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GROUND STATES FOR CHOQUARD EQUATIONS WITH DOUBLY CRITICAL EXPONENTS

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ABSTRACT. In this paper, an autonomous Choquard equation with doubly critical exponents is studied. By using the Pohožaev constraint and the perturbed method, a positive and radially symmetric ground state solution in $H^1(\mathbb{R}^N)$ is obtained. The result here extends and complements the earlier theorems obtained by Seok **[19]** and Moroz and Schaftingen **[14]**.

1. Introduction and main results. We are interested in the autonomous Choquard equation

$$-\Delta u + u = (I_\alpha * G(u))g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $g \in C(\mathbb{R}, \mathbb{R})$, $G(s) = \int_0^s g(t)\,dt$, and I_α is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma((N-\alpha)/2)}{\Gamma(\alpha/2)\pi^{N/2}2^\alpha|x|^{N-\alpha}} \tag{1.2}$$

with Γ denoting the Gamma function **[18,** page 19].

For $G(u) = |u|^p/p^{1/2}$, (1.1) is reduced to the special equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

When $N = 3$, $p = 2$ and $\alpha = 2$, (1.3) was investigated by Pekar **[16]** in the study of the quantum theory of a polaron at rest. In **[9]**, Choquard applied it as an approximation to the Hartree-Fock theory of one component plasma. It also arises in multiple particle systems **[7]** and quantum mechanics **[17]**. There are many papers devoted to the existence and multiplicity of solutions of (1.3) and their qualitative

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properties. See the survey paper [**15**] and the references therein. For $p \in ((N+\alpha)/N, (N+\alpha)/(N-2))$, Moroz and Schaftingen [**13**] established the existence, qualitative properties and decay estimates of ground states of (1.3). They also obtained some nonexistence results under the range

$$p \geq \frac{N+\alpha}{N-2} \quad \text{or} \quad p \leq \frac{N+\alpha}{N}.$$

Usually, $(N+\alpha)/N$ is called the lower critical exponent and $(N+\alpha)/(N-2)$ is the upper critical exponent for the Choquard equation.

For equation (1.1) with general nonlinearity, Moroz and Schaftingen [**14**] considered the subcritical case. In the spirit of Berestycki and Lions [**2**], they obtained the existence of ground states by using the Pohožaev-Palais-Smale sequence method under sufficient and almost necessary conditions on the nonlinearity g:

(g1) there exists a $C > 0$ such that, for every $s \in \mathbb{R}$,

$$|sg(s)| \leq C(|s|^{(N+\alpha)/N} + |s|^{(N+\alpha)/(N-2)}).$$

(g2) $\lim_{s\to 0} G(s)/|s|^{(N+\alpha)/N} = 0$ and $\lim_{|s|\to\infty} G(s)/|s|^{(N+\alpha)/(N-2)} = 0$.

(g3) There exists an $s_0 \in \mathbb{R} \setminus \{0\}$ such that $G(s_0) \neq 0$.

(g4) g is odd and has constant sign on $(0, \infty)$.

More precisely, they obtained the following results.

Theorem 1.1. *Assume that* (g1)–(g3) *hold. Then,* (1.1) *has a ground state in* $H^1(\mathbb{R}^N)$*. Furthermore, assume that* (g4) *holds. Then, every ground state of* (1.1) *has constant sign and is radially symmetric with respect to some point in* $\mathbb{R}^N$*.*

Theorem 1.2. *Assume that* (g1) *holds. Then, every solution* $u \in H^1(\mathbb{R}^N)$ *to* (1.1) *satisfies the Pohožaev identity*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 = \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u))G(u).$$

Recently, many authors considered similar equations to (1.1) for the critical case, see Alves et al. [**1**], Cassani and Zhang [**4**] for the upper critical case, Schaftingen and Xia [**21**] for the lower critical case, Gao and Yang [**5**] for the strongly indefinite critical problem, and Gao and Yang [**6**] for the Brezis-Nirenberg type critical problem. More recently, Seok [**19**] considered (1.1) with doubly critical exponents. When

$$G(u) = \frac{N}{N+\alpha}|u|^{(N+\alpha)/N} + \frac{N-2}{N+\alpha}|u|^{(N+\alpha)/(N-2)},$$

they obtained the following result.

Theorem 1.3. *Let $N \geq 5$ and $\alpha \in (0, N-4)$. Then,* (1.1) *admits a nontrivial solution $u \in H^1(\mathbb{R}^N)$ which is radially symmetric.*

In [**19**], the workspace is the radially symmetric subspace $H^1_r(\mathbb{R}^N)$ of the usual Sobolev space $H^1(\mathbb{R}^N)$. By using the mountain pass lemma, the author first obtained a $(PS)_c$ sequence $\{u_n\} \subset H^1_r(\mathbb{R}^N)$ for some suitable constant c, and then, using radial symmetry, he proved that the $(PS)_c$ sequence is relatively compact in $H^1(\mathbb{R}^N)$ and convergent to a nontrivial solution $u \in H^1_r(\mathbb{R}^N)$. The solution obtained in [**19**] may not be a ground state. A natural question arises: Can we obtain a ground state? The answer is yes, if we can obtain a $(PS)_c$ sequence $\{u_n\} \subset H^1_r(\mathbb{R}^N)$ with c not being larger than the ground state energy. However, it seems that this problem is not an easy issue. Fortunately, in this paper, we obtain a critical point sequence $\{u_n\} \subset H^1_r(\mathbb{R}^N)$ for a sequence of perturbed functional with some extra properties for its energy level. Based on that, we can obtain a ground state. A similar technique was used in [**11**], in which the authors obtained a positive radially symmetrical ground state for a class of Schrödinger equations.

