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# MULTIPLICITY OF POSITIVE SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS WITH NONLINEAR BOUNDARY CONDITION

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ABSTRACT. In this paper, we study the existence of multiple positive solutions to problem

$$\begin{cases} \left(a+b\displaystyle\int_{\Omega}(|\nabla u|^2+|u|^2)\,dx\right)(-\Delta u+u)=|u|^4u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}=\lambda|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $a, b > 0$, $\lambda > 0$ and $1 < q < 2$. Based on the Nehari manifold and variational methods, we prove that the problem has at least two positive solutions, and one of the solutions is a positive ground state solution.

**1. Introduction and main result.** In this paper, we are mainly interested in the existence of positive solutions of the following Kirchhoff-type equation

$$\begin{cases} \left(a+b\displaystyle\int_{\Omega}(|\nabla u|^2+|u|^2)\,dx\right)(-\Delta u+u)=|u|^4u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}=\lambda|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^3$, $a, b > 0$, $1 < q < 2$, $\partial/\partial\nu$ denotes the derivative along the outer normal and $\lambda > 0$ is a real parameter.

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It is well known that the Kirchhoff-type problem is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

proposed by Kirchhoff, see **[4]** and the references therein. There has been much research regarding the existence and multiplicity of positive solutions for Kirchhoff-type problems with a critical term on a bounded domain $\Omega \subset \mathbb{R}^3$, and interesting results may be found in [**7, 9, 10, 16, 17, 21, 24, 25, 27, 28**] and the references therein. For references to several existence results that have been obtained on the entire space $\mathbb{R}^3$, some representatives may be found in [**18, 19, 20, 26**]. Note that the main difficulty of such a type of problem is the lack of compactness of the Sobolev embedding.

In addition, many papers have been concerned with the Kirchhoff-type problem on a bounded domain $\Omega \subset \mathbb{R}^3$ involving concave and convex nonlinearities

(1.2)
$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = g(x)|u|^{p-2}u + \lambda f(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and there are some results on the multiplicity of solutions, see [**6, 5, 10, 15**]. For example, in the case where $1 < q < 2$, $4 < p < 6$, the weight functions $f, g \in C(\overline{\Omega})$ with $f^+ = \max\{f, 0\} \neq 0$, $g^+ = \max\{g, 0\} \neq 0$, based on the Nehari manifold, Chen, et al., **[6]** obtained two positive solutions for (1.2) when $\lambda > 0$ is small enough.

In **[14]**, Zhang discussed the following nonlinear boundary equation

$$\begin{cases} \left(a+b\displaystyle\int_\Omega (|\nabla u|^2+|u|^2)\, dx\right)(-\Delta u+u) = \lambda|u|^{q-2}u + f(x,u) + Q(x)u^5 \\ \hfill \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \hfill \text{on } \partial\Omega, \end{cases}$$

and studied the critical Neumann problem of Kirchhoff-type. By using the variational method and the concentration compactness argument, he obtained the existence and multiplicity of nontrivial solutions. Other Neumann problems were considered by Garcia-Azorero, et al., **[11]** and

Humberto, et al., **[13]**. In **[13]**, Humberto considered the problem with a sublinear Neumann boundary condition

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < 2 < p < \infty$, $a \in C^\alpha(\overline{\Omega})$ with $\alpha \in (0,1)$. Then, he established a global multiplicity result for positive solutions in the spirit of Ambrosetti, Brezis and Cerami **[2]** and analyzed the case where the nonlinearity is concave using a bifurcation analysis, a comparison principle and variational techniques.

It is well known that Ambrosetti, et al., **[2]** obtained two positive solutions of (1.2) in the case of $a = 1$, $b = 0$, $f(x) = g(x) = 1$ and $p = 6$. When $b > 0$, $f(x) = g(x) = 1$ and $p = 6$ in (1.2), it reduces to a class of nonlocal Kirchhoff-type problems with concave-convex nonlinearities. To the best of our knowledge, there are no results for the multiplicity of positive solutions in this case. The reason is that, by virtue of $b > 0$, the nonlocal Kirchhoff-type problem becomes more complicated to study than the case $b = 0$, i.e., it is difficult to estimate the critical value level. Thus, mainly motivated by **[2, 13, 14]**, we propose an interesting question for the Kirchhoff-type problem (1.1) with a nonlinear boundary condition. Based upon **[3]**, we provide some multiplicity results for (1.1).

Now, our main result can be described as follows.

**Theorem 1.1.** *Assume that $a, b > 0$ and $1 < q < 2$. Then, there exists a $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem* (1.1) *has at least two positive solutions, and one of the solutions is a positive ground state solution.*

This work is organized as follows. In Section 2, we present some preliminary results. In Section 3, we give the proof of Theorem 1.1.

**2. Some preliminary results.** Problem (1.1) is posed in the framework of the Sobolev space $H^1(\Omega)$ with the standard norm $\|u\|^2 = \int_\Omega (|\nabla u|^2 + |u|^2)\, dx$. In addition, we define $|u|_p^p = \int_\Omega |u|^p\, dx$ as the norm of the Sobolev space $L^p(\Omega)$. Let $S$ be the best Sobolev constant, i.e.,

$$S=\inf\left\{\frac{\|u\|^2}{|u|_6^2},\ u\in H^1(\Omega),\ u\neq 0\right\}. \tag{2.1}$$

The energy functional corresponding to problem (1.1) is given by

$$I_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{6}\int_\Omega |u|^6dx-\frac{\lambda}{q}\int_{\partial\Omega}|u|^q d\sigma,$$

where $d\sigma$ is the measure on the boundary.

Since $I_\lambda$ is not bounded below on $H^1(\Omega)$, we shall work on the Nehari manifold

$$\mathcal{N}_\lambda=\{u\in H^1(\Omega)\setminus\{0\}:\langle I'_\lambda(u),u\rangle=0\}.$$

Note that $\mathcal{N}_\lambda$ contains all nonzero solutions of (1.1), and $u\in\mathcal{N}_\lambda$ if and only if

$$a\|u\|^2+b\|u\|^4-\int_\Omega |u|^6dx-\lambda\int_{\partial\Omega}|u|^q d\sigma=0.$$

We split $\mathcal{N}_\lambda$ into three parts:

$$\mathcal{N}_\lambda^+=\left\{u\in\mathcal{N}_\lambda:(2-q)a\|u\|^2+(4-q)b\|u\|^4-(6-q)\int_\Omega|u|^6dx>0\right\},$$
$$\mathcal{N}_\lambda^0=\left\{u\in\mathcal{N}_\lambda:(2-q)a\|u\|^2+(4-q)b\|u\|^4-(6-q)\int_\Omega|u|^6dx=0\right\},$$
$$\mathcal{N}_\lambda^-=\left\{u\in\mathcal{N}_\lambda:(2-q)a\|u\|^2+(4-q)b\|u\|^4-(6-q)\int_\Omega|u|^6dx<0\right\}.$$

**Lemma 2.1.** *Suppose that $\lambda\in(0,T_1)$, where*

$$T_1=\frac{2a}{4-q}\left(\frac{a(2-q)S^3}{6-q}\right)^{(2-q)/4}C_q^{-q}.$$

*Then*:

(i) $\mathcal{N}_\lambda^\pm\neq\emptyset$;

(ii) $\mathcal{N}_\lambda^0=\emptyset$.

