|\phi\|_\infty|g_{n,j}(s) - g_{n,j}(t)| + \|g_{n,j}\|_\infty|\phi(t) - \phi(s)|\\ &\le (2\|\phi\|_\infty|g_{n,j}|_\alpha + \|g_{n,j}\|_\infty|\phi|_\alpha)|s-t|^\alpha\end{aligned}$$

since both $g_{n,j}$ and $\phi \in C^{(0,\alpha)}([a,b])$. Hence $|L_{n,j}\phi|_\alpha \le 2\|\phi\|_\infty|g_{n,j}|_\alpha + \|g_{n,j}\|_\infty|\phi|_\alpha$. It follows that

$$\begin{aligned}\|L_{n,j}\phi\|_\alpha &= \|L_{n,j}\phi\|_\infty + |L_{n,j}\phi|_\alpha\\ &\le 2\|g_{n,j}\|_\infty\|\phi\|_\infty + 2\|\phi\|_\infty|g_{n,j}|_\alpha + \|g_{n,j}\|_\infty|\phi|_\alpha\\ &\le 2\|g_{n,j}\|_\alpha\|\phi\|_\alpha.\end{aligned}$$

The lemma is now proved. □

For $0 < \alpha < 1$, one can readily show that

$$\|g_{n,j}\|_\alpha = \|g_{n,j}\|_\infty + |g_{n,j}|_\alpha = \frac{2}{h_n} + \frac{\alpha^\alpha(1-\alpha)^{(1-\alpha)}}{h_n^{(1+\alpha)}}.$$

**Proposition 14.** *$K_n$ is bounded on $C^{(0,\alpha)}([a,b])$, $\alpha \in (0,1]$.*

*Proof.* This follows directly from the last lemma. □

While the restriction of the operator $K$ to $C^{(0,\alpha)}([a,b])$ allows it to be defined, the restriction of the operator $K_n$ to $C^{(0,\alpha)}([a,b])$ does not gain us much. We still have

**Proposition 15.** *$K_n$ is not compact on $C^{(0,\alpha)}([a,b])$ for $\alpha \in (0,1]$.*

*Proof.* The proof follows in exactly the same manner as that in the $C([a,b])$ case. $\square$

Because $K_n$ are not compact on $C^{(0,\alpha)}([a,b])$, we cannot make use of the theory of collectively compact operators to prove convergence of our numerical method. We do have some type of compactness as we will see in the next proposition. However, this is mainly of academic interest only.

**Proposition 16.** *$K_n : C^{(0,\alpha)}([a,b]) \to C^{(0,\beta)}([a,b])$ is compact if $0 < \beta < \alpha \leq 1$.*

*Proof.* if we denote by $A^{\alpha,\beta}$ the map $A : C^{(0,\alpha)}([a,b]) \to C^{(0,\beta)}([a,b])$, then $K_n^{\alpha,\beta} = I^{\alpha,\beta} K_n^{\alpha,\alpha}$. Since $K_n^{\alpha,\alpha}$ is bounded and $I^{\alpha,\beta}$ is compact, $K_n^{\alpha,\beta}$ is compact. $\square$

**5. Convergence theorems for $K_n$.** As we cannot make use of the theory of collectively compact operators to prove the convergence of our numerical method, we resort to proving it directly. We will prove some pointwise convergence properties of $K_n$ after establishing several preliminary lemmas. For convenience, we define

$$\begin{aligned}
\Delta\phi(t,s) &:= \phi(t) - \phi(s) \\
g_\alpha(t,s) &:= g_{ss}(t,s)|t-s|^\alpha \\
B(t,\delta) &:= \{s \in [a,b] \mid |s-t| < \delta\} \\
F_\phi^\alpha(t,s) &:= \begin{cases} (\phi(t) - \phi(s))/(|t-s|^\alpha) & \text{for } t \neq s \\ 0 & \text{for } t = s \end{cases}
\end{aligned}$$

where, as before,

$$g_{ss}(t,s) = \frac{1}{|t-s|}.$$

**Lemma 17.** *Let $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$ and $\alpha \in (0,\beta]$. Then*

$$|F_\phi^\alpha(t,s)| \le \|\phi\|_\beta \, |t-s|^{\beta-\alpha}$$

*for all $(t,s) \in [a,b] \times [a,b]$.*

*Proof.* This follows trivially from the definition of $C^{(0,\beta)}([a,b])$.

**Lemma 18.** *Let $t \in [a,b]$ and $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$. Then, for any $\varepsilon > 0$, there exists $\delta(\varepsilon,\phi) > 0$ independent of $t$ such that*

$$\Big| \int_{B_{(t,\delta')}} g_{ss}(t,s) \Delta\phi(s,t)\, ds \Big| < \varepsilon,$$

*for all $\delta' \le \delta$.*

*Proof.* Let $t \in [x_1, x_2] \subset [a,b]$. Then

$$\begin{aligned}\Big| \int_{x_1}^{x_2} g_{ss}(t,s)\Delta\phi(s,t)\, ds \Big| &\le \int_{x_1}^{x_2} |g_\beta(t,s) F_\phi^\beta(t,s)|\, ds \\ &\le \|\phi\|_\beta \int_{x_1}^{x_2} |t-s|^{\beta-1}\, ds \\ &\le \left[\frac{2^{1-\beta}\|\phi\|_\beta}{\beta}\right](x_2-x_1)^\beta.\end{aligned}$$

Thus, the required $\delta$ can be chosen as

$$\delta = \frac{1}{2}\left(\frac{\beta\varepsilon}{2^{1-\beta}\|\phi\|_\beta}\right)^{1/\beta}. \qquad \square$$

**Lemma 19.** *Let $t \in [a,b]$ and $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$. Then for any $\varepsilon > 0$, there exists $N$ (independent of $t$), $0 \le k_{N,1} < k_{N,2}$, and $\delta > 0$ such that*

$$\Big| S_n(\tau) := \sum_{j=k_{n,1}+1}^{k_{n,2}} w_{n,j}\, g_{n,j}(\tau)\Delta_{n,j}\phi(\tau) \Big| < \varepsilon$$

*for all* $\tau \in B(t,\delta)$ *and for all* $n \geq N$, *where*

$$a + k_{n,1}h_n = a + k_{N,1}h_N =: a_1$$
$$a + k_{n,2}h_n = a + k_{N,2}h_N =: b_1$$

