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# REGULARITY OF EXTREMAL FUNCTIONS IN WEIGHTED BERGMAN AND FOCK TYPE SPACES

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ABSTRACT. We discuss the regularity of extremal functions in certain weighted Bergman and Fock type spaces. Given an appropriate analytic function $k$, the corresponding extremal function is the function with unit norm maximizing $\operatorname{Re} \int_\Omega f(z)\overline{k(z)}\,\nu(z)\,dA(z)$ over all functions $f$ of unit norm, where $\nu$ is the weight function and $\Omega$ is the domain of the functions in the space. We consider the case where $\nu(z)$ is a decreasing radial function satisfying some additional assumptions, and where $\Omega$ is either a disc centered at the origin or the entire complex plane. We show that, if $k$ grows slowly in a certain sense, then $f$ must grow slowly in a related sense. We also discuss a relation between the integrability and growth of certain log-convex functions and apply the result to obtain information about the growth of integral means of extremal functions in Fock type spaces.

This article deals with the regularity of solutions to extremal problems in certain weighted Bergman spaces in discs, as well as in Fock spaces (which are also called Fischer Spaces or Segal-Bargmann spaces). Our results also apply to other spaces of entire functions that are similar to Fock spaces but that are defined using a measure other than $e^{-\alpha|z|^2}dA$. For information on Bergman spaces, see **[9, 16]**. For information on Fock spaces, see, for example **[30]**. See also the important papers **[20, 21, 22]** by Newman and Shapiro. (Newman and Shapiro use the term Fischer spaces, presumably since Fischer used the (non-closed) subspace of the multivariable Fischer/Fock space consisting of polynomials to study a certain question, see **[12, 13]**. His work was extended by Newman and Shapiro.)

For Hardy spaces, which have many similarities with Bergman spaces but are often simpler to study, extremal problems have been extensively

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investigated (see **[5]** for references). Extremal problems in Bergman spaces are an area of active research. For example, see **[1, 19, 28, 29]**. One important application of extremal problems in Bergman spaces is the study of canonical divisors, which appear as solutions to certain extremal problems and play a role in Bergman spaces similar to Blaschke products in Hardy spaces, see **[6, 7, 8, 14, 15]**. Extremal functions in Fock spaces have been studied in **[2]**.

Several results concerning regularity of solutions to extremal problems in Bergman spaces are known, although there are many open questions. In **[25]**, Ryabykh obtained an important result on the subject, see **[10]** for a simplified proof. The articles **[11, 17, 18, 27]** also deal with the regularity of solutions to extremal problems in Bergman spaces.

We now discuss the subject of this paper in more detail. Let $0 < R \le \infty$, and let $\mathbb{D}_R$ be the open disc of radius $R$ centered at the origin (if $R = \infty$, then $\mathbb{D}_R$ is the entire complex plane). Let $\nu$ be a non-negative measurable function on $\mathbb{D}_R$ that is different from zero on a set of positive measure, and let $A^p_R(\nu)$ be the space of all analytic functions in $\mathbb{D}_R$ such that

$$\|f\| = \left\{ \int_{\mathbb{D}_R} |f(z)|^p \nu(z)\, dA(z) \right\}^{1/p} < \infty.$$

Throughout the paper, we make the assumption that $1 < p < \infty$, unless otherwise noted. For certain functions $\nu$, the space $A^p_R(\nu)$ is a Banach space with norm $\|\cdot\|$. When $R < \infty$, the space is known as a weighted Bergman space, whereas, when $R = \infty$, the space will be called a Fock type space. The standard Fock spaces correspond to the case where $R = \infty$ and $\nu = e^{-\alpha|z|^2}$, where $\alpha > 0$.

Let $\nu$ be a function such that $A^p_R(\nu)$ is a Banach space, and suppose that $k \in A^{p'}_R(\nu)$, where $1/p + 1/p' = 1$. Then,

$$f \longmapsto \int_{\mathbb{D}_R} f(z)\overline{k(z)}\, \nu(z)\, dA(z)$$

defines a linear functional $\Phi_k$ on $A^p_R(\nu)$, with norm at most $\|k\|_{A^{p'}_R(\nu)}$. We let $\|k\|^*$ denote the norm of $\Phi_k$. Thus, $\|k\|^* \le \|k\|_{A^{p'}_R(\nu)}$ for all functions $k \in A^{p'}_R(\nu)$.

We seek a function $f \in A^p_R(\nu)$ such that

$$\|f\|_{A^p_R(\nu)} = 1 \tag{0.1}$$

and

$$\operatorname{Re}\Phi_k(f) = \sup_{\|g\|_{A^p_R(\nu)}=1} \operatorname{Re}\int_{\mathbb{D}_R} g(z)\overline{k(z)}\,\nu(z)\,dA(z).$$

We say that $k$ is the integral kernel for the extremal problem and that $f$ is the corresponding extremal function. Since the space $L^p(\nu)$ is uniformly convex, there always exists a unique solution to this extremal problem, see [**10,** Theorem 1.4]. In the case where $\nu = 1$ and $R < \infty$, it is known that, if $k$ has some suitable additional regularity beyond being in the space $A^{p'}_R(\nu)$, then $f$ will also have some additional regularity. For example, see [**10, 11, 18, 25**].

For the case where $\nu$ is not constant, we note that the techniques in [**18**] should be able to be easily extended to show that solutions of weighted Bergman space extremal problems where $k$ is analytic in the closed unit disc, and the weight is real analytic in the closed unit disc extend continuously to the closed unit disc. In fact, a similar extension of the results in [**3**] should show that, if $k$ is analytic in the closed unit disc, and nonvanishing on the boundary of the unit disc, and the weight is both real analytic in the closed unit disc and non-vanishing on the unit circle, then there are numbers $\alpha$ and $\beta$ depending on $k$ and the weight such that $\alpha < 2 < \beta$ and that, if $\alpha < p < \beta$, then the extremal function has only a finite number of zeros in the closed unit disc. In [**18**], it is conjectured that this holds for all $p$ strictly between 1 and $\infty$ (at least when $k$ is a rational function and the weight is constant).

In what follows, we generalize the results of Ryabykh in [**25**] to certain non-constant measures $\nu$, and we obtain results for both the cases $R < \infty$ and $R = \infty$.

The outline of this article is as follows. In Section 1, we discuss some preliminary results. In Section 2, we discuss regularity for extremal functions in weighted Bergman spaces, and, in Section 3, we discuss regularity for weighted Fock type spaces. In Section 4, we give results which throw further light on some of the quantities appearing in the statement of the main theorem of Section 3. To do this, we find a relation between the integrability and growth of certain log-convex functions and apply the result to obtain information about the growth of the integral means of extremal functions in Fock type spaces. In Section 5, we discuss the density of polynomials in various weighted Bergman and Fock type spaces and present various auxiliary results, which are needed for the main results of the paper.

**1. Some preliminary results.** Let $0 < R \leq \infty$, and let $\mathbb{D}_R$ denote the open disc centered at the origin with radius $R$ (where $\mathbb{D}_\infty = \mathbb{C}$). Let $dA$ represent the area measure.

For $\nu(z)$ a non-negative measurable function defined on $\mathbb{D}_R$ that is not identically zero (in the 'almost everywhere' sense), we let $A^p_R(\nu)$ be the space of all functions analytic in $\mathbb{D}_R$ that are also in $L^p(\nu\, dA)$. We take the norm of $A^p_R(\nu)$ to be the same as the norm of $L^p(\nu\, dA)$. Note that, while $A^p_R(\nu)$ is a subspace of $L^p(\nu\, dA)$, it is not necessarily a closed subspace. However, for all of the measures with which we deal, $A^p_R(\nu)$ will be a closed subspace of $L^p(\nu\, dA)$.