*Proof.*

(i) Let $u \in H^1(\Omega) \setminus \{0\}$, define $\Phi, \Phi_1 \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\Phi(t) = at^{-4}\|u\|^2 + bt^{-2}\|u\|^4 - \lambda t^{q-6} \int_{\partial\Omega} |u|^q d\sigma,$$

and

$$\Phi_1(t) = at^{-4}\|u\|^2 - \lambda t^{q-6} \int_{\partial\Omega} |u|^q d\sigma.$$

Then,

$$\Phi_1'(t) = -4at^{-5}\|u\|^2 - \lambda(q-6)t^{q-7} \int_{\partial\Omega} |u|^q d\sigma,$$

and letting $\Phi_1'(t) = 0$, the following holds:

$$t_{\max} = \left[\frac{\lambda(6-q)\int_{\partial\Omega} |u|^q d\sigma}{4a\|u\|^2}\right]^{1/(2-q)}.$$

Simple computation shows that $\Phi_1'(t) > 0$ for all $0 < t < t_{\max}$ and $\Phi_1'(t) < 0$ for all $t > t_{\max}$, and $\Phi_1(t)$ attains its maximum at $t_{\max}$, that is,

$$\Phi_1(t_{\max}) = \frac{2-q}{4}\left[\frac{4a}{6-q}\right]^{(6-q)/(2-q)} \frac{\|u\|^{(2(6-q))/(2-q)}}{\left(\lambda \int_{\partial\Omega} |u|^q d\sigma\right)^{4/(2-q)}}.$$

By the Sobolev embedding theorem, the following holds

$$\int_{\partial\Omega} |u|^q d\sigma \le C_q^q \|u\|^q,$$

where $C_q^q$ is a constant. Then, from (2.1), we obtain

$$\begin{aligned}
&\Phi(t_{\max}) - \int_\Omega |u|^6 dx \\
&\quad \ge \Phi_1(t_{\max}) - \int_\Omega |u|^6 dx \\
&\quad > \frac{2-q}{4}\left[\frac{4a}{6-q}\right]^{(6-q)/(2-q)} \frac{\|u\|^{(2(6-q))/(2-q)}}{(\lambda \int_{\partial\Omega} |u|^q d\sigma)^{4/(2-q)}} - \int_\Omega |u|^6 dx \\
&\quad = \left\{\frac{2-q}{4}\left[\frac{4a}{6-q}\right]^{(6-q)/(2-q)} \left(\frac{1}{\lambda C_q^q}\right)^{4/(2-q)} \left(\frac{\|u\|^2}{|u|_6^2}\right)^3 - 1\right\}|u|_6^6
\end{aligned}$$

$$\geq \left\{\frac{2-q}{4}\left[\frac{4a}{6-q}\right]^{(6-q)/(2-q)}\left(\frac{1}{\lambda C_q^q}\right)^{4/(2-q)} S^3 - 1\right\}|u|_6^6$$
$$> 0.$$

The last inequality holds, provided $0 < \lambda < T_1$. It follows that there exist two positive numbers denoted by $t^{\pm}$ such that $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$, $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$.

(ii) We prove that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, T_1)$. To the contrary, suppose that there exists a $u_0 \neq 0$ such that $u_0 \in \mathcal{N}_\lambda^0$, and the following hold:

$$a\|u_0\|^2 + b\|u_0\|^4 = \int_\Omega |u_0|^6 dx + \lambda \int_{\partial\Omega} |u_0|^q d\sigma, \tag{2.2}$$

and

$$4a\|u_0\|^2 + 2b\|u_0\|^4 = \lambda(6-q)\int_{\partial\Omega} |u_0|^q d\sigma. \tag{2.3}$$

Equations (2.2) and (2.3) imply that

$$\begin{aligned} \lambda \int_{\partial\Omega} |u_0|^q d\sigma &= \frac{2a}{4-q}\|u_0\|^2 + \frac{2}{4-q}\int_\Omega |u_0|^6 dx \\ &> \frac{2a}{4-q}\|u_0\|^2. \end{aligned} \tag{2.4}$$

On one hand, the strict inequality $\|u_0\|^2 > S|u|_6^2$ holds for $u_0 \in \mathcal{N}_\lambda^0 \setminus \{0\}$. Here, it is convenient to use a parameter $\Theta$, i.e., let

$$\begin{aligned} \Theta &= C_q^{(4q)/(2-q)} \frac{\|u_0\|^{(2(6-q))/(2-q)}}{(\int_{\partial\Omega} |u_0|^q d\sigma)^{4/(2-q)}} - \|u_0\|^6 \\ &> C_q^{(4q)/(2-q)} \frac{\|u_0\|^{(2(6-q))/(2-q)}}{C_q^{(4q)/(2-q)}\|u_0\|^{(4q)/(2-q)}} - \|u_0\|^6 \\ &= \|u_0\|^6 - \|u_0\|^6 \\ &= 0. \end{aligned}$$

On the other hand, by (2.4), the following holds:

$$\Theta = C_q^{4q/(2-q)} \lambda^{4/(2-q)} \frac{\|u_0\|^{(2(6-q))/(2-q)}}{\left(\lambda \int_{\partial\Omega} |u_0|^q d\sigma\right)^{4/(2-q)}} - \|u_0\|^6$$

$$
\begin{aligned}
&< C_q^{4q/(2-q)} \lambda^{4/(2-q)} \frac{\|u_0\|^{(2(6-q))/(2-q)}}{\left(\lambda \int_{\partial\Omega} |u_0|^q d\sigma\right)^{4/(2-q)}} - S^3 |u_0|_6^6 \\
&\leq C_q^{4q/(2-q)} \lambda^{4/(2-q)} \frac{\|u_0\|^{(2(6-q))/(2-q)}}{(2a/4-q)^{4/(2-q)} \|u_0\|^{8/(2-q)}} \\
&\quad - \frac{a(2-q)S^3}{6-q} \|u_0\|^2 - \frac{b(4-q)S^3}{6-q} \|u_0\|^4 \\
&< C_q^{4q/(2-q)} \lambda^{4/(2-q)} \left(\frac{4-q}{2a}\right)^{4/(2-q)} \|u_0\|^2 - \frac{a(2-q)S^3}{6-q} \|u_0\|^2 \\
&\leq \frac{a(2-q)S^3}{6-q} \|u_0\|^2 \left[ C_q^{4q/(2-q)} \lambda^{4/(2-q)} \left(\frac{4-q}{2a}\right)^{4/(2-q)} \frac{6-q}{a(2-q)S^3} - 1 \right] \\
&< 0,
\end{aligned}
$$

a contradiction, where the last inequality holds when $\lambda < T_1$. This completes the proof of Lemma 2.1. $\square$

