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HARTMAN-GROBMAN THEOREM FOR ITERATED FUNCTION SYSTEMS

MEHDI FATEHI NIA AND FATEMEH REZAEI

ABSTRACT. In this paper, for iterated function systems, we define the classic concept of the dynamical systems: topological conjugacy of diffeomorphisms. We generalize the Hartman-Grobman theorem for one-dimensional iterated function systems on $\mathbb{R}$ Also, we introduce the basic concept of structural stability for an iterated function system, and therefore, we investigate the necessary condition for structural stability of an iterated function system on $\mathbb{R}$

1. Introduction. This section includes three subsections. In the first subsection, we provide an almost perfect review of the literature on studies which have been done on iterated function systems. Also, we introduce their applications to understand the importance of studying the IFSs. In the second, we describe the history of the creation and importance of one of the essential theorems of the local dynamic that is named the Hartman-Grobman theorem and, in what follows, we study the research done on the generalization and extension of this theorem. In the third subsection, we define a very important concept of the local dynamic, which is related to this theorem and is called structural stability. Moreover, we briefly state the history of this concept.

1.1. The concept of the iterated function systems was applied in 1981 by Hutchinson [**37**]. Moreover, the mathematical basis of the iterated function system was established by him; but this term was presented by Barnsley, briefly, as IFS [**36**]. We know that an IFS includes a set Λ and some functions f_λ, $\lambda \in \Lambda$, on an arbitrary space M. Since, in an IFS, the set Λ can be finite or infinite (countable) or its functions can

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be special, different IFSs have been investigated. Most studies on the finite IFSs were done by Barnsley [**4, 7, 8, 10, 11**]. We can see the generated countable IFSs in some articles, like [**41**]. Various functions were considered in the study of the IFSs, such as affine transformations on Euclidean spaces [**4**], onto transformations on real projective spaces [**16**], Mobius transformations on a complex plane (or equivalently, Riemann sphere) [**58**]. Although, generally, [**60**], IFSs on manifolds have been studied, IFSs can be found that are considered on Hilbert spaces, which are called Perry IFSs [**50**].

We know that an attractor of an IFS is something that is called fractal. But, what is a fractal? It is understood that an accurate description of the geometric structure of many natural things, like clouds, forests, mountains, flowers, galaxies, etc., through the use of classical geometry, has not been attainable. Mandelbrot, in 1982, changed this perspective by which he extended classical geometry into, so-called, fractal geometry. In fact, "fractal" is comprised of iterating the functions in a set, called the iterated functions system. It has been proved in the literature that a fractal is the attractor of an IFS. The IFS model is a base for different applications, such as computer graphics, image compression, learning automata, neural nets and statistical physics [**28**]. Thus, study of the fractal is important, and therefore, from one point of view, the study of an IFS as the means of generating a fractal is important [**11**]. The existence and uniqueness of the attractor of a finite IFS was proven by Hata in 1985 [**35**], also see [**27**].

Abundant studies have shown the context of the topological properties (such as dimension, measure, separation properties) of an attractor, for example, [**17, 24, 26, 41, 42, 45**]. The attractor of the affine IFS has many applications, for example, image compression [**9, 15, 30, 38, 54, 62**] and geometric modeling [**20, 57, 61**]. Moreover, IFSs that are said to be recurrent IFSs [**12**] have applications in the generation of digital images since these images have curves that are not generatable using standard techniques [**14**]. In addition, IFSs can be used as tools for filtering and transforming digital images [**13**].

IFSs have also been studied from a dynamic point of view, for example, the stability and the hyperbolicity of an IFS in [**1**] and the asymptotic stability of a countable IFS (presentation of sufficient condition) in [**39**], the asymptotic behavior of a finite IFS with contraction and positive, continuous, place-dependent probability functions in [**40**]. Here, we also study a dynamic property, structural stability, for the finite IFSs that have not yet been investigated.

1.2. We begin here by presenting the concept of "topologically conjugate." Occasionally, it has been seen that two systems seemingly are different, but, if we investigate these systems, dynamically, we find that they have the same behavior [**43**], in other words, the two systems are "equivalent," that is, studying one system will provide dynamic information about the other system. Thus, such systems allow us to look for some (approximately) simple system or to identify one equivalent to the complicated system. In this context, concepts were proposed, called topological equivalence and topological conjugacy. In this paper, we also define these concepts for IFSs and determine a special class of IFSs that has such properties.

A fundamental theorem for studying the local behavior of a system which is a strong tool in dynamical systems is well known as the Hartman-Grobman theorem or linearization theorem. This theorem examines the local behavior of a system around hyperbolic fixed points; specifically, the theorem states that the dynamical behavior of a system is the same as the dynamical behavior of its linearization near the hyperbolic fixed points. Thus, we can locally draw the phase space around these special points. In particular, this matter is important when the given system is nonlinear [**63**]. Formation of this critical theorem was a question asked by Peixoto. You can see this question and its answer in [**33**]. In 1959, Hartman answered the question [**33**]. In addition, according to the literature [**34**], Grobman [**31**], perhaps separately, provided a demonstration of the theorem; therefore, this theorem is well known as the Hartman-Grobman theorem (H-G-T). In 1968, Pails extended the H-G-T for maps to the infinite Banach space and, for this case, gave a short proof [**46**]. However, the general state of the H-G-T for maps in Banach spaces was proven by Quandt in [**53**]. The H-G-T has been expanded to different systems, such as non-autonomous systems, [**48**], and discrete and random dynamical systems, the first linearization of which was in 1994 [**59**]. Then, its generalization came in [**23**], with continuous and random dynamical systems [**22**] and control systems with special inputs [**6**]. Since the linearization theorem has many applications, in particular, in theme partial differential equations on Banach spaces, the extension of H-

G-T on Banach spaces is important. Some preeminent researchers were Barreira and Valls in 2005, who researched Banach spaces for the non-uniformly hyperbolic dynamic [**18**]. Sola-Morales and Rodrigues generalized the H-G-T for infinite spaces with special conditions such as Hilbert spaces [**55, 56**]. More research was performed on Banach spaces in order to expand the H-G-T for maps, Rayskin and Belitskii [**19**].

It is well known that the Hartman-Grobman theorem states that any C^1-diffeomorphism is topologically equivalent to its linear part in the neighborhood of the hyperbolic fixed points. We also generalize this theorem for IFSs. In fact, we show that, if the origin is a hyperbolic fixed point of the C^1-diffeomorphisms of IFSs, $\mathcal{F}$ and $\mathcal{G}$ and all the derivatives of these functions at zero belong to the same interval $(0,1)$ (or $(-1,0)$ or $(1,+\infty)$ or $(-\infty,-1)$). Then, these two IFSs have the same dynamical behaviors.

1.3. Sometimes, although these systems look alike, they, seemingly, have completely different dynamical behaviors such as bifurcation, chaos, etc. Therefore, this leads to the creation of another concept, called "structural stability." The concept of structural stability was introduced by Peixoto [**21**]. In fact, this concept is a generalization of the concept of grosser, or rough, systems [**2**]. Andronov has been interested in the preservation of the qualitative properties of flows under small perturbations and introduced the problem [**3**]. Indeed, Peixoto, in 1959, introduced the concept of the structural stability using corrections of mistakes of the article [**5**].

We say that C^k-diffeomorphism f is *structurally stable* if there exists a neighborhood of f in the C^k-topology such that f is topologically conjugate to every function at this neighborhood; other words, a C^k-diffeomorphism f is structurally stable if, for any $\epsilon > 0$, there is a neighborhood $U(\epsilon)$ of f in the C^k-topology such that any C^k-diffeomorphism $f_1 \in U(\epsilon)$ is topologically conjugate to f [**51**]. We consider the distance between two IFSs as the maximum distance between the functions of two IFSs, and thus, nearby IFSs makes sense. Thenceforth, we define the concept of structural stability for IFSs. Moreover, we demonstrate that the necessary condition for an IFS to be structurally stable is that all of the fixed points of IFS functions should be hyperbolic.

2. A preliminary lemma. We know that the diffeomorphisms $f, g : \mathbb{R}^m \to \mathbb{R}^m$ are topologically conjugate if there exists a homeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ such that $h \circ f = g \circ h$, or equivalently, $f = h^{-1} \circ g \circ h$. The function h is said to be a topological conjugacy. Also, for given $\epsilon > 0$, we say that the diffeomorphisms f and g are ε-topologically conjugate if there exists a topological conjugacy h such that $\|x - h(x)\| < \epsilon$ for every $x \in \mathbb{R}^m$. ($\|\cdot\|$ is the norm on $\mathbb{R}^m$ **[52]**.)

The next lemma has been proved in some literature like **[47]** (it was given in **[25]** as an example). However, since this lemma has a critical role in the demonstration of some of the theorems in this paper, we give a proof with complete details, especially since these details are remarkable. This lemma states that the contractive functions on $\mathbb{R}$ are topologically conjugate.