Many of our results focus on the case where $\nu(z) = \omega(|z|^2)$, where $\omega$ is a positive, decreasing and non-constant function on $[0, R^2)$ that is analytic in some complex neighborhood of $[0, R^2)$. The space $A^p_R(\omega(|z|^2))$ has norm defined by

$$\|f\|_{A^p_R(\omega(|z|^2))} = \left( \int_{\mathbb{D}_R} |f(z)|^p \omega(|z|^2)\, dA(z) \right)^{1/p},$$

where $dA$ represents the area measure. We note that the space in question is indeed a Banach space, by Proposition 5.1.

Next, we recall the Cauchy-Green theorem, which we state for convenience since it will be used several times.

**Theorem A.** *Let $\Omega$ be a $C^1$ domain in $\mathbb{C}$, and let $f \in C^1(\overline{\Omega})$. Then:*

$$\frac{1}{2i} \int_{\partial\Omega} f(z)\, dz = \int_\Omega \frac{\partial}{\partial \overline{z}} f(z)\, dA,$$

*and*

$$\frac{i}{2} \int_{\partial\Omega} f(z)\, d\overline{z} = \int_\Omega \frac{\partial}{\partial z} f(z)\, dA.$$

We will also need the following theorem, which gives a characterization of extremal functions. It can be found in [**26,** page 55].

**Theorem B.** *Let $\sigma$ be a measure, let $1 < p < \infty$, let $X$ be a closed subspace of $L^p(\sigma)$, and let $\phi \in X^*$ be the dual space of $X$. Assume that $\phi$ is not identically 0. A function $F \in X$ with $\|F\| = 1$ satisfies*

$$\operatorname{Re}\phi(F) = \sup_{g \in X, \|g\|=1} \operatorname{Re}\phi(g) = \|\phi\|_{X^*}$$

*if and only if $\phi(F) > 0$, and*

$$\int h|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = 0$$

*for all $h \in X$ with $\phi(h) = 0$. If $F$ satisfies the above conditions, then*

$$\int h|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = \frac{\phi(h)}{\|\phi\|_{X^*}}$$

*for all $h \in X$.*

Lastly, we note that $L^p$ spaces are uniformly convex for $1 < p < \infty$ (see [**4**] for a definition of uniform convexity and a proof of this result), and thus, the $A^p_R$ spaces under consideration are uniformly convex since any closed subspace of a uniformly convex space is uniformly convex. Using [**10,** Theorem 3.1] and the fact that $\|k\|^* \leq \|k\|_{A^{p'}_R(\nu)}$ for all functions $k \in A^{p'}_R(\nu)$, we have the following theorem.

**Theorem C.** *Suppose that $A^p_R(\nu)$ is a Banach space and that $k$ is a non-zero function in $A^{p'}_R(\nu)$. Then, there is a unique solution to the extremal problem (0.1) with integral kernel $k$. Let $k_n$ be a sequence of functions approaching $k$ in the $A^{p'}_R(\nu)$ norm, let $f_n$ be the extremal functions corresponding to $k_n$, and let $f$ be the extremal function corresponding to $k$. Then, $f_n \to f$ in the $A^p_R(\nu)$ norm.*

By [**10,** Theorem 4.1], we have the following result. When we apply it, we will let $X_n$ be the space of polynomials of degree at most $n$.

**Theorem D.** *Suppose that $A^p_R(\nu)$ is a Banach space, that $f$ is the solution to the extremal problem (0.1) with integral kernel $k$, and that*

*$X_1 \subset X_2 \subset \cdots$ are closed subspaces of $A^p_R(\nu)$ such that $\overline{\cup_{n=1}^\infty X_n} = A^p_R(\nu)$. Let $f_n$ be the solution to the extremal problem* (0.1) *posed over the space $X_n$ instead of the space $A^p_R(\nu)$. Then, $f_n$ exists and is unique, and $f_n \to f$ in the $A^p_R(\nu)$ norm as $n \to \infty$. In addition, $\|\Phi_k|_{X_n}\| \to \|\Phi_k\|$ as $n \to \infty$.*

**2. Regularity of extremal functions in weighted Bergman spaces.** Let $R < \infty$. We suppose that $\omega$ is analytic in a neighborhood of $[0, R^2)$ and that $\omega$ is positive and decreasing on $[0, R^2)$. This implies that $\omega$ has a limit from the left at $R^2$; thus, we may assume, without loss of generality, that it is continuous from the left at $R^2$. By Proposition 5.3, the polynomials are dense in $A^p_R(\omega(|z|^2))$. Now, suppose that $f$ is analytic in the disk $\mathbb{D}_R$ and is in $C^1(\overline{\mathbb{D}_R})$. Consider the integral

$$\frac{R^2}{2}\int_0^{2\pi} |f(\mathrm{Re}^{i\theta})|^p\,\omega(R^2)\,d\theta.$$

We let $z = \mathrm{Re}^{i\theta}$ and change variables in the above integral by substituting $R^2\,d\theta = iz\,d\overline{z}$. Next, we apply the Cauchy-Green theorem to the resulting integral. After rearrangement, we see that

$$\begin{aligned}\frac{R^2}{2}\int_0^{2\pi} |f(\mathrm{Re}^{i\theta})|^p\omega(R^2)\,d\theta &- \int_{\mathbb{D}_R} |z|^2|f(z)|^p w'(|z|^2)\,dA \\ &= \int_{\mathbb{D}_R}\left(\frac{p}{2}zf'(z) + f(z)\right)|f(z)|^{p-1}(\operatorname{sgn}\overline{f(z)})\omega(|z|^2)\,dA.\end{aligned} \tag{2.1}$$

Note that the left-hand side of equation (2.1) is non-negative since the first integral in the expression is non-negative and the second integral in the expression is non-positive. This is due to the assumption that $\omega$ is decreasing.

Now, consider the right hand side of equation (2.1). Let $k$ be a fixed function analytic in $\mathbb{D}_R$ and in $C^1(\overline{\mathbb{D}})$. Let $f_n$ be the solution to the extremal problem of maximizing the real part of $\int_{\mathbb{D}_R} g(z)\overline{k}(z)\,\omega(|z|^2)\,dA(z)$ over all polynomials $g$ of degree at most $n$ such that $\|g\|_{A^p_R(\omega(|z|^2))} = 1$. Call the maximum $\|k\|_n^*$. By Theorem B applied to the space of polynomials of degree $n$ considered as a subspace of $A^p_R(\omega(|z|^2))$, we have

$$\int_{\mathbb{D}_R} \left(\frac{p}{2} z f_n'(z) + f_n(z)\right) |f_n(z)|^{p-1} (\operatorname{sgn} \overline{f_n(z)}) \omega(|z|^2)\, dA$$
$$= \frac{1}{\|k\|_n^*} \int_{\mathbb{D}_R} \left(\frac{p}{2} z f_n'(z) + f_n(z)\right) \overline{k}(z) \omega(|z|^2)\, dA$$

since $z f_n'(z)$ is also a polynomial of degree $n$.