*and*

$$B(t,\delta) \subset [a_1, b_1].$$

*Proof.* Let $\varepsilon > 0$ be given. Let $N = \max\{N_1, N_2, N_3\}$, where $N_1, N_2, N_3$ are specified below. For any positive integer $n$, define

$$I_{n,j} := \begin{cases} [t_{n,j-1}, t_{n,j}) & \text{if } 1 \leq j < k_n \\ [t_{n,j-1}, t_{n,j}] & \text{if } j = k_n \end{cases}$$

Since $[a,b]$ is the disjoint union of $\{I_{n,j}\}_{j=1}^{j=k_n}$, $t \in I_{n,j_n^*(t)}$ for a unique $j_n^*(t)$, $1 \leq j_n^*(t) \leq k_n$. We define $[a_1, b_1] := [t_{N,j_N^*(t)-1}, t_{N,j_N^*(t)}]$. It follows that $[a_1, b_1] \supseteq I_{n,j_n^*(t)}$ and $k_{n,1} < j_n^*(t) \leq k_{n,2}$ for all $n \geq N$. For ease of presentation, we assume $t \in (a_1, b_1)$. (If, for example, $t = t_{N,j_N^*-1}$ and $t \neq a$, we can increase $N$ by 1 and let $[a_1, b_1] := [t_{N,j_N^*-2}, t_{N,j_N^*}]$.) In the following, we will always assume $\tau \in (a_1, b_1) = (k_{n,1}h_n, k_{n,2}h_n)$ and $n \geq N$. Then

$$S_n(\tau) = S_{n,1}(\tau) + S_{n,2}(\tau) + S_{n,3}(\tau)$$

where

$$\begin{aligned} S_{n,1}(\tau) &= \sum_{k_{n,1} < j < j_n^*(\tau)} w_{n,j}\, g_{n,j}(\tau)\Delta_{n,j}\phi(\tau) \\ S_{n,2}(\tau) &= w_{n,j_n^*(\tau)}\, g_{n,j_n^*(\tau)}(\tau)\Delta_{n,j_n^*(\tau)}\phi(\tau) \\ S_{n,3}(\tau) &= \sum_{j_n^*(\tau) < j \leq k_{n,2}} w_{n,j}\, g_{n,j}(\tau)\Delta_{n,j}\phi(\tau). \end{aligned}$$

We shall show that each of the above can be made arbitrarily small uniformly for $\tau \in (a_1, b_1)$ for $n$ sufficiently large. Indeed,

$$\begin{aligned} |S_{n,2}(\tau)| &= 2|\phi(t^*_{n,j_n^*(\tau)}) - \phi(\tau)| \\ &\leq 2\|\phi\|_\beta\, |t^*_{n,j_n^*(\tau)} - \tau|^\beta \\ &\leq 2\|\phi\|_\beta\, h_N^\beta \\ &\leq \frac{\varepsilon}{3} \end{aligned}$$

provided

$$n \geq N_2 := \max\left\{ \left\lceil \frac{1}{\beta} \log_2 \left( \frac{6}{\varepsilon} (b-a)^\beta \|\phi\|_\beta \right) \right\rceil, 1 \right\}.$$

We will now derive a bound for $|S_{n,1}(\tau)|$. If $j_n^*(\tau) = k_{n,1} + 1$, then $S_{n,1}(\tau) = 0$. Otherwise, if $j_n^*(\tau) > k_{n,1} + 1$, then

$$\begin{aligned} |S_{n,1}(\tau)| &\leq h_n \sum_{k_{n,1} < j < j_n^*(\tau)} \frac{|\phi(t_{n,j}^*) - \phi(\tau)|}{|t_{n,j}^* - \tau|} \\ &\leq h_n \|\phi\|_\beta \sum_{k_{n,1} < j < j_n^*(\tau)} (\tau - t_{n,j}^*)^{\beta-1} \\ &= h_n \|\phi\|_\beta \sum_{m=0}^{L_N} (x + m h_n)^{\beta-1} \end{aligned}$$

where $x := \tau - t^*_{n, j_n^*(\tau)-1}$ and $L := j_n^*(\tau) - k_{n,1} - 2$. From the definitions of $t_{n,j}^*$ and $j_n^*(\tau)$, we must have $(h_n/2) \leq x \leq (3h_n/2)$. Let $y := (x/h_n)$. Then $(1/2) \leq y \leq (3/2)$ and

$$\begin{aligned} |S_{n,1}(\tau)| &\leq h_n^\beta \|\phi\|_\beta \sum_{m=0}^{L_n} (y+m)^{\beta-1} \\ &= h_n^\beta \|\phi\|_\beta \, y^{\beta-1} + h_n^\beta \|\phi\|_\beta \sum_{m=1}^{L_n} (y+m)^{\beta-1}. \end{aligned}$$

Since $\beta \in (0,1]$ and $(1/2) \leq y \leq (3/2)$,

$$\begin{aligned} |S_{n,1}(\tau)| &\leq 2^{1-\beta} h_n^\beta \|\phi\|_\beta + h_n^\beta \|\phi\|_\beta \sum_{m=1}^{L_n} m^{\beta-1} \\ &\leq 2^{1-\beta} h_n^\beta \|\phi\|_\beta + h_n^\beta \|\phi\|_\beta \left( 1 + \frac{1}{\beta} L_n^\beta \right). \end{aligned}$$

From $j_n^*(\tau) \leq k_{n,2}$ and $L_n = j_n^*(\tau) - k_{n,1} - 2$, we have

$$\begin{aligned} L_n h_n &< (k_{n,2} - k_{n,1}) h_n \\ &= b_1 - a_1, \end{aligned}$$

and so

$$\begin{aligned}|S_{n,1}(\tau)| &\le (2^{1-\beta}+1)h_n^{\beta}\|\phi\|_{\beta} + \frac{1}{\beta}\,\|\phi\|_{\beta}(b_1-a_1)^{\beta}\\ &\le \frac{\varepsilon}{3},\end{aligned}$$

provided

$$n \ge N_1 := \max\{N_{1,a}, N_{1,b}\}$$

where

$$\begin{aligned}N_{1,a} &= \max\left\{\left\lceil \frac{1}{\beta}\,\log_2\left(\frac{18}{\varepsilon}\,(b-a)^{\beta}\|\phi\|_{\beta}\right)\right\rceil, 1\right\}\\ N_{1,b} &= \max\left\{\left\lceil \frac{1}{\beta}\,\log_2\left(\frac{6}{\varepsilon\beta}\,(b-a)^{\beta}\|\phi\|_{\beta}\right)\right\rceil, 1\right\}.\end{aligned}$$

Estimating a bound for $|S_{n,3}(\tau)|$ is similar to that for $|S_{n,1}(\tau)|$, producing $N_3$ similar to $N_1$. The lemma is proved by choosing $\delta = \min\{|t-a_1|, |t-b_1|\}$. $\square$

*Remark.* It follows from the proof of the last lemma that

$$\left|\sum_{j=m_1}^{m_2} w_{n,j}\, g_{n,j}(\tau)\Delta_{n,j}\phi(\tau)\right| < \varepsilon,$$

as long as $k_{n,1}+1 \le m_1 \le m_2 \le k_{n,2}$.