More precisely, in this paper, we consider the equation in $\mathbb{R}^N$

$$-\Delta u + \lambda u = (I_\alpha * (\mu|u|^{p_*} + \omega|u|^{p^*}))(\mu p_*|u|^{p_*-2}u + \omega p^*|u|^{p^*-2}u), \tag{1.4}$$

where $N \geq 3$, $\alpha \in (0, N)$, λ, μ, $\omega > 0$ are constants, $p_* = (N+\alpha)/N$ and $p^* = (N+\alpha)/(N-2)$. The main result of this paper is as follows.

Theorem 1.4. *Let $N \geq 5$ and $\alpha \in (0, N-4)$. Then, for every λ, μ, $\omega > 0$,* (1.4) *admits a positive ground state solution $u \in H^1(\mathbb{R}^N)$ which is radially symmetric.*

At the end of this section, we outline the methods used in this paper. To prove Theorem 1.4, inspired by **[11, 19]** (see also **[8, 12]**), we consider the equation

$$-\Delta u + \lambda u = (I_\alpha * (\mu|u|^{p_*+a} + \omega|u|^{p^*-a})) \times (\mu(p_* + a)|u|^{p_*+a-2}u + \omega(p^* - a)|u|^{p^*-a-2}u) \quad \text{in } \mathbb{R}^N \tag{1.5}$$

with $a \in [0, a_0]$ and $a_0 = (p^* - p_*)/4$. For $a = 0$, equation (1.5) is reduced to (1.4), and, for $a > 0$, equation (1.5) is subcritical, which was studied in **[14]**.

From the Hardy-Littlewood-Sobolev inequality and the Sobolev embedding theorem, the functional $I_a : H^1(\mathbb{R}^N) \to \mathbb{R}$ of (1.5) is defined as

$$I_a(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda|u|^2 - \frac{1}{2}\int_{\mathbb{R}^N} \{(I_\alpha * (\mu|u|^{p_*+a} + \omega|u|^{p^*-a})) \times (\mu|u|^{p_*+a} + \omega|u|^{p^*-a})\} \tag{1.6}$$

and

$$\langle I_a'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + \lambda uv - \int_{\mathbb{R}^N} \{(I_\alpha * (\mu|u|^{p_*+a} + \omega|u|^{p^*-a})) \times (\mu(p_* + a)|u|^{p_*+a-2}u + \omega(p^* - a)|u|^{p^*-a-2}u)v\} \tag{1.7}$$

for any $u,\ v \in H^1(\mathbb{R}^N)$, that is, any critical point of I_a in $H^1(\mathbb{R}^N)$ is a weak solution of (1.5). A nontrivial solution $u \in H^1(\mathbb{R}^N)$ of (1.5) is called a *ground state* if

$$I_a(u) = c_a^g := \inf\{I_a(v) : v \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ and } I_a'(v) = 0\}. \tag{1.8}$$

To prove Theorem 1.4, we define

$$c_a = \inf\{I_a(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ and } P_a(u) = 0\}, \tag{1.9}$$

where

$$P_a(u) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} \lambda|u|^2 - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} \{(I_\alpha * (\mu|u|^{p_*+a} + \omega|u|^{p^*-a})) \times (\mu|u|^{p_*+a} + \omega|u|^{p^*-a})\}.$$

By Lemma 2.6, c_a is well defined and $c_a < +\infty$. By Remark 2.7, $c_a \le c_a^g$ for $a \in [0, a_0]$ and $c_a = c_a^g$ for $a \in (0, a_0]$. Let $a_n \in (0, a_0]$ be a sequence satisfying $\lim_{n\to\infty} a_n = 0$. Theorem 1.1, Theorem 1.2 and Remark 2.7 imply that there exists a positive sequence $\{u_n\} \subset H_r^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$(1.10) \qquad I'_{a_n}(u_n) = 0, \qquad I_{a_n}(u_n) = c_{a_n} \quad \text{and} \quad P_{a_n}(u_n) = 0.$$

It can be shown that $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ is an almost critical point sequence of I_0 with $0 < \inf_n I_{a_n}(u_n) \le \sup_n I_{a_n}(u_n) < c_0$. By using these properties, $\{u_n\}$ is shown to converge to a nontrivial ground state of (1.4), see Section 3.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.4.

1.1. Basic notation. Throughout this paper, we assume that $N \ge 3$. $C_c^\infty(\mathbb{R}^N)$ denotes the space of infinitely differentiable functions with compact support in $\mathbb{R}^N$. $L^r(\mathbb{R}^N)$ with $1 \le r < \infty$ denotes the Lebesgue space with the norms

$$\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r\right)^{1/r}.$$

$H^1(\mathbb{R}^N)$ is the usual Sobolev space with norm

$$\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\right)^{1/2}.$$

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-2)}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}.$$

$$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\}.$$

2. Preliminaries. In this section, we give some preliminary lemmas. The following, well known Hardy-Littlewood-Sobolev inequality can be found in [**10**].