If we take equation (2.1) with $f_n$ in place of $f$, and use the above equation along with the fact that

$$z f_n'(z) \omega(|z|^2) = \partial_z [z f_n(z) \omega(|z|^2)] - z f_n(z) \omega'(|z|^2) \overline{z} - f_n(z) \omega(|z|^2),$$

we see that

$$\frac{R^2}{2} \int_0^{2\pi} |f_n(Re^{i\theta})|^p \omega(R^2)\, d\theta - \int_{\mathbb{D}_R} |z|^2 |f_n(z)|^p \omega'(|z|^2)\, dA$$
$$= \frac{p}{2\|k\|_n^*} \left[ \int_{\mathbb{D}_R} \partial_z [z f_n(z) \omega(|z|^2) \overline{k(z)}\, dA - \int_{\mathbb{D}_R} |z|^2 f_n(z) \overline{k(z)} \omega'(|z|^2)\, dA \right]$$
$$+ \frac{1}{\|k\|_n^*} \left(1 - \frac{p}{2}\right) \int_{\mathbb{D}_R} f_n(z) \overline{k(z)} \omega(|z|^2)\, dA.$$

This equals

$$\frac{1}{\|k\|_n^*} \left\{ \frac{p}{2} \int_{\mathbb{D}_R} \partial_z [z f_n(z) \omega(|z|^2) \overline{k(z)}]\, dA - \frac{p}{2} \int_{\mathbb{D}_R} |z|^2 f_n(z) \overline{k(z)} \omega'(|z|^2)\, dA \right.$$
$$+ \frac{2-p}{2} \left[ \int_{\mathbb{D}_R} \partial_{\overline{z}} [f_n(z) \overline{z K(z)} \omega(|z|^2)]\, dA \right.$$
$$\left.\left. - \int_{\mathbb{D}_R} |z|^2 f_n \overline{K} \omega'(|z|^2)\, dA \right] \right\},$$

where $K(z) = (1/z) \int_0^z k(\zeta)\, d\zeta$. Applying the Cauchy-Green theorem again and changing the variable of integration to $\theta$ in the integrals over the boundary of the disc shows that

(2.2)
$$\frac{R^2}{2} \int_0^{2\pi} |f_n(\mathrm{Re}^{i\theta})|^p \omega(R^2)\, d\theta - \int_{\mathbb{D}_R} |z|^2 |f_n(z)|^p \omega'(|z|^2)\, dA$$
$$= \frac{p}{2\|k\|_n^*} \frac{R^2}{2} \int_0^{2\pi} f_n(\mathrm{Re}^{i\theta}) \overline{k(\mathrm{Re}^{i\theta})} \omega(R^2)\, d\theta$$
$$- \frac{p}{2\|k\|_n^*} \int_{\mathbb{D}_R} |z|^2 f_n(z) \overline{k(z)} \omega'(|z|^2)\, dA$$

$$+\frac{2-p}{2\|k\|_n^*}\left\{\frac{R^2}{2}\int_0^{2\pi} f_n(z)\overline{K(z)}\omega(R^2)\,d\theta - \int_{\mathbb{D}_R}|z|^2 f_n\overline{K}\omega'(|z|^2)\,dA\right\}.$$

Now, define the $p$th integral mean of an analytic function $f$ at radius $r < R$ by

$$M_p(r,f) = \left\{\int_0^{2\pi} |f(\mathrm{r}e^{i\theta})|^p d\theta\right\}^{1/p},$$

and define $M_p(R,f) = \lim_{r\to R^-} M_p(r,f)$. Note that this differs by a factor of $(2\pi)^{-1/p}$ from the usual definition. For $0 < r \le R$, let

$$D_p(r,f;\omega) = \left\{-\int_{\mathbb{D}_r}|z|^2|f(z)|^p\omega'(|z|^2)\,dA\right\}^{1/p}. \tag{2.3}$$

We write $D_p(r,f)$ for $D_p(r,f;\omega)$ when it is clear what the function $\omega$ is. It is clear that $D_p(r,f)$ is non-decreasing with $r$, and it is well known that the same is true for $M_p(r,f)$, see [**5,** page 9]. We note in passing that, in at least one case, $D_p(r,f)$ can be given a physical interpretation. A function $f$ in the Fock space for $p = 2$ can represent the state of a quantum harmonic oscillator, in which case $D_2(\infty, f)$ represents a quantity related to the expected energy of the oscillator.

Let $q$ be the conjugate exponent to $p$ so that $1/p + 1/q = 1$. Also, note that

$$\begin{aligned}
M_q(r,zK) &= \left\{\int_0^{2\pi} |\,\mathrm{r}e^{i\theta}\,K(\mathrm{r}e^{i\theta})|^q d\theta\right\}^{1/q}\\
&= \left\{\int_0^{2\pi}\left|\int_0^r k(\rho e^{i\theta})e^{i\theta}d\rho\right|^q d\theta\right\}^{1/q}\\
&\le \int_0^r\left\{\int_0^{2\pi}|k(\rho e^{i\theta})|^q d\theta\right\}^{1/q} d\rho\\
&= \int_0^r M_q(\rho,k)\,d\rho \le rM_q(r,k).
\end{aligned}$$

Thus, $M_q(r,K) \le M_q(r,k)$, which also implies that $D_q(r,K) \le D_q(r,k)$ since the measure $|z|^2\omega'(|z|^2)$ is a radial measure.

Let $\widehat{p} = \max(p-1, 1)$. Returning to equation (2.2) and using Hölder's inequality, we see that

$$\begin{aligned}
&\frac{R^2}{2}\omega(R^2)M_p^p(R,f_n) + D_p^p(R,f_n) \\
&\quad \le \frac{1}{\|k\|_n^*}\Bigg\{\frac{p}{2}\frac{R^2}{2}\omega(R^2)M_p(R,f_n)M_q(R,k) + \frac{p}{2}D_p(R,f_n)D_q(R,k) \\
&\qquad\qquad + \left|1-\frac{p}{2}\right|\Bigg[\frac{R^2}{2}\omega(R^2)M_p(R,f_n)M_q(R,K) \\
&\qquad\qquad\qquad\qquad + D_p(R,f_n)D_q(R,K)\Bigg]\Bigg\} \\
&\quad \le \frac{1}{\|k\|_n^*}\left\{\widehat{p}\frac{R^2}{2}\omega(R^2)M_p(R,f_n)M_q(R,k) + \widehat{p}D_p(R,f_n)D_q(R,k)\right\}.
\end{aligned}$$

For ease of notation, define $N_p(r,g) = (r^2/2)^{1/p}\omega(r^2)^{1/p}M_p(r,g)$ for any analytic function $g$. Then, the right side of the last displayed inequality is at most

$$\begin{aligned}
&\frac{\widehat{p}}{\|k\|_n^*}\left[\left(\frac{R^2}{2}\omega(R^2)\right)^{1/p}M_p(R,f_n) + D_p(R,f_n)\right] \\
&\qquad \times \left[\left(\frac{R^2}{2}\omega(R^2)\right)^{1/q}M_q(R,k) + D_q(R,k)\right] \\
&\quad = \frac{\widehat{p}}{\|k\|_n^*}[N_p(R,f_n) + D_p(R,f_n)][N_q(R,k_n) + D_q(R,k)] \\
&\quad \le \frac{2^{1/q}\widehat{p}}{\|k\|_n^*}[N_p^p(R,f_n) + D_p^p(R,f_n)]^{1/p}[N_q(R,k) + D_q(R,k)].
\end{aligned}$$

And, thus, we have

$$(N_p^p(R,f_n) + D_p^p(R,f_n))^{1/q} \le \frac{2^{1/q}\widehat{p}}{\|k\|_n^*}[N_q(R,k) + D_q(R,k)].$$

If $r < R$, this implies that

$$N_p^p(r,f_n) + D_p^p(r,f_n) \le \frac{2^{1/q}\widehat{p}}{\|k\|_n^*}[N_q(R,k) + D_q(R,k)]^q. \tag{2.4}$$