Lemma 2.1. *Suppose that real value functions f and g on $\mathbb{R}$ are defined with the criteria*

$$f(x) = kx \quad \textit{and} \quad g(x) = mx,$$

where $0 < k$, $m < 1$. Then, f and g are topologically conjugate.

Proof. Let a be an arbitrary positive real number. We know that there exists a homeomorphism h from $[f(a), a]$ to $[g(a), a]$ such that $h(f(a)) = g(a)$ and $h(a) = a$. Suppose that $x \in \mathbb{R}$ is arbitrary and greater than a. A criterion for function f is that, as n increases, the value $f^n(x)$ approaches the origin. As a result, there exists an $n \in \mathbb{N}$ such that $f^n(x) < a$, assuming that n_x is the first n with this property, that is, $k^{n_x} x < a < k^{n_x - 1} x$. By considering the inequality $a < k^{n_x - 1} x$, we have $ka < k^{n_x} x$; thus, $ka < k^{n_x} x < a$, that is, $f^{n_x}(x) = k^{n_x} x \in [f(a), a]$. Now, for every $x > a$, we define the function h as:

$$\begin{cases} h : (a, +\infty) \longrightarrow (a, +\infty) \\ x \longmapsto g^{-n_x}(h(f^{n_x}(x))). \end{cases} \tag{2.1}$$

Firstly, h is well defined by considering the way n_x is chosen. Moreover, the range of the function h is $(a, +\infty)$ since, for every $x > a$, $f^{n_x}(x) \in [f(a), a]$; hence, in the definition of the function h and its continuity, we have $h(f^{n_x}(x)) \in [g(a), a]$, that is,

$$ma = g(a) < h(f^{n_x}(x)) < a,$$

and, since the function g^{-n_x} is strictly increasing, we obtain the following relation:

$$g^{-n_x}(ma) < g^{-n_x}(h(f^{n_x}(x))) \Longrightarrow \left(\frac{1}{m}\right)^{n_x} \cdot ma < g^{-n_x}(h(f^{n_x}(x)))$$
$$\Longrightarrow \left(\frac{1}{m}\right)^{n_x-1} a < g^{-n_x}(h(f^{n_x}(x))).$$

We know that $n_x \in \mathbb{N}$, and since $0 < m < 1$, so $1/m > 1$; consequently, we have $a < g^{-n_x}(h(f^{n_x}(x)))$, that is, $h(x) > a$. Secondly, the function h is a homeomorphism since it is a composition of the homeomorphisms.

Now, suppose $0 < x < f(a)$, that is, $0 < x < ka$ so $0 < x/k < a$, and this means that $0 < f^{-1}(x) < a$ is based upon the criterion of the function f^{-1}. As n increases, the value of $f^{-n}(x)$ moves far away from the origin; thus, there exists an $n \in \mathbb{N}$ such that $f^{-n}(x) > f(a)$. Assume that n_x is the first n with this property, that is,

$$\left(\frac{1}{k}\right)^{n-1} x < ka < \left(\frac{1}{k}\right)^{n} x.$$

By considering the inequality $(1/k)^{n-1}x < ka$, we obtain $(1/k)^n x < a$; thus, $ka < (1/k)^n x < a$, that is,

$$f^{-n}(x) = \left(\frac{1}{k}\right)^{n} x \in [f(a), a].$$

For every $x \in (0, f(a))$, we define the function h from $(0, f(a))$ to $(0, g(a))$ with the criterion $h(x) = g^{n_x}(h(f^{-n_x}(x)))$. Clearly, this function is well defined, and the range of the function h is $(0, g(a))$ since, for every $x \in (0, f(a))$, $f^{-n_x}(x) \in [f(a), a]$. Thus, $h(f^{-n_x}(x)) \in [g(a), a]$, that is, $ma = g(a) < h\big(f^{-n_x}(x)\big) < a$, and, since the function g^{n_x} is strictly increasing, then

$$g^{n_x}(g(a)) < g^{n_x}(h(f^{-n_x}(x))) < g^{n_x}(a)$$
$$\Longrightarrow 0 < m^{n_x+1}a < g^{n_x}(h(f^{-n_x}(x)))$$
$$< m^{n_x}a = m^{n_x-1}(ma),$$

that is, $h(x) \in (0, g(a))$. Also, the function h is a homeomorphism since it is a composition of the homeomorphisms.

Hence, the function h was defined on $(0, +\infty)$. We define $h(x) = -h(-x)$ for each $x \in (-\infty, 0]$.

Now, we show that $h \circ f = g \circ h$. Suppose that $x > a$ is arbitrary. We show that there exists an $n_x \in \mathbb{N}$ such that $f^{n_x}(x) \in [f(a), a]$, that is, $f(a) \leq k^{n_x} x \leq a$, so $f(a) \leq k^{n_x - 1}(kx) \leq a$. In other words, $f(a) \leq k^{n_x-1}(f(x)) \leq a$, meaning that $f^{n_x-1}(f(x)) \in [f(a), a]$. We claim that $n_{f(x)} = n_x - 1$. We prove this claim with a demonstration by contradiction. Assume that there exists a natural number $m < n_x - 1$ such that $f^m(f(x)) \in [f(a), a]$, that is, $f(a) \leq f^{m+1}(x) \leq a$. We also know that $m + 1 < n_x$; this is contradictory with the smallest number of n_x for x, and thus, the claim is proven. Therefore, for $x > a$, we have:

$$\begin{aligned} h(f(x)) &= g^{-n_{f(x)}}(h(f^{n_{f(x)}}(f(x)))) \\ &= g^{-n_x+1}(h(f^{n_x-1}(f(x)))) \\ &= g(g^{-n_x}(h(f^{n_x}(x)))) \\ &= g(h(x)). \end{aligned}$$

Now, let $x \in (0, f(a))$. We show that there exists an $n_x \in \mathbb{N}$ such that $f^{-n_x}(x) \in [f(a), a]$, that is, $f(a) \leq f^{-n_x}(x) \leq a$. Thus, $f(a) \leq f^{-(n_x+1)}(f(x)) \leq a$. We claim that $n_{f(x)} = n_x + 1$. We prove this claim with a demonstration by contradiction. Assume that there exists a natural number $m < n_x + 1$ such that $f(a) \leq f^{-m}(f(x)) \leq a$. Then, $f(a) \leq f^{-(m-1)}(x) \leq a$, and we have $m - 1 < n_x$; however, this is contradictory by way of the choice n_x for x, and thus, the claim is proven. Therefore, for $x \in (0, f(a))$, we have

$$\begin{aligned} h(f(x)) &= g^{n_{f(x)}}(h(f^{-n_{f(x)}}(f(x)))) \\ &= g^{n_x+1}(h(f^{-n_x-1}(f(x)))) \\ &= g^{n_x+1}(h(f^{-n_x}(x))) \\ &= g(g^{n_x}(h(f^{-n_x}(x)))) \\ &= g(h(x)). \end{aligned}$$

By considering the criteria of the functions f, g and h, we observe that these functions are odd functions. Thus, for each $x \in (-\infty, 0]$, we obtain $h(f(x)) = h(-f(-x)) = -h(f(-x)) = -g(h(-x)) = g(-h(-x)) = g(h(x))$. Hence, we have found the homeomorphism h

from $\mathbb{R}$ to $\mathbb{R}$ such that $h \circ f = g \circ h$; that is, f and g are topologically conjugate. □

By considering the previous lemma, we can say that the expansive functions on $\mathbb{R}$ are topologically conjugate.

Corollary 2.2. *Suppose that*

$$\begin{cases} f : \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto kx \end{cases} \quad \textit{and} \quad \begin{cases} g : \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto mx, \end{cases}$$

where $k, m > 1$. Then, f and g are topologically conjugate.

Proof. Clearly, the functions f and g are invertible; we have

$$\begin{cases} f^{-1} : \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto (1/k)x \end{cases} \quad \text{and} \quad \begin{cases} g^{-1} : \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto (1/m)x \end{cases}$$

such that $0 < 1/k$, $1/m < 1$. Thus, by using Lemma 2.1, the functions f^{-1} and g^{-1} are topologically conjugate, that is, there exists a homeomorphism h from $\mathbb{R}$ to $\mathbb{R}$ such that $h \circ f^{-1} = g^{-1} \circ h$, implying that $f^{-1} = h^{-1} \circ g^{-1} \circ h$. Therefore, we obtain $f \circ h^{-1} = h^{-1} \circ g$ and, since h^{-1} is a homeomorphism, hence, f and g are topologically conjugate. □

We see that, if k and m both belong to the interval $(0, 1)$ or $(1, +\infty)$, then f and g are topologically conjugate. Moreover, similarly, this statement is proven when k and m belong to the interval $(-1, 0)$ or $(-\infty, -1)$. Note that, if k and m do not belong to the same interval, then f and g are not topologically conjugate. Some examples are introduced next to illustrate this matter.