**Lemma 2.2.** *$I_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$.*

*Proof.* Suppose $u \in \mathcal{N}_\lambda$. Then, by (2.1), we obtain

$$
\begin{aligned}
I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{q}\int_{\partial\Omega} |u|^q d\sigma \\
&= \frac{a}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 - \lambda\left(\frac{1}{q} - \frac{1}{6}\right)\int_{\partial\Omega} |u|^q d\sigma \\
&\geq \frac{a}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 - \lambda\left(\frac{1}{q} - \frac{1}{6}\right) C_q^q \|u\|^q
\end{aligned}
$$

since $1 < q < 2$, and it follows that $I_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$. $\square$

We remark that, by Lemma 2.1, we have $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0, T_1)$. Due to $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^- \neq \emptyset$ and Lemma 2.2, we may define

$$
\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u).
$$

**Lemma 2.3.** $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

*Proof.* Suppose that $u \in \mathcal{N}_\lambda^+$. The following holds:

$$\int_\Omega |u|^6 dx < \frac{2-q}{6-q} a\|u\|^2 + \frac{4-q}{6-q} b\|u\|^4,$$

and thus,

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{q}\int_{\partial\Omega} |u|^q d\sigma \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) a\|u\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) b\|u\|^4 + \left(\frac{1}{q} - \frac{1}{6}\right) \int_\Omega |u|^6 dx \\ &< \left(\frac{a}{2} - \frac{1}{q}\right) a\|u\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) b\|u\|^4 \\ &\quad + \left(\frac{1}{q} - \frac{1}{6}\right)\left(\frac{2-q}{6-q} a\|u\|^2 + \frac{4-q}{6-q} b\|u\|^4\right) \\ &= \frac{1}{3}\left(1 - \frac{2}{q}\right) a\|u\|^2 + \frac{1}{3} \leq \left(\frac{1}{4} - \frac{1}{q}\right) b\|u\|^4 \\ &< 0. \end{aligned}$$

Hence, from the definitions of $\alpha_\lambda$ and $\alpha_\lambda^+$, we can deduce that $\alpha_\lambda \leq \alpha_\lambda^+ < 0$. □

**Lemma 2.4.** *For every $u \in \mathcal{N}_\lambda$, there exist an $\varepsilon > 0$ and a continuously differentiable function $f = f(w) > 0$, $w \in H^1(\Omega)$, $\|w\| < \varepsilon$, satisfying*

$$f(0) = 1, \quad f(w)(u+w) \in \mathcal{N}_\lambda \quad \textit{for all } w \in H^1(\Omega), \quad \|w\| < \varepsilon.$$

*Proof.* For $u \in \mathcal{N}_\lambda$, define $F : \mathbb{R} \times H^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} F(t,w) &= t^{2-q} a \int_\Omega (|\nabla(u+w)|^2 + |u+w|^2)\, dx - t^{6-q} \int_\Omega |u+w|^6 dx \\ &\quad + t^{4-q} b\left(\int_\Omega (|\nabla(u+w)|^2 + |u+w|^2)\, dx\right)^2 - \lambda \int_{\partial\Omega} |u|^q d\sigma. \end{aligned}$$

Since $u \in \mathcal{N}_\lambda$, $F(1,0) = 0$ and

$$F_t(1,0) = (2-q)a\|u\|^2 + (4-q)b\|u\|^4 - (6-q)\int_\Omega |u|^6 dx$$

are easily obtained. As $u \neq 0$, by Lemma 2.1, we know that $F_t(1,0) \neq 0$. Thus, we apply the implicit function theorem at the point $(0,1)$ and obtain $\varepsilon > 0$ and a continuously differentiable function $f : B(0,\varepsilon) \subset H^1(\Omega) \to \mathbb{R}^+$, satisfying that

$$f(0) = 1, \qquad f(w) > 0, \qquad f(w)(u+w) \in \mathcal{N}_\lambda,$$

for all $w \in H^1(\Omega)$ with $\|w\| < \varepsilon$. This completes the proof of Lemma 2.4. $\square$

**Lemma 2.5.** *For every $u \in \mathcal{N}_\lambda^-$, there exist an $\varepsilon > 0$ and a continuously differentiable function $\widetilde{f} = \widetilde{f}(v) > 0$, $v \in H^1(\Omega)$, $\|v\| < \varepsilon$, satisfying that*

$$\widetilde{f}(0) = 1, \qquad \widetilde{f}(v)(u+v) \in \mathcal{N}_\lambda^-$$

*for all $v \in H^1(\Omega)$, $\|v\| < \varepsilon$.*

*Proof.* The proof is similar to the argument in Lemma 2.4. For $u \in \mathcal{N}_\lambda^-$, define $\widetilde{F} : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}\widetilde{F}(t,v) = t^{2-q} a \int_\Omega (|\nabla(u+v)|^2 + |u+v|^2)\,dx - t^{6-q} \int_\Omega |u+v|^6 dx \\ + t^{4-q} b \bigg( \int_\Omega (|\nabla(u+v)|^2 + |u+v|^2)\,dx \bigg)^2 - \lambda \int_{\partial\Omega} |u|^q d\sigma.\end{aligned}$$

Since $u \in \mathcal{N}_\lambda^-$, we obtain that $\widetilde{F}(1,0) = 0$ and $\widetilde{F}_t(1,0) < 0$. Therefore, we can apply the implicit function theorem at the point $(0,1)$ and obtain the result. This completes the proof of Lemma 2.5. $\square$

**Lemma 2.6.** *If $\{u_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence of $I_\lambda$ for any $\varphi \in H^1(\Omega)$, then*

$$-\frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n} \le \langle I_\lambda'(u_n), \varphi\rangle \le \frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n}. \tag{2.5}$$

*Proof.* By Lemma 2.2, let $\{u_n\} \in \mathcal{N}_\lambda$ be a minimizing sequence for $I_\lambda$. Clearly, $|u_n| \in \mathcal{N}_\lambda$ and $I_\lambda(|u_n|) = I_\lambda(u_n)$. For this reason, we immediately assume that $u_n \ge 0$ almost everywhere in $\Omega$ for all $n$. Then, applying Ekeland's variational principle **[8]**, the following holds:

$$I_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \qquad I_\lambda(v) - I_\lambda(u_n) \ge -\frac{1}{n}\|v - u_n\| \quad \text{for all } v \in \mathcal{N}_\lambda. \tag{2.6}$$

Obviously, Lemma 2.2 suggests that $\{u_n\}$ is bounded in $H^1(\Omega)$. Thus, there exist a subsequence, still denoted $\{u_n\}$, and $u_*$ in $H^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } H^1(\Omega), \\ u_n \to u_* & \text{strongly in } L^p(\Omega), \qquad 1 \le p < 6, \\ u_n(x) \to u_*(x) & \text{almost everywhere in } \Omega. \end{cases}$$

