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SPECTRAL THEOREMS ASSOCIATED WITH THE RIEMANN-LIOUVILLE-WIGNER LOCALIZATION OPERATORS

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ABSTRACT. We introduce the notion of localization operators associated with the Riemann-Liouville-Wigner transform, and we give a trace formula for the localization operators associated with the Riemann-Liouville-Wigner transform as a bounded linear operator in the trace class from $L^2(d\nu_\alpha)$ into $L^2(d\nu_\alpha)$ in terms of the symbol and the two admissible wavelets. Next, we give results on the boundedness and compactness of localization operators associated with the Riemann-Liouville-Wigner transform on $L^p(d\nu_\alpha)$, $1 \leq p \leq \infty$.

1. Introduction. The spherical means are of great importance in many ways and have been widely studied. For example, harmonic functions are characterized by the fact that they coincide with their spherical mean values. We can also view the spherical mean value operator as a generalized Radon transform that is self dual in the context of Helgason's double fibration. The spherical mean operator has many important physical applications, namely, in image processing of synthetic aperture radar data and acoustics [**10, 15**].

The study of spherical means has a very long history. The classic work of John [**16**] dealt with various applications of the spherical means to the theory of partial differential equations. Fourier analysis was utilized, along with the renowned theorem of Stein on the spherical analogue of the Lebesgue differentiation theorem.

In [**22**], the second author generalized the spherical mean operator on $\mathbb{R}^2$ by introducing, for the first time in the literature, the permutation operator which commutes with some partial differential operators. In the same paper, Trimèche studied the harmonic analysis as-

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sociated with this permutation operator. We note that the authors in [**1, 2, 13, 14, 19**] used this operator under the name *Riemann-Liouville* and the harmonic analysis associated to it.

Time-frequency localization operators are a mathematical tool for defining a restriction of functions to a region in the time-frequency plane that is compatible with the uncertainty principle and to extract time-frequency features. In this sense, they have been introduced and studied by Daubechies [**6, 7**] and Ramanathan and Topiwala [**20**], and they are now extensively investigated as an important mathematical tool in signal analysis and other applications [**5, 8, 9, 12, 23**].

Since the harmonic analysis associated with the Riemann-Liouville operator has known remarkable development, it is a natural question whether there exists an equivalent of the theory of localization operators in the framework of the theory associated with the Riemann-Liouville operator.

In our paper, we are mainly concerned with the Wigner transform under the Riemann-Liouville operator setting. More precisely, our main aim is to expose and study the boundedness and compactness of two-wavelet localization operators associated with the Riemann-Liouville-Wigner transform.

The reason for the extension from one wavelet to two wavelets comes from the extra degree of flexibility in signal analysis and imaging when the localization operators are used as time-varying filters. It turns out that localization operators with two admissible wavelets have a richer mathematical structure than the one-wavelet analogues.

The remainder of this paper is arranged as follows. In Section 2, we recall the main results of the harmonic analysis associated with the Riemann-Liouville operator and Schatten-von Neumann classes. In Section 3, we introduce and study the localization operators associated with the Riemann-Liouville-Wigner transform. More precisely, the Schatten-von Neumann properties of these two localization wavelet operators are established, and for trace class Riemann-Liouville two-wavelet localization operators, the traces and the trace class norm inequalities are presented. Section 4 is devoted to giving results on the L^p boundedness and compactness of these two-wavelet localization operators, under suitable conditions on the symbols and two admissible wavelets.

2. Preliminaries. This section gives an introduction to the harmonic analysis associated with the Riemann-Liouville operator, Schatten-von Neumann classes, and the wavelet transform associated with the Riemann-Liouville operator. The main references are [**14, 22, 23**].

2.1. Harmonic analysis associated with the Riemann-Liouville operator. We denote by

• $C_*(\mathbb{R}^2)$ the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable;

• $C_{*,c}(\mathbb{R}^2)$ the subspace of $C_*(\mathbb{R}^2)$ formed by functions with compact support;

• $\mathcal{E}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable;

• $\mathcal{S}_*(\mathbb{R}^2)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$, even with respect to the first variable;

• S^1 the unit sphere in $\mathbb{R}^2$,

$$S^1 = \{(\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 = 1\};$$

• $\mathbb{R}^2_+ = \{(r, x) \in \mathbb{R}^2 : r \geq 0\}$.

It is well known [**22**] that, for all $(\mu, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) = -i\lambda u(r,x), \\ \Delta_2 u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1, \ (\partial u/\partial r)(0,x) = 0 \quad \text{for all } x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\big(r\sqrt{\mu^2+\lambda^2}\big)e^{-i\lambda x},$$

where Δ_1 and Δ_2 are the singular partial differential operators, given by

$$\Delta_1 = \frac{\partial}{\partial x},$$

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x) \in \mathbb{R}^2_+, \ \alpha \geqslant 0,$$

and j_α is the normalized Bessel function defined by, for all $z \in \mathbb{C}$,

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\alpha)} (z/2)^{2k}.$$

Definition 2.1. The Riemann-Liouville operator is defined on $C_*(\mathbb{R}^2)$ by, for all $(r,x) \in \mathbb{R}_+^2$,

$$\mathcal{R}_\alpha f(r,x) = \begin{cases} (\alpha/\pi) \int_{-1}^{1} \int_{-1}^{1} f\big(rs\sqrt{1-t^2}, x+rt\big)(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1} dt\, ds, & \text{if } \alpha > 0, \\ (1/\pi) \int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)(1-t^2)^{-1/2} dt, & \text{if } \alpha = 0. \end{cases}$$

Remark 2.2.

(i) The function $\varphi_{\mu,\lambda}$, $(\mu,\lambda) \in \mathbb{C}^2$, can be written as, for all $(r,x) \in \mathbb{R}_+^2$,

$$\varphi_{\mu,\lambda}(r,x) = \mathcal{R}_\alpha(\cos(\mu\cdot)e^{-i\lambda\cdot})(r,x).$$

(ii) For all $\nu \in \mathbb{N}^2$, $(r,x) \in \mathbb{R}_+^2$ and $z = (\mu,\lambda) \in \mathbb{C}^2$,

$$|D_z^\nu \varphi_{\mu,\lambda}(r,x)| \leq ||(r,x)||^{|\nu|} \exp(2||(r,x)||\,||\mathrm{Im} z||), \tag{2.1}$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \partial z_2^{\nu_2}} \quad \text{and} \quad |\nu| = \nu_1 + \nu_2.$$

In particular, for all $\nu \in \mathbb{N}^2$, $(r,x) \in \mathbb{R}_+^2$ and $z = (\mu,\lambda) \in \mathbb{C}^2$

$$|\varphi_{\mu,\lambda}(r,x)| \leq 1. \tag{2.2}$$

Now, let Γ be the set

$$\Gamma := \{(\mu,\lambda) \in \mathbb{C} \times \mathbb{R} : \mu \in \mathbb{R}, \text{ or } \mu = it, \ t \in \mathbb{R}, |t| \leq |\lambda|\}$$

and Γ_+ the subset of Γ, given by

$$\Gamma_+ := \{(\mu,\lambda) \in \mathbb{C} \times \mathbb{R} : \mu \geq 0, \text{or} \mu = it, \ t \in \mathbb{R}, 0 \leq t \leq |\lambda|\}.$$

We have, for all $(\mu, \lambda) \in \Gamma$,

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

In the following, we denote by

• $d\nu_\alpha(r,x)$ the measure defined on $\mathbb{R}^2_+$ by

$$d\nu_\alpha(r,x) = k_\alpha r^{2\alpha+1} dr \otimes dx,$$

with

$$k_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)(2\pi)^{1/2}};$$

• for $p \in [1,\infty]$, p' denotes, as in all that follows, the conjugate exponent of p;

• $L^p(d\nu_\alpha)$, $1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}^2_+$, satisfying

$$\|f\|_{L^p(d\nu_\alpha)} = \left(\int_{\mathbb{R}^2_+} |f(r,x)|^p d\nu_\alpha(r,x) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(d\nu_\alpha)} = \operatorname*{ess\,sup}_{(r,x)\in\mathbb{R}^2_+} |f(r,x)| < \infty, \quad p = \infty;$$

for $p = 2$, we provide this space $L^2(d\nu_\alpha)$ with the scalar product

$$\langle f, g\rangle_{L^2(d\nu_\alpha)} = \int_{\mathbb{R}^2_+} f(r,x)\overline{g(r,x)} d\nu_\alpha(r,x);$$

• $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+} = \{\theta^{-1}(B) : \quad B \in \mathcal{B}_{\mathrm{Bor}}(\mathbb{R}^2_+)\},$$

where θ is defined on the set Γ_+ by $\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2}, \lambda)$;

• $d\gamma_\alpha$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by, for all $A \subset \mathcal{B}_{\Gamma_+}$,

$$\gamma_\alpha(A) = \nu_\alpha(\theta(A));$$

• $L^p(d\gamma_\alpha)$, $1 \leq p \leq \infty$, the space of measurable functions on Γ_+, satisfying

$$\|f\|_{L^p(d\gamma_\alpha)} = \left(\int_{\Gamma_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(d\gamma_\alpha)} = \operatorname*{ess\ sup}_{(\mu,\lambda)\in\Gamma_+} |f(\mu,\lambda)| < \infty, \quad p = \infty.$$

We have the following properties.

Proposition 2.3.

(i) *For every nonnegative measurable function f on Γ_+, we have*

$$\int_{\Gamma_+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = k_\alpha\Big[\int_{\mathbb{R}^2_+} f(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu\, d\mu_\alpha\, d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu\, d\mu_\alpha\, d\lambda\Big].$$

(ii) *For every nonnegative measurable function f on $\mathbb{R}^2_+$ (respectively, integrable on $\mathbb{R}^2_+$ with respect to the measure $d\nu_\alpha$), $f\circ\theta$ is a measurable nonnegative function on Γ_+ (respectively, integrable on Γ_+ with respect to the measure $d\gamma_\alpha$), and we have*

$$\int_{\Gamma_+} f\circ\theta(\mu,\lambda)\, d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)\, d\nu_\alpha(r,x). \tag{2.3}$$

Remark 2.4. The eigenfunction $\varphi_{\mu,\lambda}$, satisfies the following product formula

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi \varphi_{\mu,\lambda}\big(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\big)\sin^{2\alpha}\theta\, d\theta.$$

This allows us to define the translation operator as follows.

Definition 2.5. Let f be in $L^p(d\nu_\alpha)$, $p\in[1,\infty]$, for all $(r,x)\in\mathbb{R}^2_+$. We define the generalized translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville operator by

(2.4)

$$\tau_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi f\Big(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\Big)\sin^{2\alpha}\theta\, d\theta,$$

for all $(s,y)\in\mathbb{R}_+^2$.

