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PRACTICAL STABILITY ANALYSIS WITH RESPECT TO MANIFOLDS AND BOUNDEDNESS OF DIFFERENTIAL EQUATIONS WITH FRACTIONAL-LIKE DERIVATIVES

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ABSTRACT. In this paper, a new approach for studying the practical stability and boundedness with respect to a manifold of the solutions of a class of fractional differential equations is applied. The technique is based on the recently defined "fractional-like derivative" of Lyapunov-type functions. Sufficient conditions using vector Lyapunov functions are established. Examples are also presented to illustrate the theory.

1. Introduction. Practical stability is one of the most important concepts of the stability theory of differential equations. Introduced in [**12**], it has been developed for different types of dynamical systems, and numerous important results have been reported in the literature. See, for example [**9, 11, 23, 24, 26**] and the references cited therein. The practical stability is related to the study of the behavior of solutions of differential equations close to a certain trajectory, given in advance the domain where the initial data change, and the domain where the trajectories of solutions should remain when the independent variable changes over a fixed interval (finite or infinite). The desired trajectory may be unstable in the sense of Lyapunov, but a solution of the system may oscillate sufficiently near this trajectory so that its behavior is acceptable. As such, practical stability and Lyapunov stability are quite independent concepts, and, in general, neither imply nor exclude each other. In some cases, although a system is stable or asymptotically stable in the Lyapunov sense, it is actually useless in practice

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due to undesirable transient characteristics (e.g., the stability domain or the attraction domain is not sufficiently large to allow the desired deviation to cancel out). Due to its theoretical and applied significance, the practical stability concept is applied to many mechanics and engineering problems [**20, 27**]. For example, it is very useful in estimating the worst-case transient and steady-state responses and in verifying pointwise in-time constraints imposed on the state trajectories [**26**].

At the present time, the stability theory of fractional differential equations is undergoing rapid development. Using different definitions of fractional derivatives, many results on the stability and boundedness of the solutions of such equations have been obtained (see [**5, 9, 17, 18, 25**] and the bibliography therein). The most popular definitions used are of the type of Caputo, Riemann-Liouville and Grunwald-Letnikov definitions.

In recent years, there has also been a growing interest in the application of Lyapunov's direct method to the stability analysis of fractional differential equations. Many interesting results have been obtained by using Lyapunov-type functions, mainly for equations with Caputo fractional derivatives of the state vector. However, the main difficulty in the application of the second method of Lyapunov to equations with fractional derivatives is the absence of a simple chain rule formula [**3, 15, 25**]. This difficulty motivates researchers to introduce new definitions that will avoid the restrictions of the existing ones, and will offer opportunities to establish stability results similar to those in the classical Lyapunov stability analysis [**13, 14, 28**].

A fruitful technique that has gained increasing significance is related to the application of the recently defined limit-based "conformable fractional" and related derivatives [**1, 4, 7, 8, 19, 21**]. In the recent paper [**16**], the authors called the "conformable fractional derivative" a "fractional-like derivative" (FLD) since this notion expresses their understanding of the nature of a fractional derivative. We also introduced a fractional-like derivative of a Lyapunov-type function and applied it to derive sufficient conditions for stability, asymptotic stability and instability of the trivial solution of equations of perturbed motion with a fractional-like derivative of the state vector. In addition, we showed that the fractional-like derivative of a Lyapunov-type function is an upper bound of the Caputo fractional derivative of this Lyapunov function.

The main aim of the present paper is to apply the newly defined FLD to practical stability with respect to a manifold of fractional differential equations with FLDs of the state vector. In addition, since boundedness properties of solutions of differential equations are of significant importance for the existence of periodic and almost periodic solutions [**12, 22, 24**] and have many applications in physics, biological population management and control, we use the FLD to investigate the boundedness of the solutions of such equations.

The paper is organized as follows. In Section 2, the problem is stated, the main definitions and properties of a fractional-like derivative are given, and the notion of practical stability with respect to a manifold is presented. In addition, the fractional-like derivative of a Lyapunov-type function is introduced. Section 3 is devoted to our main practical stability results. The newly defined FLD of Lyapunov functions allows us to obtain efficient criteria for practical stability, uniform practical stability and asymptotic practical stability with respect to a manifold for equations with fractional-like derivatives of the state vector. The comparison principle for vector Lyapunov's functions is the basis of the proofs. In Section 4, the problems of boundedness with respect to a manifold of the solutions of such systems are considered. Three examples are given in Section 5 to illustrate the theory. Some discussions are also performed. Finally, in Section 6, we present our concluding remarks.

2. Statement of the problem. Preliminaries. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $||\cdot||$, and let $\mathbb{R}_+ = [0, \infty)$.

Definition 2.1 ([**8, 19**])**.** For any $t \geq t_0$, $t_0 \in \mathbb{R}_+$, the *fractional-like derivative of order* q, $0 < q \leq 1$, with the lower limit t_0 for a continuous function $x(t) : [t_0, \infty) \to \mathbb{R}$, is defined as

$$\mathcal{D}^q_{t_0}(x(t)) = \lim \left\{ \frac{x(t+\theta(t-t_0)^{1-q}) - x(t)}{\theta},\ \theta \to 0 \right\}.$$

In the case $t_0 = 0$, we have [**8**]

$$\mathcal{D}^q_0(x(t)) = \lim \left\{ \frac{x(t+\theta t^{1-q}) - x(t)}{\theta},\ \theta \to 0 \right\}.$$

If the fractional-like derivative of order q for a function $x(t)$ exists on a point t, $t \in \mathbb{R}_+$, then we will say that the function $x(t)$ is q-differentiable at that point.