Lemma 2.1. *Let* $p,\ r > 1$ *and* $0 < \alpha < N$ *with* $1/p + (N-\alpha)/N + 1/r = 2$. *Let* $u \in L^p(\mathbb{R}^N)$ *and* $v \in L^r(\mathbb{R}^N)$. *Then, there exists a sharp constant* $C(N, \alpha, p)$, *independent of* u *and* v, *such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x-y|^{N-\alpha}} \right| \leq C(N, \alpha, p)\|u\|_p \|v\|_r.$$

If $p = r = 2N/(N+\alpha)$, *then*

$$C(N, \alpha, p) = C_\alpha(N) = \pi^{(N-\alpha)/2} \frac{\Gamma(\alpha/2)}{\Gamma((N+\alpha)/2)} \left\{ \frac{\Gamma(N/2)}{\Gamma(N)} \right\}^{-\alpha/N}.$$

Remark 2.2. By the Hardy-Littlewood-Sobolev inequality above, for any $v \in L^s(\mathbb{R}^N)$ with $s \in (1, (N/\alpha))$, $I_\alpha * v \in L^{Ns/(N-\alpha s)}(\mathbb{R}^N)$ and

$$\|I_\alpha * v\|_{Ns/(N-\alpha s)} \leq A_\alpha(N) C(N, \alpha, s)\|v\|_s.$$

The following Strauss inequality is used to construct a dominated function for radically symmetric function, see [**22,** Lemma 4.5] for its proof.

Lemma 2.3. *If* $N \geq 2$, *then there exists a* $C_N > 0$ *independent of* u *such that, for every* $u \in H_r^1(\mathbb{R}^N)$,

$$|u(x)| \leq C_N \|u\|_2^{1/2} \|\nabla u\|_2^{1/2} |x|^{(1-N)/2} \text{ almost everywhere on } \mathbb{R}^N.$$

The following lemma can be found in [**3, 23**].

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^N$ *be a domain, and* $q \in (1, \infty)$ *and* $\{u_n\}$ *a bounded sequence in* $L^q(\Omega)$. *If* $u_n \to u$ *almost everywhere on* Ω, *then* $u_n \rightharpoonup u$ *weakly in* $L^q(\Omega)$.

The following lemma will be frequently used in this paper. For convenience, we give its short proof.

Lemma 2.5. *Let $N \geq 3$, $q \in [2, 2N/(N-2)]$ and $u \in H^1(\mathbb{R}^N)$. Then, there exists a positive constant C independent of q and u such that*

$$\|u\|_q \leq C\|u\|_{H^1(\mathbb{R}^N)}.$$

Proof. By the Hölder inequality and the Sobelev imbedding theorem,

$$\begin{aligned}\|u\|_q \leq \|u\|_2^\theta \|u\|_{2N/(N-2)}^{1-\theta} &\leq (C_1\|u\|_{H^1(\mathbb{R}^N)})^\theta (C_2\|u\|_{H^1(\mathbb{R}^N)})^{1-\theta} \\ &\leq \max\{C_1, C_2\}\|u\|_{H^1(\mathbb{R}^N)},\end{aligned}$$

where $1/q = \theta/2 + (1-\theta)/[2N/(N-2)]$. The proof is complete. □

Define u_τ by

$$u_\tau(x) = \begin{cases} u(x/\tau) & \tau > 0, \\ 0 & \tau = 0. \end{cases} \tag{2.1}$$

The following lemma shows that c_a is well defined, where c_a is defined in (1.9).

Lemma 2.6. *Let $N \geq 3$, $\alpha \in (0, N)$ and $a \in [0, a_0]$. For any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $\tau_0 > 0$ such that $P_a(u_{\tau_0}) = 0$. Moreover, $I_a(u_{\tau_0}) = \max_{\tau \geq 0} I_a(u_\tau)$.*

Proof. Set $\varphi(\tau) = I_a(u_\tau)$. Direct calculation gives that

$$\varphi(\tau) = \frac{\tau^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\tau^N}{2}\lambda\int_{\mathbb{R}^N} |u|^2 - \frac{\tau^{N+\alpha}}{2}\int_{\mathbb{R}^N} (I_\alpha * G(u,a))G(u,a), \tag{2.2}$$

where $G(u,a) = \mu|u|^{p_*+a} + \omega|u|^{p^*-a}$. Thus, $\varphi(\tau)$ has a unique critical point τ_0 which corresponds to its maximum, that is, $I_a(u_{\tau_0}) = \max_{\tau \geq 0} I_a(u_\tau)$ and

$$\begin{aligned}0 = \varphi'(\tau_0) = \frac{N-2}{2}\tau_0^{N-3}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2}\tau_0^{N-1}\lambda\int_{\mathbb{R}^N} |u|^2 \\ - \frac{N+\alpha}{2}\tau_0^{N+\alpha-1}\int_{\mathbb{R}^N} (I_\alpha * G(u,a))G(u,a).\end{aligned}$$

Hence, $P_a(u_{\tau_0}) = 0$. The proof is complete. □

The following is a series of lemmas and remarks concerning the properties of c_a.

Remark 2.7. Theorem 1.2 implies that $c_a \leq c_a^g$ for $a \in [0, a_0]$. By using the results of **[14]**, we can further obtain that $c_a = c_a^g$ for $a \in (0, a_0]$. Indeed, **[14]** yields that

$$c_a^g = c_a^{mp} := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_a(\gamma(t)),$$

where the set of paths is defined as

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0,\ I_a(\gamma(1)) < 0\}.$$

For any $u \in H^1(\mathbb{R}^N) \backslash \{0\}$, with $P_a(u) = 0$, let u_τ be defined as in (2.1). By (2.2), there exists a $\tau_0 > 0$ large enough such that $I_a(u_{\tau_0}) < 0$. Lemma 2.6 implies that

$$c_a^{mp} \leq \max_{\tau \geq 0} I_a(u_\tau) = I_a(u).$$

Since u is arbitrary, $c_a^g = c_a^{mp} \leq c_a$. Hence, $c_a = c_a^g$ for $a \in (0, a_0]$.