Proposition 2.6. *For every $f\in L^p(d\nu_\alpha)$, $1\leqslant p\leqslant\infty$ and $(r,x)\in\mathbb{R}_+^2$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$, and we have*

$$\|\tau_{(r,x)}(f)\|_{L^p(d\nu_\alpha)}\leqslant\|f\|_{L^p(d\nu_\alpha)}. \tag{2.5}$$

Definition 2.7. The generalized convolution product of $f,g\in L^1(d\nu_\alpha)$ is defined by

(2.6)

$$f*_\alpha g(r,x)=\int_{\mathbb{R}_+^2}\tau_{(r,x)}(\check{f})(s,y)g(s,y)\,d\nu_\alpha(s,y)\quad\text{for all }(r,x)\in\mathbb{R}_+^2,$$

where $\check{f}(s,y)=f(s,-y)$.

Proposition 2.8. *Let $1\leq p,\,q,r\leq\infty$, be such that $1/p+1/q-1/r=1$. If f is a function in $L^p(d\nu_\alpha)$ and g an element of $L^q(d\nu_\alpha)$, then $f*_\alpha g$ belongs to $L^r(d\nu_\alpha)$, and we have*

$$\|f*_\alpha g\|_{L^r(d\nu_\alpha)}\leq\|f\|_{L^p(d\nu_\alpha)}\|g\|_{L^q(d\nu_\alpha)}. \tag{2.7}$$

We consider the generalized Fourier transform $\mathcal{F}_\alpha$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$.

Definition 2.9. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by, for all $(\mu,\lambda)\in\Gamma$,

$$\mathcal{F}_\alpha(f)(\mu,\lambda)=\int_{\mathbb{R}_+^2}f(r,x)\varphi_{\mu,\lambda}(r,x)\,d\nu_\alpha(r,x). \tag{2.8}$$

In the following, we recall some properties on the Fourier transform $\mathcal{F}_\alpha$.

For all $f \in L^1(d\nu_\alpha)$,

$$||\mathcal{F}_\alpha(f)||_{L^\infty(d\gamma_\alpha)} \leq ||f||_{L^1(d\nu_\alpha)}. \tag{2.9}$$

For $f \in L^1(d\nu_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(d\gamma_\alpha)$, we have the inversion formula for $\mathcal{F}_\alpha$: for almost every $(r,x) \in \mathbb{R}^2_+$,

$$f(r,x) = \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda). \tag{2.10}$$

Theorem 2.10.

(i) **(Plancherel's formula)**. *For every f in $\mathcal{S}_*(\mathbb{R}^2)$, we have*

$$\int_{\Gamma_+} |\mathcal{F}_\alpha(f)(\lambda,\mu)|^2 d\gamma_\alpha(\lambda,\mu) = \int_{\mathbb{R}^2_+} |f(r,x)|^2 d\nu_\alpha(r,x). \tag{2.11}$$

In particular, the Fourier transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.

(ii) **(Parseval's formula for $\mathcal{F}_\alpha$)**. *For all f,g in $L^2(d\nu_\alpha)$, we have*

$$\int_{\Gamma_+} \mathcal{F}_\alpha(f)(\lambda,\mu)\overline{\mathcal{F}_\alpha(g)(\lambda,\mu)}\, d\gamma_\alpha(\lambda,\mu) = \int_{\mathbb{R}^2_+} f(r,x)\overline{g(r,x)}\, d\nu_\alpha(r,x). \tag{2.12}$$

2.2. The Riemann-Liouville-Wigner transform. In this subsection, we recall some results introduced and proven in [**14**].

In the following, we denote by

• $\mathcal{S}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ rapidly decreasing together with all their derivatives, even with respect to the first variable;

• $\mathcal{S}_*(\Gamma)$ the space of functions $f : \Gamma \to \mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, i.e., for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\Gamma} (1+|\mu|^2+|\lambda|^2)^{k_1} \left| \left(\frac{\partial}{\partial\mu}\right)^{k_2} \left(\frac{\partial}{\partial\lambda}\right)^{k_3} f(\mu,\lambda) \right| < \infty,$$

where

$$\frac{\partial}{\partial\mu}f(\mu,\lambda)=\begin{cases}\dfrac{\partial}{\partial r}f(r,\lambda) & \text{if } \mu=r\in\mathbb{R},\\ \dfrac{1}{i}\dfrac{\partial}{\partial t}f(it,\lambda) & \text{if } \mu=it,\ |t|\leq|\lambda|;\end{cases}$$

• $\mathcal{S}_*(\mathbb{R}^2\times\mathbb{R}^2)$ the space of infinitely differentiable functions $f(r,x;s,y)$ on $\mathbb{R}^2\times\mathbb{R}^2$ even with respect to the variables r and s, and rapidly decreasing together with all their derivatives;

• $\mathcal{S}_*(\mathbb{R}^2\times\Gamma)$ the space of infinitely differentiable functions $f(r,x;\mu,\lambda)$ on $\mathbb{R}^2\times\Gamma$ even with respect to the variables r and μ, and rapidly decreasing together with all their derivatives.

Each of these spaces is equipped with its usual topology.

• $d\mu_\alpha(r,x;\mu,\lambda):=d\nu_\alpha(r,x)\,d\gamma_\alpha(\mu,\lambda)$, for all $(r,x;\mu,\lambda)\in\mathbb{R}^2_+\times\Gamma$;

• $L^p(d\mu_\alpha)$, $1\leq p\leq\infty$, the space of measurable functions f on $\mathbb{R}^2_+\times\Gamma$ satisfying

$$\|f\|_{L^p(d\mu_\alpha)}=\left(\int_\Gamma\int_{\mathbb{R}^2_+}|f(r,x;\mu,\lambda)|^p d\nu_\alpha(r,x)\,d\gamma_\alpha(\mu,\lambda)\right)^{1/p}<\infty,$$
$$1\leq p<\infty,$$
$$\|f\|_{L^\infty(d\mu_\alpha)}=\operatorname*{ess\,sup}_{(r,x;\mu,\lambda)\in\mathbb{R}^2_+\times\Gamma}|f(r,x;\mu,\lambda)|<\infty,\quad p=\infty\cdot$$

Definition 2.11. The Riemann-Liouville-Wigner transform $\mathcal{V}$ is defined on $\mathcal{S}_*(\mathbb{R}^2)\times\mathcal{S}_*(\mathbb{R}^2)$ by, for all $(r,x;\mu,\lambda)\in\mathbb{R}^2_+\times\Gamma$,

$$\begin{aligned}\mathcal{V}(f,h)(r,x;\mu,\lambda)&=\int_{\mathbb{R}^2_+}\varphi_{\mu,\lambda}(s,y)f(s,y)\tau_{(r,x)}h(s,y)\,d\nu_\alpha(s,y)\\&=\langle f,\overline{\varphi_{\mu,\lambda}\,\tau_{(r,x)}h}\rangle_{L^2(d\nu_\alpha)}.\end{aligned}\tag{2.13}$$

Remark 2.12. The transform $\mathcal{V}$ can also be written in the following form:

$$\mathcal{V}(f,h)(r,x;\mu,\lambda)=\mathcal{F}_\alpha(f\tau_{(r,x)}h)(\mu,\lambda)=h*_\alpha((\check{f}\check{\varphi}_{\mu,\lambda}))(r,x),\tag{2.14}$$

where $\check{g}$ is the function defined by, for all $(r,x)\in\mathbb{R}^2_+$,

$$\check{g}(r,x)=g(r,-x).$$

Proposition 2.13.

(i) *The transform* $\mathcal{V}$ *is a bilinear mapping from* $\mathcal{S}_*(\mathbb{R}^2) \times \mathcal{S}_*(\mathbb{R}^2)$ *into* $\mathcal{S}_*(\mathbb{R}^2 \times \Gamma)$.

(ii) *For all* f, h *in* $L^2(d\nu_\alpha)$, *we have* $\mathcal{V}(f,h) \in L^2(d\mu_\alpha) \bigcap L^\infty(d\mu_\alpha)$, *and*

$$\|\mathcal{V}(f,h)\|_{L^2(d\mu_\alpha)} \leq \|f\|_{L^2(d\nu_\alpha)} \|h\|_{L^2(d\nu_\alpha)}, \tag{2.15}$$

$$\|\mathcal{V}(f,h)\|_{L^\infty(d\mu_\alpha)} \leq \|f\|_{L^2(d\nu_\alpha)} \|h\|_{L^2(d\nu_\alpha)}. \tag{2.16}$$

Remark 2.14.

(i) If $h \in L^p(d\nu_\alpha)$, and $f \in L^{p'}(d\nu_\alpha)$, $p \in [1, \infty]$, we define the Riemann-Liouville-Wigner transform $\mathcal{V}(f,h)$ by the relation (2.14).

(ii) Let $h \in L^p(d\nu_\alpha)$, $p \in [1, \infty]$. Then, from relations (2.7) and (2.14), for all f in $L^{p'}(d\nu_\alpha)$, we have

$$\|\mathcal{V}(f,h)\|_{L^\infty(d\mu_\alpha)} \leq \|f\|_{L^{p'}(d\mu_\alpha)} \|h\|_{L^p(d\mu_\alpha)}. \tag{2.17}$$

We proceed as in **[17]** and prove the following.

Proposition 2.15.

(i) *For all* f *in* $L^p(d\nu_\alpha)$ *and* h *in* $L^{p'}(d\nu_\alpha)$, $p > 2$, *the Riemann-Liouville-Wigner transform* $\mathcal{V}(f,h)$ *belongs to* $L^p(d\mu_\alpha)$ *and satisfies the following inequality*

$$\|\mathcal{V}(f,h)\|_{L^p(d\mu_\alpha)} \leq \|f\|_{L^p(d\mu_\alpha)} \|h\|_{L^{p'}(d\mu_\alpha)}. \tag{2.18}$$

(ii) *Let* $r > 2$. *Suppose that* f *in* $L^p(d\nu_\alpha)$, h *in* $L^{p'}(d\nu_\alpha)$, *and* $r' \leq p, p' \leq r$. *Then,* $\mathcal{V}(f,h)$ *belongs to* $L^r(d\mu_\alpha)$ *and satisfies the following inequality*

$$\|\mathcal{V}(f,h)\|_{L^r(d\mu_\alpha)} \leq \|f\|_{L^p(d\mu_\alpha)} \|h\|_{L^{p'}(d\mu_\alpha)}. \tag{2.19}$$

2.3. Schatten-von Neumann classes. In the following, we denote by

• $l^p(\mathbb{N})$, $1 \leq p \leq \infty$, the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$, such that

$$||x||_p := \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} < \infty \quad \text{if } 1 \leq p < \infty,$$

$$||x||_\infty := \sup_{j\in\mathbb{N}} |x_j| < \infty.$$

For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x, y\rangle_2 := \sum_{j=1}^{\infty} x_j \overline{y_j}.$$

- $B(L^p(d\nu_\alpha))$, $1 \leq p \leq \infty$, the space of bounded operators from $L^p(d\nu_\alpha)$ into itself.