Now, let $f$ denote the solution of our extremal problem over the full space. Observe that, as $n \to \infty$, we have $f_n \to f$ in $A_R^p(\omega(|z|^2))$ and $\|k\|_n^* \to \|k\|^*$ by Theorem D. Thus, $f_n \to f$ uniformly on compact subsets of $\mathbb{D}_R$ by Proposition 5.1. Hence, $M_p(r, f_n) \to M_p(r, f)$ and $D_p(r, f_n) \to D_p(r, f)$ as $n \to \infty$. Also, recall that $M_p(r, f)$ and $D_p(r, f)$ are increasing with $r$. Thus, in inequality (2.4), if we first let $n \to \infty$ and then $r \to \infty$, we have

$$N_p^p(R, f) + D_p^p(R, f) \leq \frac{2^{1/q}\widehat{p}}{\|k\|^*}[N_q(R, k) + D_q(R, k)]^q. \tag{2.5}$$

Now, suppose that $k$ is not in $C^1(\overline{D})$ but that $M_q(R, k) < \infty$. It is well known that there is a sequence of polynomials $k_n$ such that $M_q(R, k - k_n) \to 0$ as $n \to \infty$ (this follows from [**5,** Theorem 2.6]. Now, since $M_q(r, g)$ increases with $r$ for any analytic function $g$, and since

$$\begin{aligned} D_q^q(R, k - k_n) &= -\int_0^R r^2 M_q^q(r, k - k_n)\omega'(r) r\, dr \\ &\leq \left(-\int_0^R \omega'(r^2) r\, dr\right) M_q^q(R, k - k_n) R^2 \\ &= \frac{1}{2}(\omega(0) - \omega(R^2)) M_q^q(R, k - k_n) R^2, \end{aligned}$$

we have that $D_q(R, k - k_n) \to 0$ as $n \to \infty$. Thus, by Minkowski's inequality, we have that $M_q(R, k) - M_q(R, k_n)$ and $D_q(R, k) - D_q(R, k_n)$ both approach 0 as $n \to \infty$. Now, for $r < R$, we have

$$N_p^p(r, f_n) + D_p^p(r, f_n) \leq \frac{2^{1/q}\widehat{p}}{\|k\|^*}[N_q(R, k_n) + D_q(R, k_n)]^q.$$

By Theorem C, as $n \to \infty$ we have $f_n \to f$ in the $A_R^p(\omega(|z|^2))$ norm, and thus $f_n \to f$ uniformly in $\{|z| \leq r\}$ by Proposition 5.1. Therefore, $D_p(f_n, r) \to D_p(f, r)$ and $M_p(f_n, r) \to M_p(f, r)$. Thus, we have that

$$N_p^p(r, f) + D_p^p(r, f) \leq \frac{2^{1/q}\widehat{p}}{\|k\|^*}[N_q(R, k) + D_q(R, k)]^q.$$

Letting $r \to R$ shows that inequality (2.5) still holds. We summarize our results in a theorem.

**Theorem 2.1.** *Let $1 < p < \infty$, and let $0 < R < \infty$. Let the function $\omega$ be analytic in a neighborhood of $[0, R^2)$, and let $\omega$ be positive, non-increasing and non-constant on $[0, R^2)$. Suppose that $f$ is the extremal* *function in* $A^p_R(\omega(|z|^2))$ *for the integral kernel* $k$. *Then*:

$$\frac{R^2}{2}\omega(R^2)M_p^p(R,f)+D_p^p(R,f) \\ \leq \frac{2^{1/q}\widehat{p}}{\|k\|^*}\left[\left(\frac{R^2}{2}\omega(R^2)\right)^{1/q}M_q(R,k)+D_q(R,k)\right]^q.$$

**3. Regularity of extremal functions in Fock type spaces.** In this section, we consider the regularity of extremal functions in the Fock type spaces $A^p_\infty(\nu)$. The measures we consider are those which satisfy our previous assumptions, for which $\lim_{r\to\infty} r^n\omega(r^2)=0$ and $\lim_{r\to\infty} r^n\omega'(r^2)=0$ for all integers $n$, and for which the polynomials are dense in $A^p_\infty(\omega(|z|^2))$ and $A^q_\infty(\omega(|z|^2)-|z|^2\omega'(|z|^2))$, where $q$ is the conjugate exponent to $p$. In Section 5, we give some sufficient conditions for polynomials to be dense in these spaces. For example, for $1<p<\infty$, the polynomials are dense in the Fock space (for which $\omega(z)=e^{-\alpha z}$).

Recall that equation (2.1) shows, for $f$ in $C^1(\overline{\mathbb{D}_R})$ and analytic in $\mathbb{D}_R$, and for $0<R<\infty$, that

$$\frac{R^2}{2}\int_0^{2\pi}|f(\mathrm{Re}^{i\theta})|^p\omega(R^2)\,d\theta-\int_{\mathbb{D}_R}|z|^2|f(z)|^pw'(|z|^2)\,dA \\ =\int_{\mathbb{D}_R}\left(\frac{p}{2}zf'(z)+f(z)\right)|f(z)|^{p-1}(\operatorname{sgn}\overline{f(z)})\omega(|z|^2)\,dA.$$

Suppose that $f$ is a polynomial. Then, letting $R\to\infty$ in equation (2.1) gives

$$-\int_{\mathbb{C}}|z|^2|f(z)|^pw'(|z|^2)\,dA \\ =\int_{\mathbb{C}}\left(\frac{p}{2}zf'(z)+f(z)\right)|f(z)|^{p-1}(\operatorname{sgn}\overline{f(z)})\omega(|z|^2)\,dA.$$

Consider the extremal problem for the space $A^p_\infty(\omega(|z|^2))$ with kernel $k$, where $k$ is a polynomial. Denote the solution to the extremal problem (0.1) over polynomials of degree at most $n$ by $f_n$.

As before, the right hand side of equation (2.1) equals

$$\frac{1}{\|k\|_n^*}\int_{\mathbb{C}}\left(\frac{p}{2}zf_n'(z)+f_n(z)\right)\overline{k(z)}\omega(|z|^2)\,dA,$$

and the same computation as before gives

$$\begin{aligned}
&-\int_{\mathbb{C}}|z|^2|f_n(z)|^p\omega'(|z|^2)\,dA\\
&=\lim_{R\to\infty}\left(\frac{1}{\|k\|_n^*}\left\{\frac{p}{2}\int_{\mathbb{D}_R}\partial_z[zf_n(z)\omega(|z|^2)\overline{k(z)}]\,dA\right.\right.\\
&\qquad\qquad\left.-\frac{p}{2}\int_{\mathbb{D}_R}|z|^2f_n(z)\overline{k(z)}\omega'(|z|^2)\,dA\right\}\\
&\qquad+\frac{2-p}{2\|k\|_n^*}\left[\int_{\mathbb{D}_R}\partial_{\overline{z}}[f_n(z)\overline{zK(z)}\omega(|z|^2)]\,dA\right.\\
&\qquad\qquad\left.\left.-\int_{\mathbb{D}_R}|z|^2f_n\overline{K}\omega'(|z|^2)\,dA\right]\right),
\end{aligned}$$

where $K(z)=(1/z)\int_0^z k(\zeta)\,d\zeta$, as before. We now use the Cauchy-Green theorem to see that the above displayed expressions equal

$$\begin{aligned}
&\lim_{R\to\infty}\left(\frac{1}{\|k\|_n^*}\left\{\frac{p}{2}\frac{i}{2}\int_{\partial\mathbb{D}_R}zf_n(z)\overline{k(z)}\,\omega(|z|^2)\,d\overline{z}\right.\right.\\
&\qquad\qquad\left.-\frac{p}{2}\int_{\mathbb{D}_R}|z|^2f_n(z)\overline{k(z)}\omega'(|z|^2)\,dA\right\}\\
&+\left.\frac{2-p}{2\|k\|_n^*}\left[\frac{1}{2i}\int_{\mathbb{D}_R}f_n(z)\overline{K(z)}\omega(|z|^2)\overline{z}\,dz-\int_{\mathbb{D}_R}|z|^2f_n\overline{K}\omega'(|z|^2)\,dA(z)\right]\right).
\end{aligned}$$