Let $t > 0$ be small enough, let $\varphi \in H^1(\Omega)$, and set $u = u_n, w = t\varphi \in H^1(\Omega)$ in Lemma 2.4. Hence, we get $f_n(t) = f_n(t\varphi)$ satisfying $f_n(0) = 1$ and $f_n(t)(u_n + t\varphi) \in \mathcal{N}_\lambda$. Note that

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_\Omega u_n^6\,dx - \lambda \int_{\partial\Omega} u_n^q\,d\sigma = 0. \tag{2.7}$$

Then, (2.6) implies that

$$\begin{aligned} \frac{1}{n}[|f_n(t) - 1| \cdot \|u_n\| + t f_n(t)\|\varphi\|] &\ge \frac{1}{n}\|f_n(t)(u_n + t\varphi) - u_n\| \\ &\ge I_\lambda(u_n) - I_\lambda[f_n(t)(u_n + t\varphi)], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &I_\lambda(u_n) - I_\lambda[f_n(t)(u_n + t\varphi)] \\ &= \frac{1 - f_n^2(t)}{2} a\|u_n\|^2 + \frac{1 - f_n^4(t)}{4} b\|u_n\|^4 \\ &\quad + \frac{f_n^6(t) - 1}{6} \int_\Omega (u_n + t\varphi)^6 dx + \lambda \frac{f_n^q(t) - 1}{q} \int_{\partial\Omega} (u_n + t\varphi)^q d\sigma \\ &\quad + \frac{f_n^2(t)}{2}\left(a + \frac{f_n^2(t)}{2} b(\|u_n\|^2 + \|u_n + t\varphi\|^2)\right)(\|u_n\|^2 - \|u_n + t\varphi\|^2) \\ &\quad + \frac{1}{6}\int_\Omega ((u_n + t\varphi)^6 - u_n^6)\,dx + \frac{\lambda}{q}\int_{\partial\Omega} ((u_n + t\varphi)^q - u_n^q)\,d\sigma. \end{aligned}$$

Combining this with (2.7) and (2.8), dividing by $t$ and letting $t \to 0$, we obtain

$$\begin{aligned} \frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n} &\ge -f_n'(0)a\|u_n\|^2 + f_n'(0)b\|u_n\|^4 + f_n'(0)\int_\Omega u_n^6\,dx \\ &\quad + \lambda f_n'(0)\int_{\partial\Omega} u_n^q\,d\sigma - (a + b\|u_n\|^2) \\ &\quad \cdot \int_\Omega (\nabla u_n \cdot \nabla\varphi + u_n\varphi)\,dx + \int_\Omega u_n^5\varphi\,dx + \lambda\int_{\partial\Omega} u_n^{q-1}\varphi\,d\sigma \end{aligned}$$

$$
\begin{aligned}
&= -f_n'(0)\Big(a\|u_n\|^2+b\|u_n\|^4-\int_\Omega u_n^6\,dx-\lambda\int_{\partial\Omega}u_n^q\,d\sigma\Big)\\
&\quad-(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi+u_n\varphi)\,dx\\
&\quad+\int_\Omega u_n^5\varphi\,dx+\lambda\int_{\partial\Omega}u_n^{q-1}\varphi\,d\sigma;
\end{aligned}
$$

consequently,

$$
\begin{aligned}
-\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}&\le(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi+u_n\varphi)\,dx\\
&\quad-\int_\Omega u_n^5\varphi\,dx-\lambda\int_{\partial\Omega}u_n^{q-1}\varphi\,d\sigma,
\end{aligned}\tag{2.9}
$$

for any $\varphi\in H_0^1(\Omega)$. Since (2.9) also holds for $-\varphi$, we obtain

$$
\begin{aligned}
\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}&\ge(a+b\|u_n\|^2)\int_\Omega(\nabla u_n\cdot\nabla\varphi+u_n\varphi)\,dx\\
&\quad-\int_\Omega u_n^5\varphi\,dx-\lambda\int_{\partial\Omega}u_n^{q-1}\varphi\,d\sigma.
\end{aligned}
$$

Then,

$$
-\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}\le\langle I_\lambda'(u_n),\varphi\rangle\le\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n},
$$

for every $\varphi\in H^1(\Omega)$. Thus, (2.5) holds. Moreover, Lemma 2.4 suggests that there exists a constant $C>0$ such that $|f_n'(0)|\le C$ for all $n\in N$. Therefore, passing to the limit as $n\to\infty$ in (2.5), we get

$$
\Big(a+b\lim_{n\to\infty}\|u_n\|^2\Big)\int_\Omega(\nabla u_*\cdot\nabla\varphi+u_*\varphi)\,dx-\int_\Omega u_*^5\varphi\,dx-\lambda\int_{\partial\Omega}u_*^{q-1}\varphi\,d\sigma=0\tag{2.10}
$$

for all $\varphi\in H^1(\Omega)$. This completes the proof of Lemma 2.6. □

Let $S_{\rm sob}$ be the best Sobolev constant for the embedding $H_0^1(\Omega)\hookrightarrow L^6(\Omega)$, namely,

$$
S_{\rm sob}=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2dx}{(\int_\Omega|u|^6dx)^{1/3}}.
$$

The proof of the following concentration-compactness lemma is standard, see **[22, 23]** for details.

**Lemma 2.7.** *Let $\{u_n\}$ be a sequence in $H^1(\Omega)$, such that*

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } H^1(\Omega), \\ u_n \to u \text{ strongly in } L^p(\Omega), \\ |\nabla u_n|_2^2 \rightharpoonup d\mu \geq |\nabla u|_2^2 + \sum_{j\in J} \mu_j \delta_{x_j}, \quad 1 \leq p < 6, \\ |u_n|_6^6 \to d\eta = |u|_6^6 + \sum_{j\in J} \eta_j \delta_{x_j}, \end{cases}$$

*where $J$ is an at most countable index set, $\delta_{x_j}$ is the Dirac mass at $x_j$, and $x_j \in \Omega$ supports $\mu, \eta$. Then*

$$\mu_j \geq S_{\mathrm{sob}} \eta_j^{1/3}.$$

Define

$$\Lambda = \frac{abS_{\mathrm{sob}}^3}{4} + \frac{b^3 S_{\mathrm{sob}}^6}{24} + \frac{(b^2 S_{\mathrm{sob}}^4 + 4aS_{\mathrm{sob}})^{3/2}}{24}.$$

**Lemma 2.8.** *Assume that $1 < q < 2$, and let $\{u_n\} \subset \mathcal{N}_\lambda^-$ be a minimizing sequence of $I_\lambda$ with*

$$\alpha_\lambda^- < \Lambda - D\lambda^{2/(2-q)} \quad \text{where } D = \left(\frac{(4-q)}{4q} C_q^q\right)^{2/(2-q)} \left(\frac{2q}{a}\right)^{q/(2-q)}.$$

