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PROPAGATION OF SINGULAR SURFACES IN A NON-IDEAL GAS WITH AXIAL MAGNETIC FIELD

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ABSTRACT. In this paper, a system of equations describing the motion of a one-dimensional, cylindrically symmetrical, inviscid, non-ideal gas with a constant axial magnetic field is considered and the singular surface theory is used to study different modes of wave propagation and its culmination into a shock wave. The transport equation for the jump in the velocity gradient is derived and solved numerically to study the affects of the ratio of specific heat, non ideal parameter and initial magnetic fields on jump in the velocity gradient.

1. Introduction. In the case of the study of nonlinear wave propagation, we often encounter discontinuities such as shock waves, acceleration waves, singular surfaces, etc. Due to nonlinearity, any perturbation in which the flow variables such as density, pressure, etc., increase gradually, will steepen before decreasing. The steepening results in the formation of a discontinuity, i.e., a shock wave. The flow variables or their derivatives across a discontinuity or shock are interrelated, and the relations are called compatibility conditions. The Rankine Hugoniot jump conditions, which are derived from the conservation form of the system of equations, relate the flow variables across the shock and are known as the first set of compatibility conditions. Compatibility conditions of the first order are the relations which connect the first order derivatives of the flow variables on both sides of the discontinuity surface and are known as the second set of compatibility conditions [**8**]. The geometrical and kinematical compatibility conditions of first and second order for a singular surface are developed in [**14**]. The evolutionary behavior of shock using singular surface theory is discussed in [**1, 3, 5, 7, 9, 12, 13**]. The singular surface theory is used in [**4, 10, 11**] to study steepening of waves and variation of the jump in

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first order derivatives of flow variables in different mediums such as radiative magnetohydrodynamics, non-ideal gas with dust particles and non-ideal relaxing gas.

In the present work, an attempt is made to determine the steepening of shock waves under the influence of an axial magnetic field in a non ideal gas. We consider a system of equations describing the motion of a one-dimensional, cylindrically symmetrical, inviscid, non-ideal gas with an axial magnetic field. An axial magnetic field is assumed to exist initially in the conducting gas. The transport equation for the gradient of flow variables in terms of jump in velocity gradient is derived. The affects of initial magnetic field and the non ideal parameter on the jump in velocity gradient are studied.

2. Basic equations. In the case when the magnetic Raynold's number $R_m \gg 1$, i.e., the gas is ionized, the system of equations describing the motion of a one-dimensional, cylindrically symmetrical, inviscid, non-ideal gas with a constant axial magnetic field is given by **[2, 15, 16]**

$$\begin{aligned} &\rho_t + u\,\rho_x + \rho\,u_x + \frac{\rho\,u}{x} = 0,\\ &u_t + u\,u_x + \frac{p_x}{\rho} + \frac{B\,B_x}{\mu\,\rho} = 0,\\ &p_t + u\,p_x - a^2\,(\rho_t + u\,\rho_x) = 0,\\ &B_t + B\,u_x + u\,B_x + \frac{B\,u}{x} = 0, \end{aligned} \tag{2.1}$$

with the van der Walls equation of state

$$p = \frac{\rho R T}{1 - b\rho}.$$

Here, x is the spatial coordinate, t is the time, ρ is the density, u is the particle velocity, p is the pressure, γ is the ratio of specific heats,

$$a = \left(\frac{\gamma p}{(1 - b\rho)\rho}\right)^{1/2}$$

is the equilibrium speed of sound, R is the specific gas constant, T is the translational temperature and b is the van der Waals excluded volume which lies in the range $0.9 \times 10^{-3} \le b \le 1.1 \times 10^{-3}$. The quantity B is the axial magnetic induction at distance x from the axis of symmetry, and μ is the magnetic permeability. A variable as a subscript indicates partial differentiation with respect to that variable, if not otherwise defined. The case $b = 0$ corresponds to the ideal gas.

