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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF THE HÉNON EQUATION

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ABSTRACT. We investigate the radial positive solutions of the Hénon equation. It is known that this equation has three different types of radial solutions: the M-solutions (singular at $r = 0$), the E-solutions (regular at $r = 0$) and the F-solutions (whose existence begins away from $r = 0$). For the M-solutions and E-solutions, by virtue of some prior estimates, we adopt a circulating iterative method, step-by-step, to derive their precise asymptotic expansions. In particular, the M-solution has an extremely plentiful structure, and its asymptotic expansions are more complicated. In contrast to previous research **[2, 9]**, our results are more accurate.

1. Introduction. In this paper, we consider the radial positive solutions to the following semilinear elliptic equation

$$\Delta\phi + |x|^\sigma \phi^p = 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $p > 1,\ \sigma > -2,\ N \geq 3,\ \Delta = \Sigma_{i=1}^N \partial^2/\partial x_i^2$ and $|x| = (\Sigma_{i=1}^N x_i^2)^{1/2}$. Equation (1.1) arises both in physics and geometry and is a model of semilinear problem. Since the radial positive solutions are of particular interest, we mainly study those of equation (1.1), which fulfill

$$\frac{1}{r^{N-1}}(r^{N-1}\phi')' + r^\sigma \phi^p = 0, \quad p > 1,\ \sigma > -2,\ N \geq 3, \tag{1.2}$$

where $r = |x|$ and $'$ denotes differentiation with respect to the variable r.

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In the past three decades, the positive solution of (1.1) has been investigated by many authors. For instance, Gidas and Spruck **[9]** showed that, if $1 < p < (N+\sigma)/(N-2)$, any positive solution of (1.1) with $0 < |x| \le R$ either has a removable singularity at $x = 0$, or there exist positive constants c_1, c_2 such that

$$\frac{c_1}{|x|^{N-2}} \le \phi(x) \le \frac{c_2}{|x|^{N-2}} \quad \text{near } x = 0.$$

If $(N+\sigma)/(N-2) < p < (N+2)/(N-2)$, but $p \neq (N+2+2\sigma)/(N-2)$, every positive solution of (1.1) has either a removable singularity at $x = 0$, or

$$|x|^{(\sigma+2)/(p-1)}\phi(x) \longrightarrow c_3 \quad \text{as } |x| \longrightarrow 0$$

for some constant $c_3 > 0$. For the case of $p > (N+\sigma)/(N-2)$, Bidaut-Véron and Véron **[4]** demonstrated that, if there is some constant $c_4 > 0$ such that $|x|^{(\sigma+2)/(p-1)}\phi(x) \le c_4$ in $0 < |x| \le R$, then the positive solution of (1.1) either has a removable singularity at $x = 0$, or

$$|x|^{(\sigma+2)/(p-1)}\phi(x) \longrightarrow \omega(x) \quad \text{as } |x| \to 0$$
$$\text{uniformly in } \theta = \frac{x}{|x|} \in S^{N-1},$$

where $\omega(x)$ solves

$$\Delta_{S^{N-1}}\omega - \frac{\sigma+2}{p-1}\left(N - \frac{2p+\sigma}{p-1}\right)\omega + \omega^p = 0 \quad \text{on } S^{N-1}.$$

In the exterior region $|x| > R$, under the same conditions, the positive solution of (1.1) either satisfies

$$|x|^{N-2}\phi(x) \longrightarrow c_5 \quad \text{as } |x| \longrightarrow \infty \quad \text{for some } c_5 > 0,$$

or

$$|x|^{(\sigma+2)/(p-1)}\phi(x) \longrightarrow \omega(x) \quad \text{as } |x| \longrightarrow \infty$$
$$\text{uniformly in } \theta = \frac{x}{|x|} \in S^{N-1}.$$

Recently, Bidaut-Véron and Véron's results have been generalized to the more general domain for p by Dance, et al., **[5]**; more relevant works regarding equation (1.1) can be found in **[8, 23]**.