**Proposition 20.** *For any* $\phi \in C^{(0,\beta)}([a,b])$, $\beta \in (0,1]$,

$$\lim_{n\to\infty} \|(K-K_n)\phi\|_{\infty} = 0.$$

*Proof.* Let $\varepsilon > 0$ be given. By the compactness of $[a,b]$, it suffices to show that for any $t \in [a,b]$, there exist $N_t$ and $\delta_t$ such that $|(K-K_n)\phi(\tau)| < \varepsilon$ for all $n > N_t$ and for all $\tau \in B(t,\delta_t)$. Now for any $\gamma_1$ and $\gamma_2$ with $0 \le \gamma_1 \le t-a$, and $0 \le \gamma_2 \le b-t$, and for any $m_1$ and $m_2$ with $1 \le m_1 < m_2 \le k_n$,

$$\begin{aligned}(K-K_n)\phi(\tau) &= T_1(\tau,a,t-\gamma_1) + T_1(\tau,t-\gamma_1,t+\gamma_2) + T_1(\tau,t+\gamma_2,b)\\ &\quad - T_2(\tau,n,1,m_1) - T_2(\tau,n,m_1+1,m_2)\\ &\quad - T_2(\tau,n,m_2+1,k_n)\end{aligned}$$

where

$$T_1(\tau, x, y) = \int_x^y g_{ss}(t,s)\Delta\phi(s,t)\,ds$$
$$T_2(\tau, n, m, k) = \sum_{j=m}^{k} w_{n,j}\, g_{n,j}(\tau)\Delta_{n,j}\phi(\tau).$$

It follows readily from the last two lemmas that there exist $N_1$ and $\delta_1 > 0$ such that by letting $\gamma_1 = t - t_{N_1,k_{N_1},1} \geq 0$ and $\gamma_2 = t_{N_1,k_{N_1},2} - t \geq 0$, we have

$$|T_1(\tau, t-\gamma_1, t+\gamma_2)| \leq \frac{\varepsilon}{3}$$
$$|T_2(\tau, n, k_{n,1}+1, k_{n,2})| \leq \frac{\varepsilon}{3}$$

for all $n \geq N_1$ and $\tau \in B(t, \delta_1)$. Let $f(t,s) := g_{ss}(t,s)\Delta\phi(s,t)$. Since $f(t,s)$ is continuous and therefore uniformly continuous on $[\,[t-\delta_1, t+\delta_1] \cap [a,b]\,] \times [a, t-\gamma_1]$, there exist $N_2$ and $\delta_2 > 0$ such that

$$\begin{aligned} &|T_1(\tau, a, t-\gamma_1) - T_2(\tau, n, 1, k_{n,1})| \\ &\qquad = \left| \sum_{j=1}^{k_{n,1}} \int_{t_{n,j-1}}^{t_{n,j}} [f(\tau,s) - f(\tau, t_{n,j}^*)]\,ds \right| \\ &\qquad \leq \frac{\varepsilon}{6} \end{aligned}$$

for all $n \geq N_2$ and $\tau \in B(t, \delta_2)$. Similarly, $N_3$ and $\delta_3 > 0$ exist such that

$$\begin{aligned} &|T_1(\tau, t+\gamma_2, b) - T_2(\tau, n, k_{n,2}+1, k_n)| \\ &\qquad = \left| \sum_{j=k_{n,2}+1}^{k_n} \int_{t_{n,j-1}}^{t_{n,j}} [f(\tau,s) - f(\tau, t_{n,j}^*)]\,ds \right| \\ &\qquad \leq \frac{\varepsilon}{6} \end{aligned}$$

for all $n \geq N_3$ and $\tau \in B(t, \delta_3)$. The proposition is proved by letting $N_t = \max\{N_1, N_2, N_3\}$ and $\delta_t = \min\{\delta_1, \delta_2, \delta_3\}$. □

It follows immediately from the definition of $X_0$ that

**Corollary 21.** *For any* $\phi \in X_0$,

$$\lim_{n\to\infty} \|(K-K_n)\phi\|_\infty = 0.$$

*Remark.* If $\phi \in C^{(0,\beta)}([a,b]), \beta \in (0,1]$, then we already know that both $K\phi$ and $K_n\phi \in C^{(0,\alpha)}([a,b])$, for $\alpha \in (0,\beta)$. One may suspect the convergence in the last proposition is also true in $C^{(0,\alpha)}([a,b])$, i.e., $\lim_{n\to\infty} \|(K-K_n)\phi\|_\alpha = 0$ for $0<\alpha<\beta$. Unfortunately, we have not been able to prove it.