3. Transport equation for a singular surface. Let ψ be the wave front with equation $x = X(t)$ across which the flow variables ρ, u, p and B are essentially continuous but discontinuities exist in their derivatives. The wave front ψ is called a singular surface and propagates with the speed $V = dX/dt$. If W is any of the flow variables ρ, u, p, and B, then the geometrical and kinematical compatibility conditions of the first and second order for a singular surface are given by **[14]**

$$\begin{aligned} |[W_x]| &= A, \qquad |[W_t]| = -VA, \\ |[W_{xx}]| &= \overline{A}, \qquad |[W_{xt}]| = V\left(\frac{dA}{dx} - \overline{A}\right), \end{aligned} \tag{3.1}$$

where A and $\overline{A}$ are the quantities defined on ψ. Here, $|[Z]| = Z - Z_0$ represents the jump, where Z_0 denotes the value just ahead of ψ and Z; the variable without a subscript denotes the value just behind ψ. Assuming that

$$\begin{aligned} |[\rho_x]| &= \zeta, & |[u_x]| &= \lambda, \\ |[p_x]| &= \xi, & |[B_x]| &= \eta, \\ |[\rho_t]| &= -V\,\zeta, & |[u_t]| &= -V\,\lambda, \\ |[p_t]| &= -V\,\xi, & |[B_x]| &= -V\,\eta, \end{aligned} \tag{3.2}$$

and then evaluating (2.1) on the inner boundary of ψ and using (3.2), we get

$$\begin{aligned} (V - u_0)\zeta &= \lambda\rho_0, \\ (V - u_0)\lambda &= \frac{1}{\rho_0}\left(\xi + \frac{B_0\eta}{\mu}\right), \\ (V - u_0)\xi &= \lambda\rho_0 a_0^2, \\ (V - u_0)\eta &= \lambda B_0. \end{aligned} \tag{3.3}$$

Since V is positive for an advancing wave, from equation (3.3), we obtain the following relations

$$V = u_0 + C_0, \qquad \lambda = \frac{C_0}{\rho_0}\,\zeta = \frac{C_0}{\rho_0\,a_0^2}\,\xi = \frac{C_0}{B_0}\,\eta, \tag{3.4}$$

where C is the effective speed of sound given by $C^2 = a^2 + B^2/\mu\,\rho$.

Differentiating (2.1) with respect to x and then taking the jump along ψ we get

$$|[\rho_{xt}]| + 2\,|[u_x\,\rho_x]| + u_0\,|[\rho_{xx}]| + \rho_0\,|[u_{xx}]| + \frac{u_0}{x}\,|[\rho_x]| + \frac{\rho_0}{x}\,|[u_x]| = 0,$$

$$\begin{aligned} &|[u_{xt}]| + |[(u_x)^2]| + u_0\,|[u_{xx}]| \\ &\qquad + \frac{|[p_{xx}]|}{\rho_0} - \frac{|[\rho_x\,p_x]|}{\rho_0^2} + \frac{|[(B_x)^2]|}{\rho_0\,\mu} \\ &\qquad\qquad + \frac{B_0}{\rho_0\,\mu}|[B_{xx}]| - \frac{B_0}{\mu\,\rho_0^2}\,|[B_x\,\rho_x]| = 0, \end{aligned}$$

$$\begin{aligned} &|[p_{xt}]| + \left(1 + \frac{\rho_0\,a_0^2}{p_0}\right)|[u_x\,p_x]| + u_0\,|[p_{xx}]| \\ &\qquad + \rho_0\,a_0^2\left(|[u_{xx}]| + \frac{1}{x}\,|[u_x]|\right) + \frac{b\,\rho_0\,a_0^2}{1 - b\,\rho_0}\,|[\rho_x\,u_x]| \\ &\qquad\qquad + \frac{u_0}{x}\left(\frac{\rho_0\,a_0^2}{p_0}\,|[p_x]| + \frac{b\,\rho_0\,a_0^2}{1 - b\,\rho_0}\,|[\rho_x]|\right) = 0, \end{aligned} \tag{3.5}$$

$$|[B_{xt}]| + 2\,|[B_x\,u_x]| + 2\,|[B\,u_{xx}]| + |[u\,B_{xx}]| + \left|\left[\frac{B_x\,u}{x}\right]\right| + \left|\left[\frac{B\,u_x}{x}\right]\right| = 0.$$

From relations (3.1), (3.2) and (3.4) we get the following relations

$$\begin{aligned} |[\rho_{xx}]| &= \overline{\zeta}, & |[u_{xx}]| &= \overline{\lambda}, \\ |[p_{xx}]| &= \overline{\xi}, & |[B_{xx}]| &= \overline{\eta}, \\ |[\rho_{xt}]| &= V\,(\zeta_x - \overline{\zeta}), & |[u_{xt}]| &= V\,(\lambda_x - \overline{\lambda}), \\ |[p_{xt}]| &= V\,(\xi_x - \overline{\xi}), & |[B_{xt}]| &= V\,(\eta_x - \overline{\eta}). \end{aligned} \tag{3.6}$$