Example 2.3. Consider $f(x) = 2x$ and $g(x) = x/2$. Suppose that f and g are topologically conjugate; thus, they have the same behavior. However, note that we have $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} 2^n x = \pm\infty$ and $\lim_{n\to\infty} g^n(x) = \lim_{n\to\infty}(1/2)^n x = 0$. Hence, we may conclude that f and g are not topologically conjugate.

Example 2.4. Assume that $f(x) = 3x$ and $g(x) = -3x$. The criterion of the function f holds its direction, but the function g reverses direction. This means that f and g do not have the same behavior; thus, they are not topologically conjugate.

Example 2.5. Consider $f(x) = -4x$ and $g(x) = -x/4$. Clearly, we have $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty}(-4)^n x = \pm\infty$ and $\lim_{n\to\infty} g^n(x) = \lim_{n\to\infty}(-1/4)^n x = 0$, that is, f and g do not have the same behavior. Thus, they cannot be topologically conjugate.

Example 2.6. Assume that $f(x) = x/5$ and $g(x) = -x/5$. Since the function f holds its direction but the function g reverses direction, they do not have the same behavior. Therefore, we deduce that f and g are not topologically conjugate.

3. Essential definitions and theorems about topological conjugacy of IFSs. Now, we define the concepts of IFS and contractive IFS accurately and formally [**44**].

Definition 3.1. Let (M, d) be a complete metric space and $\mathcal{F}$ a family of continuous mapping $f_\lambda : M \to M$ for every $\lambda \in \Lambda$, where Λ is a finite nonempty set, that is,

$$\mathcal{F} = \{f_\lambda,\, M : \lambda \in \Lambda = \{1, 2, \ldots, N\}\}.$$

We call this family an *iterated function system*, IFS.

Definition 3.2. IFS $\mathcal{F} = \{f_\lambda,\, M : \lambda \in \Lambda\}$ is called *contractive* if each function f_λ, $\lambda \in \Lambda$, is a contractive function, that is, there exists a positive real number $0 < s_\lambda < 1$ such that, for every $x, y \in M$, $d(f_\lambda(x), f_\lambda(y)) \leq s_\lambda\, d(x, y)$.

Let $T = \mathbb{Z}$ or $T = \mathbb{N}$. The set of all infinite sequences $\{\lambda_i\}_{i\in T}$ is denoted by Λ^T, where λ_i is an arbitrary element of Λ. If $T = \mathbb{N}$, then every element $\Lambda^{\mathbb{N}}$ can be shown as $\sigma = \{\lambda_1, \lambda_2, \ldots\}$. Also, the notation F_{σ_n} stands for $F_{\sigma_n} = f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \ldots \circ f_{\lambda_2} \circ f_{\lambda_1}$, for every $n \in \mathbb{N}$.

In this paper, we shall define the concept of topological conjugacy for the IFSs. Previously, this concept was defined in [**29**]; however, we give a comprehensive definition which includes the previous definition. Thus, we call it a weakly topological conjugate. The previous definition is as follows.

Definition 3.3. Suppose that $\mathcal{F} = \{f_\lambda, M : \lambda \in \Lambda\}$ and $\mathcal{G} = \{g_\lambda, M : \lambda \in \Lambda\}$ are two IFSs. The IFSs $\mathcal{F}$ and $\mathcal{G}$ are said to be topologically conjugate if there exists a homeomorphism $h : M \to M$ such that $f_\lambda \circ h = h \circ g_\lambda$ for every $\lambda \in \Lambda$.

Our definition is as follows.

Definition 3.4. Suppose that $\mathcal{F} = \{f_\lambda, M : \lambda \in \Lambda\}$ and $\mathcal{G} = \{g_\lambda, M : \lambda \in \Lambda\}$ are two IFSs. For a given $\sigma \in \Lambda^{\mathbb{N}}$, we say that $\mathcal{F}$ and $\mathcal{G}$ are *weakly topological conjugate* if, for every $n \in \mathbb{N}$, there is a homeomorphism $h : M \to M$ such that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$.

A comparison of the two definitions shows that, if any two IFSs are topologically conjugate, then they will be weakly topological conjugates. The main problem with the first definition is in the presentation of a homeomorphism h for all $\lambda \in \Lambda$, which is a very difficult task. We solve this problem by providing a new definition.

Hereafter, we will investigate IFSs. In the following, we show that, if the model of every two IFSs is $\{ax, bx, \mathbb{R}\}$, where both a and b belong to the same interval $(0, 1)$ (or $(-1, 0)$ or $(1, +\infty)$ or $(-\infty, -1)$), then they are weakly topological conjugates.

Theorem 3.5. *Suppose that* $\mathcal{F} = \{k_1x, k_2x, \mathbb{R}\}$ *and* $\mathcal{G} = \{m_1x, m_2x, \mathbb{R}\}$ *are two* IFSs *where* $0 < k_i, m_i < 1$, $i = 1, 2$. *Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates.*

Proof. Put $f_i(x) = k_ix$ and $g_i(x) = m_ix$ for $i = 1, 2$. Assume that $\sigma = \{\lambda_1, \lambda_2, \dots\}$ is an arbitrary sequence from indices $\Lambda = \{1, 2\}$. Let $n \in \mathbb{N}$. We know that $F_{\sigma_n} = f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_2} \circ f_{\lambda_1}$, so, for every $x \in \mathbb{R}$, we have

$$\begin{aligned}
F_{\sigma_n}(x) &= f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_2} \circ f_{\lambda_1}(x) \\
&= f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_2}(f_{\lambda_1}(x)) \\
&= f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_2}(k_{\lambda_1}x) \\
&= f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_3}(k_{\lambda_2} \cdot k_{\lambda_1}x) \\
&= \dots = k_{\lambda_n} \cdot k_{\lambda_{n-1}} \cdots k_{\lambda_2} \cdot k_{\lambda_1}x.
\end{aligned}$$

Put $k^*_{\sigma_n} = k_{\lambda_n} \cdot k_{\lambda_{n-1}} \cdots k_{\lambda_2} \cdot k_{\lambda_1}$. Clearly, $0 < k^*_{\sigma_n} < 1$ since every k_{λ_i}, $i = 1, 2, \ldots, n$, is a value between zero and one. Also, in the same manner as for the IFS $\mathcal{G}$, we obtain

$$\begin{aligned} G_{\sigma_n}(x) &= g_{\lambda_n} \circ g_{\lambda_{n-1}} \circ \cdots \circ g_{\lambda_2} \circ g_{\lambda_1}(x) \\ &= g_{\lambda_n} \circ g_{\lambda_{n-1}} \circ \cdots \circ g_{\lambda_2}(g_{\lambda_1}(x)) \\ &= g_{\lambda_n} \circ g_{\lambda_{n-1}} \circ \cdots \circ g_{\lambda_2}(m_{\lambda_1} x) \\ &= g_{\lambda_n} \circ g_{\lambda_{n-1}} \circ \cdots \circ g_{\lambda_3}(m_{\lambda_2} \cdot m_{\lambda_1} x) \\ &= \cdots = m_{\lambda_n} \cdot m_{\lambda_{n-1}} \cdots m_{\lambda_2} \cdot m_{\lambda_1} x. \end{aligned}$$

Now, we set $m^*_{\sigma_n} = m_{\lambda_n} \cdot m_{\lambda_{n-1}} \cdots m_{\lambda_2} \cdot m_{\lambda_1}$. Clearly, $0 < m^*_{\sigma_n} < 1$ since every m_{λ_i}, $i = 1, 2, \ldots, n$, is a value between zero and one.