Lemma 2.8. *Let $N \geq 3$, $\alpha \in (0, N)$ and $a \in [0, a_0]$. Then, $c_a \geq 0$.*

Proof. Let $\{v_n\} \subset H^1(\mathbb{R}^N) \setminus \{0\}$ be a sequence satisfying

$$\lim_{n \to \infty} I_a(v_n) = c_a \quad \text{and} \quad P_a(v_n) = 0.$$

Then, we have

$$\begin{aligned} I_a(v_n) &= I_a(v_n) - \frac{1}{N+\alpha} P_a(v_n) \\ &= \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)}\right) \int_{\mathbb{R}^N} |\nabla v_n|^2 \\ &\quad + \left(\frac{1}{2} - \frac{N}{2(N+\alpha)}\right) \lambda \int_{\mathbb{R}^N} |v_n|^2 \\ &\geq 0, \end{aligned}$$

which implies that $c_a \geq 0$. □

Lemma 2.9. *Let $N \geq 3$, $\alpha \in (0, N)$ and $a \in (0, a_0]$. Then,* $\limsup_{a \to 0} c_a \leq c_0$.

Proof. For any $\epsilon \in (0,1)$, there exists a $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ with $P_0(u) = 0$ such that $I_0(u) < c_0 + \epsilon$. By (2.2), there exists a $\tau_0 > 0$ large enough such that $I_0(u_{\tau_0}) \le -2$. By the Young inequality, we have

$$(2.3)\qquad \begin{aligned} |u|^{p_*+a} &\le \frac{p^*-p_*-a}{p^*-p_*}|u|^{p_*} + \frac{a}{p^*-p_*}|u|^{p^*}, \\ |u|^{p^*-a} &\le \frac{a}{p^*-p_*}|u|^{p_*} + \frac{p^*-p_*-a}{p^*-p_*}|u|^{p^*}, \end{aligned}$$

and, by the Hardy-Littlewood-Sobolev inequality and the Sobolev embedding theorem, there exist C_1, $C_2 > 0$, independent of u, such that

$$(2.4)\qquad \begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * |u|^{p_*})|u|^{p_*} \le C_1\|u\|_2^{2p_*} \le C_2\|u\|_{H^1(\mathbb{R}^N)}^{2p_*}, \\ &\int_{\mathbb{R}^N} (I_\alpha * |u|^{p^*})|u|^{p^*} \le C_1\|u\|_{2N/(N-2)}^{2p^*} \le C_2\|u\|_{H^1(\mathbb{R}^N)}^{2p^*}, \\ &\int_{\mathbb{R}^N} (I_\alpha * |u|^{p_*})|u|^{p^*} \le C_1\|u\|_2^{p_*}\|u\|_{2N/(N-2)}^{p^*} \le C_2\|u\|_{H^1(\mathbb{R}^N)}^{p_*+p^*}. \end{aligned}$$

Hence, the Lebesgue dominated convergence theorem implies that

$$\frac{\tau^{N+\alpha}}{2}\int_{\mathbb{R}^N} (I_\alpha * (\mu|u|^{p_*+a} + \omega|u|^{p^*-a}))(\mu|u|^{p_*+a} + \omega|u|^{p^*-a})$$

is continuous on $a \in [0, a_0]$ uniformly with $\tau \in [0, \tau_0]$. Thus, there exists a $\delta > 0$ such that

$$|I_a(u_\tau) - I_0(u_\tau)| < \epsilon$$

for $0 < a < \delta$ and $0 \le \tau \le \tau_0$, which implies that $I_a(u_{\tau_0}) \le -1$ for all $0 < a < \delta$. Since $I_a(u_\tau) > 0$ for τ small enough and $I_a(u_0) = 0$ for any $a \in [0, a_0]$, there exists a $\tau_a \in (0, \tau_0)$ such that $(d/d\tau)I_a(u_\tau)|_{\tau=\tau_a} = 0$, and then, $P_a(u_{\tau_a}) = 0$. By Lemma 2.6, $I_0(u_{\tau_a}) \le I_0(u)$. Hence,

$$c_a \le I_a(u_{\tau_a}) \le I_0(u_{\tau_a}) + \epsilon \le I_0(u) + \epsilon < c_0 + 2\epsilon$$

for any $0 < a < \delta$. Thus, $\limsup_{a\to 0} c_a \le c_0$. □

Lemma 2.10. *Let $N \ge 3$, $\alpha \in (0, N)$, $a_n \to 0^+$ and $\{u_n\} \subset H_r^1(\mathbb{R}^N) \setminus \{0\}$ satisfy* (1.10). *Then, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\liminf_{n\to\infty} c_{a_n} > 0$.*

Proof. By Lemma 2.9, for n large enough, we have

$$
\begin{aligned}
c_0+1 \geq c_{a_n} &= I_{a_n}(u_n) - \frac{1}{N+\alpha}P_{a_n}(u_n) \\
&= \left(\frac{1}{2}-\frac{N-2}{2(N+\alpha)}\right)\int_{\mathbb{R}^N}|\nabla u_n|^2 \\
&\quad + \left(\frac{1}{2}-\frac{N}{2(N+\alpha)}\right)\lambda\int_{\mathbb{R}^N}|u_n|^2,
\end{aligned} \tag{2.5}
$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