Definition 2.16.

(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2(d\nu_\alpha))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.

(ii) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \left(\sum_{n=1}^{\infty} (s_n(A))^p\right)^{1/p}. \tag{2.20}$$

Remark 2.17. We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Definition 2.18. The trace of an operator A in S_1 is defined by

$$\text{tr}(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n\rangle_{L^2(d\nu_\alpha)}, \tag{2.21}$$

where $(v_n)_n$ is any orthonormal basis of $L^2(d\nu_\alpha)$.

Remark 2.19. If A is positive, then

$$\text{tr}(A) = ||A||_{S_1}. \tag{2.22}$$

Moreover, a compact operator A on the Hilbert space $L^2(d\nu_\alpha)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then,

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = \mathrm{tr}(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2(d\nu_\alpha)} \tag{2.23}$$

for any orthonormal basis $(v_n)_n$ of $L^2(d\nu_\alpha)$.

Definition 2.20. We define $S_\infty := B(L^2(d\nu_\alpha))$, equipped with the norm

$$||A||_{S_\infty} := \sup_{v\in L^2(d\nu_\alpha):||v||_{L^2(d\nu_\alpha)}=1} ||Av||_{L^2(d\nu_\alpha)}. \tag{2.24}$$

Remark 2.21. It is obvious that $S_p \subset S_q$, $1 \le p \le q \le \infty$.

3. Localization operator associated with the Riemann-Liouville-Wigner transform. In this section, we will derive a host of sufficient conditions for the boundedness and Schatten class of the Riemann Liouville two-wavelet localization operators associated with the Riemann-Liouville-Wigner transform in terms of properties of the symbol σ and the windows u and v.

3.1. Preliminaries.

Definition 3.1. Let u and v be measurable functions on $\mathbb{R}^2_+$, σ a measurable function on $\mathbb{R}^2_+ \times \Gamma$, and we define the localization operator associated with the Riemann-Liouville-Wigner transform noted by $\mathcal{L}_{u,v}(\sigma)$, on $L^p(d\nu_\alpha)$, $1 \le p \le \infty$, by

$$\begin{aligned} &\mathcal{L}_{u,v}(\sigma)(f)(s,y) \\ &= \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda)\mathcal{V}(f,u)(r,x;\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)\tau_{s,y}v(r,x)}\, d\mu_\alpha(r,x;\mu,\lambda), \\ &\qquad (s,y) \in \mathbb{R}^2_+. \end{aligned} \tag{3.1}$$

In accordance with the different choices of the symbols σ and the different continuities required, we need to impose different conditions on u and v. Then, we obtain an operator on $L^p(d\nu_\alpha)$.

It is often more convenient to interpret the definition of $\mathcal{L}_{u,v}(\sigma)$ in a weak sense, that is, for f in $L^p(d\nu_\alpha)$, $p \in [1,\infty]$, and g in $L^{p'}(d\nu_\alpha)$,

(3.2)

$$\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)} = \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda)\mathcal{V}(f,u)(r,x;\mu,\lambda)\overline{\mathcal{V}(g,v)(r,x;\mu,\lambda)}\, d\mu_\alpha(r,x;\mu,\lambda).$$

In what follows, the operator $\mathcal{L}_{u,v}(\sigma)$ will be named the *localization operator* for the sake of simplicity.

Proposition 3.2. *Let $p \in [1,\infty)$. Formally, we assume that we have*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha).$$

Then, its adjoint is the linear operator $\mathcal{L}_{v,u}(\overline{\sigma}) : L^{p'}(d\nu_\alpha) \to L^{p'}(d\nu_\alpha)$.

Proof. For all f in $L^p(d\nu_\alpha)$ and g in $L^{p'}(d\nu_\alpha)$, it immediately follows from (3.2) that

$$\begin{aligned}
&\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)} \\
&\quad = \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda)\mathcal{V}(f,u)(r,x;\mu,\lambda)\overline{\mathcal{V}(g,v)(r,x;\mu,\lambda)}\, d\mu_\alpha(r,x;\mu,\lambda) \\
&\quad = \overline{\int_\Gamma \int_{\mathbb{R}^2_+} \overline{\sigma(r,x;\mu,\lambda)\mathcal{V}(f,u)(r,x;\mu,\lambda)}\mathcal{V}(g,v)(r,x;\mu,\lambda)\, d\mu_\alpha(r,x;\mu,\lambda)} \\
&\quad = \overline{\langle \mathcal{L}_{v,u}(\overline{\sigma})(g), f\rangle_{L^2(d\nu_\alpha)}} = \langle f, \mathcal{L}_{v,u}(\overline{\sigma})(g)\rangle_{L^2(d\nu_\alpha)}.
\end{aligned}$$

Thus, we get

$$\mathcal{L}^*_{u,v}(\sigma) = \mathcal{L}_{v,u}(\overline{\sigma}). \tag{3.3}$$

□

In the sequel of this section, u and v will be any functions in $L^2(d\nu_\alpha)$ such that

$$\|u\|_{L^2(d\nu_\alpha)} = \|v\|_{L^2(d\nu_\alpha)} = 1.$$

3.2. Boundedness for $\mathcal{L}_{u,v}(\sigma)$ on S_∞. The main result of this subsection is to prove that the linear operators

$$\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

are bounded for all symbols $\sigma \in L^p(d\mu_\alpha)$, $1 \leq p \leq \infty$. We consider first this problem for σ in $L^1(d\mu_\alpha)$ and next in $L^\infty(d\mu_\alpha)$, and then, we conclude by using interpolation theory.

Proposition 3.3. *Let σ be in $L^1(d\mu_\alpha)$. Then, the localization operator $\mathcal{L}_{u,v}(\sigma)$ is in S_∞, and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}. \tag{3.4}$$

Proof. For every function f and g in $L^2(d\nu_\alpha)$, from (3.2), we have

$$\begin{aligned} &|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \\ &\leqslant \int_\Gamma \int_{\mathbb{R}^2_+} |\sigma(r, x; \mu, \lambda)| \\ &\times |\mathcal{V}(f, u)(r, x; \mu, \lambda)\overline{\mathcal{V}(g, v)(r, x; \mu, \lambda)}| \, d\mu_\alpha(r, x; \mu, \lambda) \\ &\leqslant \|\mathcal{V}(f, u)\|_{L^\infty(d\mu_\alpha)} \|\mathcal{V}(g, v)\|_{L^\infty(d\mu_\alpha)} \|\sigma\|_{L^1(d\mu_\alpha)}. \end{aligned}$$

Using relation (2.16), we get

$$\|\mathcal{V}(f, u)\|_{L^\infty(d\mu_\alpha)} \leq \|u\|_{L^2(d\nu_\alpha)} \|f\|_{L^2(d\nu_\alpha)},$$
$$\|\mathcal{V}(g, v)\|_{L^\infty(d\mu_\alpha)} \leq \|v\|_{L^2(d\nu_\alpha)} \|g\|_{L^2(d\nu_\alpha)}.$$

Hence, we deduce that

$$|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \leqslant \|f\|_{L^2(d\nu_\alpha)} \|g\|_{L^2(d\nu_\alpha)} \|\sigma\|_{L^1(d\mu_\alpha)}.$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}.$$ □

Proposition 3.4. *Let σ be in $L^\infty(d\mu_\alpha)$. Then, the localization operator $\mathcal{L}_{u,v}(\sigma)$ is in S_∞, and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty(d\mu_\alpha)}.$$

Proof. For all functions f and g in $L^2(d\nu_\alpha)$, we have, from Cauchy-Schwarz's inequality,

$$\begin{aligned} &|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \\ &\leqslant \int_\Gamma \int_{\mathbb{R}^2_+} |\sigma(r, x; \mu, \lambda)||\mathcal{V}(f, u)(r, x; \mu, \lambda)| \end{aligned}$$

$$\times |\overline{\mathcal{V}(g,v)(r,x;\mu,\lambda)}|\, d\mu_\alpha(r,x;\mu,\lambda)$$
$$\leqslant \|\sigma\|_{L^\infty(d\mu_\alpha)}\|\mathcal{V}(f,u)\|_{L^2(d\mu_\alpha)}\|\mathcal{V}(g,v)\|_{L^2(d\mu_\alpha)}.$$

Using formula (2.15), we get

$$|\langle\mathcal{L}_{u,v}(\sigma)(f),g\rangle_{L^2(d\nu_\alpha)}| \leqslant \|\sigma\|_{L^\infty(d\mu_\alpha)}\|f\|_{L^2(d\nu_\alpha)}\|g\|_{L^2(d\nu_\alpha)}.$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty(d\mu_\alpha)}.$$ □

We can now associate a localization operator

$$\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

to every symbol σ in $L^p(d\mu_\alpha)$, $1 \leq p \leq \infty$ and prove that $\mathcal{L}_{u,v}(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 3.5. *Let σ be in $L^p(d\mu_\alpha)$, $1 \leq p \leq \infty$. Then, there exists a unique bounded linear operator $\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$, such that*

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^p(d\mu_\alpha)}.$$

Proof. Let f be in $L^2(d\nu_\alpha)$. We consider the following operator

$$\mathcal{T} : L^1(d\mu_\alpha)\bigcap L^\infty(d\mu_\alpha) \longrightarrow L^2(d\nu_\alpha),$$

given by

$$\mathcal{T}(\sigma) := \mathcal{L}_{u,v}(\sigma)(f).$$

Then, by Proposition 3.3 and Proposition 3.4,

$$||\mathcal{T}(\sigma)||_{L^2(d\nu_\alpha)} \leqslant ||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)} \tag{3.5}$$

and

$$||\mathcal{T}(\sigma)||_{L^2(d\nu_\alpha)} \leq ||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^\infty(d\mu_\alpha)}. \tag{3.6}$$

Therefore, by (3.5), (3.6) and the Riesz-Thorin interpolation theorem (see [**21,** Theorem 2] and [**23,** Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear operator on $L^p(d\mu_\alpha)$, $1 \leq p \leq \infty$, and we have

$$||\mathcal{L}_{u,v}(\sigma)(f)||_{L^2(d\nu_\alpha)} = ||\mathcal{T}(\sigma)||_{L^2(d\nu_\alpha)} \leq ||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^p(d\mu_\alpha)}. \tag{3.7}$$

Since (3.7) is true for arbitrary functions f in $L^2(d\nu_\alpha)$, we then obtain the desired result. □

3.3. Traces of $\mathcal{L}_{u,v}(\sigma)$. The main result of this subsection is to prove that the localization operator $\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ is in the Schatten class S_p.