*Then, there exists a $u \in H^1(\Omega)$ such that $u_n \to u$ in $L^6(\Omega)$.*

*Proof.* Let $\{u_n\} \subset \mathcal{N}_\lambda^-$ be a minimizing sequence of $I_\lambda$. Then

$$I_\lambda(u_n) \longrightarrow \alpha_\lambda^- \quad \text{as } n \to \infty. \tag{2.11}$$

By Lemma 2.2, it is easily obtained that $\{u_n\}$ is bounded in $H^1(\Omega)$. Passing to a subsequence, if necessary, there exists a $u \in H^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H^1(\Omega), \\ u_n \to u & \text{strongly in } L^p(\Omega), \qquad 1 \leq p < 6, \\ u_n(x) \to u(x) & \text{almost everywhere in } \Omega. \end{cases}$$

Furthermore, by the concentration compactness principle, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned}|\nabla u_n|_2^2 &\rightharpoonup d\mu \geq |\nabla u|_2^2 + \sum_{j\in J} \mu_j \delta_{x_j},\\ |u_n|_6^6 &\longrightarrow d\eta = |u|_6^6 + \sum_{j\in J} \eta_j \delta_{x_j},\end{aligned}$$

and

$$\mu_j,\ \eta_j \geq 0, \quad \mu_j \geq S_{\mathrm{sob}} \eta_j^{1/3}. \tag{2.12}$$

For any $\varepsilon > 0$ small, let $\psi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at $x_j$ such that $0 \leq \psi_{\varepsilon,j}(x) \leq 1$,

$$\begin{aligned}\psi_{\varepsilon,j}(x) &= 1 \quad \text{in } B\left(x_j, \frac{\varepsilon}{2}\right),\\ \psi_{\varepsilon,j}(x) &= 0 \quad \text{in } B(x_j, \varepsilon),\\ |\nabla \psi_{\varepsilon,j}(x)| &\leq \frac{4}{\varepsilon}.\end{aligned}$$

From Hölder's inequality, we have

$$\begin{aligned}\int_\Omega |\nabla(\psi_{\varepsilon,j} u_n)|^2 dx &= \int_\Omega |u_n \nabla \psi_{\varepsilon,j} + \psi_{\varepsilon,j} \nabla u_n|^2 dx\\ &\leq \frac{c_1}{\varepsilon^2} \int_{B(x_j,\varepsilon)} |u_n|^2 dx + \frac{c_2}{\varepsilon} \int_{B(x_j,\varepsilon)} u_n |\nabla u_n|\, dx + \|u_n\|^2\\ &\leq \frac{c_3}{\varepsilon^2} \|u_n\|^2 \varepsilon^2 + \|u_n\|^2 + \frac{c_4}{\varepsilon} \|u_n\|^3 \varepsilon\\ &= (c_3 + 1)\|u_n\|^2 + c_4 \|u_n\|^3,\end{aligned}$$

where $c_i$, $i = 1, 2, 3, 4$, are positive constants. Since $\{f_n'(0)\}$ and $\{u_n\}$ are bounded in $H_0^1(\Omega)$, we obtain

$$\lim_{n\to\infty} \frac{|f_n'(0)| \|u_n\| + \|\psi_{\varepsilon,j} u_n\|}{n} = 0,$$

so that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{|f_n'(0)| \|u_n\| + \|\psi_{\varepsilon,j} u_n\|}{n} = 0.$$

Setting $\varphi = \psi_{\varepsilon,j} u_n$ in (2.5) and taking $\varepsilon \to 0$, the following holds:

$$
\begin{aligned}
0 &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \langle I'_\lambda(u_n), \psi_{\varepsilon,j} u_n \rangle \\
&= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\{ (a + b\|u_n\|^2) \int_\Omega (\nabla u_n \cdot \nabla(\psi_{\varepsilon,j} u_n) + \psi_{\varepsilon,j} u_n^2)\, dx \right. \\
&\qquad\qquad \left. - \int_\Omega u_n^5 \psi_{\varepsilon,j} u_n \, dx - \lambda \int_{\partial\Omega} u_n^{q-1} \psi_{\varepsilon,j}(x) u_n \, d\sigma \right\} \\
&\geq \left( a + b \int_\Omega d\mu \right) \int_\Omega \psi_{\varepsilon,j} \, d\mu - \int_\Omega \psi_{\varepsilon,j} \, d\eta,
\end{aligned}
$$

so that

$$\eta_j \geq (a + b\mu_j)\mu_j.$$

By (2.12), we deduce that

$$\eta_j^{2/3} \geq a S_{\text{sob}} + b S_{\text{sob}}^2 \eta_j^{1/3}, \quad \text{or} \quad \eta_j = \mu_j = 0. \tag{2.13}$$

Let $X = \eta_j^{1/3}$. It follows from (2.13) that

$$X^2 \geq a S_{\text{sob}} + b S_{\text{sob}}^2 X,$$

which means

$$X \geq \frac{b S_{\text{sob}}^2 + \sqrt{b^2 S_{\text{sob}}^4 + 4a S_{\text{sob}}}}{2},$$

so that

$$\mu_j \geq S_{\text{sob}} X \geq \frac{b S_{\text{sob}}^3 + \sqrt{b^2 S_{\text{sob}}^6 + 4a S_{\text{sob}}^3}}{2} \triangleq K.$$

Next, we show that

$$\mu_j \geq \frac{b S_{\text{sob}}^3 + \sqrt{b^2 S_{\text{sob}}^6 + 4a S_{\text{sob}}^3}}{2}$$

is impossible. Therefore, the set $J$ is empty. Assume to the contrary that there exists some $j_0 \in J$ such that

$$\mu_{j_0} \geq \frac{b S_{\text{sob}}^3 + \sqrt{b^2 S_{\text{sob}}^6 + 4a S_{\text{sob}}^3}}{2}.$$