**6. Convergence theorems for** $I-\gamma K_n$**.** In studying the properties of the operator $A_n := I - \gamma K_n$ associated with the Simple method (see equation (19)), the matrix $B_n$ defined below will be useful. For any $\gamma \in C([a,b])$ with $\gamma(t) > 0$ in $(a,b)$, let

$$c_{n,j}(t) := \gamma(t) w_{n,j}\, g_{n,j}(t) > 0 \quad \text{on } (a,b), \quad j = 1, \dots, k_n$$

and

$$b_n(t) := 1 + \sum_{j=1}^{k_n} c_{n,j}(t) > 0 \quad \text{on } [a,b].$$

Then

$$B_n := (b_{i,j})$$

where

$$b_{i,j} = b_n(t^*_{n,i})\delta_{i,j} - c_{n,j}(t^*_{n,i}), \quad 0 \le i,j \le k_n,$$

$B_n$ is simply the discretized version of $A_n$ in the sense that

$$B_n(\phi(t^*_{n,i}))_{i=1,\dots,k_n} = (A_n\phi(t^*_{n,i}))_{i=1,\dots,k_n}.$$

**Lemma 22.** $B_n$ *is invertible.*

*Proof.* Let $\Lambda_i := \sum_{j\ne i} c_{n,j}(t^*_{n,i})$, $i = 1, \dots, k_n$. One can directly verify that $b_{i,i} = 1 + \Lambda_i$, $i = 1, \dots, k_n$. Thus, applying Gerschgorin Circle theorem [**12**], all eigenvalues of $B_n$ are contained in the union of the disks $|z - b_{i,i}| \le \Lambda_i$, $1 \le i \le k_n$. It follows that all the eigenvalues must have absolute values $\ge 1$. Hence $B_n$ is invertible. $\square$

**Proposition 23.** *Let $\gamma \in C([a,b])$ and $\gamma(t) > 0$ on $(a,b)$. Then, for each positive integer $n$, $A_n$ maps $C(J)$ 1–1 onto $C(J)$, where $J := [a,b]$.*

*Proof.* $A_n$ is clearly defined on $C(J)$. For a given $\chi \in C(J)$, we can define, because of the invertibility of $B_n$,

$$\vec{\chi}_n := (\chi(t^*_{n,1}), \ldots, \chi(t^*_{n,N}))^t$$
$$(\phi^\chi_i) := B_n^{-1}\,\vec{\chi}_n.$$

Then it can readily be shown that $A_n\phi = \chi$, where

$$\phi(t) = b_n(t)^{-1}\left[\chi(t) + \sum_{j=1}^{k_n} c_{n,j}(t)\phi^\chi_j\right]. \tag{20}$$

Hence $A_n$ is onto. If $A_n\phi = 0$, then $A_n\phi(t^*_{n,i}) = 0$, $i = 1, \ldots, k_n$. Since $B_n$ is invertible, $\phi(t^*_{n,i}) = 0$, $i = 1, \ldots, k_n$. Subsequently, $\phi(t) = 0$, since $b_n > 0$. □

**Corollary 24.** *$A_n^{-1}$ is bounded for each $n$.*

*Proof.* Clearly $A_n = I - \gamma K_n$ is bounded on the Banach space $C([a,b])$. The boundedness of $A_n^{-1}$ is a consequence of the Open Mapping theorem [**2**]. □

Other properties of the matrix $B_n$ that we will need are contained in the following lemmas.

**Lemma 25.** *$B_n$ is irreducible.*

*Proof.* Because $B_n$ is a full matrix with no nonzero entries, it is irreducible [**12**].

**Lemma 26.** *$B_n^{-1} > 0$, i.e., all entries are positive.*

*Proof.* $B_n$ is real, irreducible, diagonally dominant with

$$b_{i,j} = \begin{cases} < 0 & i \neq j \\ > 0 & i = j. \end{cases} \tag{21}$$

The lemma now follows from a theorem in Varga [**12**, p. 85]. $\square$

**Lemma 27.** *Each row-sum of $B_n$ is 1.*

*Proof.* The $i$th row-sum of $B_n$ is $\sum_{j=1}^{N} b_{i,j}$. From the definition of $b_{i,j}$, each is seen to be one.

*Remark.* Since the entries $b_{i,j}$ of $B_n$ do not all have the same signs,

$$\|B_n\|_\infty = \max_{1\le i\le k_n} \sum_{1=j}^{k_n} |b_{i,j}| \neq 1.$$

**Lemma 28.** *If each row-sum of a nonsingular $n\times n$ matrix $A$ is 1, then its inverse has the same property.*

*Proof.* Let $\mathbf{e}$ be the vector whose components are all ones. Then $A\mathbf{e}=\mathbf{e}$. Therefore, $A^{-1}\mathbf{e}=\mathbf{e}$. (This short proof was suggested by one of the reviewers.) $\square$

**Corollary 29.** $\|B_n^{-1}\|_\infty = 1$ *for all positive integer $n$.*

*Proof.* This follows from the last lemma and the fact that all entries in $B_n^{-1}$ are positive.

**Proposition 30.** $(A_n^{-1})_{n=1}^\infty$ *is uniformly bounded on $C([a,b])$.*

*Proof.* Let $\chi\in C([a,b])$. Then, using the notations in equation (20), we have

$$A_n^{-1}\chi(t) = b_n(t)^{-1}[\chi(t) + \vec{C}_n(t)B_n^{-1}\,\vec{\chi}_n]$$

where

$$\vec{C}_n(t) := (c_{n,1}(t),\dots,c_{n,N}(t)),\quad N=k_n.$$

From the definition of $b_n(t)$, it follows immediately that

$$|b_n(t)^{-1}| < 1$$

and

$$\|b_n(t)^{-1}\,\vec{C}_n(t)\|_1 \le 1$$

for all $t \in [a,b]$ and $n > 0$. Also,

$$\begin{aligned}\|B_n^{-1}\,\vec{\chi}_n\|_\infty &\le \|B_n^{-1}\|_\infty \|\vec{\chi}_n\|_\infty \\ &= \|\vec{\chi}_n\|_\infty \\ &\le \|\chi\|_\infty.\end{aligned}$$

It readily follows that

$$\|A_n^{-1}\,\chi\|_\infty \le 2\|\chi\|_\infty$$

for each $\chi$ in $C([a,b])$ and for all $n > 0$. Hence $\|A_n^{-1}\|_\infty \le 2$ for all $n$.
□

**Theorem 31.** *Let $\chi \in C([a,b])$ and assume $(I - \gamma K)\phi = \chi$ has a unique solution $\phi \in X_0$. For each positive integer $n$, let $\phi_n$ be the solution of*

$$A_n\phi_n := (I - \gamma K_n)\phi_n = \chi.$$

*Then*

$$\|\phi - \phi_n\|_\infty \longrightarrow 0$$

*as $n \to \infty$.*

*Proof.* Following the standard arguments, we have

$$\begin{aligned}0 &= (I - \gamma K)\phi - (I - \gamma K_n)\phi_n \\ &= \phi - \phi_n - \gamma(K\phi - K_n\phi_n) \\ &= \phi - \phi_n - \gamma(K\phi - K_n\phi + K_n\phi - K_n\phi_n) \\ &= \phi - \phi_n - \gamma((K - K_n)\phi + K_n(\phi - \phi_n)) \\ &= (I - \gamma K_n)(\phi - \phi_n) - \gamma(K - K_n)\phi \\ &= A_n(\phi - \phi_n) - \gamma(K - K_n)\phi.\end{aligned}$$