Proposition 3.6. *Let σ be in $L^1(d\mu_\alpha)$. Then, the localization operator $\mathcal{L}_{u,v}(\sigma)$ is in S_2, and we have*

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_2} \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}.$$

Proof. Let $\{\phi_j, j = 1, 2, \ldots\}$ be an orthonormal basis for $L^2(d\nu_\alpha)$. Then, by (3.2), Fubini's theorem, Parseval's identity and (3.3), we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} ||\mathcal{L}_{u,v}(\sigma)(\phi_j)||^2_{L^2(d\nu_\alpha)} \\
&\quad = \sum_{j=1}^{\infty} \langle \mathcal{L}_{u,v}(\sigma)(\phi_j), \mathcal{L}_{u,v}(\sigma)(\phi_j)\rangle_{L^2(d\nu_\alpha)} \\
&\quad = \sum_{j=1}^{\infty} \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda)\langle \phi_j, \overline{\varphi_{\mu,\lambda}\tau_{(r,x)}u}\rangle_{L^2(d\nu_\alpha)} \\
&\quad \times \overline{\langle \mathcal{L}_{u,v}(\sigma)(\phi_j), \overline{\varphi_{\mu,\lambda}\,\tau_{(r,x)}v}\rangle}_{L^2(d\nu_\alpha)}\, d\mu_\alpha(r,x;\mu,\lambda) \\
&\quad = \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda) \sum_{j=1}^{\infty} \langle \mathcal{L}^*_{u,v}(\sigma)(\overline{\varphi_{\mu,\lambda}\,\tau_{(r,x)}v}), \phi_j\rangle_{L^2(d\nu_\alpha)} \\
&\quad \times \langle \phi_j, \overline{\tau_{(r,x)}u\,\varphi_{\mu,\lambda}}\rangle_{L^2(d\nu_\alpha)}\, d\mu_\alpha(r,x;\mu,\lambda) \\
&\quad = \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda) \\
&\quad \times \langle \mathcal{L}^*_{u,v}(\sigma)(\overline{\tau_{(r,x)}v\,\varphi_{\mu,\lambda}}), \overline{\varphi_{\mu,\lambda}\,\tau_{(r,x)}u}\rangle_{L^2(d\nu_\alpha)}\, d\mu_\alpha(r,x;\mu,\lambda).
\end{aligned}$$

Thus, from (3.4), we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty} ||\mathcal{L}_{u,v}(\sigma)(\phi_j)||^2_{L^2(d\nu_\alpha)} \\
&\qquad \leq \int_\Gamma \int_{\mathbb{R}^2_+} |\sigma(r,x;\mu,\lambda)|\, ||\mathcal{L}^*_{u,v}(\sigma)||_{S_\infty} d\mu_\alpha(r,x;\mu,\lambda) \\
&\qquad \leq \|\sigma\|^2_{L^1(d\mu_\alpha)} < \infty.
\end{aligned} \tag{3.8}$$

Hence, by (3.8) and [**23,** Proposition 2.8],

$$\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in the Hilbert-Schmidt class S_2, and hence, compact. □

Proposition 3.7. *Let σ be a symbol in $L^p(d\mu_\alpha)$, $1 \leqslant p < \infty$. Then, the localization operator $\mathcal{L}_{u,v}(\sigma)$ is compact.*

Proof. Let σ be in $L^p(d\mu_\alpha)$, and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(d\mu_\alpha)\bigcap L^\infty(d\mu_\alpha)$ such that $\sigma_n \to \sigma$ in $L^p(d\mu_\alpha)$ as $n \to \infty$. Then, by Theorem 3.5,

$$||\mathcal{L}_{u,v}(\sigma_n) - \mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq ||\sigma_n - \sigma||_{L^p(d\mu_\alpha)}.$$

Hence, $\mathcal{L}_{u,v}(\sigma_n) \to \mathcal{L}_{u,v}(\sigma)$ in S_∞ as $n \to \infty$. On the other hand, as by Proposition 3.6, $\mathcal{L}_{u,v}(\sigma_n)$ is in S_2, hence compact, it follows that $\mathcal{L}_{u,v}(\sigma)$ is compact. □

Theorem 3.8. *Let σ be in $L^1(d\mu_\alpha)$. Then,*

$$\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_1, and we have

$$\|\widetilde{\sigma}\|_{L^1(d\mu_\alpha)} \leqslant \|\mathcal{L}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}, \tag{3.9}$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(r,x;\mu,\lambda) = \langle \mathcal{L}_{u,v}(\sigma)\, \varphi_{\mu,\lambda}\, \tau_{(r,x)}u,\ \varphi_{\mu,\lambda}\, \tau_{(r,x)}v\rangle_{L^2(d\nu_\alpha)},$$
$$(r,x;\mu,\lambda) \in \mathbb{R}_+^2 \times \Gamma.$$

Proof. Since σ is in $L^1(d\mu_\alpha)$, by Proposition 3.6, $\mathcal{L}_{u,v}(\sigma)$ is in S_2. Using [**23,** Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\ j = 1, 2, \ldots\}$ for the orthogonal complement of the kernel of the operator $\mathcal{L}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathcal{L}_{u,v}(\sigma)|$, and $\{h_j,\ j = 1, 2, \ldots\}$ an orthonormal set in $L^2(d\nu_\alpha)$, such that

$$\mathcal{L}_{u,v}(\sigma)(f) = \sum_{j=1}^{\infty} s_j \langle f, \phi_j\rangle_{L^2(d\nu_\alpha)} h_j, \tag{3.10}$$

where s_j, $j = 1, 2, \ldots$, are the positive singular values of $\mathcal{L}_{u,v}(\sigma)$ corresponding to ϕ_j. Then, we get

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} \langle \mathcal{L}_{u,v}(\sigma)(\phi_j), h_j \rangle_{L^2(d\nu_\alpha)}.$$

Thus, by (2.2), (2.5), (2.13), (2.16), (3.2), Fubini's theorem, Parseval's identity, Bessel's inequality, Cauchy-Schwarz's inequality, and $||u||_{L^2(d\nu_\alpha)} = ||v||_{L^2(d\nu_\alpha)} = 1$, we obtain

$$\begin{aligned}
&\|\mathcal{L}_{u,v}(\sigma)\|_{S_1} \\
&= \sum_{j=1}^{\infty} \langle \mathcal{L}_{u,v}(\sigma)(\phi_j), h_j \rangle_{L^2(d\nu_\alpha)} \\
&= \sum_{j=1}^{\infty} \int_\Gamma \int_{\mathbb{R}_+^2} \sigma(r, x; \mu, \lambda) \mathcal{V}(\phi_j, u)(r, x; \mu, \lambda) \\
&\times \overline{\mathcal{V}(h_j, v)(r, x; \mu, \lambda)} \, d\mu_\alpha(r, x; \mu, \lambda) \\
&= \int_\Gamma \int_{\mathbb{R}_+^2} \sigma(r, x; \mu, \lambda) \sum_{j=1}^{\infty} \langle \phi_j, \overline{\varphi_{\mu,\lambda} \, \tau_{(r,x)} u} \rangle_{L^2(d\nu_\alpha)} \\
&\times \langle \overline{\varphi_{\mu,\lambda} \, \tau_{(r,x)} v}, h_j \rangle_{L^2(d\nu_\alpha)} \, d\mu_\alpha(r, x; \mu, \lambda) \\
&\leq \int_\Gamma \int_{\mathbb{R}_+^2} |\sigma(r, x; \mu, \lambda)| \, ||\varphi_{\mu,\lambda} \, \tau_{(r,x)} u||_{L^2(d\nu_\alpha)} \\
&\times ||\varphi_{\mu,\lambda} \, \tau_{(r,x)} v||_{L^2(d\nu_\alpha)} \, d\mu_\alpha(r, x; \mu, \lambda) \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}.
\end{aligned}$$

Hence, $\|\mathcal{L}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}$.

We now prove that $\mathcal{L}_{u,v}(\sigma)$ satisfies the first member of (3.9). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1(d\mu_\alpha)$, and, using formula (3.10), we get

$$\begin{aligned}
&|\widetilde{\sigma}(r, x; \mu, \lambda)| \\
&= |\langle \mathcal{L}_{u,v}(\sigma)(\varphi_{\mu,\lambda} \, \tau_{(r,x)} u), \ \varphi_{\mu,\lambda} \, \tau_{(r,x)} v \rangle_{L^2(d\nu_\alpha)}| \\
&= \left| \sum_{j=1}^{\infty} s_j \langle \ \varphi_{\mu,\lambda} \, \tau_{(r,x)} u, \phi_j \rangle_{L^2(d\nu_\alpha)} \langle h_j, \varphi_{\mu,\lambda} \, \tau_{(r,x)} v \rangle_{L^2(d\nu_\alpha)} \right| \\
&\leqslant \frac{1}{2} \sum_{j=1}^{\infty} s_j (|\langle \varphi_{\mu,\lambda} \, \tau_{(r,x)} u, \phi_j \rangle_{L^2(d\nu_\alpha)}|^2 + |\langle \ \varphi_{\mu,\lambda} \, \tau_{(r,x)} v, h_j \rangle_{L^2(d\nu_\alpha)}|^2).
\end{aligned}$$

By Fubini's theorem, we obtain

$$\begin{aligned}\int_\Gamma\int_{\mathbb{R}_+^2} &|\widetilde{\sigma}(r,x;\mu,\lambda)|d\mu_\alpha(r,x;\mu,\lambda)\\ &\le \frac{1}{2}\sum_{j=1}^{\infty} s_j\bigg(\int_\Gamma\int_{\mathbb{R}_+^2} |\langle\, \varphi_{\mu,\lambda}\,\tau_{(r,x)}u,\phi_j\rangle_{L^2(d\nu_\alpha)}|^2 d\mu_\alpha(r,x;\mu,\lambda)\\ &\qquad + \int_\Gamma\int_{\mathbb{R}_+^2} |\langle\, \varphi_{\mu,\lambda}\,\tau_{(r,x)}v,h_j\rangle_{L^2(d\nu_\alpha)}|^2 d\mu_\alpha(r,x;\mu,\lambda)\bigg).\end{aligned}$$

Thus, using the formula given by relation (2.15), we have

$$\begin{aligned}\int_\Gamma\int_{\mathbb{R}_+^2} |\widetilde{\sigma}(r,x;\mu,\lambda)|d\mu_\alpha(r,x;\mu,\lambda) &\le \frac{\|u\|^2_{L^2(d\nu_\alpha)}+\|v\|^2_{L^2(d\nu_\alpha)}}{2}\sum_{j=1}^{\infty} s_j\\ &\le \|\mathcal{L}_{u,v}(\sigma)\|_{S_1}.\end{aligned}$$