Hence, $F_{\sigma_n}(x) = k^*_{\sigma_n} x$ and $G_{\sigma_n}(x) = m^*_{\sigma_n} x$, where $0 < k^*_{\sigma_n}, m^*_{\sigma_n} < 1$; thus, for every $n \in \mathbb{N}$, by using Lemma 2.1, the functions F_{σ_n} and G_{σ_n} are topological conjugates. This means that, for every $n \in \mathbb{N}$, there exists a homeomorphism h from $\mathbb{R}$ to $\mathbb{R}$ such that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$, and this shows that the two IFSs, $\mathcal{F}$ and $\mathcal{G}$, are weakly topological conjugates. □

Theorem 3.6. *Suppose that* $\mathcal{F} = \{-k_1 x, -k_2 x, \mathbb{R}\}$ *and* $\mathcal{G} = \{-m_1 x, -m_2 x, \mathbb{R}\}$ *are two* IFSs, *where* $0 < k_i, m_i < 1$, $i = 1, 2$. *Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates.*

Proof. Put $f_i(x) = -k_i x$ and $g_i(x) = -m_i x$ for $i = 1, 2$. Assume that $\sigma = \{\lambda_1, \lambda_2, \ldots\}$ is an arbitrary sequence from indices $\Lambda = \{1, 2\}$. Analogously to the proof of Theorem 3.5, for every $n \in \mathbb{N}$, we obtain $F_{\sigma_n}(x) = (-1)^n k_{\lambda_n} \cdot k_{\lambda_{n-1}} \cdots k_{\lambda_2} \cdot k_{\lambda_1} x$ and $G_{\sigma_n}(x) = (-1)^n m_{\lambda_n} \cdot m_{\lambda_{n-1}} \cdots m_{\lambda_2} \cdot m_{\lambda_1} x$, for all $x \in \mathbb{R}$, where $0 < k_{\lambda_i}, m_{\lambda_i} < 1$ for $i = 1, 2, \ldots, n$. Put $\mathcal{F}^* = \{k_1 x, k_2 x, \mathbb{R}\}$ and $\mathcal{G}^* = \{m_1 x, m_2 x, \mathbb{R}\}$. From Theorem 3.5, for given σ as above and for every $n \in \mathbb{N}$, there exists a homeomorphism h^* on $\mathbb{R}$ such that $h^* \circ F^*_{\sigma_n} = G^*_{\sigma_n} \circ h^*$. Put $h = -h^*$. We claim that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$.

First, note that, for each $x \in \mathbb{R}$, we have $F_{\sigma_n}(x) = (-1)^n F^*_{\sigma_n}(x)$ and $G_{\sigma_n}(x) = (-1)^n G^*_{\sigma_n}(x)$, and also, the homeomorphism h^* is an odd function of Lemma 2.1. Moreover, the functions $F^*_{\sigma_n}$ and $G^*_{\sigma_n}$ are clearly odd. Suppose that $x \in \mathbb{R}$ is arbitrary. We prove the claim for two states: when n is odd and when n is even. If n is an odd number, then we have:

$$\begin{aligned} h(F_{\sigma_n}(x)) &= h(-F^*_{\sigma_n}(x)) = -h^*(-F^*_{\sigma_n}(x)) = h^*(F^*_{\sigma_n}(x)) \\ &= G^*_{\sigma_n}(h^*(x)) = G^*_{\sigma_n}(-h^*(-x)) = -G^*_{\sigma_n}(h^*(-x)) \\ &= -G^*_{\sigma_n}(-h^*(x)) = G_{\sigma_n}(h(x)). \end{aligned}$$

Similarly, when n is an even number, we have:

$$\begin{aligned} h(F_{\sigma_n}(x)) &= h(F^*_{\sigma_n}(x)) = -h^*(F^*_{\sigma_n}(x)) \\ &= -G^*_{\sigma_n}(h^*(x)) = G^*_{\sigma_n}(-h^*(x)) = G_{\sigma_n}(h(x)). \end{aligned}$$

Hence, for every $n \in \mathbb{N}$, we have the homeomorphism h (since h^* is the homeomorphism) such that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$, that is, $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates. □

Corollary 3.7. *Suppose that* $\mathcal{F}=\{k_1x,\, k_2x,\, \mathbb{R}\}$ *and* $\mathcal{G}=\{m_1x,\, m_2x,\, \mathbb{R}\}$, *where* k_i *and* m_i *are more than* 1 *or both are less than* -1 *for each* $i = 1, 2$*. Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates.*

Proof. First, we suppose that k_i and m_i are more than 1 for each $i = 1, 2$. Put $f_i(x) = k_ix$ and $g_i(x) = m_ix$ for $i = 1, 2$. Assume that $\sigma = \{\lambda_1, \lambda_2, \ldots\}$ is an arbitrary sequence from indices $\Lambda = \{1, 2\}$. Analogously to the proof of Theorem 3.5, for every n, we obtain $F_{\sigma_n}(x) = k^*_{\sigma_n}x$, where $k^*_{\sigma_n} = k_{\lambda_n} \cdots k_{\lambda_1}$. Clearly, $k^*_{\sigma_n} > 1$, and also, $G_{\sigma_n}(x) = m^*_{\sigma_n}x$, where $m^*_{\sigma_n} = m_{\lambda_n} \cdots m_{\lambda_1}$ and $m^*_{\sigma_n} > 1$. Thus, for every $n \in \mathbb{N}$, F_{σ_n} and G_{σ_n} are topologically conjugate from Corollary 2.2, that is, for every $n \in \mathbb{N}$, there exists a homeomorphism h on $\mathbb{R}$ such that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$. Hence, $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates. Now, suppose that k_i and m_i are less than -1 for each $i = 1, 2$. Corollary 3.7 can be proven in a similar manner to the proof of Theorem 3.6. □

4. Extension of the Hartman-Grobman theorem for IFSs.

It has been shown in the literature that nonlinear systems sometimes "look like" their linearizations near a hyperbolic fixed point (for example, [**43, 47, 49**]). The theorem found therein is well known as the Hartman-Grobman theorem.

Hartman-Grobman theorem ([32]). *Suppose that* x_0 *is a hyperbolic fixed point of the local* C^1 *diffeomorphism* f *defined on a neighborhood* U *of* x_0 *in* $\mathbb{R}^m$*. Let* $L = Df(x_0)$*. Then, there exists a neighborhood*

$U_1 \subseteq U$ of x_0 and a homeomorphism h from U_1 into $\mathbb{R}^m$ such that $h(x_0) = 0$ and $hf(x) = Lh(x)$ for $x \in U_1 \cap f^{-1}(U_1)$ (or $hfh^{-1}(y) = L(y)$ for $h^{-1}(y) \in U_1 \cap f^{-1}(U_1)$).

We record here the necessary theorems and their corollaries for the purpose of extending the Hartman-Grobman theorem for IFSs.

Theorem 4.1. *Suppose that $\mathcal{F} = \{k_1 I + \varphi_1, k_2 I + \varphi_2; \mathbb{R}\}$ and $\mathcal{G} = \{m_1 I + \psi_1, m_2 I + \psi_2; \mathbb{R}\}$ are two* IFSs *where I is the identity map on $\mathbb{R}$ and for $i = 1, 2$, k_i and m_i all of the same sign, $0 < |k_i|$, $|m_i| < 1$ and, in addition, the functions φ_i and ψ_i, $i = 1, 2$, are Lipschitz functions with Lipschitz constant at most ϵ which contain the conditions $\varphi_i(0) = \psi_i(0) = 0$ and $0 < |k_i| + \epsilon$, $|m_i| + \epsilon < 1$ for each $i = 1, 2$. Then*:

(i) *the functions $k_i I + \varphi_i$ and $m_i I + \psi_i$ are contractions for $i = 1, 2$;*
(ii) *$\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates.*

Proof.

(i) Consider the usual norm $\| \cdot \|$ on $\mathbb{R}$. Since the functions φ_i and ψ_i, $i = 1, 2$, are Lipschitz, for every $x, y \in \mathbb{R}$, we have $\|\varphi_i(x) - \varphi_i(y)\| < \epsilon\|x - y\|$ and $\|\psi_i(x) - \psi_i(y)\| < \epsilon\|x - y\|$ for $i = 1, 2$. Then,

$$\begin{aligned} \|(k_i I + \varphi_i)(x) - (k_i I + \varphi_i)(y)\| &= \|k_i x + \varphi_i(x) - k_i y - \varphi_i(y)\| \\ &= \|k_i(x - y) + \varphi_i(x) - \varphi_i(y)\| \\ &\le \|k_i(x - y)\| + \|\varphi_i(x) - \varphi_i(y)\| \\ &< |k_i|\|x - y\| + \epsilon\|x - y\| \\ &= (|k_i| + \epsilon)\|x - y\|. \end{aligned}$$

Therefore, by considering the hypothesis of the theorem, that is, $0 < k_i + \epsilon < 1$ for $i = 1, 2$, the previous relation shows that the function $k_i I + \varphi_i$ is a contraction, and similarly, we obtain that the function $m_i I + \psi_i$ is a contraction, for each $i = 1, 2$. Thus, the first statement has been proved.

(ii) Assume that $\sigma = \{\lambda_1, \lambda_2, \ldots\}$ is an arbitrary sequence from indices $\Lambda = \{1, 2\}$. First, we show that $\{|F_{\sigma_n}|\}_{n=1}^{\infty}$ is a strictly decreasing sequence, and the sequence $\{F_{\sigma_n}\}_{n=1}^{\infty}$ is convergent to zero.