The fractional-like integral of order $0 < q \leq 1$ with a lower limit t_0 is defined by (see **[8]**)

$$I_{t_0}^{q} x(t) = \int_{t_0}^{t} (s - t_0)^{q-1} x(s)\, ds.$$

Throughout this paper, we will use the following properties of fractional-like derivatives.

Lemma 2.2 (**[19]**). *Let $l(y(t)) : (t_0, \infty) \to \mathbb{R}$. If $l(\cdot)$ is differentiable with respect to $y(t)$ and $y(t)$ is q-differentiable, where $0 < q \leq 1$, then, for any $t \in \mathbb{R}_+$, $t \neq t_0$ and $y(t) \neq 0$,*

$$\mathcal{D}_{t_0}^{q} l(y(t)) = l'(y(t)) \mathcal{D}_{t_0}^{q}(y(t)),$$

where $l'(t)$ is a partial derivative of $l(\cdot)$.

Lemma 2.3 (**[1, 8]**). *Let the function $x(t) : (t_0, \infty) \to \mathbb{R}$ be q-differentiable for $0 < q \leq 1$. Then, for all $t > t_0$,*

$$I_{t_0}^{q}(\mathcal{D}_{t_0}^{q} x(t)) = x(t) - x(t_0).$$

For more results on fractional-like derivatives, we refer the reader to **[1, 4, 7, 8, 16, 19, 21]**.

In this paper, we consider a system of differential equations with a fractional-like derivative of the state vector

$$\mathcal{D}_{t_0}^{q} x(t) = f(t, x(t)), \tag{2.1}$$

where $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \geq 0$.

Let $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$. Denote by $x(t) = x(t; t_0, x_0)$ the solution of system (2.1), satisfying the initial condition

$$x(t_0) = x_0. \tag{2.2}$$

We will further assume that, for $(t_0, x_0) \in \mathrm{int}(\mathbb{R}_+ \times \mathbb{R}^n)$, the solution $x(t, t_0, x_0)$ of the initial value problem (IVP) (2.1)–(2.2) exists on $[t_0, \infty)$, and $x(t, t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$, $t \geq t_0$. In addition, it is assumed that $f(t, 0) = 0$ for all $t \geq t_0$.

Let $h : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^k$ $(k \leq n)$ be a continuous function. We introduce the sets:

$$M_t(n-k) = \{x \in \mathbb{R}^n : \ h(t, x) = 0, \ t \in [t_0, \infty)\},$$

$$M_t(n-k)(\varepsilon) = \{x \in \mathbb{R}^n : \ ||h(t, x)|| < \varepsilon, \ t \in [t_0, \infty)\}, \quad \varepsilon > 0,$$

$$M_t(n-k)(\overline{\varepsilon}) = \{x \in \mathbb{R}^n : \ ||h(t, x)|| \leq \varepsilon, \ t \in [t_0, \infty)\}.$$

In the remainder of the paper, we will assume that the set $M_t(n-k)$ is an $(n-k)$-dimensional manifold in $\mathbb{R}^n$.

We shall introduce the following definitions of practical stability of system (2.1) with respect to the function h which are generalizations of the definitions given in **[10]**.

Definition 2.4. System (2.1) is said to be:

(a) *practically stable* with respect to the function h, if, given (λ, A) with $0 < \lambda < A$, we have $x_0 \in M_{t_0}(n-k)(\lambda)$ implies $x(t; t_0, x_0) \in M_t(n-k)(A)$, $t \geq t_0$ for some $t_0 \in \mathbb{R}_+$;

(b) *uniformly practically stable* with respect to the function h, if (a) holds for every $t_0 \in \mathbb{R}_+$;

(c) *practically asymptotically stable* with respect to the function h, if (a) holds and

$$\lim_{t \to \infty} ||h(t, x(t; t_0, x_0))|| = 0.$$

(d) *practically exponentially stable* with respect to the function h, if, given (λ, A) with $0 < \lambda < A$, we have $x_0 \in M_{t_0}(n-k)(\lambda)$ implies

$$x(t; t_0, x_0) \in M_t(n-k)(A + \mu||h(t_0, x_0)||E_q(-\kappa, t - t_0)), \quad t \geq t_0,$$

for some $t_0 \in \mathbb{R}_+$, where $0 < q < 1$, $\mu, \kappa > 0$, and $E_q(\nu, s)$ is the fractional-like exponential function given as **[1, 21]**

$$E_q(\nu, s) = \exp\left(\nu \frac{s^q}{q}\right), \quad \nu \in \mathbb{R}, \ s \in \mathbb{R}_+.$$

Let $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$, $r > 0$. We shall use the class of continuous vector Lyapunov-like functions:

$$C_q^m = \{V : V \in C^q(\mathbb{R}_+ \times B_r \to \mathbb{R}^m),\ V \text{ is locally Lipschitzian in } x \in B_r \text{ and } V(t,0) = 0 \text{ for all } t \in \mathbb{R}_+\},$$

$$V(t,x) = (V_1(t,x_1), \ldots, V_m(t,x_m)).$$

Definition 2.5 ([**16**])**.** Let $V \in C_q^m$ be a continuous and q-differentiable function (scalar or vector). Then, for $(t,x) \in \mathbb{R}_+ \times B_r$, the expression:

$$(2.3)\quad {}^{+}\mathcal{D}_{t_0}^q\ V(t,x) = \limsup\left\{\frac{V(t+\theta(t-t_0)^{1-q},\ x(t+\theta(t-t_0)^{1-q}, t, x)) - V(t,x)}{\theta},\ \theta \to 0^+\right\},$$

is the upper right fractional-like derivative of the Lyapunov function,

$${}_{+}\mathcal{D}_{t_0}^q\ V(t,x) = \liminf\left\{\frac{V(t+\theta(t-t_0)^{1-q},\ x(t+\theta(t-t_0)^{1-q}, t, x)) - V(t,x)}{\theta},\ \theta \to 0^+\right\},$$