The proof is complete. □

Corollary 3.9. *For σ in $L^1(d\mu_\alpha)$, we have the following trace formula*

(3.11)

$$\begin{aligned}&\operatorname{tr}(\mathcal{L}_{u,v}(\sigma))\\ &\quad = \int_\Gamma\int_{\mathbb{R}_+^2} \sigma(r,x;\mu,\lambda)\langle\varphi_{\mu,\lambda}\,\tau_{(r,x)}u,\varphi_{\mu,\lambda}\,\tau_{(r,x)}v\rangle_{L^2(d\nu_\alpha)}\, d\mu_\alpha(r,x;\mu,\lambda).\end{aligned}$$

Proof. Let $\{\phi_j, j=1,2,\ldots\}$ be an orthonormal basis for $L^2(d\nu_\alpha)$. From Theorem 3.8, the localization operator $\mathcal{L}_{u,v}(\sigma)$ belongs to S_1. Then, by the definition of the trace given by relation (2.21), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned}&\operatorname{tr}(\mathcal{L}_{u,v}(\sigma))\\ &\quad = \sum_{j=1}^{\infty}\langle\mathcal{L}_{u,v}(\sigma)(\phi_j),\phi_j\rangle_{L^2(d\nu_\alpha)}\\ &\quad = \sum_{j=1}^{\infty}\int_\Gamma\int_{\mathbb{R}_+^2} \sigma(r,x;\mu,\lambda)\langle\phi_j,\,\overline{\varphi_{\mu,\lambda}\,\tau_{(r,x)}u}\rangle_{L^2(d\nu_\alpha)}\\ &\quad \times \overline{\langle\phi_j,\,\overline{\varphi_{\mu,\lambda}\,\tau_{(r,x)}v}\rangle}_{L^2(d\nu_\alpha)} d\mu_\alpha(r,x;\mu,\lambda)\end{aligned}$$

$$= \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda) \sum_{j=1}^{\infty} \langle \phi_j, \overline{\varphi_{\mu,\lambda}\, \tau_{(r,x)} u} \rangle_{L^2(d\nu_\alpha)}$$
$$\times \overline{\langle \varphi_{\mu,\lambda}\, \tau_{(r,x)} v, \phi_j \rangle}_{L^2(d\nu_\alpha)} \, d\mu_\alpha(r,x;\mu,\lambda)$$
$$= \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda) \langle \overline{\varphi_{\mu,\lambda}\, \tau_{(r,x)} v}, \overline{\varphi_{\mu,\lambda}\, \tau_{(r,x)} u} \rangle_{L^2(d\nu_\alpha)} \, d\mu_\alpha(r,x;\mu,\lambda),$$

and the proof is complete. □

In the following, we give the main result of this section.

Corollary 3.10. *Let σ be in $L^p(d\mu_\alpha)$, $1 \leqslant p \leqslant \infty$. Then, the localization operator*

$$\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_p, and we have

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_p} \leqslant \|\sigma\|_{L^p(d\mu_\alpha)}.$$

Proof. The result follows from Proposition 3.4, Theorem 3.8, and by interpolation [**23,** Theorems 2.10, 2.11]. □

Remark 3.11. If $u = v$, and, if σ is a real-valued and nonnegative function in $L^1(d\mu_\alpha)$, then

$$\mathcal{L}_{u,v}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a positive operator. Thus, by (2.22) and Corollary 3.9,

$$||\mathcal{L}_{u,v}(\sigma)||_{S_1} = \int_\Gamma \int_{\mathbb{R}^2_+} \sigma(r,x;\mu,\lambda) ||\varphi_{\mu,\lambda}\, \tau_{(r,x)} u||^2_{L^2(d\nu_\alpha)} d\mu_\alpha(r,x;\mu,\lambda). \tag{3.12}$$

Now, we state a result concerning the trace of products of localization operators.

Corollary 3.12. *Let σ_1 and σ_2 be any real-valued and non-negative functions in $L^1(d\mu_\alpha)$. We assume that $u = v$ is a function in $L^2(d\nu_\alpha)$ such that $||u||_{L^2(d\nu_\alpha)} = 1$. Then, the localization operators $\mathcal{L}_{u,v}(\sigma_1)$,*

$\mathcal{L}_{u,v}(\sigma_2)$ *are positive trace class operators, and*

$$\begin{aligned}||(\mathcal{L}_{u,v}(\sigma_1)\,\mathcal{L}_{u,v}(\sigma_2))^n||_{S_1} &= \mathrm{tr}(\mathcal{L}_{u,v}(\sigma_1)\,\mathcal{L}_{u,v}(\sigma_2))^n\\ &\leq (\mathrm{tr}(\mathcal{L}_{u,v}(\sigma_1)))^n(\mathrm{tr}(\mathcal{L}_{u,v}(\sigma_2)))^n\\ &= ||\mathcal{L}_{u,v}(\sigma_1)||^n_{S_1}||\mathcal{L}_{u,v}(\sigma_2)||^n_{S_1},\end{aligned}$$

for all natural numbers n.

Proof. By [**18,** Theorem 1], we know that, if A and B are in the trace class S_1 and are positive operators, then, for all $n \in \mathbb{N}$,

$$\mathrm{tr}(AB)^n \leq (\mathrm{tr}(A))^n(\mathrm{tr}(B))^n.$$

Thus, if we take $A = \mathcal{L}_{u,v}(\sigma_1)$, $B = \mathcal{L}_{u,v}(\sigma_2)$ and we invoke the previous remark, the proof is complete. □

4. L^p Boundedness and compactness of $\mathcal{L}_{u,v}(\sigma)$.

4.1. Boundedness for symbols in $L^p(d\mu_\alpha)$. For $1 \leq p \leq \infty$, let $\sigma \in L^1(d\mu_\alpha)$, $v \in L^p(d\nu_\alpha)$ and $u \in L^{p'}(d\nu_\alpha)$.

We shall show that $\mathcal{L}_{u,v}(\sigma)$ is a bounded operator on $L^p(d\nu_\alpha)$. We start with the following propositions.

Proposition 4.1. *Let σ be in $L^1(d\mu_\alpha)$, $u \in L^\infty(d\nu_\alpha)$ and $v \in L^1(d\nu_\alpha)$. Then, the localization operator $\mathcal{L}_{u,v}(\sigma) : L^1(d\nu_\alpha) \to L^1(d\nu_\alpha)$ is a bounded linear operator, and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^1(d\nu_\alpha))} \leqslant \|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.$$

Proof. For every function f in $L^1(d\nu_\alpha)$, we have from the relations (2.2), (2.5) (2.17) and (3.1),

$$\begin{aligned}&||\mathcal{L}_{u,v}(\sigma)(f)||_{L^1(d\nu_\alpha)}\\ &\quad\leqslant \int_{\mathbb{R}^2_+\times\Gamma}\int_{\mathbb{R}^2_+}|\sigma(r,x;\mu,\lambda)|\,|\mathcal{V}(f,u)(r,x;\mu,\lambda)|\\ &\quad\times|\tau_{(r,x)}v(s,y)\,\varphi_{\mu,\lambda}(s,y)|\,d\mu_\alpha(r,x;\mu,\lambda)\,d\nu_\alpha(s,y)\\ &\quad\leqslant \|f\|_{L^1(d\nu_\alpha)}||u||_{L^\infty(d\nu_\alpha)}||v||_{L^1(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.\end{aligned}$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^1(d\nu_\alpha))} \leqslant ||u||_{L^\infty(d\nu_\alpha)}||v||_{L^1(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.$$ □

Proposition 4.2. *Let σ be in $L^1(d\mu_\alpha)$, $u \in L^1(d\nu_\alpha)$ and $v \in L^\infty(d\nu_\alpha)$. Then, the localization operator $\mathcal{L}_{u,v}(\sigma) : L^\infty(d\nu_\alpha) \to L^\infty(d\nu_\alpha)$ is a bounded linear operator, and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^\infty(d\nu_\alpha))} \leqslant \|u\|_{L^1(d\nu_\alpha)}\|v\|_{L^\infty(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.$$

Proof. Let f in $L^\infty(d\nu_\alpha)$. As above, from relations (2.2), (2.5), (2.17) and (3.1), for all $(s,y) \in \mathbb{R}^2_+$,

$$\begin{aligned}
&|\mathcal{L}_{u,v}(\sigma)(f)(s,y)| \\
&\quad \leqslant \int_\Gamma \int_{\mathbb{R}^2_+} |\sigma(r,x;\mu,\lambda)| \\
&\quad \times |\mathcal{V}(f,u)(r,x;\mu,\lambda)|\, |\tau_{(r,x)}v(s,y)\, \varphi_{\mu,\lambda}(s,y)|\, d\mu_\alpha(r,x;\mu,\lambda) \\
&\quad \leqslant \|f\|_{L^\infty(d\nu_\alpha)}||u||_{L^1(d\nu_\alpha)}||v||_{L^\infty(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.
\end{aligned}$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^\infty(d\nu_\alpha))} \leqslant ||u||_{L^1(d\nu_\alpha)}||v||_{L^\infty(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}. \qquad \square$$

Remark 4.3. Proposition 4.2 is also a corollary of Proposition 4.1 since the adjoint of

$$\mathcal{L}_{v,u}(\overline{\sigma}) : L^1(d\nu_\alpha) \longrightarrow L^1(d\nu_\alpha)$$

is $\mathcal{L}_{u,v}(\sigma) : L^\infty(d\nu_\alpha) \to L^\infty(d\nu_\alpha)$.

Using an interpolation of Propositions 4.1 and 4.2, we obtain the following result.

Theorem 4.4. *Let u and v be functions in $L^1(d\nu_\alpha) \cap L^\infty(d\nu_\alpha)$. Then, for all σ in $L^1(d\mu_\alpha)$, there exists a unique bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha), \quad 1 \leq p \leq \infty,$$

such that

$$\begin{aligned}
&||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \\
&\quad \leqslant \|u\|^{1/p'}_{L^1(d\nu_\alpha)}\|v\|^{1/p}_{L^1(d\nu_\alpha)}\|u\|^{1/p}_{L^\infty(d\nu_\alpha)}\|v\|^{1/p'}_{L^\infty(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.
\end{aligned}$$

With a Schur technique, we can obtain an L^p-boundedness result as in the previous theorem, but the estimate for the norm $||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))}$ is cruder.