Since $f_n$ and $k$ are polynomials, our assumptions on $\omega$ imply that

$$\begin{aligned}
-\int_{\mathbb{C}}|z|^2|f_n(z)|^p\omega'(|z|^2)\,dA&=-\frac{p}{2\|k\|_n^*}\int_{\mathbb{C}}|z|^2f_n(z)\overline{k(z)}\omega'(|z|^2)\,dA\\
&\quad-\left(1-\frac{p}{2}\right)\int_{\mathbb{D}_R}|z|^2f_n\overline{K}\omega'(|z|^2)\,dA(z).
\end{aligned}$$

Applying Hölder's inequality, we see that

$$\begin{aligned} D_p^p(\infty, f_n) &\leq \frac{p}{2\|k\|_n^*} D_p(\infty, f_n) D_q(\infty, k) \\ &\quad + \frac{1}{\|k\|_n^*} \left|1 - \frac{p}{2}\right| D_p(\infty, f_n) D_q(\infty, K) \\ &\leq \frac{\widehat{p}}{\|k\|_n^*} D_p(\infty, f_n) D_q(\infty, k), \end{aligned}$$

so that

$$D_p(\infty, f_n) \leq \left[\frac{\widehat{p}}{\|k\|_n^*} D_q(\infty, k)\right]^{1/(p-1)}.$$

Therefore,

$$D_p(R, f_n) \leq \left[\frac{\widehat{p}}{\|k\|_n^*} D_q(\infty, k)\right]^{1/(p-1)},$$

where $R > 0$ is arbitrary.

Let $f$ denote the solution to the extremal problem over the full space. Then, $f_n \to f$ in $A^p_\infty(\omega(|z|^2))$ and $\|k\|_n^* \to \|k\|^*$ by Theorem D. Also, by Proposition 5.1, $f_n \to f$ uniformly on $|z| \leq R$. Letting $n \to \infty$, and then letting $R \to \infty$ in the above displayed inequality, gives

$$D_p(\infty, f) \leq \left[\frac{\widehat{p}}{\|k\|^*} D_q(\infty, k)\right]^{1/(p-1)}.$$

Lastly, if $k$ is not a polynomial, we let $k_n$ be a sequence of polynomials approaching $k$ in $A^q_\infty(\omega(|z|^2))$, such that $D_q(\infty, k_n) \to D_q(\infty, k)$ as $n \to \infty$. This can be done since polynomials are dense in the space $A^q_\infty(\omega(|z|^2) - |z|^2\omega'(|z|^2))$. Let $f_n$ be the solution to the extremal problem with kernel $k_n$. Then, we have, for fixed $R > 0$, that

$$D_p(R, f_n) \leq \left[\frac{\widehat{p}}{\|k_n\|^*} D_q(\infty, k_n)\right]^{1/(p-1)}.$$

Now, $f_n \to f$ in $A^p_\infty(\omega(|z|^2))$ by Theorem C, and thus, uniformly for $|z| \leq R$ by Proposition 5.1. Also, $\|k_n\|^* \to \|k\|^*$ as $n \to \infty$, since $\|k - k_n\|^*$ is bounded above by $\|k - k_n\|_{A^q_\infty(\omega(|z|^2))}$, which approaches zero. Therefore, we have that

$$D_p(R, f) \leq \left[\frac{\widehat{p}}{\|k\|^*} D_q(\infty, k)\right]^{1/(p-1)}.$$

Letting $R \to \infty$ gives

$$D_p(\infty, f) \leq \left[ \frac{\widehat{p}}{\|k\|^*} D_q(\infty, k) \right]^{1/(p-1)}.$$

Again, we state our results in a theorem.

**Theorem 3.1.** *Let $1 < p < \infty$. Let the function $\omega$ be analytic in a neighborhood of $[0, \infty)$, and let $\omega$ be positive, non-increasing and non-constant on $[0, \infty)$. Also, suppose that $\lim_{r\to\infty} r^n \omega(r^2) = \lim_{r\to\infty} r^n \omega'(r^2) = 0$ for all integers $n$, and that the polynomials are dense in $A^p_\infty(\omega(|z|^2))$ and in $A^q_\infty(\omega(|z|^2) - |z|^2\omega'(|z|^2))$. Suppose that $f$ is the extremal function in $A^p_R(\omega(|z|^2))$ for the integral kernel $k$. Then:*

$$D_p(\infty, f) \leq \left[ \frac{\widehat{p}}{\|k\|^*} D_q(\infty, k) \right]^{1/(p-1)}.$$

It could be questioned whether the condition $D_p(\infty, f) < \infty$ is implied by the condition $f \in A^p_\infty(\omega(|z|)^2)$, in which case the above theorem would be less interesting. However, in general, $f \in A^p_\infty(\omega(|z|)^2)$ does not imply $D_p(\infty, f) < \infty$. For example, consider the Fock space with measure $e^{-\alpha|z|^2}$. In this case, the statement that $D_p(\infty, f) < \infty$ is equivalent to $f$ being in the space $A^p_\infty(|z|^2 e^{-\alpha|z|^2})$. Now, the norm of $z^n$ in the original Fock space is

$$\left[ \frac{\pi}{\alpha^{np/2}} \Gamma\left( \frac{np}{2} + 1 \right) \right]^{1/p},$$

while its norm in the second space is

$$\left[ \frac{\pi}{\alpha^{np/2+1}} \Gamma\left( \frac{np}{2} + 2 \right) \right]^{1/p}.$$

The ratio of the second norm to the first is $((np+2)/(2\alpha))^{1/p}$, which is unbounded in $n$. If every element in the Fock space were in $A^p_\infty(|z|^2 e^{-\alpha|z|^2})$, then, by the closed graph theorem, the identity map from the Fock space into $A^p_\infty(|z|^2 e^{-\alpha|z|^2})$ would be bounded, which contradicts the above analysis of the norms of the monomials. Thus, $f \in A^p_\infty(e^{-\alpha|z|^2})$ does not imply that $f \in A^p_\infty(|z|^2 e^{-\alpha|z|^2})$.

**4. Growth of integral means and log-convex functions.** Theorem 3.1 does not bound the quantity $\lim_{r\to\infty} r^2\omega(r^2)M_p^p(r,f)$, although a similar term was bounded in Theorem 2.1. Thus, by analogy with Theorem 2.1, it is natural to ask whether $\omega(r^2)M_p^p(r,f)$ is bounded as $r\to\infty$ if $r^2\omega(r^2)M_q^q(r,k)$ is bounded as $r\to\infty$, and if $D_q(\infty,k)$ is bounded. For certain measures, we show that this is the case. In fact, if $D_q(\infty,k)$ is bounded, so is $D_p(\infty,f)$ by Theorem 3.1, and in this section, we show that, for certain measures $\omega$, the condition $D_p(f,\infty)<\infty$ implies that $r^3\omega(r^2)M_p^p(r,f)\to 0$ as $r\to\infty$ for any entire function $f$.