is the lower right fractional-like derivative of the Lyapunov function,

$${}^{-}\mathcal{D}_{t_0}^q\ V(t,x) = \limsup\left\{\frac{V(t+\theta(t-t_0)^{1-q},\ x(t+\theta(t-t_0)^{1-q}, t, x)) - V(t,x)}{\theta},\ \theta \to 0^-\right\},$$

is the upper left fractional-like derivative of the Lyapunov function, and

$${}_{-}\mathcal{D}_{t_0}^q\ V(t,x) = \liminf\left\{\frac{V(t+\theta(t-t_0)^{1-q},\ x(t+\theta(t-t_0)^{1-q}, t, x)) - V(t,x)}{\theta},\ \theta \to 0^-\right\},$$

is the lower left fractional-like derivative of the Lyapunov function.

Let $x(t, t_0, x_0)$ be the solution of the IVP (2.1)–(2.2), which exists and is defined on $\mathbb{R}_+ \times B_r$. Then, [**16**], the fractional-like derivative of the function $V(t,x)$ with respect to the solution $x(t, t_0, x_0)$ is defined by

(2.3*) $\quad {}^{+}\mathcal{D}^{q}_{t_0}\, V(t,x)$

$$=\limsup\left\{\frac{V(t+\theta(t-t_0)^{1-q},\, x+\theta(t-t_0)^{1-q}f(t,x))-V(t,x)}{\theta},\ \theta\to 0^{+}\right\}.$$

If $V(t,x(t)) = V(x(t)),\ 0 < q \leq 1$, the function V is differentiable on x, and the function $x(t)$ is q -differentiable on t for $t > t_0$, then

$$ {}^{+}\mathcal{D}^{q}_{t_0}\, V(t,x) = V'(x(t))\ \mathcal{D}^{q}_{t_0}x(t), $$

where V' is a partial derivative of the function V.

From (2.3) and (2.3*), we derive the following result

$$ {}^{+}\mathcal{D}^{q}_{t_0}\, V(t, x(t,t_0,x_0)) = {}^{+}\mathcal{D}^{q}_{t_0}\, V(t,x)\,|_{(2.1)}\ . $$

Together with system (2.1), we consider the following comparison vector fractional-like equation

$$ \mathcal{D}^{q}_{t_0}\ u(t) = H(t,u), \tag{2.4} $$

where $0 < q \leq 1,\ H : \mathbb{R}_+ \times \mathbb{R}^m_+ \to \mathbb{R}^m$.

Let $u_0 \in \mathbb{R}^m_+$. Denote by $u^+(t) = u^+(t;t_0,u_0)$ the maximal solution of equation (2.4), which satisfies the initial condition

$$ u^+(t_0;t_0,u_0) = u_0. \tag{2.5} $$

Let $e \in \mathbb{R}^m$ be the vector $(1,1,\dots,1)$. Furthermore, we shall consider only such solutions $u(t)$ of system (2.4) for which $u(t) \geq 0$, which is why the following stability definitions are used.

Definition 2.6. The system (2.4) is said to be:

(a) *practically stable* with respect to (λ, A), if, given (λ, A) with $0 < \lambda < A$, we have $u_0 < \lambda e$ implies $u^+(t;t_0,u_0) < Ae$, $t \geq t_0$ for some $t_0 \in \mathbb{R}_+$;

(b) *uniformly practically stable* with respect to (λ, A), if (a) holds for every $t_0 \in \mathbb{R}_+$;

(c) *practically asymptotically stable* with respect to (λ, A), if (a) holds and $\lim_{t\to\infty} u^+(t;t_0,u_0) = 0$;

(d) *practically exponentially stable* with respect to (λ, A), if, given (λ, A) with $0 < \lambda < A$, we have $u_0 < \lambda e$ implies

$$u^+(t; t_0, u_0) < Ae + \mu u_0 E_q(-\kappa, t - t_0), \quad t \geq t_0, \ \mu, \kappa > 0, \ 0 < q < 1,$$

for some $t_0 \in \mathbb{R}_+$.

In our further considerations, we use quasimonotonic, non-decreasing vector functions **[9, 10, 11]** and the following comparison lemma.

Lemma 2.7 ([**16**]). *Assume that* $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^m_+)$ *is* q*-differentiable,*

$$^+\mathcal{D}^q_{t_0}(V(t, x(t))) \leq H(t, V(t, x(t))),$$

where $H \in C(\mathbb{R}_+ \times \mathbb{R}^m_+, \mathbb{R}^m)$, $H(t; u)$ *is quasimonotonic and non-decreasing with respect to* u*, and, for all* $t \geq t_0$*, there exists a maximal solution* $u^+(t) = u^+(t; t_0, u_0)$ *of the fractional-like equation* (2.4) *for values* $0 < q \leq 1$*. Then,* $V(t_0, x_0) \leq u_0$ *implies*

$$V(t, x(t)) \leq u^+(t), \quad t \geq t_0.$$

Corollary 2.8 ([**16**]). *If, in Lemma* 2.7*, the upper bound function*

$$H(t, V(t, x)) \leq -\kappa V(t, x), \quad \kappa = \text{const} > 0,$$

then

$$V(t, x(t)) \leq V(t_0, x_0) \exp\left(- \kappa \frac{(t - t_0)^q}{q} \right)$$

for all $t \geq t_0$ *and any values of* $0 < q \leq 1$.