Theorem 4.5. *Let σ be in $L^1(d\mu_\alpha)$, u and v in $L^1(d\nu_\alpha) \cap L^\infty(d\nu_\alpha)$. Then, there exists a unique bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha), \quad 1 \leq p \leq \infty$$

such that

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant \max(\|u\|_{L^1(d\nu_\alpha)}\|v\|_{L^\infty(d\nu_\alpha)}, \|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)})\|\sigma\|_{L^1(d\mu_\alpha)}.$$

Proof. Let $\mathcal{K}_\alpha$ be the function defined on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ by

$$\mathcal{K}_\alpha(s,y;t,z) = \int_\Gamma \int_{\mathbb{R}_+^2} \sigma(r,x;\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)\tau_{(s,y)}v(r,x)}\varphi_{\mu,\lambda}(t,z) \times \tau_{(r,x)}u(t,z)\, d\mu_\alpha(r,x;\mu,\lambda). \tag{4.1}$$

We have

$$\mathcal{L}_{u,v}(\sigma)(f)(s,y) = \int_{\mathbb{R}_+^2} \mathcal{K}_\alpha(y,z)f(t,z)\, d\nu_\alpha(t,z).$$

By simple calculations, it is easy to see that

$$\int_{\mathbb{R}_+^2} |\mathcal{K}_\alpha(s,y;t,z)|\, d\nu_\alpha(s,y) \leq \|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}, \quad (t,z) \in \mathbb{R}_+^2,$$

and

$$\int_{\mathbb{R}_+^2} |\mathcal{K}_\alpha(s,y;t,z)| d\nu_\alpha(t,z) \leq \|u\|_{L^1(d\nu_\alpha)}\|v\|_{L^\infty(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}, \quad (s,y) \in \mathbb{R}_+^2.$$

Thus, by the Schur lemma, cf., **[11]**, we can conclude that

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$

is a bounded linear operator for $1 \leq p \leq \infty$, and we have

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant \max(\|u\|_{L^1(d\nu_\alpha)}\|v\|_{L^\infty(d\nu_\alpha)}, \|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)})\|\sigma\|_{L^1(d\mu_\alpha)}. \square$$

Remark 4.6. Theorem 4.5 tells us that the unique bounded linear operator on $L^p(d\nu_\alpha)$, $1 \leq p \leq \infty$, obtained by interpolation in Theorem 4.4, is, in fact, the integral operator on $L^p(d\nu_\alpha)$ with kernel $\mathcal{K}_\alpha$ given by (4.1).

We can give another version of the L^p-boundedness. Firstly, we generalize and improve Proposition 4.2.

Proposition 4.7. *Let σ be in $L^1(d\mu_\alpha)$, $v \in L^p(d\nu_\alpha)$ and $u \in L^{p'}(d\nu_\alpha)$, for $1 < p \leq \infty$. Then, the localization operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$

is a bounded linear operator, and we have

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant \|u\|_{L^{p'}(d\nu_\alpha)}\|v\|_{L^p(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.$$

Proof. For any $f \in L^p(d\nu_\alpha)$, consider the linear functional

$$\mathcal{I}_f : L^{p'}(d\nu_\alpha) \longrightarrow \mathbb{C}$$
$$g \longmapsto \langle g, \mathcal{L}_{u,v}(\sigma)(f)\rangle_{L^2(d\nu_\alpha)}.$$

From relation (3.2),

$$|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \leqslant \int_\Gamma\int_{\mathbb{R}^2_+} |\sigma(r,x;\mu,\lambda)||\mathcal{V}(f,u)(r,x;\mu,\lambda)||\mathcal{V}(g,v)(r,x;\mu,\lambda)|d\mu_\alpha(r,x;\mu,\lambda)$$
$$\leqslant \|\sigma\|_{L^1(d\mu_\alpha)}\|\mathcal{V}(f,u)\|_{L^\infty(d\mu_\alpha)}\|\mathcal{V}(g,v)\|_{L^\infty(d\mu_\alpha)}.$$

Using relation (2.17), we get

$$|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \leqslant \|\sigma\|_{L^1(d\mu_\alpha)}\|u\|_{L^{p'}(d\nu_\alpha)}\|v\|_{L^p(d\nu_\alpha)}\|f\|_{L^p(d\nu_\alpha)}\|g\|_{L^{p'}(d\nu_\alpha)}.$$

Thus, the operator $\mathcal{I}_f$ is a continuous linear functional on $L^{p'}(d\nu_\alpha)$, and the operator norm

$$||\mathcal{I}_f||_{B(L^{p'}(d\nu_\alpha))} \leqslant \|u\|_{L^{p'}(d\nu_\alpha)}\|v\|_{L^p(d\nu_\alpha)}\|f\|_{L^p(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.$$

As $\mathcal{I}_f(g) = \langle g, \mathcal{L}_{u,v}(\sigma)(f)\rangle_{L^2(d\nu_\alpha)}$, by the Riesz representation theorem, we have

$$\begin{aligned}||\mathcal{L}_{u,v}(\sigma)(f)||_{L^p(d\nu_\alpha)} &= ||\mathcal{I}_f||_{B(L^{p'}(d\nu_\alpha))} \\ &\leqslant \|u\|_{L^{p'}(d\nu_\alpha)}\|v\|_{L^p(d\nu_\alpha)}\|f\|_{L^p(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)},\end{aligned}$$

which establishes the proposition. □

Combining Proposition 4.1 and Proposition 4.7, we have the following theorem.

Theorem 4.8. *Let σ be in $L^1(d\mu_\alpha)$, $v \in L^p(d\nu_\alpha)$ and $u \in L^{p'}(d\nu_\alpha)$, for $1 \le p \le \infty$. Then, the localization operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$

is a bounded linear operator, and we have

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant \|u\|_{L^{p'}(d\nu_\alpha)}\|v\|_{L^p(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}.$$

We can now state and prove the main results in this subsection.

Theorem 4.9. *Let σ be in $L^r(d\mu_\alpha)$, $r \in [1,2]$, and u, v belong to $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha) \cap L^\infty(d\nu_\alpha)$. Then, there exists a unique bounded linear operator $\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \to L^p(d\nu_\alpha)$ for all $p \in [r, r']$, and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant C_1^t C_2^{1-t}\|\sigma\|_{L^r(d\mu_\alpha)}, \tag{4.2}$$

where

$$\begin{aligned}C_1 &= (\|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)})^{2/r-1}(\|u\|_{L^2(d\nu_\alpha)}\|v\|_{L^2(d\nu_\alpha)})^{1/r'}, \\ C_2 &= (\|u\|_{L^1(d\nu_\alpha)}\|v\|_{L^\infty(d\nu_\alpha)})^{2/r-1}(\|u\|_{L^2(d\nu_\alpha)}\|v\|_{L^2(d\nu_\alpha)})^{1/r'},\end{aligned}$$

and

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

Proof. Consider the linear functional

$$\begin{aligned}\mathcal{I} : (L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)) \times (L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)) &\longrightarrow L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha) \\ (\sigma, f) &\longmapsto \mathcal{L}_{u,v}(\sigma)(f).\end{aligned}$$

Then, by Proposition 4.1 and Theorem 3.5,

$$(4.3)\quad ||\mathcal{I}(\sigma,f)||_{L^1(d\nu_\alpha)} \leqslant \|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)}||f||_{L^1(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)}$$

and

$$(4.4)\quad ||\mathcal{I}(\sigma,f)||_{L^2(d\nu_\alpha)} \leq \|u\|_{L^2(d\nu_\alpha)}\|v\|_{L^2(d\nu_\alpha)}||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^2(d\mu_\alpha)}.$$

Therefore, by (4.3), (4.4) and the multi-linear interpolation theory [**4,** subsection 10.1], we obtain a unique bounded linear operator $\mathcal{I} : L^r(d\mu_\alpha) \times L^r(d\nu_\alpha) \to L^r(d\nu_\alpha)$ such that

$$(4.5)\qquad ||\mathcal{I}(\sigma,f)||_{L^r(d\nu_\alpha)} \leq C_1||f||_{L^r(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)},$$

where

$$C_1 = (\|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)})^\theta(\|u\|_{L^2(d\nu_\alpha)}\|v\|_{L^2(d\nu_\alpha)})^{(1-\theta)/2}$$

and

$$\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{r}.$$

By the definition of $\mathcal{I}$, we have

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^r(d\nu_\alpha))} \leqslant C_1\|\sigma\|_{L^r(d\mu_\alpha)}.$$

As the adjoint of $\mathcal{L}_{u,v}(\sigma)$ is $\mathcal{L}_{v,u}(\overline{\sigma})$, so $\mathcal{L}_{u,v}(\sigma)$ is a bounded linear map on $L^{r'}(d\nu_\alpha)$ with its operator norm

$$(4.6)\quad ||\mathcal{L}_{u,v}(\sigma)||_{B(L^{r'}(d\nu_\alpha))} = ||\mathcal{L}_{v,u}(\overline{\sigma})||_{B(L^r(d\nu_\alpha))} \leq C_2\|\sigma\|_{L^r(d\mu_\alpha)},$$

where

$$C_2 = (\|u\|_{L^1(d\nu_\alpha)}\|v\|_{L^\infty(d\nu_\alpha)})^\theta(\|u\|_{L^2(d\nu_\alpha)}\|v\|_{L^2(d\nu_\alpha)})^{(1-\theta)/2}.$$

Using an interpolation of (4.5) and (4.6), we have that, for any p belongs to $[r, r']$,

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant C_1^t C_2^{1-t}\|\sigma\|_{L^r(d\mu_\alpha)},$$

with

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}. \qquad \square$$

In the remainder of this subsection, we prove different versions of L^p boundedness for the localization operators.