Suppose that $x \in \mathbb{R}$ is arbitrary. Note the following terms of the sequence $\{|F_{\sigma_n}|\}_{n=1}^{\infty}$:

$$\begin{aligned}
|F_{\sigma_1}(x)| &= |f_{\lambda_1}(x)| = |k_{\lambda_1}x + \varphi_{\lambda_1}(x)| \\
|F_{\sigma_2}(x)| &= |f_{\lambda_2}(f_{\lambda_1}(x))| \\
&= |k_{\lambda_2}(k_{\lambda_1}x + \varphi_{\lambda_1}(x)) + \varphi_{\lambda_2}(k_{\lambda_1}x + \varphi_{\lambda_1}(x))| \\
&\le |k_{\lambda_2}(k_{\lambda_1}x + \varphi_{\lambda_1}(x))| + |\varphi_{\lambda_2}(k_{\lambda_1}x + \varphi_{\lambda_1}(x))|.
\end{aligned}$$

By using the suppositions of the theorem, that is, for $i = 1, 2$, $\varphi_i(0) = 0$ and $0 < |k_i| + \epsilon < 1$, as well as the functions φ_i are Lipschitz with constant at most ϵ, we can write the previous relation as follows:

$$\begin{aligned}
|F_{\sigma_2}(x)| &< |k_{\lambda_2}||k_{\lambda_1}x + \varphi_{\lambda_1}(x)| + \epsilon|k_{\lambda_1}x + \varphi_{\lambda_1}(x)| \\
&= (|k_{\lambda_2}| + \epsilon)|k_{\lambda_1}x + \varphi_{\lambda_1}(x)| \\
&< |k_{\lambda_1}x + \varphi_{\lambda_1}(x)| = |F_{\sigma_1}(x)|.
\end{aligned}$$

Generally, for every n, we have:

$$\begin{aligned}
F_{\sigma_n}(x) &= f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_2} \circ f_{\lambda_1}(x) = f_{\lambda_n}(f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_2} \circ f_{\lambda_1}(x)) \\
&= f_{\lambda_n}(F_{\sigma_{n-1}}(x)) = k_{\lambda_n}(F_{\sigma_{n-1}}(x)) + \varphi_{\lambda_n}(F_{\sigma_{n-1}}(x)).
\end{aligned}$$

Thus,

$$\begin{aligned}
|F_{\sigma_n}(x)| &= |k_{\lambda_n}(F_{\sigma_{n-1}}(x)) + \varphi_{\lambda_n}(F_{\sigma_{n-1}}(x))| \\
&\le |k_{\lambda_n}||F_{\sigma_{n-1}}(x)| + |\varphi_{\lambda_n}(F_{\sigma_{n-1}}(x))| \\
&< |k_{\lambda_n}||F_{\sigma_{n-1}}(x)| + \epsilon|F_{\sigma_{n-1}}(x)| \\
&= (|k_{\lambda_n} + \epsilon|)|F_{\sigma_{n-1}}(x)| < |F_{\sigma_{n-1}}(x)|.
\end{aligned}$$

Hence, the sequence $\{|F_{\sigma_n}|\}_{n=1}^{\infty}$ is strictly decreasing and, since it is bounded from below (for every $x \in \mathbb{R}$, $|F_{\sigma_n}(x)| > 0$), thus, it is convergent. Setting $k = \text{Max}\{|k_1| + \epsilon, |k_2| + \epsilon\}$, clearly, $0 < k < 1$. Now, for each $i = 1, 2$, we obtain

$$\|(k_iI + \varphi_i)(x) - (k_iI + \varphi_i)(y)\| < k\|x - y\|.$$

In addition, $(k_iI + \varphi_i)(0) = 0$; thus we have:

$$\begin{aligned}
|F_{\sigma_n}(x)| &= |k_{\lambda_n}(F_{\sigma_{n-1}}(x)) + \varphi_{\lambda_n}(F_{\sigma_{n-1}}(x))| \\
&< k|F_{\sigma_{n-1}}(x)| \\
&= k|k_{\lambda_{n-1}}(F_{\sigma_{n-2}}(x)) + \varphi_{\lambda_{n-1}}(F_{\sigma_{n-2}}(x))| \\
&< k \cdot k|F_{\sigma_{n-2}}(x)| = k^2|F_{\sigma_{n-2}}(x)|.
\end{aligned}$$

Continuing in this way, we finally obtain:

$$|F_{\sigma_n}(x)| < k^{n-1}|F_{\sigma_1}(x)| = k^{n-1}|k_{\lambda_1}x + \varphi_{\lambda_1}(x)| < k^{n-1} \cdot k|x| = k^n|x|.$$

Hence, for every $x \in \mathbb{R}$, $|F_{\sigma_n}(x)| < k^n|x|$ where $0 < k^n < 1$ and, since the functions of $\mathcal{F}$ are contractions, Theorem 4.1 (i), this implies that the sequence $\{F_{\sigma_n}\}_{n=1}^{\infty}$ is convergent to zero. Similarly, the sequence $\{G_{\sigma_n}\}_{n=1}^{\infty}$ is convergent to zero. Thus, these two IFSs have the same behavior as the two qualified IFSs in Theorem 3.5; consequently, $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates. □

Corollary 4.2. *Suppose that* $\mathcal{F} = \{f_1, f_2, \mathbb{R}\}$ *is an* IFS *where the functions* f_1 *and* f_2 *are diffeomorphisms on* $\mathbb{R}$. *The origin is a fixed point of the functions* f_1 *and* f_2. *Also assume that the derivative values of these functions at the origin have the same sign and* $0 < |\acute{f}_1(0)|, |\acute{f}_2(0)| < 1$. *Consider* IFS $\mathcal{G} = \{\acute{f}_1(0)I, \acute{f}_2(0)I, \mathbb{R}\}$. *Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates in a neighborhood of zero.*

Proof. Suppose that $\epsilon > 0$ is a number such that $0 < \epsilon + |\acute{f}_i(0)| < 1$ for each $i = 1, 2$. For every $i = 1, 2$, using [**47,** Lemma (4.4)], for given $\epsilon > 0$, there exists a neighborhood U_i of zero and an extension of $f_i|_{U_i}$ to $\mathbb{R}$ of the form $\acute{f}_i(0)I + \varphi_i$, where φ_i is a bounded continuous map from $\mathbb{R}$ to $\mathbb{R}$ which has a Lipschitz constant at most ϵ. Since zero is a fixed point of f_i and the functions f_i and $\acute{f}_i(0)I + \varphi_i$ are equal on U, it follows that $\varphi_i(0) = f_i(0) = 0$. Now, put $U = U_1 \cap U_2$ and $\mathcal{F}^* = \{\acute{f}_1(0)I + \varphi_1, \acute{f}_2(0)I + \varphi_2, \mathbb{R}\}$. From Theorem 4.1, we conclude that $\mathcal{F}^*$ and $\mathcal{G}$ are weakly topological conjugates and, since IFS $\mathcal{F}$ has the same behavior as IFS $\mathcal{F}^*$ on U (since the functions of IFS $\mathcal{F}^*$ are extensions of the functions IFS $\mathcal{F}$ on U), so the IFSs $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates on U. □

Corollary 4.3. *Suppose that we have the assumptions of Corollary* 4.2, *except that* $|\acute{f}_1(0)|$ *and* $|\acute{f}_2(0)| > 1$. *Consider* IFS $\mathcal{G} = \{\acute{f}_1(0)I, \acute{f}_2(0)I, \mathbb{R}\}$. *Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates in a neighborhood of zero.*

Proof. Since the functions of IFS $\mathcal{F}$ are diffeomorphisms, thus, we can consider IFS $\mathcal{F}^* = \{{f_1}^{-1}, {f_2}^{-1}, \mathbb{R}\}$. Clearly, the origin is a fixed point of the functions ${f_1}^{-1}$ and ${f_2}^{-1}$. We know that $({f_i}^{-1})'(0) =$

$1/\acute{f}_i(0)$; therefore, the values $({f_1}^{-1})'(0)$ and $({f_2}^{-1})'(0)$ have the same sign and $0 < |({f_1}^{-1})'(0)|,\ |({f_2}^{-1})'(0)| < 1$. Thus, $\mathcal{F}^*$ and IFS $\mathcal{G}^* = \{({f_1}^{-1})'(0)I,\ ({f_2}^{-1})'(0)I, \mathbb{R}\}$ are weakly topological conjugates on neighborhood U of zero, of the previous corollary, that is, for every σ and $n \in \mathbb{N}$, there exists a homeomorphism h on U such that $h \circ F^*_{\sigma_n} = G^*_{\sigma_n} \circ h$. Now, for $n \in \mathbb{N}$, put $\sigma^* = \{\lambda_n, \lambda_{n-1}, \ldots, \lambda_1, \lambda_{n+1}, \ldots\}$. For this n and σ^*, there exists a homeomorphism h on U such that $h \circ F^*_{\sigma^*_n} = G^*_{\sigma^*_n} \circ h$. Thus, we have

$$F^*_{\sigma^*_n}(x) = f^{-1}_{\lambda_1} \circ f^{-1}_{\lambda_2} \circ \cdots \circ f^{-1}_{\lambda_n}(x) = (f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_1})^{-1}(x) = F^{-1}_{\sigma_n}(x)$$

and $G^*_{\sigma^*_n}(x) = k^* x$, where $0 < |k^*| < 1$, and clearly, $G^*_{\sigma^*_n}(x) = G^{-1}_{\sigma_n}(x)$. Hence, we can obtain that $h \circ F^{-1}_{\sigma_n} = G^{-1}_{\sigma_n} \circ h$, and subsequently, $G_{\sigma_n} \circ h = h \circ F_{\sigma_n}$, that is, $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates in neighborhood U of zero. □