It will simplify matters if we introduce some notation. Let $\lambda(x)$ be a positive, increasing, smooth function defined for $x\geq R$, where $R\geq 0$. Now, let $X=\log x$ and $Y=\log y$. Define $\nu(X)=\log(\lambda(x))=\log(\lambda(e^X))$. Let $g(x)$ be differentiable and a log-convex function of $\log x$, i.e., let $\log g(x)=\log g(e^X)$ be a convex function of $X$. Let $\widetilde{g}(X)=g(e^X)$ and $\widetilde{\nu}(X)=\nu(e^X)$.

Now, suppose that, for some $x_1>0$, we have $g(x_1)=\lambda(x_1)$, but that $g(x)\leq\lambda(x)$ for some $x<x_1$. Then, for some $x_0$ such that $x<x_0\leq x_1$, we must have $g(x_0)=\lambda(x_0)$ and $g'(x_0)\geq\lambda'(x_0)$, which implies that

$$\frac{d\log\widetilde{g}(X)}{dX}(X_0)\geq\frac{d\widetilde{\nu}}{dX}(X_0).$$

Now, let

$$Y=\widetilde{\nu}(X_0)+\left(\frac{d\widetilde{\nu}}{dX}(X_0)\right)\cdot(X-X_0).$$

For $X\geq X_0$, the function $Y$ lies below the line tangent to the function $\log\widetilde{g}(X)$ at $X_0$. Since $\log\widetilde{g}(X)$ is a convex function of $X$, this means that $Y\leq\log\widetilde{g}(X)$ for all $X\geq X_0$.

We compute that

$$\frac{d\widetilde{\nu}}{dX}=\frac{d\log\lambda(e^X)}{dX}=\frac{e^X\lambda'(e^X)}{\lambda(e^X)}=\frac{x\lambda'(x)}{\lambda(x)}.$$

Let $e^Y=y$. Then, we have

$$y=\lambda(x_0)\left(\frac{x}{x_0}\right)^{x_0\lambda'(x_0)/\lambda(x_0)}.$$

Now, let

$$S(x_0,\lambda) = \int_{x_0}^{\infty} y\lambda(x)^{-1}dx = \int_{x_0}^{\infty} \frac{\lambda(x_0)}{\lambda(x)}\left(\frac{x}{x_0}\right)^{x_0\lambda'(x_0)/\lambda(x_0)} dx.$$

Note that $S(x_0, \alpha\lambda) = S(x_0, \lambda)$ for $\alpha \neq 0$. From the above discussion, we have the following lemma.

**Lemma 4.1.** *Let $g$ and $\lambda$ be as above. Suppose that there is some $x_1$ such that $g(x_1) = \lambda(x_1)$ and that there is some $x_2$ such that $R \leq x_2 < x_1$ and such that $g(x_2) < \lambda(x_2)$. Then, there is some $x_0$ such that $x_2 < x_0 \leq x_1$ and $g(x_0) = \lambda(x_0)$, and such that*

$$\int_{x_0}^{\infty} g(x)\lambda(x)^{-1}dx \geq S(x_0, \lambda).$$

We now show that, if

$$\int_0^{\infty} g(x)\lambda(x)^{-1}dx < \infty,$$

and, if $\liminf_{x\to\infty} S(x,\lambda) > 0$, we have $\lim_{x\to\infty} g(x)\lambda(x)^{-1} = 0$. Suppose, for the sake of contradiction, that $\limsup_{x\to\infty} g(x)\lambda(x)^{-1} = 2k > 0$. From the fact that the above integral is finite, we must have $\liminf_{x\to\infty} g(x)\lambda(x)^{-1} = 0$. Thus, there must be some sequence of points $a_1 < b_1 < a_2 < b_2 \cdots$ such that $g(a_j)\lambda(a_j)^{-1} < k$ and $g(b_j)\lambda(b_j)^{-1} = k$, and such that $b_n \to \infty$ and $a_n \to \infty$ as $n \to \infty$. Thus, if we define $\lambda_k = k\lambda$, the previous lemma shows that there is some sequence of points $x_n \to \infty$ such that

$$\int_{x_n}^{\infty} g(x)\lambda_k(x)^{-1}dx \geq S(x_n, \lambda_k)$$

for all $n$. As $n \to \infty$, the left hand side of the previous inequality must approach 0. Thus, if we can show for all $k > 0$ that $\liminf_{x\to\infty} S(x, \lambda_k) > 0$, we will obtain a contradiction, showing, in fact, that $k = 0$. However, $S(x, \lambda_k) = S(x, \lambda)$ for $k \neq 0$. Thus, we have obtained a contradiction between our assumptions and the supposition that $k \neq 0$. In summary, we have the next theorem.

**Theorem 4.2.** *Suppose that* $\liminf_{x\to\infty} S(x,\lambda) > 0$. *Then, if* $g(x)$ *is a log-convex function of* $\log x$ *such that*

$$\int_0^\infty g(x)\lambda(x)^{-1}dx < \infty,$$

*we have that* $\lim_{x\to\infty} g(x)\lambda^{-1}(x) = 0$.

To apply this theorem in our situation, we choose $\lambda(x) = -1/\omega'(x^2)$. Recall that, if $f$ is an analytic function, then $M_p(r,f)$ is a log convex function of $\log r$, see [**5,** page 9]. Now, let $g(x) = x^3 M_p^p(x,f)$. Letting $X = \log x$, we have

$$\log g(e^X) = 3X + p\log M_p(e^X, f).$$

Since both $X$ and $\log M_p(e^X,f)$ are convex functions of $X$, so is $g(x)$. Suppose that

$$D_p^p(\infty, f; \omega) = \int_{\mathbb{C}} |z|^2|f(z)|^p\lambda(|z|)^{-1} < \infty,$$

which means that

$$\int_0^\infty x^3 M_p^p(x,f)\lambda(x)^{-1} = \int_0^\infty g(x)\lambda(x)^{-1} < \infty.$$

If $\liminf_{x\to\infty} S(x,\lambda) > 0$, then, $\lim_{r\to\infty} r^3 M_p^p(r,f)\omega'(r^2) = 0$ by Theorem 4.2. Thus, we have the following theorem.

**Theorem 4.3.** *Suppose that* $\liminf_{x\to\infty} S(x, -1/\omega'(r^2)) > 0$ *and that there is some positive constant* $C$ *such that* $-\omega'(r) \geq C\omega(r)$ *for all sufficiently large* $r$. *If* $D_p(\infty, f;\omega) < \infty$, *then* $\lim_{r\to\infty} r^3 M_p^p(r,f)\omega(r^2) = 0$.

**Example 4.4.** Suppose that $\omega(|z|^2) = (1/\alpha)e^{-\alpha|z|^2}$ so that we are dealing with a Fock space. (The $1/\alpha$ factor is there as a convenience so an extra factor of $\alpha$ does not appear in the definition of $\lambda$.) Note that

$$\begin{aligned} S(x_0,\lambda) &= \int_{x_0}^\infty e^{\alpha x_0^2} e^{-\alpha x^2}\left(\frac{x}{x_0}\right)^{2\alpha x_0^2} dx \\ &= e^{\alpha x_0^2}(\alpha x_0^2)^{-\alpha x_0^2}\int_{x_0}^\infty e^{-\alpha x^2}(\alpha x^2)^{\alpha x_0^2} dx \end{aligned}$$

$$\begin{aligned}
&= (1/[2\alpha^{1/2}])e^{\alpha x_0^2}(\alpha x_0^2)^{-\alpha x_0^2}\int_{\alpha x_0^2}^{\infty} e^{-u}u^{\alpha x_0^2-(1/2)}du \\
&= (1/[2\alpha^{1/2}])e^{\alpha x_0^2}(\alpha x_0^2)^{-\alpha x_0^2}\Gamma(\alpha x_0^2+(1/2),\alpha x_0^2) \\
&\geq (1/[2\alpha^{1/2}])e^{\alpha x_0^2}(\alpha x_0^2)^{-\alpha x_0^2}\Gamma(\alpha x_0^2+(1/2),\alpha x_0^2+(1/2)),
\end{aligned}$$