Theorem 4.10. *Let σ be in $L^r(d\mu_\alpha)$, $r \in [1,2)$, and u, v belong to $L^r(d\nu_\alpha)\bigcap L^{r'}(d\nu_\alpha)$. Then, there exists a unique bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$

for all $p \in [r, r']$, and we have

(4.7)

$$\begin{aligned}&||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))}\\ &\quad\leqslant (\|u\|_{L^r(d\nu_\alpha)}\|v\|_{L^{r'}(d\nu_\alpha)})^t(\|u\|_{L^{r'}(d\nu_\alpha)}\|v\|_{L^r(d\nu_\alpha)})^{1-t}\|\sigma\|_{L^r(d\mu_\alpha)},\end{aligned}$$

where

$$t = \frac{r-p}{p(r-2)}.$$

Proof. For every function f in $L^{r'}(d\nu_\alpha)$ and g in $L^r(d\nu_\alpha)$, from (3.2) we have

$$\begin{aligned}&|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}|\\ &\quad\leqslant \int_\Gamma\int_{\mathbb{R}^2_+}|\sigma(r,x;\mu,\lambda)|\,|\mathcal{V}(f,u)(r,x;\mu,\lambda)\overline{\mathcal{V}(g,v)(r,x;\mu,\lambda)}|\,d\mu_\alpha(r,x;\mu,\lambda)\\ &\quad\leqslant \|\mathcal{V}(f,u)\|_{L^{r'}(d\mu_\alpha)}\|\mathcal{V}(g,v)\|_{L^\infty(d\mu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)}.\end{aligned}$$

Using relations (2.17) and (2.18), we get

$$\begin{aligned}\|\mathcal{V}(f,u)\|_{L^{r'}(d\mu_\alpha)} &\leq \|u\|_{L^r(d\nu_\alpha)}\|f\|_{L^{r'}(d\nu_\alpha)},\\ \|\mathcal{V}(g,v)\|_{L^\infty(d\mu_\alpha)} &\leq \|v\|_{L^{r'}(d\nu_\alpha)}\|g\|_{L^r(d\nu_\alpha)}.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}&|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}|\\ &\quad\leqslant \|u\|_{L^r(d\nu_\alpha)}\|v\|_{L^{r'}(d\nu_\alpha)}\|f\|_{L^{r'}(d\nu_\alpha)}\|g\|_{L^r(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)}.\end{aligned}$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^{r'}(d\nu_\alpha))} \leqslant \|u\|_{L^r(d\nu_\alpha)}\|v\|_{L^{r'}(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)}. \tag{4.8}$$

As the adjoint of $\mathcal{L}_{v,u}(\overline{\sigma})$ is $\mathcal{L}_{u,v}(\sigma)$, so $\mathcal{L}_{u,v}(\sigma)$ is a bounded linear map on $L^r(d\nu_\alpha)$ with its operator norm

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^r(d\nu_\alpha))} \leq \|u\|_{L^{r'}(d\nu_\alpha)}\|v\|_{L^r(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)}. \tag{4.9}$$

Using an interpolation of (4.8) and (4.9), we deduce the result. □

Proposition 4.11. *Let $p, r \in [1, \infty]$ be such that $p \in [2r/(r+1), 2]$. Let σ be in $L^r(d\mu_\alpha)$, u belongs to $L^2(d\nu_\alpha) \cap L^\infty(d\nu_\alpha)$, and v belongs to $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$. Then, there exists a unique bounded linear operator $\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \to L^p(d\nu_\alpha)$, and we have*

(4.10)
$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant \|u\|^{1-t}_{L^\infty(d\nu_\alpha)} \|v\|^{1-t}_{L^1(d\nu_\alpha)} \|u\|^{t}_{L^2(d\nu_\alpha)} \|v\|^{t}_{L^2(d\nu_\alpha)} \|\sigma\|_{L^r(d\mu_\alpha)},$$

where
$$t = \frac{(r-1)q}{(q-1)r}, \quad \textit{with } q = \frac{(2p-2)r}{p-(2-p)r}.$$

Proof. The proof follows from Theorem 4.8 with $p = 1$, Theorem 3.5 with q instead of p, and interpolation theory. □

Theorem 4.12. *Let σ be in $L^r(d\mu_\alpha)$, $r \in [1, \infty]$, and u, v belong to $L^1(d\nu_\alpha) \cap L^\infty(d\nu_\alpha)$. Then, there exists a unique bounded linear operator*
$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$
for all $p \in [2r/(r+1), 2r/(r-1)]$, and we have

(4.11)
$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant C_3^{t/r} C_4^{(1-t)/r} \|u\|^{1/r'}_{L^2(d\nu_\alpha)} \|v\|^{1/r'}_{L^2(d\nu_\alpha)} \|\sigma\|_{L^r(d\mu_\alpha)},$$

where
$$C_3 = \|v\|_{L^\infty(d\nu_\alpha)} \|u\|_{L^1(d\nu_\alpha)},$$
$$C_4 = \|v\|_{L^1(d\nu_\alpha)} \|u\|_{L^\infty(d\nu_\alpha)},$$

and
$$t = \frac{r+1}{2} - \frac{r}{p}.$$

In order to prove this theorem, we need the following lemmas.

Lemma 4.13. *Let σ be in $L^r(d\mu_\alpha)$, $r \in [1, \infty]$, $u \in L^2(d\nu_\alpha) \cap L^\infty(d\nu_\alpha)$, and v belongs in $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$. Then, there exists a unique bounded linear operator*
$$\mathcal{L}_{u,v}(\sigma) : L^{2r/(r+1)}(d\nu_\alpha) \longrightarrow L^{2r/(r+1)}(d\nu_\alpha),$$

and we have

$$\begin{aligned}(4.12)\quad &||\mathcal{L}_{u,v}(\sigma)||_{B(L^{2r/(r+1)}(d\nu_\alpha))}\\ &\leqslant \|u\|^{1/r}_{L^\infty(d\nu_\alpha)}\|v\|^{1/r}_{L^1(d\nu_\alpha)}\|u\|^{1/r'}_{L^2(d\nu_\alpha)}\|v\|^{1/r'}_{L^2(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)}.\end{aligned}$$

Proof. Consider the linear functional

$$\begin{aligned}\mathcal{I} : L^1(d\mu_\alpha)\cap L^\infty(d\mu_\alpha) &\longrightarrow B(L^1(d\nu_\alpha))\cap B(L^2(d\nu_\alpha))\\ \sigma &\longmapsto \mathcal{L}_{u,v}(\sigma).\end{aligned}$$

Then, by Proposition 4.1 and Theorem 3.5,

$$(4.13)\qquad ||\mathcal{I}||_{B(L^1(d\mu_\alpha),B(L^1(d\nu_\alpha)))} \leqslant \|u\|_{L^\infty(d\nu_\alpha)}\|v\|_{L^1(d\nu_\alpha)}$$

and

$$(4.14)\qquad ||\mathcal{I}||_{B(L^\infty(d\mu_\alpha),B(L^2(d\nu_\alpha)))} \leqslant \|u\|_{L^2(d\nu_\alpha)}\|v\|_{L^2(d\nu_\alpha)},$$

where $||\ ||_{B(L^p(d\mu_\alpha),B(L^q(d\nu_\alpha)))}$ denotes the norm in the Banach space of the bounded linear operators from $L^p(d\mu_\alpha)$ into $B(L^q(d\nu_\alpha))$, with $1 \leq p,\ q \leq \infty$. Using an interpolation of (4.13) and (4.14), we obtain the result. □

Lemma 4.14. *Let σ be in $L^r(d\mu_\alpha)$, $r \in [1,\infty]$, $v \in L^2(d\nu_\alpha)\bigcap L^\infty(d\nu_\alpha)$, and u belongs in $L^1(d\nu_\alpha)\bigcap L^2(d\nu_\alpha)$. Then, there exists a unique bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^{2r/(r-1)}(d\nu_\alpha) \longrightarrow L^{2r/(r-1)}(d\nu_\alpha),$$

and we have

$$\begin{aligned}(4.15)\quad &||\mathcal{L}_{u,v}(\sigma)||_{B(L^{2r/(r-1)}(d\nu_\alpha))}\\ &\leqslant \|v\|^{1/r}_{L^\infty(d\nu_\alpha)}\|u\|^{1/r}_{L^1(d\nu_\alpha)}\|u\|^{1/r'}_{L^2(d\nu_\alpha)}\|v\|^{1/r'}_{L^2(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)}.\end{aligned}$$

Proof. As the adjoint of $\mathcal{L}_{v,u}(\overline{\sigma})$ is $\mathcal{L}_{u,v}(\sigma)$, so the result follows from duality and the previous lemma. □

Proof of Theorem 4.12*.* Using an interpolation of (4.12) and (4.15), we have that, for any $p \in [2r/(r+1), 2r/(r-1)]$,

$$||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \leqslant C_3^{t/r}C_4^{(1-t)/r}\|u\|^{1/r'}_{L^2(d\nu_\alpha)}\|v\|^{1/r'}_{L^2(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)},$$

with
$$t = \frac{r+1}{2} - \frac{r}{p}. \qquad \square$$

Proposition 4.15. *Let σ be in $L^r(d\mu_\alpha)$, $r > 2$, and let u belongs to $L^{(\theta' r')'}(d\nu_\alpha)$ and $v \in L^{\theta r'}(d\nu_\alpha)$, where $\theta \in [1, \infty]$ is such that $\min\{\theta r', \theta' r'\} > 2$. Then, there exists a unique bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^{\theta' r'}(d\nu_\alpha) \longrightarrow L^{\theta r'}(d\nu_\alpha),$$

and we have

$$(4.16) \qquad \begin{aligned} &||\mathcal{L}_{u,v}(\sigma)||_{B(L^{\theta' r'}(d\nu_\alpha), L^{\theta r'}(d\nu_\alpha))} \\ &\quad \leqslant \|u\|_{L^{(\theta' r')'}(d\nu_\alpha)} \|v\|_{L^{\theta r'}(d\nu_\alpha)} \|\sigma\|_{L^r(d\mu_\alpha)}, \end{aligned}$$

where $|| \ ||_{B(L^{\theta' r'}(d\nu_\alpha), L^{\theta r'}(d\nu_\alpha))}$ is the norm in the Banach space of all bounded linear operators from $L^{\theta' r'}(d\nu_\alpha)$ into $L^{\theta r'}(d\nu_\alpha)$.

Proof. For every function f in $L^{\theta' r'}(d\nu_\alpha)$ and g in $L^{(\theta r')'}(d\nu_\alpha)$, from Hölder's inequality and (3.2), we have

$$\begin{aligned} &|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \\ &\quad \leqslant \int_\Gamma \int_{\mathbb{R}^2_+} |\sigma(r, x; \mu, \lambda)| \, |\mathcal{V}(f, u)(r, x; \mu, \lambda) \overline{\mathcal{V}(g, v)(r, x; \mu, \lambda)}| \, d\mu_\alpha(r, x; \mu, \lambda) \\ &\quad \leqslant \|\mathcal{V}(f, u)\|_{L^{\theta' r'}(d\mu_\alpha)} \|\mathcal{V}(g, v)\|_{L^{\theta r'}(d\mu_\alpha)} \|\sigma\|_{L^r(d\mu_\alpha)}. \end{aligned}$$

As $\min\{\theta r', \theta' r'\} > 2$, from Proposition 2.15,

$$\mathcal{V}(\cdot, u) : L^{\theta' r'}(d\nu_\alpha) \longrightarrow L^{\theta' r'}(d\mu_\alpha)$$

and

$$\mathcal{V}(\cdot, v) : L^{(\theta r')'}(d\nu_\alpha) \longrightarrow L^{\theta r'}(d\mu_\alpha),$$

and, by relation (2.19), we get

$$\begin{aligned} &|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \\ &\quad \leqslant \|u\|_{L^{(\theta' r')'}(d\nu_\alpha)} \|v\|_{L^{\theta r'}(d\nu_\alpha)} \|f\|_{L^{\theta' r'}(d\nu_\alpha)} \|g\|_{L^{(\theta r')'}(d\nu_\alpha)} \|\sigma\|_{L^r(d\mu_\alpha)}, \end{aligned}$$

and the proof is complete. $\square$

Now, by the same scheme as above, by duality and interpolation arguments, we obtain the following result.