In view of (2.3) and (2.4), and by the Cauchy inequality, there exist $C_3,\ C_4>0$, independent of n, such that

$$
\begin{aligned}
0 &= P_{a_n}(u_n) \\
&= \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 + \frac{N}{2}\lambda\int_{\mathbb{R}^N}|u_n|^2 \\
&\quad - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\{(I_\alpha * (\mu|u_n|^{p_*+a_n}+\omega|u_n|^{p^*-a_n})) \\
&\qquad\qquad \times (\mu|u_n|^{p_*+a_n}+\omega|u_n|^{p^*-a_n})\} \\
&\geq C_3\|u_n\|^2_{H^1(\mathbb{R}^N)} - C_4(\|u_n\|^{2p_*}_{H^1(\mathbb{R}^N)} + \|u_n\|^{2p^*}_{H^1(\mathbb{R}^N)}),
\end{aligned}
$$

which implies that there exists a $C_5>0$, independent of n, such that

$$
\|u_n\|_{H^1(\mathbb{R}^N)} \geq C_5. \tag{2.6}
$$

Combining (2.5) and (2.6), we obtain that $\liminf_{n\to\infty} c_{a_n} > 0$. □

By Lemmas 2.9 and 2.10, we have $c_0>0$. In the following, we give an upper estimate of c_0. Towards this end, we define

$$
S_1 = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|u|^2}{(\int_{\mathbb{R}^N}(I_\alpha * |u|^{p_*})|u|^{p_*})^{1/p_*}} \tag{2.7}
$$

and

$$
S_2 = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2}{(\int_{\mathbb{R}^N}(I_\alpha * |u|^{p^*})|u|^{p^*})^{1/p^*}}. \tag{2.8}
$$

It is known that

$$
U(x) = \frac{A}{(1+|x|^2)^{N/2}} \quad \text{and} \quad V(x) = \frac{B}{(1+|x|^2)^{(N-2)/2}}
$$

are the extremal functions of S_1 and S_2, respectively, see [19]. In the following, we choose A and B such that

$$\int_{\mathbb{R}^N}(I_\alpha * |U|^{p_*})|U|^{p_*} = 1 \quad \text{and} \quad \int_{\mathbb{R}^N}(I_\alpha * |V|^{p^*})|V|^{p^*} = 1.$$

By direct calculation, we have the following result.

Lemma 2.11. *Assume that* $N \geq 5$ *and* $\alpha \in (0, N-4)$. *Then,*

$$c_0 < \min\left\{\frac{2+\alpha}{2(N+\alpha)}\left(\frac{N-2}{(N+\alpha)\omega^2}\right)^{(N-2)/(2+\alpha)} S_2^{(N+\alpha)/(2+\alpha)}, \right.$$
$$\left. \frac{\alpha}{2(N+\alpha)}\left(\frac{N}{(N+\alpha)\mu^2}\right)^{N/\alpha}(\lambda S_1)^{(N+\alpha)/\alpha}\right\}.$$

Proof. For δ, $\epsilon > 0$, define $u_\delta(x) = \delta^{N/2}U(\delta x)$ and $v_\epsilon(x) = \epsilon^{(2-N)/2}V(x/\epsilon)$. For $N \geq 5$, $v_\epsilon(x) \in H^1(\mathbb{R}^N)$. In the following, we use u_δ and v_ϵ to estimate c_0. By Lemma 2.6, there exists a unique τ_δ such that $P_0((u_\delta)_{\tau_\delta}) = 0$ and $I_0((u_\delta)_{\tau_\delta}) = \sup_{\tau\geq 0} I_0((u_\delta)_\tau)$. Thus, $c_0 \leq \sup_{\tau \geq 0} I_0((u_\delta)_\tau)$. By direct calculation, we have

$$\begin{aligned}
&I_0((u_\delta)_\tau)\\
&= \frac{\tau^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla u_\delta|^2 + \frac{\tau^N}{2}\lambda\int_{\mathbb{R}^N}|u_\delta|^2\\
&\quad - \frac{\tau^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * (\mu|u_\delta|^{p_*} + \omega|u_\delta|^{p^*}))(\mu|u_\delta|^{p_*} + \omega|u_\delta|^{p^*})\\
&= \frac{\tau^{N-2}}{2}\delta^2\int_{\mathbb{R}^N}|\nabla U|^2 + \frac{\tau^N}{2}\lambda\int_{\mathbb{R}^N}|U|^2\\
&\quad - \frac{\tau^{N+\alpha}}{2}\mu^2\int_{\mathbb{R}^N}(I_\alpha * |U|^{p_*})|U|^{p_*}\\
&\quad - \frac{\tau^{N+\alpha}}{2}\omega^2\delta^{[2(N+\alpha)]/(N-2)}\int_{\mathbb{R}^N}(I_\alpha * |U|^{p^*})|U|^{p^*}\\
&\quad - \tau^{N+\alpha}\mu\omega\delta^{(N+\alpha)/(N-2)}\int_{\mathbb{R}^N}(I_\alpha * |U|^{p_*})|U|^{p^*}.
\end{aligned} \tag{2.9}$$

We claim that there exist τ_0, $\tau_1 > 0$, independent of δ, such that $\tau_\delta \in [\tau_0, \tau_1]$ for $\delta > 0$ small. Suppose, by contradiction, that $\tau_\delta \to 0$ or

$\tau_\delta \to \infty$ as $\delta \to 0$. Equation (2.9) implies that $c_0 \leq 0$ as $\delta \to 0$, which contradicts $c_0 > 0$. Thus, the claim holds.