Using equations (3.2), (3.4) and (3.6) in (3.5), we obtain

$$V\,\zeta_x - C_0\,\overline{\zeta} + \rho_0\,\overline{\lambda} + 2((\zeta + \rho_{0x})\,\lambda + \zeta\,u_{0x}) + \frac{1}{x}(u_0\,\zeta + \rho_0\,\lambda) = 0,$$

$$
\begin{aligned}
(3.7)\quad & V\lambda_x - C_0\,\overline{\lambda} + \lambda^2 + 2\,u_{0x}\lambda \\
&+ \frac{1}{\rho_0^2}\left(\rho_0\,\overline{\xi} - \zeta\,(\xi + p_{0x}) - \rho_{0x}\,\xi\right) + \frac{1}{\rho_0\,\mu}\,(\eta^2 + 2\,B_{0x}\,\eta)\frac{B_0\,\overline{\eta}}{\rho_0\,\mu} \\
&\qquad - \frac{B_0}{\mu\,\rho_0^2}(\eta\,\zeta + B_{0x}\,\zeta + \eta\,\rho_{0x}) = 0,
\end{aligned}
$$

$$
\begin{aligned}
& V\xi_x - C_0\,\overline{\xi} + \lambda\,(\xi + p_{0x}) + u_{0x}\,\xi + \rho_0\,a_0^2\left(\overline{\lambda} + \frac{\lambda}{x}\right) \\
&+ \frac{\gamma\,b\,p_0}{(1 - b\,\rho_0)^2}\left(\left(\lambda + u_{0x} + \frac{u_0}{x}\right)\zeta + \lambda\,\rho_{0x}\right) \\
&\qquad + \frac{\gamma}{1 - b\,\rho_0}\left(\left(\lambda + u_{0x} + \frac{u_0}{x}\right)\xi + p_{0x}\,\lambda\right) = 0,
\end{aligned}
$$

$$
V\eta_x - V\,\overline{\eta} + 2\,\eta\,\lambda + 2\,B_{0x}\,\lambda + 2\,u_{0x}\,\eta + B_0\,\overline{\lambda} + u_0\,\overline{\eta} + \frac{u_0}{x}\,\eta + \frac{B_0}{x}\,\lambda = 0.
$$

Elimination of $\overline{\lambda}$, $\overline{\eta}$ and $\overline{\xi}$ from (3.7) leads to the following Bernoulli-type transport equation for λ

$$
2\,\frac{d\,\lambda}{dt} + \Theta_1\,\lambda + \Theta_2\,\lambda^2 = 0, \tag{3.8}
$$

where

$$
\begin{aligned}
\Theta_1 &= u_{0x}\left(\frac{a_0^2}{C_0^2} + \frac{(\gamma + b\,\rho_0)\,a_0^2}{C_0^2(1 - b\,\rho_0)} + 2\,\frac{B_0^2}{C_0^2\,\rho_0\,\mu} + 2\right) + p_{0x}\,\frac{a_0^2}{C_0\,p_0} \\
&+ \rho_{0x}\left(\frac{b\,a_0^2}{C_0(1 - b\,\rho_0)} - \frac{a_0^2}{C_0\,\rho_0} - \frac{B_0^2}{\mu\,C_0\rho_0^2}\right) + 3\,B_{0x}\frac{B_0}{\mu} \\
&+ \frac{1}{x}\left(\frac{a_0^2}{C_0} + \frac{b\,a_0^2\,u_0\,\rho_0}{C_0^2\,(1 - b\,\rho_0)} + \frac{u_0\,a_0^4\,\rho_0}{C_0^2\,p_0}\right) + \frac{B_0^2\,u_0}{x\,\mu\rho_0\,C_0^2} + \frac{B_0^2}{\mu\,\rho_0\,C_0}, \\
\Theta_2 &= \frac{a_0^2}{C_0^2(1 - b\,\rho_0)}(\gamma + b\,\rho_0) + 1 + \frac{2\,B_0^2}{C_0^2\,\mu\,\rho_0}
\end{aligned}
$$

It may be noted that the amplitude of the discontinuity waves, which is also known as acceleration waves and are characterized by a discontinuity in a normal derivative of the field itself, satisfy the transport equation of Bernoulli type.