Theorem 4.16. *Let σ be in $L^r(d\mu_\alpha)$, $r \in [1,\infty]$, and u, v belong to $L^{2r/(r+1)}(d\nu_\alpha)\bigcap L^{2r/(r-1)}(d\nu_\alpha)$. Then, there exists a unique bounded linear operator $\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \to L^p(d\nu_\alpha)$ for all $p \in [2r/(r+1), 2r/(r-1)]$, and we have*

$$
\begin{aligned}
(4.17)\qquad & ||\mathcal{L}_{u,v}(\sigma)||_{B(L^p(d\nu_\alpha))} \\
& \leqslant \|u\|^{1-t}_{L^{2r/(r+1)}(d\nu_\alpha)}\|u\|^{t}_{L^{2r/(r-1)}(d\nu_\alpha)}\|v\|^{1-t}_{L^{2r/(r-1)}(d\nu_\alpha)} \\
& \times \|v\|^{t}_{L^{2r/(r+1)}(d\nu_\alpha)}\|\sigma\|_{L^r(d\mu_\alpha)},
\end{aligned}
$$

where

$$t = \frac{r}{p} - \frac{r-1}{2}.$$

Remark 4.17. Theorem 4.16 gives the same boundedness result as in Theorem 4.12, but under slightly less restrictive conditions on the windows. On the other hand, it does not cover Theorem 4.12 because of the presence of a different constant in the estimate.

4.2. Compactness of $\mathcal{L}_{u,v}(\sigma)$.

Proposition 4.18. *Under the same hypotheses as Theorem* 4.4*, the localization operator*

$$\mathcal{L}_{u,v}(\sigma) : L^1(d\nu_\alpha) \longrightarrow L^1(d\nu_\alpha)$$

is compact.

Proof. Let $(f_n)_{n\in\mathbb{N}} \in L^1(d\nu_\alpha)$ be such that $f_n \rightharpoonup 0$ weakly in $L^1(d\nu_\alpha)$ as $n \to \infty$. It is sufficient to prove that

$$\lim_{n\to\infty} ||\mathcal{L}_{u,v}(\sigma)(f_n)||_{L^1(d\nu_\alpha)} = 0.$$

We have

(4.18)

$$
\begin{aligned}
||\mathcal{L}_{u,v}(\sigma)(f_n)||_{L^1(d\nu_\alpha)} \leqslant & \int_{\mathbb{R}^2_+}\int_\Gamma\int_{\mathbb{R}^2_+} |\sigma(r,x;\mu,\lambda)|\,|\langle f_n, \tau_{(r,x)}u\,\varphi_{\mu,\lambda}\rangle_{L^2(d\nu_\alpha)}| \\
& \times |\tau_{(r,x)}v(s,y)\,\overline{\varphi_{\mu,\lambda}(s,y)}|\,d\mu_\alpha(r,x;\mu,\lambda)\,d\nu_\alpha(s,y).
\end{aligned}
$$

Now, using the fact that $f_n \rightharpoonup 0$ weakly in $L^1(d\nu_\alpha)$, we deduce that, for all $(r,x),(s,y) \in \mathbb{R}^2_+, (\mu,\lambda) \in \Gamma$,

(4.19)

$$\lim_{n\to\infty} |\sigma(r,x;\mu,\lambda)|\,|\langle f_n, \varphi_{\mu,\lambda}\, \tau_{(r,x)}u\rangle_{L^2(d\nu_\alpha)}|\,|\tau_{(r,x)}v(s,y)\,\overline{\varphi_{\mu,\lambda}(s,y)}| = 0.$$

On the other hand, since $f_n \rightharpoonup 0$ weakly in $L^1(d\nu_\alpha)$ as $n \to \infty$, then there exists a positive constant C such that

$$||f_n||_{L^1(d\nu_\alpha)} \leq C.$$

Hence, by simple calculations, we get, for all $(r,x),(s,y) \in \mathbb{R}^2_+, (\mu,\lambda) \in \Gamma$,

$$\begin{aligned}(4.20)\quad |\sigma(r,x;\mu,\lambda)|\,|\langle f_n, \varphi_{\mu,\lambda}\, \tau_{(r,x)}u\rangle_{L^2(d\nu_\alpha)}|\,|\tau_{(r,x)}v(s,y)\,\overline{\varphi_{\mu,\lambda}(s,y)}| \\ \leq C|\sigma(r,x;\mu,\lambda)|\,||u||_{L^\infty(d\nu_\alpha)}\,|v(s,y)|.\end{aligned}$$

Moreover, from Fubini's theorem and relation (2.17), we have

$$\begin{aligned}(4.21)\qquad &\int_{\mathbb{R}^2_+}\int_\Gamma\int_{\mathbb{R}^2_+} |\sigma(r,x;\mu,\lambda)|\,|\langle f_n, \varphi_{\mu,\lambda}\tau_{(r,x)}u\rangle_{L^2(d\nu_\alpha)}| \\ &\times |\tau_{(r,x)}v(s,y)\,\overline{\varphi_{\mu,\lambda}(s,y)}|\, d\mu_\alpha(r,x;\mu,\lambda)\, d\nu_\alpha(s,y) \\ &\leq C||u||_{L^\infty(d\nu_\alpha)} \int_\Gamma\int_{\mathbb{R}^2_+} |\sigma(r,x;\mu,\lambda)| \\ &\times \int_{\mathbb{R}^2_+} |v(s,y)|\, d\nu_\alpha(s,y)\, d\mu_\alpha(r,x;\mu,\lambda) \\ &\leq C||u||_{L^\infty(d\nu_\alpha)}||v||_{L^1(d\nu_\alpha)}\|\sigma\|_{L^1(d\mu_\alpha)} < \infty.\end{aligned}$$

Thus, from relations (4.18), (4.19), (4.20), (4.21) and the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} ||\mathcal{L}_{u,v}(\sigma)(f_n)||_{L^1(d\nu_\alpha)} = 0,$$

and the proof is complete. □

In the following, we give two results for compactness of the localization operator.

Theorem 4.19. *Under the hypotheses of Theorem* 4.4, *the bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$

is compact for $1 \leq p \leq \infty$.

Proof. From the previous proposition, we only need show that the conclusion holds for $p = \infty$. In fact, the operator

$$\mathcal{L}_{u,v}(\sigma) : L^\infty(d\nu_\alpha) \longrightarrow L^\infty(d\nu_\alpha)$$

is the adjoint of the operator $\mathcal{L}_{v,u}(\overline{\sigma}) : L^1(d\nu_\alpha) \to L^1(d\nu_\alpha)$, which is compact by the previous proposition. Thus, by the duality property, $\mathcal{L}_{u,v}(\sigma) : L^\infty(d\nu_\alpha) \to L^\infty(d\nu_\alpha)$ is compact. Finally, by an interpolation of the compactness on $L^1(d\nu_\alpha)$ and on $L^\infty(d\nu_\alpha)$ such as that given in [**3,** pages 202, 203], the proof is complete. □

The following result is an analogue of Theorem 4.9 for compact operators.

Theorem 4.20. *Under the hypotheses of Theorem* 4.9, *the bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^p(d\nu_\alpha) \longrightarrow L^p(d\nu_\alpha)$$

is compact for all $p \in [r, r']$.

Proof. The result is an immediate consequence of an interpolation of Corollary 3.10 and Proposition 4.18. See, again, [**3,** pages 202, 203] for the interpolation used. □

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REFERENCES

1. C. Baccar and N.B. Hamadi, *Localization operators of the wavelet transform associated to the Riemann-Liouville operator*, Int. J. Math. **27** (2016).

2. C. Baccar, N.B. Hamadi and L.T. Rachdi, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators*, Int. J. Math. Math. Sci. **2006** (2006), 1–26.

3. C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, New York, 1988.

4. J.P. Calderon, *Intermediate spaces and interpolation, The complex method*, Stud. Math. **24** (1964), 113–190.

5. E. Cordero and K. Gröchenig, *Time-frequency analysis of localization operators*, J. Funct. Anal. **205** (2003), 107–131.

6. I. Daubechies, *Time-frequency localization operators: A geometric phase space approach*, IEEE Trans. Info. Th. **34** (1988), 605–612.

7. ———, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Info. Th. **36** (1990), 961–1005.

8. F. De Mari, H. Feichtinger and K. Nowak, *Uniform eigenvalue estimates for time-frequency localization operators*, J. London Math. Soc. **65** (2002), 720–732.

9. F. De Mari and K. Nowak, *Localization type Berezin-Toeplitz operators on bounded symmetric domains*, J. Geom. Anal. **12** (2002), 9–27.

10. J.A. Fawcett, *Inversion of N-dimensional spherical means*, SIAM. J. Appl. Math. **45** (1983), 336–341.

11. G.B. Folland, *Introduction to partial differential equations*, Princeton University Press, Princeton 1995.

12. K. Gröchenig, *Foundations of time-frequency analysis*, Springer Sci. Bus. Media, New York, 2001.

13. N.B. Hamadi, *Localization operators for the windowed Fourier transform associated with singular partial differential operators*, Rocky Mountain J. Math. **47** (2017), 2179–2195.

14. N.B. Hamadi and L.T. Rachdi, *Weyl transforms associated with the Riemann-Liouville operator*, Int. J. Math. Math. Sci. **2006** (2006).

15. H. Helesten and L.E. Andersson, *An inverse method for the processing of synthetic aperture radar data*, Inv. Prob. **3** (1987), 111–124.

16. F. John, *Plane waves and spherical means applied to partial differential equations*, Interscience, New York, 1955.

17. E.H. Lieb, *Integral bounds for radar ambiguity functions and Wigner distributions*, J. Math. Phys. **31** (1990), 594–599.

18. L. Liu, *A trace class operator inequality*, J. Math. Anal. Appl. **328** (2007), 1484–1486.

19. S. Omri and L. Rachdi, *Heisenberg-Pauli-Weyl uncertainty principle for the Riemann-Liouville operator*, J. Inequal. Pure Appl. Math. **9** (2008), 23.

20. J. Ramanathan and P. Topiwala, *Time-frequency localization via the Weyl correspondence*, SIAM J. Math. Anal. **24** (1993), 1378–1393.

21. E.M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492.

22. K. Trimèche, *Permutation operators and the central limit theorem associated with partial differential operators*, in *Probability measures on groups* **10** (1991), 395–424.

23. M.W. Wong, *Wavelet transforms and localization operators*, Springer Sci. Bus. Med. **136** (2002).

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