From (2.1), (2.11) and Young's inequality, we obtain

(2.14)

$$
\begin{aligned}
\alpha_\lambda^- &= \lim_{n\to\infty} I_\lambda(u_n) \\
&= \lim_{n\to\infty} \bigg\{ I_\lambda(u_n) - \frac{1}{4}\bigg( a\|u_n\|^2 + b\|u_n\|^4 \\
&\qquad\qquad - \int_\Omega |u_n|^6 dx - \lambda \int_{\partial\Omega} |u_n|^q d\sigma \bigg) \bigg\} \\
&\geq \lim_{n\to\infty} \bigg\{ \left(\frac{1}{2} - \frac{1}{4}\right) a\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{4}\right)\|u_n\|^4 \\
&\qquad\qquad + \left(\frac{1}{4} - \frac{1}{6}\right) \int_\Omega u_n^6 dx - \lambda\left(\frac{1}{q} - \frac{1}{4}\right) \int_{\partial\Omega} |u_n|^q d\sigma \bigg\} \\
&\geq \bigg\{ \left(\frac{1}{2} - \frac{1}{4}\right) a\bigg(\|u\|^2 + \sum_{j\in J}\mu_j\bigg) + b\left(\frac{1}{4} - \frac{1}{4}\right)\bigg(\|u\|^2 + \sum_{j\in J}\mu_j\bigg)^2 \\
&\qquad + \left(\frac{1}{4} - \frac{1}{6}\right)\bigg(\int_\Omega u^6 dx + \sum_{j\in J}\nu_j\bigg) - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)\int_{\partial\Omega} |u|^q d\sigma \bigg\} \\
&\geq \left(\frac{1}{2} - \frac{1}{4}\right) a\mu_{j_0} + \left(\frac{1}{4} - \frac{1}{4}\right) b\mu_{j_0}^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\nu_{j_0} \\
&\qquad + \frac{a}{4}\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right) C_q^q \|u\|^q \\
&\geq \left(\frac{1}{2} - \frac{1}{4}\right) aK + \left(\frac{1}{4} - \frac{1}{4}\right) bK^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\frac{K^3}{S_{\mathrm{sob}}^3} - D\lambda^{2/(2-q)} \\
&\geq \frac{a}{2}K + \frac{b}{4}K^2 - \frac{K^3}{6S_{\mathrm{sob}}^3} - \frac{1}{4}\left(aK + bK^2 - \frac{K^3}{S_{\mathrm{sob}}^3}\right) - D\lambda^{2/(2-q)},
\end{aligned}
$$

where

$$
D = \left(\frac{4-q}{4q} C_q^q\right)^{2/(2-q)} \left(\frac{2q}{a}\right)^{q/(2-q)}.
$$

In the following, we claim that

$$
\frac{a}{2}K + \frac{b}{4}K^2 - \frac{K^3}{6S_{\mathrm{sob}}^3} = \Lambda.
$$

Indeed,

$$
\begin{aligned}
&\frac{aK}{2}+\frac{b}{4}K^2-\frac{K^3}{6S_{\text{sob}}^3}\\
&\quad=K\left(\frac{a}{2}+\frac{bK}{4}-\frac{K^2}{6S_{\text{sob}}^3}\right)\\
&\quad=K\bigg[\frac{a}{2}+\frac{b}{4}\cdot\frac{bS_{\text{sob}}^3+\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{2}\\
&\qquad\qquad-\frac{2b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3+2bS_{\text{sob}}^3\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{24S_{\text{sob}}^3}\bigg]\\
&\quad=K\bigg[\frac{a}{2}+\frac{b^2S_{\text{sob}}^3+b\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{8}\\
&\qquad\qquad-\frac{b^2S_{\text{sob}}^3+2+b\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{12}\bigg]\\
&\quad=K\left[\frac{8a+b^2S_{\text{sob}}^3+b\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{24}\right]\\
&\quad=\frac{bS_{\text{sob}}^3+\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{2}\cdot\frac{8a+b^2S_{\text{sob}}^3+b\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{24}\\
&\quad=\frac{12abS_{\text{sob}}^3+2b^3S_{\text{sob}}^6+(2b^2S_{\text{sob}}^3+8a)\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{48}\\
&\quad=\frac{abS_{\text{sob}}^3}{4}+\frac{b^3S_{\text{sob}}^6}{24}+\frac{(b^2S_{\text{sob}}^3+4a)\sqrt{b^2S_{\text{sob}}^6+4aS_{\text{sob}}^3}}{24}\\
&\quad=\frac{abS_{\text{sob}}^3}{4}+\frac{b^3S_{\text{sob}}^6}{24}+\frac{(b^2S_{\text{sob}}^4+4aS_{\text{sob}})\sqrt{b^2S_{\text{sob}}^4+4aS_{\text{sob}}}}{24}\\
&\quad=\Lambda.
\end{aligned}
$$

With simple computation, we obtain

$$aK+bK^2-\frac{K^3}{S_{\text{sob}}^3}=0.$$

Therefore, by (2.14), we obtain $\Lambda-D\lambda^{2/(2-q)}\leq\alpha_\lambda^-<\Lambda-D\lambda^{2/(2-q)}$. This is a contradiction. Consequently, $J$ is empty; thus,

$$\int_\Omega u_n^6\,dx\longrightarrow\int_\Omega u^6dx\quad\text{as } n\to\infty.$$

This completes the proof of Lemma 2.8. □

We recall the following lemma, which plays an important role in proving Lemma 2.10 below.

**Lemma 2.9** ([**3**]). *There exists a positive function $v_S \in H^1(\Omega)$ such that*

$$S = \frac{\int_\Omega (|\nabla v_S|^2 + |v_S|^2)\,dx}{(\int_\Omega |v_S|^{2^*})^{2/2^*}}.$$

By Lemma 3.4, we normalize $v_S$, imposing $\int_\Omega |v_S|^{2^*}\,dx = 1$. Then,

$$S = \int_\Omega (|\nabla v_S|^2 + |v_S|^2)\,dx. \tag{2.15}$$

**Lemma 2.10.** *Assume that $1 < q < 2$. Then there exists a $v_S \in H^1(\Omega)$ such that*

$$\sup_{t\geq 0} I_\lambda(tv_S) < \Lambda - D\lambda^{2/(2-q)},$$

*where $D$ is given in Lemma 2.7. In particular,*

$$\alpha_\lambda^- < \Lambda - D\lambda^{2/(2-q)}.$$

*Proof.* Since $\lim_{t\to+\infty} I_\lambda(tv_S) = -\infty$, there exists a $t_\lambda > 0$ such that

$$I_\lambda(t_\lambda v_S) = \sup_{t\geq 0} I_\lambda(t_\lambda v_S) \quad \text{and} \quad \frac{dI_\lambda(t_\lambda v_S)}{dt}\bigg|_{t=t_\lambda} = 0. \tag{2.16}$$

It follows from (2.16) that

$$a\|v_S\|^2 + t_\lambda^2 b\|v_S\|^4 - t_\lambda^4 - \lambda t_\lambda^{q-2}\int_{\partial\Omega} v_S^q\,d\sigma = 0, \tag{2.17}$$

and

$$a\|v_S\|^2 + 3t_\lambda^2 b\|v_S\|^4 - 5t_\lambda^4 - \lambda(q-1)t_\lambda^{q-2}\int_{\partial\Omega} v_S^q\,d\sigma < 0. \tag{2.18}$$

Hence, the combination of (2.17) and (2.18) implies that

$$\begin{aligned}(6-q)t_\lambda^4 &> (2-q)a\|v_S\|^2 + (4-q)t_\varepsilon^2 b\|v_S\|^4 \\ &> (2-q)aS.\end{aligned}$$