3.1. Particular case. We consider a particular case where the particle velocity exhibits linear dependence on the spatial co-ordinate and the flow ahead is characterized **[6]** by the following:

(3.9)
$$u_0(x,t) = f(t)\,x, \qquad \rho_0 = \rho_0(t), \qquad p_0 = p_0(t), \qquad B_0 = B_0(t).$$

In view of (3.9), we integrate (2.1) to obtain

$$\begin{aligned} \rho_0(t) &= \rho_{0c}(1 + f_c(t - t_c))^{-2}, \\ f(t) &= f_c(1 + f_c(t - t_c))^{-1}, \\ p_0(t) &= p_{0c}\left(\frac{1 - b\rho_{0c}}{(1 + f_c(t - t_c))^2 - b\rho_{0c}}\right)^{\gamma}, \\ B_0 &= B_{0c}(1 + f_c(t - t_c))^{-2}, \end{aligned} \tag{3.10}$$

where f_c, ρ_{0c}, p_{0c} and B_{0c} are the reference values of velocity, density, pressure and axial magnetic field at $t = t_c$, respectively.

3.2. Evaluation of the velocity gradient. In order to compute the velocity gradient, we consider the following dimensionless variables

$$\begin{aligned} t^* &= \frac{t}{t_c}, \qquad f^* = f\,t_c, \qquad \rho_0^* = \frac{\rho_0}{\rho_{0c}}, \qquad p_0^* = \frac{p_0}{\rho_{0c}a_{0c}^2}, \\ x^* &= \frac{x}{t_c\,a_{0c}}, \qquad b^* = \rho_{0c}\,b, \qquad \lambda^* = t_c\lambda, \qquad B_0^* = \frac{B_0}{\rho_{0c}a_{0c}^2}. \end{aligned} \tag{3.11}$$

In view of (3.11), the relations in (3.10), after suppressing the asterisk sign, can be written as

$$\begin{aligned} \rho_0(t) &= (1 + f_c(t - 1))^{-2}, \\ f(t) &= f_c(1 + f_c(t - 1))^{-1}, \\ p_0(t) &= \frac{1}{\gamma}\left(\frac{1 - b}{(1 + f_c(t - 1))^2 - b}\right)^{\gamma}, \\ B_0 &= B_{0c}(1 + f_c(t - 1))^{-2}. \end{aligned} \tag{3.12}$$

Using (3.11) and (3.12), the transport equation (3.8) can be written in the following form, after suppressing the asterisk sign:

$$2\frac{d\lambda}{dt} + \Theta_1\lambda + \Theta_2\lambda^2 = 0, \tag{3.13}$$

where

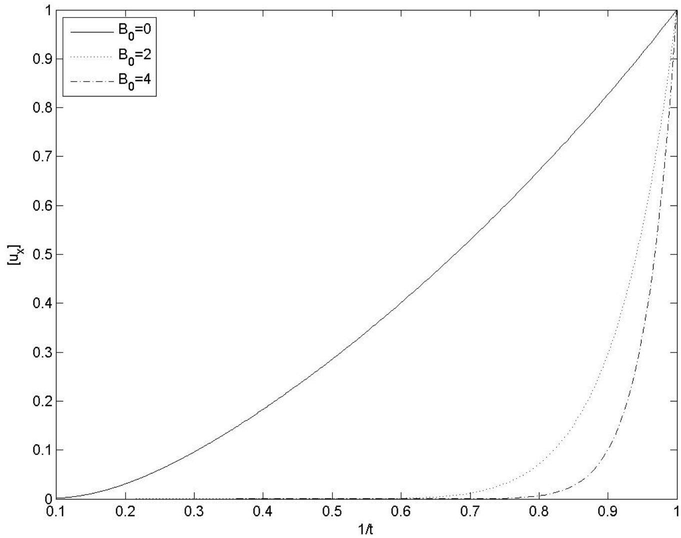

FIGURE 1. Variation of jump in velocity gradient $[u_x]$ versus $1/t$ for varying B_{0c} and $\gamma = 1.4$, $b = 1.1 \times 10^{-3}$, $f_{0c} = 0.01$, $\rho_{0c} = 0.1$, $p_{0c} = 0.1$, $\mu = 0.1$.