where $\Gamma$ is the incomplete Gamma function, as defined in [**23,** formula 8.2.2]. Now, for large $x$, we have [**23,** formula 8.11.12]

$$\Gamma(x,x) \sim x^x e^{-x} x^{-1/2}\sqrt{\pi/2},$$

which implies, for large enough $x$, that

$$\begin{aligned}
\Gamma(x+(1/2),x+(1/2)) &\geq C\left(x+\frac{1}{2}\right)^{x+(1/2)} e^{-x-(1/2)}\left(x+\frac{1}{2}\right)^{-1/2} \\
&\geq \frac{Cx^x e^{-x}}{e^{1/2}}
\end{aligned}$$

for some constant $C > 0$. Thus, we have that

$$S(x_0,\lambda) \geq \frac{C}{2\alpha^{1/2}e^{1/2}}e^{\alpha x_0^2}(\alpha x_0^2)^{-\alpha x_0^2}(\alpha x_0^2)^{\alpha x_0^2}e^{-\alpha x_0^2} = \frac{C}{2\alpha^{1/2}e^{1/2}}$$

for large enough $x_0$. Hence, we have the following result, which we state as a theorem.

**Theorem 4.5.** *Let $f$ be an entire function, and let $0 < p < \infty$. If*

$$\int_{\mathbb{C}} |f(z)|^p |z|^2 e^{-\alpha|z|^2}\, dA(z) < \infty,$$

*then*

$$\lim_{r\to\infty} r^3 M_p^p(r,f)e^{-\alpha r^2} = 0.$$

**5. Density of polynomials in various spaces.** In this section, we discuss the density of polynomials in various weighted Bergman spaces. Propositions 5.1, 5.2, 5.3 and 5.4 are well known, at least in certain cases, and we follow the standard proofs. The proof of Proposition 5.5 is similar to the proof that the polynomials are dense in the Fock space (see, e.g. [**30**]).

We say a nonnegative function $\nu$ defined on $[0, R)$ is a radial weight function if the measure $\nu(|z|)\, dA$ is in $L^p(\mathbb{D}_R)$, and it is not the case that $\nu = 0$ almost everywhere.

**Proposition 5.1.** *Let $0 < R \leq \infty$. Suppose that $\nu$ is a radial weight function, and that there is some $R'$ such that $0 \leq R' < R$ and, for each $y$ such that $R' < y < R$, the quantity $\inf\{\nu(x) : R' \leq x < y\}$ is nonzero. Then, $A^p_R(\nu(|z|))$ is a Banach space, and convergence in the norm of $A^p_R(\nu(|z|))$ implies uniform convergence on compact subsets of $\mathbb{D}_R$. Also, the point evaluations of $A^p_R(\nu(|z|))$ are bounded uniformly on compact subsets of $\mathbb{D}_R$.*

*Proof.* We first consider the case $R < \infty$.

Since the space in question is a subspace of $L^p(\nu(|z|))$, to show that it is a Banach space, we need only show that it is closed. To show this, we will show that convergence in the norm implies uniform convergence on compact subsets. Let $f \in A^p_R(\nu(|z|))$. Since $f$ is analytic, the function $|f(z)|^p$ is subharmonic, and thus,

$$|f(z)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta$$

for any $z \in \mathbb{D}_R$ with $|z| + r < R$. Suppose first that $|z| > (R' + R)/2$. Now, let $r' = (R - |z|)/2$, and define $m_z = \inf\{\nu(|w|) : |z| - r' < |w| < |z| + r'\}$. By assumption, $m_z > 0$. We multiply the last displayed inequality by $2\pi m_z r$ and integrate $r$ from 0 to $r'$ to conclude that

$$\begin{aligned} m_z |f(z)|^p \pi r'^2 &\leq \int_{|z-w|<r'} |f(w)|^p m_z \, dA(w) \\ &\leq \int_{|z-w|<r'} |f(w)|^p \nu(|w|)\, dA(w) \\ &\leq \int_{\mathbb{D}_R} |f(w)|^p \nu(|w|)\, dA(w). \end{aligned}$$

This shows that point evaluation is a bounded linear functional for $R > |z| > (R' + R)/2$, and the bound depends only on $|z|$. By the maximum principle, point evaluation is also bounded for $|z| \leq (R + R')/2$. Since $m_z r'^2$ is a decreasing positive function of $|z|$, the bound is uniform on closed discs of radius less than $R$ centered at the origin. Thus, convergence in the norm implies convergence on compact subsets. This implies that a convergent sequence of analytic functions converges to an analytic function, which shows the space is complete.

For the case $R = \infty$, we repeat the above proof, except that we first consider $z$ such that $|z| > R' + 1$, and we let $r' = 1$. □

We define the dilation of the analytic function $f$ by $f_\rho(z) = f(\rho z)$ for $0 < \rho < \infty$.

**Proposition 5.2.** *Let* $0 < R \leq \infty$, *and let* $\nu$ *be a radial weight function. If* $f$ *is an analytic function in* $A^p_R(\nu(|z|))$, *then* $f_\rho \to f$ *in* $A^p_R(\nu(|z|))$ *as* $\rho \to 1^-$.

*Proof.* Since the integral means of an analytic function are increasing, we have for $\rho < 1$ that $f_\rho \in A^p_R(\nu(|z|))$, and that

$$M_p(r, f - f_\rho) \leq M_p(r, f) + M_p(r, f_\rho) \leq 2M_p(r, f).$$

The hypothesis that $f \in A^p_R(\nu(|z|))$ implies that $rM_p(r, f) \in L^p(\nu\, dr)$. Thus,

$$\int_0^R M_p^p(r, f - f_\rho) r\, \nu(r)\, dr \longrightarrow 0$$

as $\rho \to 1^-$, by the Lebesgue dominated convergence theorem. □

**Proposition 5.3.** *Let* $R < \infty$, *and let* $0 < p < \infty$. *Assume that* $\nu$ *is a radial weight function. Then, the polynomials are dense in* $A^p_R(\nu(|z|))$.

*Proof.* Let $f \in A^p_R(\nu(|z|))$, and let $\rho < 1$. Since $f_\rho$ is analytic on $\mathbb{D}_{R/\rho}$, the Taylor series of $f_\rho$ converges to $f_\rho$ uniformly in $\mathbb{D}_R$, and thus, each dilation is in the closure of the polynomials since the integrability of $\nu(|z|)$ guarantees that uniform convergence on $\mathbb{D}_R$ implies convergence in $A^p_R(\nu(|z|))$. By Proposition 5.2, $f$ is in the closure of the set of its dilations. Thus, $f$ is in the closure of the polynomials in $A^p_R(\nu(|z|))$. □

The situation is more difficult for the case $R = \infty$. We first prove the following proposition.