3. Practical stability analysis. In our main theorems, we will use the Hahn classes of functions

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(u) \text{ is strictly increasing and } a(0) = 0\}$$

and

$$CK = \{a \in C[\mathbb{R}^2_+, \mathbb{R}_+] : a(t, u) \in K \text{ for each } t \in \mathbb{R}_+ \text{ and } a(t, u) \to \infty \text{ as } u \to \infty\}.$$

Theorem 3.1. *Assume that*:

(i) $0 < \lambda < A$ *are given.*

(ii) *conditions of Lemma* 2.7 *are met, and* $H(t,0)=0$ *for* $t \in [t_0,\infty)$.

(iii) *For the q-differentiable function* $V(t,x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^m_+)$, *the following condition holds*:

(3.1)
$$a(||h(t,x)||)e \le V(t,x) \le \gamma(t)b(||h(t,x)||)e, \ (t,x) \in [t_0,\infty) \times \mathbb{R}^n,$$

where $a, b \in K$, *and the function* $\gamma(t) \ge 1$ *is defined and continuous for* $t \in [t_0,\infty)$.

(iv) $\gamma(t_0)b(\lambda) < a(A)$.

Then:

(a) *If system* (2.4) *is practically stable, then system* (2.1) *is practically stable with respect to the function* h.

(b) *If system* (2.4) *is practically asymptotically stable, then system* (2.1) *is practically asymptotically stable with respect to the function* h.

Proof.

(a) Assume, without loss of generality, that $A < r$. From the practical stability of system (2.4) and condition (iv) of Theorem 3.1, it follows that, for $(\lambda^*, A^*) = (\gamma(t_0)b(\lambda), a(A))$ with $0 < \lambda^* < A^*$, we have

$$u_0 < \lambda^* e \quad \text{implies} \quad u^+(t;t_0,u_0) < A^* e, \quad t \ge t_0, \tag{3.2}$$

for some given $t_0 \in \mathbb{R}_+$.

Note that, since $u^+(t;t_0,u_0)$ is the maximal solution of equation (2.4), then it follows from (3.2) that

$$u_0 < \lambda^* e \quad \text{implies} \quad u(t;t_0,u_0) < A^* e$$

for some given $t_0 \in \mathbb{R}_+$, where $u(t;t_0,u_0)$ is any solution of (2.4) defined on $[t_0,\infty)$.

Let $x_0 \in M_{t_0}(n-k)(\lambda)$. Then, we have that

$$\gamma(t_0)b(||h(t_0,x_0))||) < \lambda^*.$$

From (3.1), we obtain

$$V(t_0,x_0) \le \gamma(t_0)b(||h(t_0,x_0)||)e < \lambda^* e.$$

Hence,

$$u^+(t; t_0, V(t_0, x_0)) < A^* e \tag{3.3}$$

for $t \geq t_0$.

Let $x(t) = x(t; t_0, x_0)$ be the solution of the IVP (2.1), (2.2). Since the conditions of Lemma 2.7 have been met, then

$$V(t, x(t; t_0, x_0)) \leq u^+(t; t_0, V(t_0, x_0)), \quad t \in [t_0, \infty). \tag{3.4}$$

From (3.1), (3.3) and (3.4), the inequalities

$$\begin{aligned} a(||h(t, x(t; t_0, x_0))||)e &\leq V(t, x(t; t_0, x_0)) \\ &\leq u^+(t; t_0, V(t_0, x_0)) \\ &< a(A)e, \quad t \geq t_0, \end{aligned}$$

follow. Hence, $||h(t, x(t; t_0, x_0))|| < A$ for $t \geq t_0$, i.e., system (2.1) is practically stable with respect to the function h.

(b) From (a), it follows that system (2.1) is practically stable with respect to the function h. Hence, it is sufficient to prove that every solution $x(t; t_0, x_0)$ with $x_0 \in M_{t_0}(n-k)(\eta)$ satisfies $\lim_{t\to\infty} ||h(t, x(t; t_0, x_0))|| = 0$.

Suppose that this is not true, and consider the solution $x(t, t_0, x_0)$ with initial data $t_0 \in \mathbb{R}_+$ and $x_0 \in R^n$: $||h(t_0, x_0)|| < \eta$. Let, for $t_0 < t \leq t_0 + T$ and $\sigma > 0$, where

$$T = T(t_0, \eta, \sigma), \qquad T \geq \left(\frac{q\gamma(t_0)b(\eta)}{a(\sigma)} \right)^{1/q}$$

for $x(t)$, we have $\gamma(t)b(||h(t, x(t))||) \geq a(\sigma)$. From the Lyapunov relation in Lemma 2.3 for each component of $V \in \mathbb{R}^m_+$, we obtain

$$\begin{aligned} &V_j(t, x(t)) - V_j(t_0, x_0) \\ &\quad = I^q_{t_0}({}^+\mathcal{D}^q_{t_0}(V_j(t, x(t))) \leq I^q_{t_0}(\gamma(t)b(||h(t, x(t))||)) \\ &\quad = \int_{t_0}^{t} (s - t_0)^{q-1}\gamma(s)b(||h(s, x(s))||)\, ds, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{3.5}$$

From (3.5), we get

$$(3.6)\qquad V_j(t,x(t)) \le V_j(t_0,x_0) - \int_{t_0}^{t} (s-t_0)^q \gamma(s) b(||h(s,x(s))||)\, ds$$
$$\le \gamma(t_0)b(\eta) - a(\sigma)\frac{(t-t_0)^q}{q}.$$

For $t = t_0 + T$, by (3.6), we have

$$0 < V_j(t_0+T, x(t_0+T)) \le \gamma(t_0)b(\eta) - a(\sigma)\frac{T^q}{q} \le 0, \quad j = 1, 2, \dots, m,$$

which is a contradiction.