(3.14)

$$\Theta_1 = f\left(\frac{a_0^2}{C_0^2} + \frac{a_0^2(\gamma + b\,\rho_0)}{C_0^2(1 - b\,\rho_0)} + 2 + \frac{3\,B_0^2}{C_0^2\,\rho_0\,\mu} + \frac{ba_0^2\,\rho_0}{C_0^2(1 - b\,\rho_0)} + \frac{\rho_0\,a_0^4}{p_0\,C_0^2}\right) + \frac{a_0^2}{x_0\,C_0} + \frac{B_0^2}{\mu\,\rho_0\,C_0},$$

$$\Theta_2 = \frac{a_0^2(\gamma + b\,\rho_0)}{C_0^2(1 - b\,\rho_0)} + 1 + 2\frac{B_0^2}{\mu\,\rho_0\,C_0^2}.$$

Integration of (3.12) with respect to t leads to

$$\lambda = \frac{\lambda_0\,\Upsilon(t)}{1 + \lambda_0\,\Psi(t)}, \tag{3.15}$$

where

TABLE 1. Critical time $\widehat{t}$ for various values of γ, b, f_{0c}, B_{0c}.

γ	b	f_{0c}	B_{0c}	μ	$\widehat{t}$
1	1.1×10^{-3}	0.5	0.1	0.1	1.497
1	0.9×10^{-3}	0.5	0.1	0.1	1.499
1	0	0.5	0.1	0.1	1.475
1	1.1×10^{-3}	1	0.1	0.1	1.786
1	1.1×10^{-3}	1.5	0.1	0.1	1.572
1	1.1×10^{-3}	2	0.1	0.1	1.464
1	1.1×10^{-3}	1	5	0.1	1.262
1	1.1×10^{-3}	1	10	0.1	1.172
1	1.1×10^{-3}	1	15	0.1	1.130
1.4	1.1×10^{-3}	0.5	0.1	0.1	1.628
1.4	0.9×10^{-3}	0.5	0.1	0.1	1.610
1.4	0	0.5	0.1	0.1	1.445
1.4	1.1×10^{-3}	1	0.1	0.1	1.449
1.4	1.1×10^{-3}	1.5	0.1	0.1	1.385
1.4	1.1×10^{-3}	2	0.1	0.1	1.347
1.4	1.1×10^{-3}	1	5	0.1	1.2610
1.4	1.1×10^{-3}	1	10	0.1	1.172
1.4	1.1×10^{-3}	1	15	0.1	1.130
1.7	1.1×10^{-3}	0.5	0.1	0.1	1.506
1.7	0.9×10^{-3}	0.5	0.1	0.1	1.492
1.7	0	0.5	0.1	0.1	1.398
1.7	1.1×10^{-3}	1	0.1	0.1	1.361
1.7	1.1×10^{-3}	1.5	0.1	0.1	1.326
1.7	1.1×10^{-3}	2	0.1	0.1	1.300
1.7	1.1×10^{-3}	1	5	0.1	1.258
1.7	1.1×10^{-3}	1	10	0.1	1.171
1.7	1.1×10^{-3}	1	15	0.1	1.130

$$\Psi(t) = \int_{t_0}^{t} \Theta_2\, \Upsilon(s)\, ds, \tag{3.16}$$
$$\Upsilon(t) = \exp\left(- \int_{t_0}^{t} \Theta_1\, ds\right)$$

and $\lambda_0 = \lambda(t_0)$. From equation (3.15), it is observed that the solution breaks down at some critical time $t = \widehat{t}$, where $1 + \lambda_0\, \Psi(t) = 0$. This shows that the compression wave turns into a shock at $t = \widehat{t}$, when the initial discontinuity associated with the wave exceeds a critical value.

Equation (3.12) is numerically solved for different values of initial axial magnetic field B_{0c}, non ideal parameter b and ratio of specific heats γ, and the results are depicted in Figures 1–3. It is observed that, as t increases, the value of λ decreases and tends to zero as $t \to \infty$.