It is well known (see, e.g., **[2]**) that the solutions of equation (1.2) with $N = 3$ can be divided into three different types: the M-solutions (singular at $r = 0$), the E-solutions (regular at $r = 0$) and the F-solutions (whose existence begins away from $r = 0$). The existence and uniqueness of the E and F-solutions has been established by Ni and Yotsutani **[22]** and Kwong and Li **[12]** in an annular region, respectively. The asymptotic behavior of the M-solutions near the origin was studied by Gidas and Spruck **[9]** and Batt and Pfaffelmoser **[2]** with $N = 3$. In **[27]**, Yanagida investigated the positive radial E-solutions of the Matukuma equation

$$\begin{cases} \frac{1}{r^{N-1}}(r^{N-1}\phi')' + \frac{1}{1+r^2}\phi^p = 0, \\ \phi(0) = \alpha > 0 \quad \phi'(0) = 0. \end{cases} \tag{1.3}$$

His result indicates that, when $1 < p < (N+2)/(N-2)$, there exists a unique $\alpha^* > 0$ such that, if $\alpha > \alpha^*$, (1.3) has finite zero; if $\alpha = \alpha^*$, (1.3) has a finite total mass solution; if $\alpha < \alpha^*$, (1.3) has an infinite total mass solution. Recently, by virtue of Yanagida's method, Sha and Li **[24]** extended the above results to the radial F-solutions of the generalized Matukuma equation. Deng, et al., **[6]** considered the stability of the radial solutions of

$$\begin{cases} \frac{\partial \phi}{\partial t} = \Delta\phi + K(|x|)\phi^p & (x,t) \in \mathbb{R}^N \times (0,T), \\ \phi(x,0) = \varphi(|x|) & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $p > 1$, $T > 0$, $\varphi \neq 0$ is a nonnegative continuous function. If K satisfies the following conditions:

$$\lim_{r\to 0} r^{-\sigma}K(r) = K_0 > 0, \qquad \lim_{r\to\infty} r^{-\sigma}K(r) = K_\infty > 0,$$
$$\sigma > -2, \quad (r^{-\sigma}K(r))' \leq 0,$$

then the positive radial solution of (1.4) is stable with respect to some defined norm provided $p > p_c$, where

$$p_c = \begin{cases} \frac{(N-2)^2 - 2(\sigma+2)(N+\sigma) + 2(\sigma+2)\sqrt{(N+\sigma)^2-(N-2)^2}}{(N-2)(N-10-4\sigma)} & N > 10 + 4\sigma, \\ +\infty & 3 \leq N \leq 10 + 4\sigma. \end{cases}$$

For more research regarding the positive radial solutions of general semilinear equations, the interested reader is referred to **[7, 11, 14, 15, 16, 17, 18, 19, 20, 21, 26, 28]**.

In **[1]**, Batt and Li developed a comprehensive theory of positive radial solutions $\phi(r)$ of the Matukuma equation

$$\Delta\phi + \frac{|x|^{\lambda-2}}{(1+|x|^2)^{\lambda/2}}\phi^p = 0, \quad p > 1,\ \lambda > 0 \tag{1.5}$$

in $\mathbb{R}^3$. Their results reveal that the three different types of solutions known for the Hénon equation (1.2) with $N = 3$ also exist for the Matukuma equation (1.5). Moreover, with the aid of an asymptotic expansion method **[13]**, they obtained accurate asymptotic expansions for the M- and E-solutions. Recently, Wang, et al., **[25]** investigated the M-solution of the Matukuma equation (1.5) in higher-dimensional space ($N > 3$) and obtained the precise asymptotic expansion of the M-solutions. In this paper, by employing the methods and ideas of **[1, 13, 25]** used to discuss positive radial solutions of the Matukuma equation (1.5), we systematically restudy the positive radial solutions of (1.1). We pay attention to asymptotic expansions of the M- and E-solutions near the origin. In comparison with the preceding results about the asymptotic expansions of the M- and E-solutions of (1.1) **[2, 9]**, our outcome appears to be more precise.

The rest of this paper is organized as follows. In Section 2, we include some preliminaries which shall be used throughout the entire paper. In Section 3, we present the E-solutions. Section 4 is devoted to the asymptotic expansion of the M-solutions. In Section 5, we establish a uniqueness theorem of the F-solutions.