The above contradiction shows that there exists a $t_1 \in [t_0, t_0 + T]$ such that $\gamma(t_1)b(||h(t_1,x(t_1))||) < a(\sigma)$ or $||h(t_1,x(t_1))|| < \sigma$. Hence, $||h(t,x(t))|| < \sigma$ for all $t \ge t_0 + T$, as far as $h(t_0,x_0) < \eta$, and $\lim ||h(t,x(t))|| = 0$ as $t \to \infty$ uniformly on $t_0 \in \mathbb{R}_+$, proving practical asymptotic stability with respect to the function h. □

The next theorem gives sufficient conditions, in terms of vector Lyapunov functions, for uniform practical stability properties of system (2.1) with respect to the function h.

Theorem 3.2. *If in Theorem* 3.1 *condition* (3.1) *is replaced by the condition*

$$a(||h(t,x)||)e \le V(t,x) \le b(||h(t,x)||)e, \quad a, b \in K, \ (t,x) \in [t_0,\infty) \times \mathbb{R}^n,$$

then the uniform practical stability of system (2.4) *implies the uniform practical stability of system* (2.1) *with respect to the function h.*

The proof of Theorem 3.2 is analogous to that of Theorem 3.1. In this case, we can choose λ (as well as λ^*) independent of t_0.

For nonuniform practical stability properties, we can also use functions from the class CK. The proof of the next theorem is similar to that of Theorem 3.1.

Theorem 3.3. *Assume that*:

(i) *Conditions* (i) *and* (ii) *of Theorem* 3.1 *hold.*

(ii) *There exist functions* $a \in K$ *and* $b \in CK$ *such that*

$$a(||h(t,x)||)e \leq V(t,x) \leq b(t,||h(t,x)||)e, \quad (t,x) \in [t_0,\infty) \times \mathbb{R}^n.$$

(iii) $b(t_0,\lambda) < a(A)$.

Then:

(a) *If system* (2.4) *is practically stable, then system* (2.1) *is practically stable with respect to the function* h.

(b) *If system* (2.4) *is practically asymptotically stable, then system* (2.1) *is practically asymptotically stable with respect to the function* h.

Remark 3.4. Theorems 3.1–3.3 extend the practical stability results for the integer-order systems obtained in [**10, 11, 12, 23, 24, 26**] to the fractional order case. In addition, these theorems generalize the stability results for differential equations with fractional-like derivatives [**16, 21**]. Furthermore, since the fractional-like derivatives of Lyapunov functions are upper bounds of fractional derivatives in the sense of Caputo [**16**], our results generalize some existing results of stability and practical stability of fractional differential equations with Caputo's derivatives of the state vector [**6**].

Theorem 3.5. *If, in Theorem* 3.1*, condition* (3.1) *is replaced by the condition*

$$(3.7) \quad ||h(t,x)||e \leq V(t,x) \leq b(t,||h(t,x)||)e, \quad (t,x) \in [t_0,\infty) \times \mathbb{R}^n,$$

where $b \in CK$*, and* $b(t_0,\lambda) < ||h(t_0,x_0)||$*, then the practical exponential stability of system* (2.4) *implies the practical exponential stability of system* (2.1) *with respect to the function* h.

Proof. Let $0 < \lambda < A$ and $A < r$. Since system (2.4) is practically exponentially stable with respect to $(\lambda,\ A)$, then $u_0 < \lambda e$ implies

$$u^+(t;t_0,u_0) < Ae + \mu u_0 E_q(-\kappa, t-t_0), \quad t \geq t_0,$$

for some $t_0 \in \mathbb{R}_+$, where $\mu > 0$, $\kappa > 0$.

Let $x_0 \in M_{t_0}(n-k)(\lambda)$ and $x(t) = x(t;t_0,x_0)$ be the solution of the IVP (2.1), (2.2). From Lemma 2.7, for $u_0 = V(t_0,x_0)$, we obtain (3.4).

From (3.4) and (3.7), we have

$$||h(t, x(t; t_0, \varphi_0))||e \leq V(t, x(t; t_0, \varphi_0)) \leq u^+(t; t_0, V(t_0, x_0)) < [A + \mu b(t_0, \lambda)E_q(-\kappa, t - t_0)]e, \quad t \geq t_0.$$

Hence,

$$x(t; t_0, x_0) \in M_t(n-k)(A + \mu||h(t_0, x_0)||E_q(-\kappa, t - t_0))$$

for $t \geq t_0$, $0 < q \leq 1$, i.e., system (2.1) is practically exponentially stable with respect to the function h. □

Corollary 3.6. *Assume that*:

(i) $0 < \lambda < A$ *are given.*

(ii) *The conditions of Corollary* 2.8 *are met.*

(iii) *For the q-differentiable function* $V(t, x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^m_+)$*, the following condition holds*:

(3.8)
$$(||h(t, x)|| - A)e \leq V(t, x) \leq \Lambda(r)||h(t, x)||e, \quad (t, x) \in [t_0, \infty) \times \mathbb{R}^n,$$

where the function $\Lambda(r) \geq 1$ *is defined and continuous for any* $0 < r \leq \infty$*. Then, system* (2.1) *is practically exponentially stable with respect to the function* h*.*

Proof. Let $t_0 \in \mathbb{R}_+$. For the function $V(t, x)$ and any values of $0 < q \leq 1$, we deduce from Corollary 2.8

$$V(t, x(t)) \leq V(t_0, x_0)E_q(-\kappa, t - t_0), \quad t \geq t_0. \tag{3.9}$$

From (3.8) and (3.9), we have

$$(||h(t, x(t; t_0, x_0)|| - A)e \leq V(t, x(t; t_0, x_0)) \leq V(t_0, x_0)E_q(-\kappa, t - t_0) \leq \Lambda(r)||h(t_0, x_0)||eE_q(-\kappa, t - t_0), \quad t \geq t_0.$$

Therefore,

$$x(t; t_0, x_0) \in M_t(n-k)(A + \mu_1||h(t_0, x_0)||E_q(-\kappa, t - t_0)),$$

where $\mu_1 = \text{const} > \Lambda(r)$ for any $0 < r \leq \infty$. Then, (2.1) is practically exponentially stable with respect to the function h. □

4. Boundedness criteria. In this section, we will state our boundedness results for systems of differential equations of type (2.1) with fractional-like derivatives.