Hence,

$$\phi - \phi_n = \gamma A_n^{-1}(K - K_n)\phi,$$

and

$$\|\phi - \phi_n\|_\infty \le C\|A_n^{-1}\|_\infty \|(K - K_n)\phi\|_\infty.$$

The theorem follows from the uniform boundedness of $A_n^{-1}$ and the pointwise convergence of $K_n$ to $K$. $\square$

*Remark*. It would be valuable to have regularity results on the solution $\phi(t)$ based on the regularity of the input function $\chi(t)$, analogous to the results of Giraud [**4**] or Schneider [**11**]. These results would greatly strengthen the last theorem. Presently, we have to make an unverifiable assumption on the solution. Our conjecture is that if $\chi(t) \in C^{(0,\alpha)}([a, b])$, $\alpha \in [0, 1)$, then the solution $\phi(t)$, if it exists, $\in X_\alpha$.

**7. A convergence theorem for the general case.** We now consider the general case given by equation (4). It is repeated here using a slightly different notation ($K_b = \gamma_1 K$):

$$\lambda(t)\phi(t) - K_g\phi(t) - \gamma_1 K\phi(t) = \chi_1(t) \tag{22}$$

for $t \in (a, b)$ and $\chi_1 \in C((a, b))$. Again, $\gamma_1 = 1/k_o^2$ and

$$\begin{aligned} \lambda(t) &= \frac{1}{\tau(t)} - \frac{1}{k_o^2}\,[g_t(t, a) - g_t(t, b)] \\ K_g\phi(t) &= \int_a^b g(t, s)\phi(s)\, ds \\ K\phi(t) &= \int_a^b g_{ss}(t, s)[\phi(s) - \phi(t)]\, ds. \end{aligned}$$

Defining

$$K_1\phi(t) := k_o^2 \int_a^b g(t, s)(\phi(s) - \phi(t))\, ds,$$

we can rewrite equation (22) in a form similar to equation (11):

$$\tilde{\lambda}(t)\phi(t) - \gamma_1 K_c\phi(t) = \chi_1(t) \tag{23}$$

where $K_c = K + K_1$ and $\tilde{\lambda}(t) = \lambda(t) - \int_a^b g(t, s)\, ds$. Because of the assumptions on $A_g$ and $B_g$ given in (10), we have again

$$\tilde{\lambda}(t) > c > 0 \quad \text{on } (a, b).$$

As in the special case, the general problem reduces to solving a problem of the form:

$$(I - \gamma_2 K_c)\phi = \chi_2, \tag{24}$$

where $\gamma_2 := \gamma_1/\tilde{\lambda}(t)$ and $\chi_2 := \chi_1(t)/\tilde{\lambda}(t)$. Defining $K_{c,n}, \tilde{A}_n, \tilde{c}_{n,j}, \tilde{b}_n$ and $\tilde{B}_n$, the counterparts of $K_n, A_n, c_{n,j}, b_n$ and $B_n$, by

$$\begin{aligned}
K_{c,n}\phi(t) &:= \sum_{j=1}^{k_n} w_{n,j}\,[k_o^2 g(t, t^*_{n,j}) + g_{n,j}(t)]\Delta_{n,j}\phi(t)\\
\tilde{A}_n\phi(t) &:= \phi(t) - \gamma_2(t)K_{c,n}\phi(t)\\
\tilde{c}_{n,j}(t) &:= \gamma_2(t)w_{n,j}\,[k_o^2 g(t, t^*_{n,j}) + g_{n,j}(t)]\\
\tilde{b}_n(t) &:= 1 + \sum_{j=1}^{k_n} \tilde{c}_{n,j}(t)\\
\tilde{B}_n &:= (\tilde{b}_{i,j}) := (\tilde{b}_n(t^*_{n,i})\delta_{i,j} - \tilde{c}_{n,j}(t^*_{n,i}))
\end{aligned}$$

we can prove the following

**Theorem 32.** *Let $\chi_2 \in C([a,b])$ and assume $(I - \gamma_2 K_c)\phi = \chi_2$ has a unique solution $\phi \in X_0$. For each positive integer $n$, let $\phi_n$ be the solution of*

$$\tilde{A}_n\phi_n := (I - \gamma_2 K_{c,n})\phi_n = \chi.$$

*Then*

$$\|\phi - \phi_n\|_\infty \longrightarrow 0$$

*as $n \to \infty$.*

The proof of this theorem is similar to Theorem 31 for the special case as $K_{c,n}, \tilde{A}_n, \tilde{c}_{n,j}, \tilde{b}_n$ and $\tilde{B}_n$ have properties similar to that of $K_n, A_n, c_{n,j}, b_n$, and $B_n$, respectively. In particular, since $g(t,s)$ is continuous on $[a,b]\times[a,b]$, one easily obtains the pointwise convergence of $K_{c,n}$ to $K_c$ in $X_0$. The remark following Theorem 31 clearly applies here also.

**8. Numerical estimation of the convergence rate.** As remarked earlier, we presently do not have regularity results on the solution $\phi(t)$

of equation (4) based on the regularity of the input function $\chi_1(t)$. This makes the estimation of the convergence rate of the Simple method difficult. However, to get some idea of the convergence rate we have conducted some numerical experiments.

In the Simple method we have used the simplest numerical integration algorithm possible to simplify the convergence proofs. However, any numerical integration algorithm that makes the approximate

$$\int_a^b f(t,s)\,ds \approx \sum_{j=1}^{n} w_j\, f(t,s_j)$$

would have worked. For comparison, we considered two additional methods that involved numerical integration algorithms of this type. The first alternate method, which we will call the Extended Simpson method, consists of using, for its numerical integration algorithm, the fourth order Simpson's rule in the interior intervals and a fourth order extrapolative open formula for the first and last intervals. The second alternate method, which we will call the Gauss-Legendre method, uses the Gauss-Legendre numerical integration algorithm. Both of these numerical integration algorithms are well-documented in [**8**].