Since $N > 4 + \alpha$, we have $(N + \alpha)/(N - 2) < 2$. Thus, for $\delta > 0$ small enough,

$$\begin{aligned} c_0 &< \sup_{\tau \geq 0} \left\{ \frac{\tau^N}{2} \lambda \int_{\mathbb{R}^N} |U|^2 - \frac{\tau^{N+\alpha}}{2} \mu^2 \int_{\mathbb{R}^N} (I_\alpha * |U|^{p_*}) |U|^{p_*} \right\} \\ &= \frac{\alpha}{2(N+\alpha)} \left(\frac{N}{(N+\alpha)\mu^2} \right)^{N/\alpha} (\lambda S_1)^{(N+\alpha)/\alpha}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &I_0((v_\epsilon)_\tau) \\ &\quad = \frac{\tau^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 + \frac{\tau^N}{2} \lambda \int_{\mathbb{R}^N} |v_\epsilon|^2 \\ &\qquad - \frac{\tau^{N+\alpha}}{2} \int_{\mathbb{R}^N} (I_\alpha * (\mu |v_\epsilon|^{p_*} + \omega |v_\epsilon|^{p^*}))(\mu |v_\epsilon|^{p_*} + \omega |v_\epsilon|^{p^*}) \\ &\quad = \frac{\tau^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla V|^2 + \frac{\tau^N}{2} \lambda \epsilon^2 \int_{\mathbb{R}^N} |V|^2 \\ &\qquad - \frac{\tau^{N+\alpha}}{2} \omega^2 \int_{\mathbb{R}^N} (I_\alpha * |V|^{p^*}) |V|^{p^*} \\ &\qquad - \frac{\tau^{N+\alpha}}{2} \mu^2 \epsilon^{[2(N+\alpha)]/N} \int_{\mathbb{R}^N} (I_\alpha * |V|^{p_*}) |V|^{p_*} \\ &\qquad - \tau^{N+\alpha} \mu \omega \epsilon^{(N+\alpha)/N} \int_{\mathbb{R}^N} (I_\alpha * |V|^{p_*}) |V|^{p^*} \end{aligned}$$

and

$$\begin{aligned} c_0 &< \sup_{\tau \geq 0} \left\{ \frac{\tau^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla V|^2 - \frac{\tau^{N+\alpha}}{2} \omega^2 \int_{\mathbb{R}^N} (I_\alpha * |V|^{p^*}) |V|^{p^*} \right\} \\ &= \frac{2+\alpha}{2(N+\alpha)} \left(\frac{N-2}{(N+\alpha)\omega^2} \right)^{(N-2)/(2+\alpha)} S_2^{(N+\alpha)/(2+\alpha)}. \end{aligned} \tag{2.10}$$

The proof is complete. □

3. Proof of the main result. Based on the results obtained in Section 2, we prove Theorem 1.4 in this section.

Proof of Theorem 1.4. Let $a_n \to 0^+$ as $n \to \infty$ and $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ be a positive sequence which satisfies (2.10). Lemma 2.10 shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, there exists a nonnegative function $u \in H_r^1(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^s(\mathbb{R}^N)$ for $s \in (2, 2N/(N-2))$, and $u_n \to u$ almost everywhere on $\mathbb{R}^N$. Since $a_n \to 0^+$, and $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N) \cap L^{(2N)/(N-2)}(\mathbb{R}^N)$, by Lemma 2.5, we have

(3.1)
$$\{\omega(p^* - a_n)|u_n|^{p^*-a_n-2}u_n\} \text{ is bounded in } L^{(2Np^*)/[(p^*-1)(N+\alpha)]}(\mathbb{R}^N),$$
$$\{\mu(p_* + a_n)|u_n|^{p_*+a_n-2}u_n\} \text{ is bounded in } L^{(2Np_*)/[(p_*-1)(N+\alpha)]}(\mathbb{R}^N),$$

and

$$\{\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n}\} \text{ is bounded in } L^{(2N)/(N+\alpha)}(\mathbb{R}^N). \tag{3.2}$$

By (3.1) and the Hölder inequality,

$$\begin{aligned} &\{\omega(p^* - a_n)|u_n|^{p^*-a_n-2}u_n\varphi\} \text{ is bounded in } L^{(2N)/(N+\alpha)}(\mathbb{R}^N), \\ &\{\mu(p_* + a_n)|u_n|^{p_*+a_n-2}u_n\varphi\} \text{ is bounded in } L^{(2N)/(N+\alpha)}(\mathbb{R}^N) \end{aligned} \tag{3.3}$$

and

$$\mu p_*|u|^{p_*-2}u\varphi \quad \text{and} \quad \omega p^*|u|^{p^*-2}u\varphi \in L^{(2N)/(N+\alpha)}(\mathbb{R}^N), \tag{3.4}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, and then, Remark 2.2 shows that

$$I_\alpha * (\mu p_*|u|^{p_*-2}u\varphi + \omega p^*|u|^{p^*-2}u\varphi) \in L^{(2N)/(N-\alpha)}(\mathbb{R}^N). \tag{3.5}$$

It follows from Lemma 2.4 and (3.2) that

(3.6)
$$\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n} \rightharpoonup \mu|u|^{p_*} + \omega|u|^{p^*} \quad \text{weakly in } L^{(2N)/(N+\alpha)}(\mathbb{R}^N).$$

By (3.5) and (3.6), we obtain

(3.7)
$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * (\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n}))(\mu p_*|u|^{p_*-2}u\varphi + \omega p^*|u|^{p^*-2}u\varphi) \\ &\quad = \int_{\mathbb{R}^N} \{(\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n}) \\ &\qquad\qquad \times (I_\alpha * (\mu p_*|u|^{p_*-2}u\varphi + \omega p^*|u|^{p^*-2}u\varphi))\} \end{aligned}$$

$$\longrightarrow \int_{\mathbb{R}^N} (\mu|u|^{p_*} + \omega|u|^{p^*})(I_\alpha * (\mu p_*|u|^{p_*-2}u\varphi + \omega p^*|u|^{p^*-2}u\varphi))$$
$$= \int_{\mathbb{R}^N} (I_\alpha * (\mu|u|^{p_*} + \omega|u|^{p^*}))(\mu p_*|u|^{p_*-2}u\varphi + \omega p^*|u|^{p^*-2}u\varphi)$$

as $n \to \infty$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$.