First, we shall give definitions of the boundedness of the solutions of system (2.1) with respect to the function h.

Definition 4.1. We say that the solutions of system (2.1) are:

(a) *equibounded* with respect to the function h, if

$$(\forall t_0 \in \mathbb{R}_+)\ (\forall \alpha > 0)\ (\exists \beta = \beta(t_0, \alpha) > 0)$$
$$(\forall x_0 \in M_{t_0}(n-k)(\overline{\alpha}))$$
$$(\forall t \geq t_0) : x(t; t_0, x_0) \in M_t(n-k)(\beta);$$

(b) *uniformly bounded* with respect to the function h, if the number β in (a) is independent of $t_0 \in \mathbb{R}_+$;

(c) *ultimately bounded* with respect to the function h for bound N, if

$$(\exists N > 0)\ (\forall t_0 \in \mathbb{R}_+)\ (\forall \alpha > 0)$$
$$(\exists T = T(t_0, \alpha) > 0)\ (\forall x_0 \in M_{t_0}(n-k)(\overline{\alpha}))$$
$$(\forall t \geq t_0 + T) : x(t; t_0, x_0) \in M_t(n-k)(N);$$

(d) *uniformly ultimately bounded* with respect to the function h for bound N, if the number T from (c) does not depend upon $t_0 \in \mathbb{R}_+$.

In the next boundedness situation, properties of the positive solutions of (2.4) are defined.

Definition 4.2. We say that the solutions of system (2.4) are:

(a) *equibounded*, if

$$(\forall t_0 \in \mathbb{R}_+)\ (\forall \alpha > 0)$$
$$(\exists \beta = \beta(t_0, \alpha) > 0)\ (\forall u_0 \in \mathbb{R}^m_+ : 0 \leq u_0 \leq \alpha e)$$
$$(\forall t \geq t_0) : u^+(t; t_0, u_0) < \beta e;$$

(b) *uniformly bounded*, if the number β in (a) is independent of $t_0 \in \mathbb{R}_+$;

(c) *ultimately bounded* for bound N, if

$$(\exists N > 0)\ (\forall t_0 \in \mathbb{R}_+)$$
$$(\forall \alpha > 0)\ (\exists T = T(t_0, \alpha) > 0)\ (\forall u_0 \in \mathbb{R}^m_+ : 0 \leq u_0 \leq \alpha e)$$
$$(\forall t \geq t_0 + T) : u^+(t; t_0, u_0) < Ne;$$

(d) *uniformly ultimately bounded* for bound N, if the number T from (c) does not depend upon $t_0 \in \mathbb{R}_+$.

Theorem 4.3. *Assume that conditions of Theorem* 3.1 *hold and* $a(u) \to \infty$ *as* $u \to \infty$. *Then*:

(a) *If the solutions of system* (2.4) *are equibounded, then the solutions of system* (2.1) *are equibounded with respect to the function* h.

(b) *If the solutions of system* (2.4) *are ultimately bounded for a bound* N, *then the zero solutions of system* (2.1) *are ultimately bounded for the bound* $a^{-1}(N)$, *with respect to the function* h.

Proof.

(a) Let $t_0 \in \mathbb{R}_+$ and $\alpha > 0$ be given. Set $\alpha^* = \gamma(t_0)b(\alpha)$. Then, $a(u) \to \infty$ as $u \to \infty$, implying $\alpha \to \infty$ as $\alpha^* \to \infty$.

From the equiboundedness of the solutions of system (2.4), it follows that there exists a $\beta_1 = \beta_1(t_0, \alpha)$ such that $u_0 \in \mathbb{R}^m_+$, and $0 \leq u_0 \leq \alpha^* e$ implies

$$u^+(t; t_0, u_0) < \beta_1 e, \quad t \geq t_0.$$

Set

$$\beta = \beta(t_0, \alpha) = a^{-1}(\beta_1(t_0, \alpha)).$$

Let $x_0 \in M_{t_0}(n-k)(\overline{\alpha})$. This means that $\gamma(t_0)b(||h(t_0, x_0)||) \leq \alpha^*$, and, since

$$V(t_0, x_0) \leq \gamma(t_0)b(||h(t_0, x_0)||)e,$$

then

$$V(t_0, x_0) \leq \alpha^* e.$$

Hence,

$$u^+(t; t_0, V(t_0, x_0)) < \beta_1 e, \quad t \geq t_0. \tag{4.1}$$

Let $x(t) = x(t; t_0, x_0)$ be the solution of the IVP (2.1), (2.2). From (3.1), (3.4) and (4.1), the inequalities

$$
\begin{aligned}
a(||h(t, x(t; t_0, x_0))||)e &\le V(t, x(t; t_0, x_0)) \\
&\le u^+(t; t_0, V(t_0, x_0)) < \beta_1 e, \quad t \ge t_0,
\end{aligned}
$$

follow. Hence, $||h(t, x(t; t_0, x_0))|| < a^{-1}(\beta_1) = \beta$ for $t \ge t_0$, i.e., the solutions of (2.1) are equibounded with respect to the function h.