We considered three problems with the following characteristics:

- Problem I: The true solution is $\phi(t) = 1 + (t - 1/2)^2 + (t - 1/2)^3$
- Problem II: The true solution is $\phi(t) = \sqrt{t}$
- Problem III: The input function is $\chi_1(t) = \cos(10t)$.

We considered these problems on the interval $[0, 1]$. The coefficients in the integral equation are arbitrarily chosen as if a non-magnetic body with relative permittivity 70 is incident by a 1-GHz wave. The result of applying the three methods (Simple, Extended Simpson, and Gauss-Legendre), to each of the three problems are summarized in Tables 1–3. The order is calculated using the formula

$$\log\left(\frac{\mathrm{RMS}_n}{\mathrm{RMS}_{n-1}}\right)\Big/\log\left(\frac{n-1}{n}\right),$$

where $\mathrm{RMS}_n$ is the root-mean-square error between the calculated solution $\phi_n(t)$ and the exact solution. For Problems I and II, the $\mathrm{RMS}_n$'s are evaluated at the collocation points $\{t_{n,j}\}$. As we do

TABLE 1. Order estimation: $\phi(t) = 1 + (t-1/2)^2 + (t-1/2)^3$.

| | Simple | | Extended Simpson | | Gauss-Legendre | |
|---|---|---|---|---|---|---|
| $N$ | RMS | Order | RMS | Order | RMS | Order |
| 21 | 0.299245E-02 | | 0.179426E-01 | | 0.106354E-03 | |
| 29 | 0.160181E-02 | 1.94 | 0.904380E-02 | 2.12 | 0.561217E-04 | 1.98 |
| 39 | 0.896201E-03 | 1.96 | 0.321361E-02 | 3.49 | 0.322635E-04 | 1.87 |
| 53 | 0.489313E-03 | 1.97 | 0.114327E-02 | 3.37 | 0.174202E-04 | 2.01 |
| 69 | 0.290498E-03 | 1.98 | 0.488913E-03 | 3.22 | 0.968257E-05 | 2.23 |
| 93 | 0.161246E-03 | 1.97 | 0.205825E-03 | 2.90 | 0.446552E-05 | 2.59 |
| 125 | 0.903678E-04 | 1.96 | 0.992015E-04 | 2.47 | 0.166996E-05 | 3.33 |
| 167 | 0.516632E-04 | 1.93 | 0.538077E-04 | 2.11 | 0.967447E-06 | 1.88 |

not have an analytic solution for Problem III, we used an apparently converged solution, $n = 999$, for the exact solution and used the integral equation to interpolate the values of the approximate solution at the collocation points of the 'exact' solution.

The order of convergence for each method depends on the problem. For Problem I the convergence rates for the three methods (Simple, Extended Simpson and Gauss-Legendre) are approximately (averaging orders in rows 4 to 7 in the tables) 1.97, 3.16 and 2.54, respectively. Not unexpected is the fact that the order of convergence here is different from the theoretical order of convergence (based on sufficiently smooth integrands) of the respective numerical quadrature used. Nevertheless, the Gauss-Legendre method is most accurate in this case, as its root-mean-square error is 1 to 2 orders of magnitude smaller than the other two methods.

For Problem II, the order of convergence for the three methods are approximately 1.46, 1.02 and 1.46, respectively. These orders are much smaller than the corresponding orders in Problem I, especially for the Extended Simpson and the Gauss-Legendre methods. Moreover, the estimated order of the Extended Simpson method is less than that of the Simple method. This is unexpected. The difference between Problem I and Problem II is the regularity of the solution $\phi(t)$. In Problem I, $\phi(t) \in C^{\infty}([0,1])$, whereas in Problem II, $\phi(t) \in C^{(0,1/2)}([0,1])$. This suggests the regularity of the solution may play an important role in determining the order of convergence of the method.

Finally, in Problem III, the order of convergence for the three methods

TABLE 2. Order estimation: $\phi(t) =$ square root $(t)$.

| | Simple | | Extended Simpson | | Gauss-Legendre | |
|---|---|---|---|---|---|---|
| $N$ | RMS | Order | RMS | Order | RMS | Order |
| 21 | 0.112546E-01 | | 0.250719E-01 | | 0.563858E-04 | |
| 29 | 0.694025E-02 | 1.50 | 0.647886E-02 | 4.19 | 0.608720E-04 | -0.24 |
| 39 | 0.448682E-02 | 1.47 | 0.360581E-02 | 1.98 | 0.478940E-04 | 0.81 |
| 53 | 0.286456E-02 | 1.46 | 0.297206E-02 | 0.63 | 0.328247E-04 | 1.23 |
| 69 | 0.195050E-02 | 1.46 | 0.229246E-02 | 0.98 | 0.224827E-04 | 1.43 |
| 93 | 0.126451E-02 | 1.45 | 0.161527E-02 | 1.17 | 0.141634E-04 | 1.55 |
| 125 | 0.824122E-03 | 1.45 | 0.110573E-02 | 1.28 | 0.877121E-05 | 1.62 |
| 167 | 0.542340E-03 | 1.44 | 0.749686E-03 | 1.34 | 0.541797E-05 | 1.66 |

TABLE 3. Order estimation: $\chi_1(t) = \cos(10t)$.

| | Simple | | Extended Simpson | | Gauss-Legendre | |
|---|---|---|---|---|---|---|
| $N$ | RMS | Order | RMS | Order | RMS | Order |
| 21 | 0.120568E+01 | | 0.120448E+02 | | 0.277724E+01 | |
| 29 | 0.696564E+00 | 1.70 | 0.341448E+01 | 3.91 | 0.118370E+01 | 2.64 |
| 39 | 0.448696E+00 | 1.48 | 0.116724E+01 | 3.62 | 0.624652E+00 | 2.16 |
| 53 | 0.295290E+00 | 1.36 | 0.411297E+00 | 3.40 | 0.355010E+00 | 1.84 |
| 69 | 0.209759E+00 | 1.30 | 0.247087E+00 | 1.93 | 0.231757E+00 | 1.62 |
| 93 | 0.144163E+00 | 1.26 | 0.173976E+00 | 1.18 | 0.150264E+00 | 1.45 |
| 125 | 0.100035E+00 | 1.24 | 0.126691E+00 | 1.07 | 0.101824E+00 | 1.32 |
| 167 | 0.703408E-01 | 1.22 | 0.916024E-01 | 1.12 | 0.709499E-01 | 1.25 |

are approximately 1.29, 1.90 and 1.56, respectively. A noticeable difference between this problem and the previous two are the relatively large RMS value in this problem. This is unexpected since the input function $\chi_1(t) \in C([0,1])$ in this case, but only $\in C((0,1))\backslash C([0,1])$ (by direct verification) in the previous two. Similarly, results were obtained (not shown here) for Problem III when we used a less oscillatory input $\chi_1(t) = \cos(t)$ instead.