**Proposition 5.4.** *Let $\nu$ be a radial weight function on $[0,\infty)$ such that every polynomial is in $A^2_\infty(\nu(|z|))$. Then, the polynomials are dense in $A^2_\infty(\nu(|z|))$.*

*Proof.* Suppose $f$ is a function in $A^2_\infty(\nu(|z|))$ such that $\langle f, z^n\rangle = 0$ for every $n \geq 0$, and let $\sum_{n=0}^\infty a_n z^n$ be the Taylor series of $f$. We have, by the dominated convergence theorem and Hölder's inequality, that

$$\begin{aligned}0 = \langle f, z^m\rangle &= \int_{\mathbb{C}} \left(\sum_{n=0}^\infty a_n z^n\right)\overline{z^m}\nu(|z|)\,dA(z)\\ &= \lim_{r\to\infty}\int_{\mathbb{D}_r}\left(\sum_{n=0}^\infty a_n z^n\right)\overline{z^m}\nu(|z|)\,dA(z),\end{aligned}$$

for $m \geq 0$. By the uniform convergence of the Taylor series on $\mathbb{D}_r$ and the integrability of $\nu(|z|)$, the above expressions equal

$$\begin{aligned}&\lim_{r\to\infty}\sum_{n=0}^\infty\int_{\mathbb{D}_r} a_n z^n\overline{z^m}\nu(|z|)\,dA(z)\\ &\quad= \lim_{r\to\infty}\int_{\mathbb{D}_r} a_m|z|^{2m}\nu(|z|)\,dA(z)\\ &\quad= a_m\int_{\mathbb{C}}|z|^{2m}\nu(|z|)\,dA(z).\end{aligned}$$

However, the above integral is positive, so $a_m = 0$ for each $m \geq 0$, and thus, $f$ is identically 0. This shows that the polynomials are dense in $A^2_\infty(\nu(|z|))$. □

For a radial weight function $\nu$ with the property that $A^p_\infty$ has bounded point evaluations (and thus, is a Banach space), define $m_p(z;\nu) = \sup_{\|f\|_{A^p_\infty(\nu)}=1}|f(z)|$.

**Proposition 5.5.** *Let $\nu$ be a function on $[0,\infty)$ such that $A^p_\infty(\nu(|z|))$ contains every polynomial and has point evaluations uniformly bounded on compact subsets of $\mathbb{C}$. Suppose that, for any $\rho$ such that $0<\rho<1$, there is some function $\mu$ such that $A^2_\infty(\mu(|z|))$ has point evaluations uniformly bounded on compact subsets of $\mathbb{C}$ and such that*

$$\int_{\mathbb{C}} m_p(\rho z;\nu)^2\mu(|z|)\,dA(z) = C_1^2 < \infty \tag{5.1}$$

*and*

$$\int_{\mathbb{C}} m_2(z;\mu)^p \nu(|z|)\, dA(z) = C_2^p < \infty. \tag{5.2}$$

*Then, the polynomials are dense in* $A^p_\infty(\nu(|z|))$.

*Proof.* Let $0 < \rho < 1$. Note that

$$M_p(r, f_\rho) = M_p(r\rho, f) \leq M_p(r, f)$$

so that $\|f_\rho\|_{A^p_\infty(\nu(|z|))} \leq \|f\|_{A^p_\infty(\nu(|z|))}$. Thus,

$$|f(\rho z)| \leq m_p(\rho z, \nu)\|f\|_{A^p_\infty(\nu(|z|))}.$$

Therefore, the quantity $\|f_\rho\|_{A^2_\infty(\mu(|z|))}$ is at most $C_1\|f\|_{A^p_\infty(\nu(|z|))}$ and $f_\rho \in A^2_\infty(\mu(|z|))$. Note that this implies that every polynomial is in $A^2_\infty(\mu(|z|))$, since, if $p$ is a polynomial, then $p_{1/\rho}$ is also a polynomial and is in $A^p_\infty(\nu(|z|))$, which implies that $p$ is in $A^2_\infty(\mu(|z|))$.

Now, by Proposition 5.4, there is a sequence of polynomials in $A^2_\infty(\mu(|z|))$ that approach $f_\rho$. But, for any function $g \in A^2_\infty(\mu(|z|))$, we have $\|g\|_{A^p_\infty(\nu(|z|))} \leq C_2\|g\|_{A^2_\infty(\mu(|z|))}$. Thus, there is a sequence of polynomials approaching $f_\rho$ in $A^p_\infty(\nu(|z|))$. Since the functions $f_\rho$ approach $f$ in $A^p_\infty(\nu(|z|))$, there is a sequence of polynomials approaching $f$ in $A^p_\infty(\nu(|z|))$. □

Note that the quantities $m_p(z, \nu)$ and $m_2(z, \mu)$ can often be estimated by the method used in the proof of Proposition 5.1.

The next corollary follows from Proposition 5.5.

**Corollary 5.6.** *Suppose that* $\nu$ *is a nonzero decreasing function on* $[0, \infty)$ *such that* $\nu(|z|) \in L^1(\mathbb{C})$ *and such that every polynomial is in* $A^p_\infty(\nu(|z|))$. *Also, assume that for each* $\rho$ *such that* $0 < \rho < 1$, *there is a* $\beta$ *such that* $0 < \beta < 1$ *and such that*

$$\int_{\mathbb{C}} \nu(\rho|z| + 1)^{-2/p}\nu(|z|)^{2\beta/p} dA(z) < \infty$$

*and*

$$\int_{\mathbb{C}} \nu(|z| + 1)^{-\beta}\nu(|z|)\, dA(z) < \infty.$$

*Then, the polynomials are dense in* $A^p_\infty(\nu(|z|))$.

*Proof.* Let $f$ be an entire function. By subharmonicity,

$$|f(z)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+\mathrm{re}^{i\theta})|^p d\theta$$

for any $r > 0$. If we multiply the previous displayed inequality by $2\pi r\nu(|z|+1)$ and integrate from $r=0$ to $r=1$, we find that

$$\begin{aligned}\pi\nu(|z|+1)|f(z)|^p &\leq \int_{|z-w|<1} |f(w)|^p \nu(|z|+1)\, dA(w)\\ &\leq \int_{|z-w|<1} |f(w)|^p \nu(|w|)\, dA(w)\\ &\leq \int_{\mathbb{C}} |f(w)|^p \nu(|w|)\, dA(w).\end{aligned}$$

Thus, we have that

$$|f(z)| \leq \pi^{1/p}\nu(|z|+1)^{-1/p}\|f\|_{A^p_\infty(\nu(|z|))}.$$

And, similarly,

$$|f(z)| \leq \pi^{1/2}\nu(|z|+1)^{-\beta/p}\|f\|_{A^2_\infty(\nu(|z|)^{2\beta/p})}$$

for any function $f \in A^2_\infty(\nu(|z|)^{2\beta/p})$. So, if there is some $\beta$ such that $0 < \beta < 1$ and such that

$$\int_{\mathbb{C}} \nu(\rho|z|+1)^{-2/p}\nu(|z|)^{2\beta/p} dA(z) < \infty$$

and

$$\int_{\mathbb{C}} \nu(|z|+1)^{-\beta}\nu(|z|)\, dA(z) < \infty,$$

then the result will hold by Proposition 5.5. $\square$

Note that, if $\nu$ is a bounded function that is eventually decreasing, then $A^p_\infty(\nu(|z|))$ is equivalent in norm to $A^p_\infty(\widetilde{\nu}(|z|))$, where $\widetilde{\nu}$ is decreasing and $\widetilde{\nu}(x) = \nu(x)$ for $x$ sufficiently large. Thus, the previous corollary can be applied in modified form to such functions $\nu$.

The next corollary is needed to apply the results of Section 3 to the Fock space. It follows from the above corollary by choosing $\beta$ such that $\rho < \beta < 1$.

**Corollary 5.7.** *Let $\alpha > 0$ and $0 < p < \infty$. Then the space $A^p_\infty(|z|^2 e^{-\alpha|z|^2} + e^{-\alpha|z|^2})$ is a Banach space in which the polynomials are dense.*

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