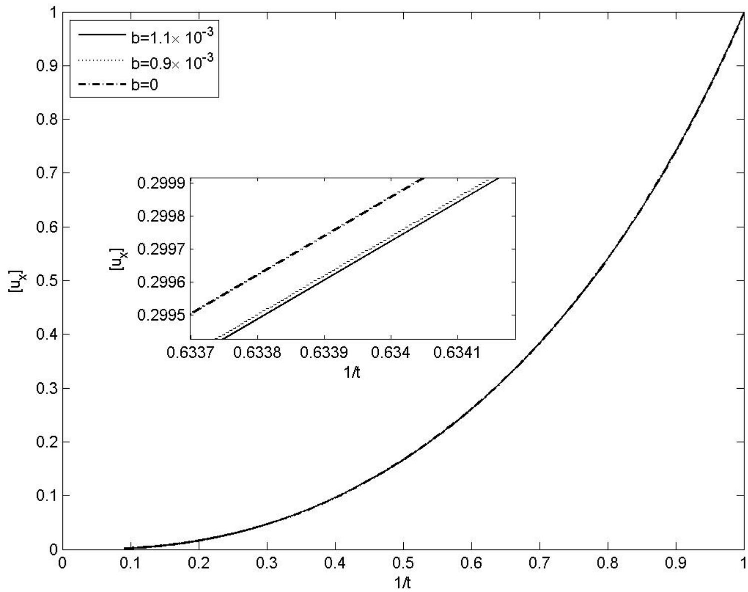

FIGURE 2. Variation of jump in velocity gradient $[u_x]$ versus $1/t$ for $\gamma = 1.4$, $B_{0c} = 2$, $f_{0c} = 0.01$, $\rho_{0c} = 0.1$, $p_{0c} = 0.1$, $\mu = 0.1$.

An increase in the initial axial magnetic field B_{0c} or ratio of specific heat γ leads to a decrease in λ, whereas an increase in the non ideal parameter b leads to an increase in λ.

4. Results and conclusion. In this paper, we considered a system of equations describing the motion of a one-dimensional, cylindrically

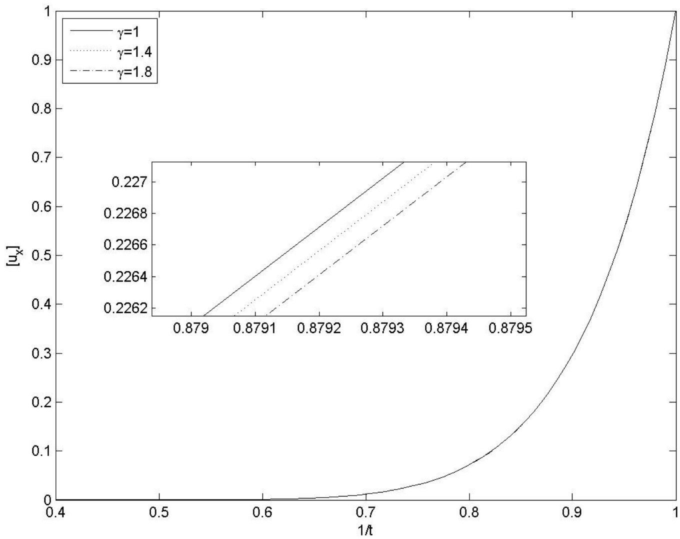

FIGURE 3. Variation of jump in velocity gradient $[u_x]$ versus $1/t$ for γ and $b = 1.1 \times 10^{-3}$, $B = 2$, $f_{0c} = 0.01$, $\rho_{0c} = 0.1$, $p_{0c} = 0.1$, $\mu = 0.1$.

symmetrical, inviscid, non-ideal gas with a constant axial magnetic field. The singular surface theory was used to study the propagation, steepening of the wave and its culmination into a shock under the influence of the axial magnetic field. The transport equation for the jump in velocity gradient, which is a Bernoulli-type equation, was derived. The affects of non ideal parameter and initial magnetic field on the jump in velocity gradient were studied by numerically solving the transport equation.

It was observed that there exists some critical time at which the solution breaks down, implying thereby that the compression wave turns into a shock at that critical time when the initial discontinuity associated with the wave exceeds a critical value. It was also observed that, as time t increases, the jump in the velocity gradient decreases and tends to zero for a large value of t. An increase in the initial axial magnetic field or the ratio of specific heat leads to a decrease in the jump in velocity gradient, whereas an increase in the non ideal parameter b leads to an increase in the jump in velocity gradient.

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