These numerical experiments tend to show (1) the order of convergence of the Simple method is in the neighborhood of 1.5, (2) one can generally (but not always) increase the order of convergence by using higher order numerical quadrature, and (3) more analyses are required to truly understand the convergence properties of numerical solutions of the integral equation that we discussed here.

**9. Conclusion.** In this paper we analyzed the numerical solution of the singular integral equation given in equation (4):

$$\lambda(t)\phi(t) - K_g\phi(t) - K_b\phi(t) = \chi_1(t) \tag{25}$$

using the Simple Nyström method described in Section 4. The distinctive feature of this equation is the integral operator $K_b = \gamma_1 K$ where

$$K\phi(t) = \int_a^b |t-s|^{-1}[\phi(s) - \phi(t)]\, ds.$$

We studied the mapping properties of the operator $K$ and found that the space $(X_0, \|\cdot\|_\infty)$ of all uniformly Hölder continuous functions, despite not being a Banach space, is a natural setting to study the unbounded operator $K$, as it $(X_0)$ is invariant under $K$.

We also studied the mapping properties of the numerical integral operators $K_n$ that arise from a Simple Nyström method. It is found that $K_n$ are bounded on $C([a,b])$ and (therefore) on $X_0$, but they are not compact on $C([a,b])$. Nevertheless, we proved a pointwise convergence theorem of $K_n$ to $K$ on $(X_0, \|\cdot\|_\infty)$. Using this and other properties of $K_n$, we proved, under appropriate conditions, the convergence of the numerical solutions of the singular integral equation (25) to its actual solution.

The convergence theorems we have given will carry over to the slightly more general case

$$K\phi(t) = \int_a^b \frac{M(t,s)}{|t-s|}\,[\phi(s) - \phi(t)]\, ds$$

in which $0 < M(t,s) \in C([a,b]\times[a,b])$, as the positivity of $\tilde{c}_{n,j}(t)$ is retained. Whether or not these theorems are true for more general $M(s,t)$ remains to be seen.

There are many other interesting issues yet to be resolved, utmost of which are the regularity of the solution of this singular integral equation and its effects on the order of convergence of the simple Nyström method or other numerical methods.

## Appendix A

Starting with equation (2),

$$\mathbf{E^s}(\mathbf{r}) = \left(\mathbf{I} + \frac{1}{k_o^2}\,\nabla\nabla\cdot\right)\int_V g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(r')\,dV'$$

one can move (with careful manipulations) the differentiations under the integral sign, resulting readily in a vector integral equation of the form [**13**]

$$\begin{aligned}\mathbf{E^s}(\mathbf{r}) = &\int_V g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(\mathbf{r}')\,dV'\\ &+\frac{1}{k_o^2}\int_{V-V_e(\mathbf{r})}\nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(\mathbf{r}')\,dV'\\ &+\frac{1}{k_o^2}\int_{V_e(\mathbf{r})}\nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')(\mathbf{F}(\mathbf{r}')-\mathbf{F}(\mathbf{r}))\,dV'\\ &+\frac{1}{k_o^2}\int_{\partial V_e(\mathbf{r})}\nabla' g_3(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}'\,dS'\mathbf{F}(\mathbf{r}),\end{aligned}$$

where $V_e(\mathbf{r})$ is a sub-volume of $V$ containing $\mathbf{r}$. There are different ways to pick $V_e(\mathbf{r})$, depending on the application. If $V_e(\mathbf{r})$ is taken to be $V$ for all $\mathbf{r}$, then this is equation becomes

$$\begin{aligned}\mathbf{E^s}(\mathbf{r}) = &\int_V g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(\mathbf{r}')\,dV'\\ &+\frac{1}{k_o^2}\int_{\partial V}\nabla' g_3(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}'\,dS'\mathbf{F}(\mathbf{r})\\ &+\frac{1}{k_o^2}\int_{V}\nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')(\mathbf{F}(\mathbf{r}')-\mathbf{F}(\mathbf{r}))\,dV'.\end{aligned}$$

In terms of the total field $\mathbf{E}(\mathbf{r})$ or, equivalently, $\mathbf{F}(r) = \tau(r)\mathbf{E}(\mathbf{r})$, the equation $\mathbf{E}(\mathbf{r}) - \mathbf{E^s}(\mathbf{r}) = \mathbf{E^i}(\mathbf{r})$ becomes

$$\begin{aligned}\overline{\mathbf{A}}(\mathbf{r})\mathbf{F}(\mathbf{r}) - &\int_V g_3(\mathbf{r},\mathbf{r}')\mathbf{F}(r')\,dV'\\ &-\frac{1}{k_o^2}\int_{V}\nabla'\nabla' g_3(\mathbf{r},\mathbf{r}')(\mathbf{F}(\mathbf{r}')-\mathbf{F}(\mathbf{r}))\,dV' = \mathbf{E^i}(\mathbf{r})\end{aligned} \tag{26}$$

where

$$\overline{\mathbf{A}}(\mathbf{r}) = \frac{1}{\tau(\mathbf{r})}\,\bar{\mathbf{I}} - \frac{1}{k_o^2}\int_{\partial V} \nabla' g_3(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}'\,dS'.$$

## Appendix B

The derivation of equation (4) is formally outlined below. We assume $\phi \in C^{(0,\alpha)}([a,b])$ for some $\alpha > 0$. We also assume

$$\int_a^b g_{ss}(t,s)\phi(s)\,ds \quad \text{is divergent}$$

but

$$\int_a^b g_{ss}(t,s)[\phi(s)-\phi(t)]\,ds \quad \text{is convergent,}$$

and $g_t(t,s) = -g_s(t,s)$. We start with the equation

$$\frac{1}{\tau(t)}\,\phi(t) = \phi_s(t) + \phi^{\text{inc}}(t) \tag{27}$$

where

$$\phi_s(t) = \left(1 + \gamma_1 \frac{d^2}{dt^2}\right)\int_a^b g(t,s)\phi(s)\,ds. \tag{28}$$