Consequently,

$$t_\lambda > \left(\frac{2-q}{6-q} aS\right)^{1/4}. \tag{2.19}$$

On the other hand, from (2.17) and (2.19), the following holds:

$$\begin{aligned} t_\lambda^2 &= \frac{a\|v_S\|^2}{t_\lambda^2} + b\|v_S\|^4 - \frac{\lambda}{t_\lambda^{4-q}} \int_{\partial\Omega} v_S^q \, d\sigma \\ &\le \frac{a\|v_S\|^2}{t_\lambda^2} + b\|v_S\|^4 \\ &\le aS\left(\frac{6-q}{aS(2-q)}\right)^{1/2} + bS^2. \end{aligned}$$

Then, we deduce that

$$\left(\frac{2-q}{6-q} aS\right)^{1/4} < t_\lambda < \left(aS\left(\frac{6-q}{aS(2-q)}\right)^{1/2} + bS^2\right)^{1/2}. \tag{2.20}$$

Set

$$A(t_\lambda v_S) = \frac{a}{2} t_\lambda^2 \|v_S\|^2 + \frac{b}{4} t_\lambda^4 \|v_S\|^4 - \frac{t_\lambda^6}{6}.$$

Firstly, we claim that $A(t_\varepsilon u_\varepsilon) \le \Lambda$. If we define

$$h(t) = \frac{a}{2} t^2 \|v_S\|^2 + \frac{b}{4} t^4 \|v_S\|^4 - \frac{t^6}{6},$$

we see that $\lim_{t\to\infty} h(t) = -\infty$, $h(0) = 0$, and $\lim_{t\to 0^+} h(t) > 0$. It follows that $\sup_{t\ge 0} h(t)$ is attained at $T > 0$, that is,

$$h'(t)|_T = aT\|v_S\|^2 + bT^3\|v_S\|^4 - T^5 = 0.$$

Observe that

$$T^4 - a\|v_S\|^2 - bT^2\|v_S\|^4 = 0,$$

so that

$$T = \left(\frac{b\|v_S\|^4 + \sqrt{b^2\|v_S\|^8 + 4a\|v_S\|^2}}{2}\right)^{1/2}.$$

Note that $h(t)$ is increasing in the interval $[0,T]$. Then, from (2.15), the following holds:

$$\begin{aligned}
h(t_\lambda v_S) &\le h(T)\\
&= \frac{a}{2}T^2\|v_S\|^2 + \frac{b}{4}T^4\|v_S\|^4 - \frac{T^6}{6}\\
&= T^2\left(\frac{a}{3}\|v_S\|^2 + \frac{b}{12}T^2\|v_S\|^4\right)\\
&= T^2\left(\frac{a}{3}\|v_S\|^2 + \frac{b^2\|v_S\|^8 + b\|v_S\|^4\sqrt{b^2\|v_S\|^8 + 4a\|v_S\|^2}}{24}\right)\\
&= \frac{ab\|v_S\|^6}{4} + \frac{b^3\|v_S\|^{12}}{24} + \frac{a\|v_S\|^2\sqrt{b^2\|v_S\|^8 + 4a\|v_S\|^2}}{6}\\
&\quad + \frac{b^2\|v_S\|^8\sqrt{b^2\|v_S\|^8 + 4a\|v_S\|^2}}{24}\\
&= \frac{abS_{\mathrm{sob}}^3}{4} + \frac{b^3S_{\mathrm{sob}}^6}{24} + \frac{(b^2S_{\mathrm{sob}}^4 + 4aS_{\mathrm{sob}})^{3/2}}{24}\\
&= \Lambda.
\end{aligned}$$

Now, we present a well-known result [**1**], that is, there exists a $\theta > 0$ dependent upon $\Omega$ such that

$$S < \frac{S_{\mathrm{sob}}}{2^{2/N}} - \theta.$$

As a result, we infer that

$$S < S_{\mathrm{sob}}.$$

In addition, due to a result by Azorero, et al., [**3**], there exists a constant $C > 0$ such that

$$\int_{\partial\Omega} |v_S|^q d\sigma \ge C.$$

Consequently, using (2.20), the following holds:

$$\begin{aligned}
\sup_{t\ge 0} I_\lambda(tv_S) &< \Lambda - \frac{\lambda t_\lambda^q}{q}\int_{\partial\Omega}|v_S|^q d\sigma\\
&\le \Lambda - \frac{C\lambda}{q}\left(\frac{2-q}{6-q}aS\right)^{q/4}.
\end{aligned}$$

For $\lambda > 0$, set $A \doteq C/q((2-q)/(6-q)aS)^{q/4}$. Let $-A\lambda \leq -D\lambda^{2/(2-q)}$, i.e.,

$$\begin{aligned} -A\lambda \leq -D\lambda^{2/(2-q)} &\iff A\lambda \geq D\lambda^{2/(2-q)} \\ &\iff \lambda^{q/(2-q)} \leq \frac{A}{D} \\ &\iff \lambda \leq \left(\frac{A}{D}\right)^{(2-q)/q}. \end{aligned}$$

Therefore, let $0 < \lambda < \lambda_0 = (A/D)^{(2-q)/q}$, we obtain that $\sup_{t\geq 0} I_\lambda(tv_S) < \Lambda - D\lambda^{2/(2-q)}$ holds for every $\lambda \in (0, \lambda_0)$. The proof is complete. $\square$

## 3. Proof of the main theorem.

*Proof of Theorem* 1.1. There exists a constant $\delta > 0$ such that $\Lambda - D\lambda^{2/(2-q)} > 0$ when $\lambda < \delta$. Set $\lambda_* = \min\{T_1, \delta, \lambda_0\}$. Then Lemmas 2.1–2.10 hold for all $0 < \lambda < \lambda_*$. By Lemma 2.6, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of $I_\lambda$. Obviously, $\{u_n\}$ is bounded in $H^1(\Omega)$, going if necessary to a subsequence, still denoted by $\{u_n\}$. There exists a $u_\lambda \in H^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } H^1(\Omega), \\ u_n \to u_\lambda & \text{strongly in } L^s(\Omega), \qquad 1 \leq s < 6, \\ u_n(x) \to u_\lambda(x) & \text{almost everywhere in } \Omega, \end{cases}$$

as $n \to \infty$.