It follows from $N \geq 5$ that $N/[[(N-1)/2](p_* - 1)]$ and

$$\frac{N}{[(N-1)/2](p^* - 1)} \in \left(\frac{2N}{N+\alpha}, \infty\right).$$

Since $a_n \to 0^+$ and $\varphi \in L^t(\mathbb{R}^N)$ for $t \in (1, \infty)$, by Lemma 2.3 and the Young inequality, there exists a constant $C > 0$ such that

(3.8)
$$||u_n|^{p_*+a_n-2}u_n\varphi|,\ ||u_n|^{p^*-a_n-2}u_n\varphi| \leq C(|u_n|^{p_*-1}|\varphi| + |u_n|^{p^*-1}|\varphi|)$$
$$\leq C(|x|^{[(1-N)/2](p_*-1)}|\varphi| + |x|^{[(1-N)/2](p^*-1)}|\varphi|) \in L^{(2N)/(N+\alpha)}(\mathbb{R}^N).$$

By (3.3), (3.4), (3.8) and the Lebesgue dominated convergence theorem,

$$A_n := \|\mu(p_* + a_n)|u_n|^{p_*+a_n-2}u_n\varphi - \mu p_*|u|^{p_*-2}u\varphi\|_{(2N)/(N+\alpha)} \longrightarrow 0$$

and

$$B_n := \|\omega(p^* - a_n)|u_n|^{p^*-a_n-2}u_n\varphi - \omega p^*|u|^{p^*-2}u\varphi\|_{(2N)/(N+\alpha)} \longrightarrow 0$$

as $n \to \infty$. Hence, the Hardy-Littlewood-Sobolev inequality implies that

(3.9)
$$\int_{\mathbb{R}^N} (I_\alpha * (\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n}))(\mu(p_* + a_n)|u_n|^{p_*+a_n-2}u_n\varphi$$
$$+ \omega(p^* - a_n)|u_n|^{p^*-a_n-2}u_n\varphi - \mu p_*|u|^{p_*-2}u\varphi - \omega p^*|u|^{p^*-2}u\varphi)$$
$$\leq C\|\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n}\|_{(2N)/(N+\alpha)}(A_n + B_n) \longrightarrow 0$$

as $n \to \infty$. By (3.7) and (3.9), for any $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$0 = \langle I'_{a_n}(u_n), \varphi\rangle$$
$$= \int_{\mathbb{R}^N} \nabla u_n \nabla\varphi + \lambda u_n\varphi - \int_{\mathbb{R}^N} \{(I_\alpha * (\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n}))$$
$$\times (\mu(p_* + a_n)|u_n|^{p_*+a_n-2}u_n\varphi + \omega(p^* - a_n)|u_n|^{p^*-a_n-2}u_n\varphi)\}$$

$$\longrightarrow \int_{\mathbb{R}^N} \nabla u \nabla \varphi + \lambda u \varphi - \int_{\mathbb{R}^N} \{(I_\alpha * (\mu|u|^{p_*} + \omega|u|^{p^*})) \times (\mu p_* |u|^{p_*-2} u\varphi + \omega p^* |u|^{p^*-2} u\varphi)\}$$

as $n \to \infty$, that is, u is a solution of (3.4).

We claim that $u \not\equiv 0$. Suppose, by contradiction, that $u \equiv 0$. Fix $\epsilon \in (0, 2/(N-2))$. In the Hardy-Littlewood-Sobolev inequality (Lemma 2.1), choosing

$$p = \frac{2N(1+\epsilon)}{N+\alpha} \quad \text{and} \quad r = \frac{2N(1+\epsilon)}{(N+\alpha)(1+2\epsilon)},$$

and noting that $u_n \to 0$ strongly in $L^s(\mathbb{R}^N)$ for $s \in (2, 2N/(N-2))$, we obtain that

(3.10)

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{p_*})|u_n|^{p^*} &\le C_1 \|u_n^{p_*}\|_p \|u_n^{p^*}\|_r \\ &= C_1 \|u_n\|_{2(1+\epsilon)}^{p_*} \|u_n\|_{[(2N)/(N-2)][(1+\epsilon)/(1+2\epsilon)]}^{p^*} \\ &= o(1).\end{aligned}$$

In view of (2.7), (2.8), (3.10), and by using $P_{a_n}(u_n) = 0$ and the Young inequality (3.3), we get that

(3.11)

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{N}{N-2}\lambda \int_{\mathbb{R}^N} |u_n|^2 \\ &\quad = \frac{N+\alpha}{N-2} \int_{\mathbb{R}^N} \{(I_\alpha * (\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n})) \\ &\qquad\qquad \times (\mu|u_n|^{p_*+a_n} + \omega|u_n|^{p^*-a_n})\} \\ &\quad \le p^*(\mu^2 \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{p_*})|u_n|^{p_*} + \omega^2 \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{p^*})|u_n|^{p^*}) + o(1) \\ &\quad \le p^* \left(\mu^2 \left(\frac{\int_{\mathbb{R}^N} |u_n|^2}{S_1} \right)^{p_*} + \omega^2 \left(\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2}{S_2} \right)^{p^*} \right) + o(1),\end{aligned}$$

which implies that either $\|u_n\|_{H^1(\mathbb{R}^N)} \to 0$ or

$$\limsup_{n\to\infty} \|\nabla u_n\|_2^2 \ge \left(\frac{S_2^{p^*}}{p^* \omega^2} \right)^{1/(p^*-1)} \tag{3.12}$$

or

$$\limsup_{n\to\infty} \|u_n\|_2^2 \geq \left(\frac{N\lambda S_1^{p_*}}{(N+\alpha)\mu^2}\right)^{1/(p_*-1)}.$$

If $\|u_n\|_{H^1(\mathbb{R}^N)} \to 0$, then (3.5) implies that $c_{a_n} \to 0$, which contradicts Lemma 2.10.