The goal is to remove the differentiations outside the integral. Concentrating on the second integral on the right side of equation (28), we have

$$\begin{aligned}
\frac{d^2}{dt^2}\int_a^b g(t,s)\phi(s)\,ds &= \frac{d}{dt}\int_a^b g_t(t,s)\phi(s)\,ds \\
&= -\frac{d}{dt}\int_a^b g_s(t,s)\phi(s)\,ds \\
&= -\frac{d}{dt}\int_a^b \frac{\partial}{\partial s}\{g(t,s)\phi(s)\} - g(t,s)\phi_s(s)\,ds \\
&= -\frac{d}{dt}\,[g(t,s)\phi(s)]_{s=a}^{s=b} + \frac{d}{dt}\int_a^b g(t,s)\phi_s(s)\,ds \\
&= -\,[g_t(t,s)\phi(s)]_{s=a}^{s=b} - \int_a^b g_s(t,s)\phi_s(s)\,ds.
\end{aligned}$$

Concentrating on the second term, we have

$$\begin{aligned}
&-\int_a^b g_s(t,s)\phi_s(s)\,ds \\
&\quad= -\int_a^b g_s(t,s)\frac{\partial}{\partial s}\left\{\phi(s)-\phi(t)\right\}ds \\
&\quad= -\int_a^b \frac{\partial}{\partial s}\left(g_s(t,s)\{\phi(s)-\phi(t)\}\right) - g_{ss}(t,s)\{\phi(s)-\phi(t)\}\,ds \\
&\quad= -\left[g_s(t,s)\{\phi(s)-\phi(t)\}\right]_{s=a}^{s=b} + \int_a^b g_{ss}(t,s)\{\phi(s)-\phi(t)\}\,ds \\
&\quad= \left[g_t(t,s)\{\phi(s)-\phi(t)\}\right]_{s=a}^{s=b} + \int_a^b g_{ss}(t,s)\{\phi(s)-\phi(t)\}\,ds.
\end{aligned}$$

Combining and simplifying the above equations, we obtain

$$\begin{aligned}
\frac{d^2}{dt^2}\int_a^b g(t,s)\phi(s)\,ds &= -\left[g_t(t,s)\phi(s)\right]_{s=a}^{s=b} + g_t(t,s)[\phi(s)-\phi(t)]_{s=a}^{s=b} \\
&\qquad + \int_a^b g_{ss}(t,s)\{\phi(s)-\phi(t)\}\,ds \\
&= -\left[g_t(t,s)\phi(t)\right]_{s=a}^{s=b} + \int_a^b g_{ss}(t,s)\{\phi(s)-\phi(t)\}\,ds.
\end{aligned}$$

The full equation becomes

$$\begin{aligned}
\frac{1}{\tau(t)}\,\phi(t) &= \phi_s(t) + \phi^{\mathrm{inc}}(t) \\
&= \int_a^b g(t,s)\phi(s)\,ds + \gamma_1[g_t(t,s)]_{s=b}^{s=a}\,\phi(t) \\
&\qquad + \gamma_1\int_a^b g_{ss}(t,s)\{\phi(s)-\phi(t)\}\,ds + \phi^{\mathrm{inc}}(t).
\end{aligned}$$

This is equivalent to equation (4).

## Appendix C

Consider the class of functions of the form

$$g(t,s) = |t-s|\left\{\ln(|t-s|) + A\right\} + B$$

defined on the square $[a,b]\times[a,b]$ for arbitrary constants $A$ and $B$. We have

$$g(t,s)=\begin{cases}(t-s)\,[\ln(t-s)+A]+B & \text{for } a\le s<t\le b\\ (s-t)\,[\ln(s-t)+A]+B & \text{for } a\le t<s\le b\\ \;0 & \text{for } t=s.\end{cases} \tag{29}$$

Clearly $g(t,s)$ is continuous everywhere on $[a,b]\times[a,b]$. It immediately follows that

$$g_t(t,s)=\begin{cases}\ln(t-s)+A+1 & \text{for } a\le s<t\le b\\ -\ln(s-t)-A-1 & \text{for } a\le t<s\le b\end{cases} \tag{30}$$

and

$$g_s(t,s)=\begin{cases}-\ln(t-s)-A-1 & \text{for } a\le s<t\le b\\ \ln(s-t)+A+1 & \text{for } a\le t<s\le b.\end{cases} \tag{31}$$

Finally,

$$g_{tt}(t,s)=\begin{cases}\dfrac{1}{t-s} & \text{for } a\le s<t\le b\\ \dfrac{1}{s-t} & \text{for } a\le t<s\le b\end{cases} \tag{32}$$

and

$$g_{ss}(t,s)=\begin{cases}\dfrac{1}{t-s} & \text{for } a\le s<t\le b\\ \dfrac{1}{s-t} & \text{for } a\le t<s\le b.\end{cases} \tag{33}$$

All derivatives are undefined for $t=s$. By inspection, we have the following properties:

$$\begin{aligned} g(u,v)&=g(v,u)\\ g_s(u,v)&=-\,g_t(u,v) \quad \text{for } u\ne v\\ g_s(u,v)&=-\,g_s(v,u) \quad \text{for } u\ne v\\ g_{ss}(u,v)&=g_{tt}(u,v)\\ &=\frac{1}{|u-v|} \quad \text{for } u\ne v.\end{aligned}$$

**Lemma.** *If* $A \leq -\ln(b-a) - 1/2$ *and* $B \leq (b-a)/8$, *then, for* $t \in (a,b)$,

$$\int_a^b g(t,s)\,ds < 0$$

*and*

$$g_t(t,a) - g_t(t,b) < 0.$$

*Proof.* By direct calculation, we can readily show

$$\begin{aligned}\int_a^b g(t,s)\,ds &= F(t-a) + F(b-t) - \frac{1}{2}\,(t-m)^2 - \frac{h^2}{8} + Bh\\ g_t(t,a) - g_t(t,b) &= \ln(C^2(t-a)(b-t)) + 2\end{aligned}$$

where

$$\begin{aligned}F(x) &= \frac{1}{2}\,x^2\ln(Cx)\\ m &= \frac{a+b}{2}\\ h &= b-a\\ C &= \exp(A).\end{aligned}$$

The lemma follows immediately.

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