Theorem 4.4. *Assume that* $\mathcal{F} = \{f_1, f_2, \mathbb{R}\}$ *is an* IFS, *where the functions* f_1 *and* f_2 *are homeomorphisms on* $\mathbb{R}$. *The origin is a fixed point of the functions* f_1 *and* f_2. *Also, suppose that* $\acute{f}_1(0)$ *and* $\acute{f}_2(0)$ *have the same sign, and* $0 < |\acute{f}_1(0)| < 1$ *and* $|\acute{f}_2(0)| > 1$. *Consider the* IFS $\mathcal{G} = \{\acute{f}_1(0)I,\ \acute{f}_2(0)I, \mathbb{R}\}$. *Let* $\sigma = \{\lambda_1, \lambda_2, \ldots\}$, *and let the number of times of* λ_i, $i \in \mathbb{N}$, *that* $\lambda_i = 1$ *is* n_1 *and let the number of times of* λ_i *that* $\lambda_i = 2$ *is* n_2 *be such that* $\lim_{n\to+\infty} n_1/n_2 = +\infty$ (*or* $\lim_{n\to+\infty} n_2/n_1 = 0$). *Then, for every* $n \in \mathbb{N}$, *there exists a homeomorphism* h *on a neighborhood of zero such that* $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$.

Proof. Put $\acute{f}_i(0) = a_i$, $i = 1, 2$. Assume that $\epsilon > 0$ is a number such that $|a_1| + \epsilon < 1$ and $|a_2| - \epsilon > 1$. Let the neighborhood U, the functions φ_1 and φ_2 and IFS $\mathcal{F}^*$ be as defined in Corollary 4.2. Put $f^*_i = a_i I + \varphi_i$, $i = 1, 2$. We prove that, for every $x \in \mathbb{R}$, the sequences $\{F^*_{\sigma_n}(x)\}_{n=1}^{\infty}$ and $\{G_{\sigma_n}(x)\}_{n=1}^{\infty}$ are convergent to zero. Suppose that $x \in \mathbb{R}$ is arbitrary. Some terms of the sequence $\{F^*_{\sigma_n}(x)\}_{n=1}^{\infty}$ are as follows:

$$\begin{aligned} |F^*_{\sigma_1}(x)| &= |f^*_{\lambda_1}(x)| = |a_{\lambda_1} x + \varphi_{\lambda_1}(x)| \\ &\leq |a_{\lambda_1}||x| + \epsilon|x| = (|a_{\lambda_1}| + \epsilon)|x| \end{aligned}$$

$$|F^*_{\sigma_2}(x)| = |f^*_{\lambda_2}(f^*_{\lambda_1}(x))| = |a_{\lambda_2}(f^*_{\lambda_1}(x)) + \varphi_{\lambda_2}(f^*_{\lambda_1}(x))|$$

$$
\begin{aligned}
&\leq |a_{\lambda_2}||f^*_{\lambda_1}(x)| + \epsilon|f^*_{\lambda_1}(x)| = (|a_{\lambda_2}| + \epsilon)|f^*_{\lambda_1}(x)| \\
&\leq (|a_{\lambda_2}| + \epsilon) \cdot (|a_{\lambda_1}| + \epsilon)|x|.
\end{aligned}
$$

Applying induction, we get

$$
|F^*_{\sigma_{n-1}}(x)| \leq (|a_{\lambda_{n-1}}| + \epsilon) \cdot (|a_{\lambda_{n-2}}| + \epsilon) \cdots (|a_{\lambda_1}| + \epsilon)|x|.
$$

Thus,

$$
\begin{aligned}
|F^*_{\sigma_n}(x)| &= |f^*_{\lambda_n} \circ f^*_{\lambda_{n-1}} \circ \cdots \circ f^*_{\lambda_2} \circ f^*_{\lambda_1}(x)| \\
&= |f^*_{\lambda_n}(f^*_{\lambda_{n-1}} \circ \cdots \circ f^*_{\lambda_2} \circ f^*_{\lambda_1}(x))| = |f^*_{\lambda_n}(F^*_{\sigma_{n-1}}(x))| \\
&= |a_{\lambda_n}(F^*_{\sigma_{n-1}}(x)) + \varphi_{\lambda_n}(F^*_{\sigma_{n-1}}(x))| \\
&\leq |a_{\lambda_n}||F^*_{\sigma_{n-1}}(x)| + \epsilon|F^*_{\sigma_{n-1}}(x)| \\
&= (|a_{\lambda_n}| + \epsilon)|F^*_{\sigma_{n-1}}(x)| \\
&\leq (|a_{\lambda_n}| + \epsilon) \cdot (|a_{\lambda_{n-1}}| + \epsilon) \cdots (|a_{\lambda_1}| + \epsilon)|x|.
\end{aligned}
$$

By utilizing supposition, we can write the previous relation as follows:

$$
|F^*_{\sigma_n}(x)| \leq (|a_1| + \epsilon)^{n_1} \cdot (|a_2| + \epsilon)^{n_2}|x|.
$$

In the basic assumption, $\lim_{n\to+\infty}(n_1/n_2) = +\infty$, and n_1 is much larger than n_2 when $n \to +\infty$, that is, n_1 gradually approaches n, then, from the relations $|a_1| + \epsilon < 1$, $|a_2| > 1$ and $|F^*_{\sigma_n}(x)| \leq (|a_1| + \epsilon)^{n_1} \cdot (|a_2| + \epsilon)^{n-n_1}|x|$, we conclude that $F^*_{\sigma_n}(x) \to 0$ as $n \to +\infty$. In addition, for IFS $\mathcal{G}$ and every $x \in \mathbb{R}$, we have

$$
\begin{aligned}
|G_{\sigma_n}(x)| &= |a_{\lambda_n} \cdot a_{\lambda_{n-1}} \cdots a_{\lambda_1} x| \\
&= |a_{\lambda_n}| \cdot |a_{\lambda_{n-1}}| \cdots |a_{\lambda_1}||x| \\
&= |a_1|^{n_1}|a_2|^{n_2}|x| = |a_1|^{n_1}|a_2|^{n-n_1}|x|.
\end{aligned}
$$

We know that $0 < |a_1| < 1$, so, with reasoning similar to the above argument, we obtain $G_{\sigma_n}(x) \to 0$ as $n \to +\infty$. Then, these two IFSs have the same behavior as the two qualified IFSs in Theorem 3.5. It follows that, for given $\sigma \in \Lambda^{\mathbb{N}}$ and every $n \in \mathbb{N}$, there exists a homeomorphism h such that $h \circ F^*_{\sigma_n} = G_{\sigma_n} \circ h$. Since the IFSs $\mathcal{F}^*$ and $\mathcal{F}$ are equal on U, for given σ and every $n \in \mathbb{N}$, there exists a homeomorphism h on U such that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$, and therefore, the statement is proven. □

Note that, hereafter, if, for a given $\sigma \in \Lambda^{\mathbb{N}}$ and every $n \in \mathbb{N}$, there exists a homeomorphism h such that $h \circ F_{\sigma_n} = G_{\sigma_n} \circ h$, then we say that the IFSs $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugate relative to σ.

Corollary 4.5. *Suppose that all the assumptions of Theorem* 4.4 *are satisfied, but, instead, we have* $\lim_{n\to+\infty} n_1/n_2 = 0$*. Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugate relative to* σ*.*

Proof. Using the relations obtained in Theorem 4.4, since $\lim_{n\to+\infty} n_1/n_2 = 0$, we get $F^*_{\sigma_n}(x) \to \infty$ and $G_{\sigma_n}(x) \to \infty$. Then, $\mathcal{F}^*$ and $\mathcal{G}$ are weakly topological conjugate relative to the σ given in Corollary 3.7. Thus, the IFSs $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugate relative to σ on U. □

Definition 4.6. Let f be a C^1 diffeomorphism from a neighborhood U of x_0 in $\mathbb{R}^n$ into $\mathbb{R}^n$. The fixed point x_0 is called *hyperbolic* if all of the eigenvalues of $Df(x_0)$ have absolute values with norm different from one [**52**].

Now, we are ready to give the generalized Hartman-Grobman theorem for IFSs.