(b) Let $t_0 \in \mathbb{R}_+$, $N > 0$ and $A > 0$ be given. Again, set $\alpha^* = \gamma(t_0)b(\alpha)$. From the ultimate boundedness of the solutions of system (2.4) for a bound N, it follows that there exists a $T = T(t_0, \alpha) > 0$, such that $u_0 \in \mathbb{R}^m_+$, and $0 \le u_0 \le \alpha^* e$ implies

$$u^+(t; t_0, u_0) < Ne, \quad t \ge t_0 + T.$$

Let $x_0 \in M_{t_0}(n-k)(\overline{\alpha})$. This means that

$$\gamma(t_0)b(||h(t_0, x_0)||) \le \alpha^*,$$

and, since

$$V(t_0, x_0) \le \gamma(t_0)b(||h(t_0, x_0)||)e,$$

then

$$V(t_0, x_0) \le \alpha^* e.$$

Hence,

$$u^+(t; t_0, V(t_0, x_0)) < Ne, \quad t \ge t_0 + T. \tag{4.2}$$

Let $x(t) = x(t; t_0, x_0)$ be the solution of the IVP (2.1), (2.2). From (3.1), (3.4) and (4.2), the inequalities

$$
\begin{aligned}
a(||h(t, x(t; t_0, x_0))||)e &\le V(t, x(t; t_0, x_0)) \\
&\le u^+(t; t_0, V(t_0, x_0)) < Ne, \quad t \ge t_0,
\end{aligned}
$$

follow. Hence, $||h(t, x(t; t_0, x_0))|| < a^{-1}(N)$ for $t \ge t_0 + T$, i.e., the solutions of (2.1) are ultimately bounded with respect to the function h for the bound $a^{-1}(N)$. □

Theorem 4.4. *Assume that the conditions of Theorem* 3.2 *hold and* $a(u) \to \infty$ *as* $u \to \infty$. *Then*:

(a) *If the solutions of system* (2.4) *are uniformly bounded, then the solutions of system* (2.1) *are uniformly bounded with respect to the function h.*

(b) *If the solutions of system* (2.4) *are uniformly ultimately bounded for a bound N, then the zero solutions of system* (2.1) *are uniformly ultimately bounded for the bound $a^{-1}(N)$, with respect to the function h.*

Proof. The proof of Theorem 4.4 is similar to that of Theorem 4.3. In this case, β and T can be chosen independently of t_0. □

Remark 4.5. Despite the great possibilities for application, the boundedness theory for fractional-order systems with different fractional derivatives is not yet well developed, as in the integer-order case. With this research, we extend and improve some existing boundedness results **[2]** to the case of boundedness with respect to a manifold.

5. Examples and discussions.

Example 5.1. Consider the following IVP for the fractional-like system

$$\begin{cases} \mathcal{D}^q_{t_0}\, x(t) = n(t)y(t) + x(t)g(t) & x(t_0) = x_0, \\ \mathcal{D}^q_{t_0}\, y(t) = n(t)x(t) + y(t)g(t) & y(t_0) = y_0, \end{cases} \tag{5.1}$$

where $0 < q \le 1$, $n(t)$ and $g(t)$ are continuous functions for all $t \ge t_0$, and $g(t) \ge 0$ for $t \ge t_0$. Applying the function $V = (V_1, V_2)^T$, where $V_1(t, x, y) = (x + y)^2$, $V_2(t, x, y) = (x - y)^2$ to system (5.1), we have

$$^{+}\mathcal{D}^q_{t_0}\, V(t, x(t), y(t)) \le H(t, V(t, x(t), y(t))) \tag{5.2}$$

with $H = (H_1, H_2)$, where

$$H_1(t, u_1, u_2) = 2[g(t) + |n(t)|]u_1,$$

$$H_2(t, u_1, u_2) = 2[g(t) + |n(t)|]u_2.$$

In addition, we have that

$$\max_{i=1,2} V_i(t, x, y) = \begin{cases} (x + y)^2 & xy > 0, \\ (x - y)^2 & xy < 0, \\ x^2 & y = 0, \\ y^2 & x = 0, \end{cases}$$

and $V(t, x, y) \to 0$ as $x^2 + y^2 \to 0$ uniformly for $t \ge t_0$.

Performing a q-integration of (5.2), we obtain the Lyapunov relation

$$V(t,x(t),y(t)) - V(t_0,x_0,y_0) \leq 4r^2 \int\limits_{t_0}^{t} \frac{[g(s)+|n(s)|]}{(s-t_0)^{1-q}}\, ds \tag{5.3}$$

on the domain $x^2+y^2 \leq r^2$, determined by the function $h(t,x,y) = \sqrt{x^2+y^2}$. Let $M = \max_{t\geq 0}(g(t)+|n(t)|)$. From (5.2) and (5.3), using the comparison system

$$\begin{cases} \mathcal{D}^q_{t_0} u_1(t) = 2[g(t)+|n(t)|]u_1(t) & t \geq t_0, \\ \mathcal{D}^q_{t_0} u_2(t) = 2[g(t)+|n(t)|]u_2(t & t \geq t_0, \end{cases} \tag{5.4}$$

we have:

(a) by Theorem 2.2 for $4r^2M(t-t_0)^q < q(A-\lambda)$ the uniform practical stability of system (5.4) implies the uniform practical stability of (5.1) with respect to the function $h(t,x,y) = \sqrt{x^2+y^2}$;

(b) By Theorem 4.4 (a) for $4r^2M(t-t_0)^q < q(\beta-\alpha)$, the uniform boundedness of the solutions of system (5.4) implies the uniform boundedness of the solutions of (5.1) with respect to the function $h(t,x,y) = \sqrt{x^2+y^2}$.