Now, we shall prove that $u_\lambda$ is a positive ground state solution of (1.1). Indeed, by Lemma 2.6, for all $\varphi \in H^1(\Omega)$, we know that

$$\left(a+b\lim_{n\to\infty}\|u_n\|^2\right)\int_\Omega (\nabla u_\lambda\cdot\nabla\varphi+u_\lambda\varphi)\,dx-\int_\Omega u_\lambda^5\varphi\,dx-\lambda\int_{\partial\Omega} u_\lambda^{q-1}\varphi\,d\sigma = 0.$$

Set $\lim_{n\to\infty}\|u_n\| = l$. Then, we have

$$(a+bl^2)\int_\Omega (\nabla u_\lambda\cdot\nabla\varphi+u_\lambda\varphi)\,dx-\int_\Omega u_\lambda^5\varphi\,dx-\lambda\int_{\partial\Omega} u_\lambda^{q-1}\varphi\,d\sigma = 0. \tag{3.1}$$

Taking the test function $\varphi = u_\lambda$ in (3.1) implies that

$$(a+bl^2)\|u_\lambda\|^2 - \int_\Omega u_\lambda^6\,dx - \lambda\int_{\partial\Omega} u_\lambda^q\,d\sigma = 0. \tag{3.2}$$

The fact that $u_n \in \mathcal{N}_\lambda$ implies

$$(a + b\|u_n\|^2)\|u_n\|^2 - \int_\Omega u_n^6\,dx - \lambda \int_{\partial\Omega} u_n^q\,d\sigma = 0.$$

Since $\alpha_\lambda < 0 < \Lambda - D\lambda^{2/(2-q)}$, by Lemma 2.8, we have

$$(a + bl^2)l^2 - \int_\Omega u_\lambda^6\,dx - \lambda \int_{\partial\Omega} u_\lambda^q\,d\sigma = 0. \tag{3.3}$$

It follows from (3.2) and (3.3) that $\|u_\lambda\| = l$. This implies that $u_n \to u_\lambda$ in $H^1(\Omega)$, and $u_\lambda$ is a solution of (1.1). Furthermore, note that $u_\lambda \in \mathcal{N}_\lambda$ and $\alpha_\lambda < 0$ (by Lemma 2.3). Then, the following holds:

$$\begin{aligned}\left(\frac{1}{q} - \frac{1}{6}\right)\lambda \int_{\partial\Omega} u_\lambda^q\,d\sigma &= \frac{a}{3}\|u_\lambda\|^2 + \frac{b}{12}\|u_\lambda\|^4 - I_\lambda(u_\lambda)\\ &\geq \frac{a}{3}\|u_\lambda\|^2 + \frac{b}{12}\|u_\lambda\|^4 - \alpha_\lambda\\ &> 0,\end{aligned}$$

which implies that $u_\lambda \not\equiv 0$. Therefore, by the strong maximum principle, $u_\lambda > 0$ in $\Omega$. Moreover, Lemma 2.8 suggests that

$$\alpha_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda). \tag{3.4}$$

Next, we show that $u_\lambda \in \mathcal{N}_\lambda^+$ and $I_\lambda(u_\lambda) = \alpha_\lambda^+$. We claim that $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, assume that $u_\lambda \in \mathcal{N}_\lambda^-$ ($\mathcal{N}_\lambda^0 = \emptyset$ for $\lambda \in (0, T_1)$). By Lemma 2.1, there exist positive numbers $t^+ < t_{\max} < t^- = 1$ such that $t^+ u \in \mathcal{N}_\lambda^+$, $t^- u \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda < I_\lambda(t^+ u_\lambda) < I_\lambda(t^- u_\lambda) = I_\lambda(u_\lambda) = \alpha_\lambda,$$

a contradiction. Thus, $u_\lambda \in \mathcal{N}_\lambda^+$. From the definition of $\alpha_\lambda^+$, we obtain $\alpha_\lambda^+ \leq I_\lambda(u_\lambda)$; thus, from Lemma 2.3 and (3.4), the following holds:

$$I_\lambda(u_\lambda) = \alpha_\lambda^+ = \alpha_\lambda < 0.$$

Consequently, $u_\lambda$ is a positive ground state solution of (1.1).

In what follows, we shall verify that problem (1.1) has a second solution $v_\lambda$, and $v_\lambda \in \mathcal{N}_\lambda^-$. Since $I_\lambda$ is also coercive on $\mathcal{N}_\lambda^-$, applying Ekeland's variational principle to the minimization problem $\alpha_\lambda^- = \inf_{v\in\mathcal{N}_\lambda^-} I_\lambda(v)$ yields a minimizing sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ of $I_\lambda$, with the following properties:

(i) $I_\lambda(v_n) < \alpha_\lambda^- + 1/n$;

(ii) $I_\lambda(u) \geq I_\lambda(v_n) - \|u - v_n\|/n$, for all $u \in \mathcal{N}_\lambda^-$.

Since $\{v_n\}$ is bounded in $H^1(\Omega)$, passing to a subsequence, if necessary, there exists a $v_\lambda \in H^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v_\lambda & \text{weakly in } H^1(\Omega), \\ v_n \to v_\lambda & \text{strongly in } L^s(\Omega), \qquad 1 \leq s < 6, \\ v_n(x) \to v_\lambda(x) & \text{almost everywhere in } \Omega, \end{cases}$$

since $n \to \infty$. Similarly, we can prove $v_n \to v_\lambda$ in $H^1(\Omega)$, and $v_\lambda$ is a nonnegative solution of (1.1).

Based on $v_n \in \mathcal{N}_\lambda^-$, the following holds:

$$\begin{aligned} a(2-q)\|v_n\|^2 &\leq (6-q)\int_\Omega v_n^6\,dx - b(4-q)\|v_n\|^4 \\ &\leq (6-q)\int_\Omega |v_n|^6 dx \\ &< (6-q)S^{-3}\|v_n\|^6 \end{aligned}$$

such that

$$\|v_n\| > \left(\frac{a(2-q)S^3}{(6-q)}\right)^{1/4} \quad \text{for all } v_n \in \mathcal{N}_\lambda^-. \tag{3.5}$$

Note that $v_n \to v_\lambda$ in $H^1(\Omega)$, along with (3.5), implies that $v_\lambda \not\equiv 0$. Therefore, from the strong maximum principle, $v_\lambda > 0$ in $\Omega$.

Next, we concentrate on proving that $v_\lambda \in \mathcal{N}_\lambda^-$. It suffices to prove that $\mathcal{N}_\lambda^-$ is closed. Indeed, by Lemmas 2.8 and 2.10, for $\{v_n\} \subset \mathcal{N}_\lambda^-$, we have

$$\lim_{n\to\infty}\int_\Omega v_n^6\,dx = \int_\Omega v_\lambda^6\,dx.$$

From the definition of $\mathcal{N}_\lambda^-$, the following holds:

$$(2-q)a\|v_n\|^2 + (4-q)b\|v_n\|^4 - (6-q)\int_\Omega v_n^6\,dx < 0;$$

thus,

$$(2-q)a\|v_\lambda\|^2 + (4-q)b\|v_\lambda\|^4 - (6-q)\int_\Omega v_\lambda^6\,dx \leq 0,$$

which implies that $v_\lambda \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$. If $\mathcal{N}_\lambda^-$ is not closed, then we have $v_\lambda \in \mathcal{N}_\lambda^0$ and, by Lemma 2.1, it follows that $v_\lambda = 0$. This contradicts $v_\lambda > 0$. Consequently, $v_\lambda \in \mathcal{N}_\lambda^-$. Note that $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, i.e., $u_\lambda$ and $v_\lambda$ are different positive solutions of (1.1). $\square$

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