Theorem 4.7. (Generalized Hartman-Grobman theorem for IFSs). *Suppose that* $\mathcal{F} = \{f_\lambda : \lambda \in \Lambda, \mathbb{R}\}$ (Λ *is a finite nonempty set*) *is an* IFS *and the origin is a hyperbolic fixed point of the homeomorphisms* f_λ *for every* $\lambda \in \Lambda$*. Consider the* IFS $\mathcal{G} = \{f'_\lambda(0)I : \lambda \in \Lambda, \mathbb{R}\}$; *we call it the "linear part of* IFS $\mathcal{F}$*." Then*:

(i) *if* $f'_\lambda(0)$ *belong to the same interval* $(0,1)$ (*or* $(-1,0)$ *or* $(1,+\infty)$ *or* $(-\infty,-1)$) *for all* $\lambda \in \Lambda$*, then* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugate on a neighborhood of zero.*

(ii) *Suppose that* $f'_\lambda(0)$*,* $\lambda \in \Lambda$*, all with the same sign. Moreover, some belong to the same interval* $(0,1)$ (*or* $(-1,0)$)*, and some belong to the same interval* $(1,+\infty)$ (*or* $(-\infty,-1)$)*. Assume that* $\sigma \in \Lambda^{\mathbb{N}}$ *is given and is the number of times of* λ_i*,* $i \in \mathbb{N}$*, that* $0 < |f'_{\lambda_i}(0)| < 1$ *is* n_1 *and is the number of times of* λ_i *that* $|f'_{\lambda_i}(0)| > 1$ *is* n_2 *such that* $\lim_{n\to+\infty} n_1/n_2 = +\infty$ (*or* $\lim_{n\to+\infty} n_1/n_2 = 0$)*. Thus,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugate relative to* σ *on a neighborhood of zero.*

Proof.

(i) The proof is similar to Theorem 4.1 and Corollaries 4.2 and 4.3 for the case that Λ is a finite nonempty set and, subsequently, the first statement is true.

(ii) In the same manner as for Corollary 4.3 and Theorem 4.4, we can see that this is also the case where Λ is a finite nonempty set; thus, the second statement is true. □

In the next theorem, we examine topological conjugacy for two IFSs.

Theorem 4.8. *Suppose that* $\mathcal{F} = \{f_\lambda, \mathbb{R} : \lambda \in \Lambda\}$ *and* $\mathcal{G} = \{g_\lambda, \mathbb{R} : \lambda \in \Lambda\}$ *are two* IFSs *where, for every* $\lambda \in \Lambda$, *the functions* f_λ *and* g_λ *are homeomorphisms. Let the origin be a fixed point of functions* IFSs $\mathcal{F}$ *and* $\mathcal{G}$. *Assume, for all* $\lambda \in \Lambda$, *that* $\acute{f_\lambda}(0)$ *and* $\acute{g_\lambda}(0)$ *belong to the same interval* $(0,1)$ $($*or* $(-1,0)$ *or* $(1,+\infty)$ *or* $(-\infty,-1))$. *Then,* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates in a neighborhood of zero.*

Proof. Let $\mathcal{F}^*$ be the linear part of the IFS, and let $\mathcal{F}$ and $\mathcal{G}^*$ be the linear part of the IFS $\mathcal{G}$. From our assumptions and the first part of the generalized Hartman-Grobman theorem for IFSs, we obtain that the IFSs $\mathcal{F}$ and $\mathcal{F}^*$ are weakly topological conjugates in neighborhood U of zero, in addition, the IFSs $\mathcal{G}$ and $\mathcal{G}^*$ are weakly topological conjugates in neighborhood V of zero. Applying the primary theorems of this paper, we can conclude that the IFSs $\mathcal{F}^*$ and $\mathcal{G}^*$ are weakly topological conjugates. Now, put $W = U \cap V$. Clearly, W is a neighborhood of zero, and thus, we obtain that the IFSs $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates on neighborhood W of zero. □

5. Topological conjugacy of m-dimensional IFSs. Now, we assume that the functions f_i of the IFS $\mathcal{F}$ are determined from $\mathbb{R}^m$ to $\mathbb{R}^m$, and we investigate the concept of "weakly topological conjugate" for some special IFSs.

Theorem 5.1. *Consider the* IFSs $\mathcal{F} = \{A, B, \mathbb{R}^m\}$ *and* $\mathcal{G} = \{C, D, \mathbb{R}^m\}$, *where* A, B, C, *and* D *are diagonal matrices, respectively, with the diagonal elements* a_{ii}, b_{ii}, c_{ii} *and* d_{ii}, $i = 1, 2, \ldots, m$. *If all of these elements belong to the same interval* $(0,1)$ $($*or* $(-1,0)$ *or* $(1,+\infty)$ *or* $(-\infty,-1))$, *then* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly topological conjugates.*

Proof. Assume that

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix},$$

and also let $\sigma \in \Lambda^{\mathbb{N}}$ be arbitrary. Since the product of diagonal matrices is a diagonal matrix, we obtain:

$$F_{\sigma_n}(X) = \begin{bmatrix} {a_{11}}^{n_1}{b_{11}}^{n_2}x_1 \\ {a_{22}}^{n_1}{b_{22}}^{n_2}x_2 \\ \vdots \\ {a_{mm}}^{n_1}{b_{mm}}^{n_2}x_m \end{bmatrix}$$

and

$$G_{\sigma_n}(X) = \begin{bmatrix} {c_{11}}^{n_1}{d_{11}}^{n_2}x_1 \\ {c_{22}}^{n_1}{d_{22}}^{n_2}x_2 \\ \vdots \\ {c_{mm}}^{n_1}{d_{mm}}^{n_2}x_m \end{bmatrix},$$

where $n_1 + n_2 = n$; in fact, n_1 is the number of times of the iteration of A (associated to which, $G_{\sigma_n}(X)$ is C) at sequence σ, and n_2 is the number of times of the iteration B (associated to which, $G_{\sigma_n}(X)$ is D) at sequence σ. Now, for $i = 1, 2, \ldots, m$, put $\mathcal{F}_i = \{a_{ii}x_iI, b_{ii}x_iI, \mathbb{R}\}$ and $\mathcal{G}_i = \{c_{ii}x_iI, d_{ii}x_iI, \mathbb{R}\}$. On the basis of the previously proven statements, for each $i = 1, 2, \ldots, m$, the IFSs $\mathcal{F}_i$ and $\mathcal{G}_i$ are weakly topological conjugates. Hence, there exists a homeomorphism $h_i : \mathbb{R} \to \mathbb{R}$ such that $h_i(F_{i,\sigma_n}(x_i)) = G_{i,\sigma_n}(h_i(x_i))$. We define the function $h : \mathbb{R}^m \to \mathbb{R}^m$ with criterion

$$h(X) = \begin{bmatrix} h_1(x_1) \\ h_2(x_2) \\ \vdots \\ h_m(x_m) \end{bmatrix}.$$

Clearly, h is a homeomorphism since, for each $i = 1, 2, \ldots, m$, the function h_i is the homeomorphism. We claim that the homeomorphism h holds such that $h(F_{\sigma_n}(X)) = G_{\sigma_n}(h(X))$ for every $X \in \mathbb{R}^m$. For each

$X \in \mathbb{R}^m$, we have

$$\begin{aligned} h(F_{\sigma_n}(X)) &= \begin{bmatrix} h_1(a_{11}{}^{n_1} b_{11}{}^{n_2} x_1) \\ h_2(a_{22}{}^{n_1} b_{22}{}^{n_2} x_2) \\ \vdots \\ h_m(a_{mm}{}^{n_1} b_{mm}{}^{n_2} x_m) \end{bmatrix} = \begin{bmatrix} h_1(F_{1,\sigma_n}(x_1)) \\ h_2(F_{2,\sigma_n}(x_2)) \\ \vdots \\ h_m(F_{m,\sigma_n}(x_m)) \end{bmatrix} \\ &= \begin{bmatrix} G_{1,\sigma_n}(h_1(x_1)) \\ G_{2,\sigma_n}(h_2(x_2)) \\ \vdots \\ G_{m,\sigma_n}(h_m(x_m)) \end{bmatrix} = \begin{bmatrix} c_{11}{}^{n_1} d_{11}{}^{n_2} h_1(x_1) \\ c_{22}{}^{n_1} d_{22}{}^{n_2} h_2(x_2) \\ \vdots \\ c_{mm}{}^{n_1} d_{mm}{}^{n_2} h_m(x_m) \end{bmatrix} \\ &= G_{\sigma_n}(h(X)). \end{aligned}$$

Therefore, our claim is proven; hence, the IFSs $\mathcal{F}$ and $\mathcal{G}$ will be weakly topological conjugates. □

Theorem 5.2. *Let $J \subseteq \mathbb{N}$ be a finite set. Consider the* IFS $\mathcal{F} = \{D_j : j \in J, \mathbb{R}^m\}$*, where D_j is a diagonal matrix for every $j \in J$. Let $\mathcal{G} = \{AD_jA^{-1} : j \in J, \mathbb{R}^m\}$, where the matrix A is an invertible matrix. Then, $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates.*

Proof. Let $\sigma \in \Lambda^{\mathbb{N}}$ be an arbitrary sequence. According to the associative property of the product of matrices, we can gain the following relation for each $X \in \mathbb{R}^m$:

$$\begin{aligned} G_{\sigma_n}(X) &= AD_{\lambda_n} \underbrace{A^{-1}A}_{I} D_{\lambda_{n-1}} A^{-1} \ldots AD_{\lambda_2} \underbrace{A^{-1}A}_{I} D_{\lambda_1} A^{-1} X \\ &= AD_{\lambda_n} D_{\lambda_{n-1}} \ldots D_{\lambda_2} D_{\lambda_1} A^{-1} X, \end{aligned}$$

and we have $F_{\sigma_n}(X) = D_{\lambda_n} D_{\lambda_{n-1}} \cdots D_{\lambda_2} D_{\lambda_1} X$.