Remark 5.2. It is well known that, in the Lyapunov stability theory of integer-order systems, it is required that the total derivative (as well as the upper right derivative) of a Lyapunov function be negative or nonpositive [**10, 11, 12**]. The same criteria was observed for the stability of fractional-order systems with fractional-like derivatives in [**16**]. However, in the practical stability analysis of fractional order systems, as we can see from Example 5.1, condition (5.2) allows the upper right fractional-like derivative of the Lyapunov function to be positive. This is an important difference between the stability and practical stability properties of fractional order systems with fractional-like derivatives.

Example 5.3. Let $0 < q \leq 1$. Consider the IVP for the system with fractional-like derivatives of the form

$$(5.5)\qquad \begin{cases} \mathcal{D}^q_{t_0}\ x(t) = [20ax(t) - 10y(t) - 1]x(t) & \\ \qquad\qquad +2x(t)[y(t) - 2ax(t)] & x(t_0) = x_0, \\ \mathcal{D}^q_{t_0}\ y(t) = y^2(t) - 4ax(t) - 5y^2(t) & y(t_0) = y_0, \end{cases}$$

where $x, y \in \mathbb{R}$, $t \geq 0$, $a \in \mathbb{R}$.

Let $h(t, x, y) = y - 4ax$ and $V(t, x, y) = |h|$. In the case $h = 0$, $x, y \in \mathbb{R}$, we have $\mathcal{D}^q_{t_0}\ h(t, x, y) = 0$, or, if $h > 0$, $x, y \in \mathbb{R}$, it follows that

$$\mathcal{D}^q_{t_0}\ |h(t, x, y)| = \mathcal{D}^q_{t_0} h(t, x, y).$$

On the other hand, if $h < 0$, $x, y \in \mathbb{R}$, then, using the properties of limits, we have

$$\mathcal{D}^q_{t_0}\ |h(t, x, y)| = -\mathcal{D}^q_{t_0} h(t, x, y).$$

Consequently, for $x, y \in \mathbb{R}$, we have

$$\mathcal{D}^q_{t_0}\ |h(t, x, y)| = \operatorname{sgn}(h(t, x, y))\mathcal{D}^q_{t_0} h(t, x, y).$$

Then, for $t \geq t_0$, for the upper right fractional-like derivative of V of order q with respect to (5.5), we have

$${}^+\mathcal{D}^q_{t_0}\ V(t, x(t), y(t)) \leq -4V^2(t, x(t), y(t)).$$

Consider the fractional-like comparison equation

$$(5.6)\qquad \mathcal{D}^q_{t_0} u(t) = -4u^2,\ u \in \mathbb{R}_+, \quad t \geq t_0.$$

After a q-integration of (5.6), we derive the estimate

$$u(t) - u_0 \leq -4 \int_{t_0}^{t} \frac{u^2(s)}{(s - t_0)^{1-q}}\, ds.$$

From the last estimate, it follows that equation (5.6) is practically asymptotically stable with respect to $\lambda = A$. Then, by Theorem 3.1 (b), system (5.5) is practically asymptotically stable with respect to the function h.

Moreover, the solutions of (5.6) are uniformly bounded and uniformly ultimately bounded. Then, from Theorem 4.4, it follows that the solutions of system (5.5) are uniformly bounded and uniformly ultimately bounded with respect to the function h.

Remark 5.4. Example 5.3 shows another important difference between stability and practical stability with respect to a manifold. Stability theory is always related to an equilibrium point, while the practical stability can be considered with respect to a function that does not necessarily define an equilibrium state of the system.

Example 5.5. Let $0 < q \leq 1$ and

$$x_1, x_2 : [t_0, \infty) \longrightarrow \mathbb{R},$$

x_1, x_2 be q-differentiable. Consider the system with fractional-like derivatives in the form

$$\begin{cases} \mathcal{D}^q_{t_0} x_1(t) = -\mu(t)x_2 - \nu(t)x_1, \\ \mathcal{D}^q_{t_0} x_2(t) = \mu(t)x_1 - \nu(t)x_2, \end{cases} \tag{5.7}$$

where $\mu(t)$ and $\nu(t)$ are continuous, single-valued functions defined on $t \geq t_0$.

Let $h(x_1, x_2) = x_1^2 + x_2^2$ and the function $V(x_1, x_2) = h/2$. For the upper right fractional-like derivative of V of order q with respect to (5.7), we get

$$^{+}\mathcal{D}^q_{t_0} V(x_1, x_2) = -2\nu(t)V(x_1, x_2).$$

If there exists a constant $\kappa > 0$ such that $\nu(t) > \kappa$ for any $t \geq t_0$, then, by Corollary 3.6, system (5.7) is practically exponentially stable with respect to the function h.

6. Conclusions. In this paper, we applied the recently defined "fractional-like derivative" of Lyapunov-type functions to study the practical stability and boundedness behavior of solutions of fractional differential equations with fractional-like derivatives. The importance of applications using the notion of practical stability is extended to practical stability with respect to a manifold. Vector Lyapunov-type functions and the comparison principle were used to prove criteria for practical stability, uniform practical stability, asymptotical practical stability and boundedness of solutions with respect to a manifold defined by a particular function. Since the fractional-like derivative of a Lyapunov-type function is an upper bound of the Caputo fractional derivative of the same Lyapunov function, our results extend and improve some existing stability and boundedness results for fractional differential equations with Caputo fractional derivatives. Fractional-like derivatives are natural extensions of the integer-order derivatives that allow us to apply the key element of the direct Lyapunov method, i.e., the opportunity to calculate in a simple manner the total derivative of a composition of functions (chain rule) corresponding to an auxiliary function under consideration and the perturbed motion fractional differential equations. As such, it provides great possibilities for developments in the qualitative theory of differential equations with fractional-like derivatives.

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