If (3.12) holds, then

$$\begin{aligned}
c_0 &\geq \limsup_{n\to\infty} c_{a_n} \\
&= \limsup_{n\to\infty} (I_{a_n}(u_n) - \frac{1}{N+\alpha} P_{a_n}(u_n)) \\
&= \limsup_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 \right. \\
&\qquad \left. + \left(\frac{1}{2} - \frac{N}{2(N+\alpha)}\right) \lambda \int_{\mathbb{R}^N} |u_n|^2 \right\} \\
&\geq \min \left\{ \frac{2+\alpha}{2(N+\alpha)} \left(\frac{N-2}{(N+\alpha)\omega^2}\right)^{(N-2)/(2+\alpha)} S_2^{(N+\alpha)/(2+\alpha)}, \right. \\
&\qquad \left. \frac{\alpha}{2(N+\alpha)} \left(\frac{N}{(N+\alpha)\mu^2}\right)^{N/\alpha} (\lambda S_1)^{(N+\alpha)/\alpha} \right\},
\end{aligned}$$

which contradicts Lemma 2.11. Thus, $u \not\equiv 0$.

By Theorem 1.2, $P_0(u) = 0$, and by the weakly lower semi-continuity of the norm, we have

$$\begin{aligned}
c_0 &\leq I_0(u) \\
&= I_0(u) - \frac{1}{N+\alpha} P_0(u) \\
&= \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)}\right) \int_{\mathbb{R}^N} |\nabla u|^2 + \left(\frac{1}{2} - \frac{N}{2(N+\alpha)}\right) \lambda \int_{\mathbb{R}^N} |u|^2 \\
&\leq \liminf_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 \right. \\
&\qquad \left. + \left(\frac{1}{2} - \frac{N}{2(N+\alpha)}\right) \lambda \int_{\mathbb{R}^N} |u_n|^2 \right\}
\end{aligned}$$

$$= \liminf_{n\to\infty}\left(I_{a_n}(u_n) - \frac{1}{N+\alpha}P_{a_n}(u_n)\right) = \liminf_{n\to\infty} c_{a_n} \le \limsup_{n\to\infty} c_{a_n} \le c_0.$$

Hence, $I_0(u) = c_0$. By the definition of c_0^g, we have $c_0^g \le I_0(u) = c_0$, which, combined with Remark 2.7, shows that $c_0^g = c_0 = I_0(u)$, that is, u is a ground state solution of (3.4). The strongly maximum principle implies that u is positive. The proof is complete. □

REFERENCES

1. C.O. Alves, F. Gao, M. Squassina and M. Yang, *Singularly perturbed critical Choquard equations*, J. Diff. Eqs. **263** (2017), 3943–3988.

2. H. Berestycki and P.L. Lions, *Nonlinear scalar field equations*, I, *Existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), 313–345.

3. V.I. Bogachev, *Measure theory*, Springer, Berlin, 2007.

4. D. Cassani and J. Zhang, *Ground states and semiclassical states of nonlinear Choquard equations involving Hardy-Littlewood-Sobolev critical growth*, `arXiv: 1611.02919`, 2016.

5. F. Gao and M. Yang, *A strongly indefinite Choquard equation with critical exponent due to the Hardy-Littlewood-Sobolev inequality*, Comm. Contemp. Math. **20** (2018), 1750037.

6. ———, *On the Brezis-Nirenberg type critical problem for nonlinear Choquard equation*, Sci. China Math., `DOI:10.1007/s11425-016-9067-5`.

7. E.P. Gross, *Physics of many-particle systems*, Gordon and Breach, New York, 1996.

8. Xinfu Li, *Groundstates for Choquard equations with the upper critical exponent*, preprint.

9. E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1977), 93–105.

10. E.H. Lieb and M. Loss, *Analysis*, Grad. Stud. Math. **14** (2001).

11. J. Liu, J.F. Liao and C.L. Tang, *Ground state solution for a class of Schrödinger equations involving general critical growth term*, Nonlinearity **30** (2017), 899–911.

12. X.Q. Liu, J.Q. Liu and Z.Q. Wang, *Ground states for quasilinear Schrödinger equations with critical growth*, Calc. Var. PDE **46** (2013), 641–669.

13. V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations*: *Existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184.

14. V. Moroz and J. Van Schaftingen, *Existence of ground states for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), 6557–6579.

15. ———, *A guide to the Choquard equation*, J. Fixed Point Th. Appl. **19** (2017), 773–813.

16. S. Pekar, *Untersuchung ber die Elektronentheorie der Kristalle*, Akad. Verlag, Berlin, 1954.

17. R. Penrose, *On gravity's role in quantum state reduction*, Gen. Rel. Grav. **28** (1996), 581–600.

18. M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. **81** (1949), 1–223.

19. J. Seok, *Nonlinear Choquard equations*: *Doubly critical case*, Appl. Math. Lett. **76** (2018), 148–156.

20. W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

21. J. Van Schaftingen and J. Xia, *Groundstates for a local nonlinear perturbation of the Choquard equations with lower critical exponent*, `arXiv:1710.03973`, 2017.

22. M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996.

23. ______, *Functional analysis*: *Fundamentals and applications*, Birkhäuser, Basel, 2013.

24. J. Zhang and W.M. Zou, *The critical case for a Berestycki-Lions theorem*, Sci. China Math. **57** (2014), 541–554.

25. J.J. Zhang and W.M. Zou, *A Berestycki-Lions theorem revisited*, Comm. Contemp. Math. **14** (2012), 1250033.

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