Now, we define

$$\begin{cases} h : \mathbb{R}^m \longrightarrow \mathbb{R}^m \\ X \longmapsto AX. \end{cases}$$

First note that, since A is an invertible matrix, the function h is a homeomorphism. For each $X \in \mathbb{R}^m$, we have:

$$\begin{aligned} h(F_{\sigma_n}(X)) &= A\,F_{\sigma_n}(X) = A\,D_{\lambda_n} D_{\lambda_{n-1}} \cdots D_{\lambda_2} D_{\lambda_1}\, X \\ &= A\,F_{\sigma_n}(X) = A\,D_{\lambda_n} D_{\lambda_{n-1}} \cdots D_{\lambda_2} D_{\lambda_1}\, \underbrace{A^{-1}\,A}_{I} X \\ &= G_{\sigma_n}(A\,X) = G_{\sigma_n}(h(X)). \end{aligned}$$

Thus, $\mathcal{F}$ and $\mathcal{G}$ are weakly topological conjugates. □

6. Necessary condition for structural stability of IFSs. Now, we shall define the concept of structural stability for IFSs. In order to define the distance of two IFSs, we need the following definitions from **[51]**. Suppose that M is a C^∞ smooth m-dimensional, closed, that is, compact and boundariless, manifold and r is a Riemannian metric on M. Let f and g be homeomorphisms on M. We define the metric ρ_0 as follows:

$$\rho_0(f,g) = \text{Max}\{r(f(x),g(x)),\, r(f^{-1}(x),g^{-1}(x));\ \text{for all } x \in M\}.$$

Now, assume that the functions f and g are C^1-diffeomorphisms on M. We define the metric ρ_1 as:

$$\rho_1(f,g) = \rho_0(f,g) + \text{Max}\{\|Df(x) - Dg(x)\|;\ \text{for all } x \in M\},$$

where

$$\begin{aligned} \|Df(x) - Dg(x)\| = \text{Max}\{&|Df(x)u - Dg(x)u|; \\ &\text{for all } x \in M \text{ and for all } u \in T_xM : |u| = 1\}. \end{aligned}$$

The set of C^1-diffeomorphisms on M with the induced topology of the metric ρ_1 is denoted by $\text{Diff}^1(M)$.

Definition 6.1. Let the IFSs $\mathcal{F} = \{f_\lambda,\, M : \lambda \in \Lambda\}$ and $\mathcal{G} = \{g_{\bar\lambda},\, M : \overline{\lambda} \in \overline{\Lambda}\}$ be subsets of $\text{Diff}^1(M)$. Let $\sigma \in \Lambda^{\mathbb{N}}$ and $\overline{\sigma} \in \overline{\Lambda}^{\mathbb{N}}$. We consider the sequences $\mathcal{F}_\sigma = \{f_{\lambda_i}\}_{i\in\mathbb{N}}$ and $\mathcal{G}_{\bar\sigma} = \{g_{\bar\lambda_i}\}_{i\in\mathbb{N}}$, where $\lambda_i \in \sigma$ and $\overline{\lambda}_i \in \overline{\sigma}$ for every $i \in \mathbb{N}$. The distance measured between the two IFSs relative to the sequences σ and $\overline{\sigma}$ will be denoted by $\mathcal{D}_1$ and is defined as follows:

$$\mathcal{D}_1(\mathcal{F}_\sigma, \mathcal{G}_{\bar\sigma}) = \sup\{\rho_1(f_{\lambda_i}, g_{\bar\lambda_i}) : \lambda_i \in \sigma,\ \overline{\lambda}_i \in \overline{\sigma} \text{ for all } i \in \mathbb{N}\}$$

Note that $\mathcal{D}_1$ is well defined and metric.

Definition 6.2. Assume that $\mathcal{F} = \{f_\lambda, M : \lambda \in \Lambda\} \subset \mathrm{Diff}^1(M)$ is an IFS. We say that the IFS $\mathcal{F}$ is *structurally stable* if, for a given $\epsilon > 0$, there is a $\delta > 0$ such that, for any IFS $\mathcal{G} = \{g_{\overline{\lambda}}, M : \overline{\lambda} \in \overline{\Lambda}\} \subset \mathrm{Diff}^1(M)$ and the sequences σ and $\overline{\sigma}$ with the condition $\mathcal{D}_1(\mathcal{F}_\sigma, \mathcal{G}_{\bar{\sigma}}) < \delta$, there exists a homeomorphism $h : M \to M$ with the following properties:

$$F_{\sigma_n} \circ h = h \circ G_{\bar{\sigma}_n} \quad \text{for all } n \in \mathbb{N},$$

$$r(x, h(x)) < \epsilon \quad \text{for all } x \in M.$$

The next theorem states the necessary conditions for structural stability of the IFSs.

Theorem 6.3. *If the* IFS $\mathcal{F} = \{f_\lambda, \mathbb{R} : \lambda \in \Lambda\}$ *as the subset of* $\mathrm{Diff}^1(\mathbb{R})$ *is structurally stable, then the fixed points of the functions* $\mathcal{F}$ *are hyperbolic.*

Proof. Assume that $\epsilon > 0$ is given. Then, there exists a $\delta > 0$ by the definition of structural stability of the IFSs. To obtain a contradiction, suppose that there exists the function f_{λ^*}, $\lambda^* \in \Lambda$, of $\mathcal{F}$ such that the fixed point p is not hyperbolic, that is, $|f'_{\lambda^*}(p)| = 1$. Put $\sigma = \{\lambda^*, \lambda^*, \ldots\}$. We consider the IFS $\mathcal{G}$ containing all of the C^1-diffeomorphisms $g_{\bar{\lambda}}$, $\overline{\lambda} \in \overline{\Lambda}$, such that $\rho_1(f_{\lambda^*}, g_{\bar{\lambda}}) < \delta$. We choose $\overline{\lambda} \in \overline{\Lambda}$, and consider $\overline{\sigma} = \{g_{\bar{\lambda}}, g_{\bar{\lambda}}, \ldots\}$. Clearly, $\mathcal{D}_1(\mathcal{F}_\sigma, \mathcal{G}_{\bar{\sigma}}) < \delta$. According to the structural stability of the IFS $\mathcal{F}$, there exists a homeomorphism h on $\mathbb{R}$ such that $F_{\sigma_n} \circ h = h \circ G_{\bar{\sigma}_n}$. Hence, for $n = 1$, we have $f_{\lambda^*} \circ h = h \circ g_{\bar{\lambda}}$. We repeat the above process for every $\overline{\lambda} \in \overline{\Lambda}$. Thus, for every $g_{\bar{\lambda}}$ with $\rho_1(f_{\lambda^*}, g_{\bar{\lambda}}) < \delta$, there exists a homeomorphism h on $\mathbb{R}$ such that $f_{\lambda^*} \circ h = h \circ g_{\bar{\lambda}}$. This means that the function f_{λ^*} is structurally stable. We know that, if a diffeomorphism is structurally stable, then its fixed points are hyperbolic; therefore, the fixed points of the function f_{λ^*} are hyperbolic. This is contradictory with $|f'_{\lambda^*}(p)| = 1$, and the statement is proven. □

7. An outline of future challenges. Some questions arise which, heretofore, have not been addressed.

Question 7.1. *How we can define the limit sets and the limit points for an* IFS*?*

Question 7.2. *What would happen if there was no value of limit* $\lim_{n\to+\infty} n_1/n_2$ *or it was not zero, in Theorem* 4.4*?*

Question 7.3. *Can we extend the concept of an* IFS *to continuous systems, and how can we generalize the Hartman-Grobman theorem to these systems. Furthermore, how can we define the structural stability?*

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Yazd University, Department of Mathematics, Yazd 89195-741, Iran
Email address: **fatehiniam@yazd.ac.ir**

Yazd University, Department of Mathematics, Yazd 89195-741, Iran
Email address: **f_rezaaei@yahoo.com**