2. Preliminaries.

2.1. Classification of positive solutions. Let K be a positive function in $C^1(\mathbb{R}^+)$ with $r^2K(r)$ bounded away from zero for $r \to \infty$ and $p > 1$. Assume that $\phi : (R_-, R) \to (0, \infty)$ is a maximal radial solution of

$$\frac{1}{r^{N-1}}(r^{N-1}\phi')' = -K(r)\phi^p, \quad N \geq 3, \tag{2.1}$$

where $0 \leq R_- < R \leq \infty$. Let $r_0 \in (R_-, R)$ and

$$H(r) := \phi'(r_0)r_0^{N-1} - \int_{r_0}^{r} s^{N-1}K(s)\phi^p(s)\,ds \quad \text{in } (R_-, R). \tag{2.2}$$

Then, $\phi'(r) = H(r)/r^{N-1}$ and $H'(r) = -r^{N-1}K(r)\phi^p(r) < 0$ in (R_-, R). Hence,

$$H_0 := \lim_{r\to R_-} H(r) \in (-\infty, +\infty]$$

exist. Indeed, if $H_0 > 0$, then $R_- > 0$. Furthermore, there exists some $R_0 \in (R_-, R)$ such that $H(R_0) = \phi'(R_0) = 0$. If $H_0 \le 0$, then $R_- = 0$. Consequently, $R_- = 0$, $\phi' < 0$ in $(0, R)$, and the limit $\lim_{r\to 0}\phi(r) \in (0,\infty]$ exists. In this case, we define $R_0 := 0$, and have

$$R_0 := \inf\{r \in (R_-, R) \mid \phi'(r) < 0\}$$

for all solutions, see [**25,** subsection 2.1] for more details.

The solutions are classified as follows:

$$H_0 > 0 \Longleftrightarrow R_0 > R_- > 0,$$

where we call ϕ an F-solution. Moreover,

$$H_0 \le 0 \Longleftrightarrow R_0 = R_- = 0:$$

if $\lim_{r\to 0}\phi(r) < \infty$, we call ϕ an E-solution, and, if $\lim_{r\to 0}\phi(r) = \infty$, we call ϕ an M-solution.

For the sake of convenience, we give a lemma which can be used to demonstrate the existence and uniqueness of E-solutions. The proof of Lemma 2.1 with dimension $N \ge 3$ can be found in [**22**].

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ and $f(r,\phi) : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ satisfy the following conditions*:

(i) $f(r,\phi) \in C^1((0,\infty)\times\mathbb{R})$;

(ii) $rf(r,\alpha) \in L^1_{\mathrm{loc}}[0,\infty)$;

(iii) *there exist a constant $\delta > 0$ and a function*

$$L_\alpha : (0,\delta) \to [0,\infty] \quad \textit{with } rL_\alpha(r) \in L^1[0,\delta]$$

such that, for every $r \in (0,\delta)$ and $\phi_1, \phi_2 \in [\alpha-\delta, \alpha+\delta]$,

$$|f(r,\phi_1) - f(r,\phi_2)| \le L_\alpha(r)|\phi_1 - \phi_2|.$$

Then, the initial value problem

$$\frac{1}{r^{N-1}}(r^{N-1}\phi')' = f(r,\phi), \quad \phi(0) = \alpha,$$

admits a unique solution.

2.2. Transformation to Lotka-Volterra systems. In this section, we consider solutions ϕ of (2.1) in their intervals $J_\phi := (R_0, R)$. Define

$$u(t) := rK(r)\frac{\phi^p(r)}{-\phi'(r)}, \qquad v(t) := r\frac{-\phi'(r)}{\phi(r)}, \qquad r := e^t. \tag{2.3}$$

Then, $\varphi := (u, v) : I_\varphi \to \mathbb{R}^+ \times \mathbb{R}^+, I_\varphi := \ln J_\phi$ is a maximal solution of the system

$$\begin{cases} \dot{u} = u\left(N + r\frac{K'(r)}{K(r)} - u - pv\right), \\ \dot{v} = v(-N + 2 + u + v), \end{cases} \tag{2.4}$$

where $\cdot$ denotes differentiation with regard to the variable t. $\mathbb{R}^+ \times \mathbb{R}^+$ is an invariant set of this system (the positive u- and v- axes are invariant). The inverse is

$$\phi(r) = \left[\frac{u(\ln r)v(\ln r)}{r^2 K(r)}\right]^{1/(p-1)}. \tag{2.5}$$

In particular, for $K(r) = r^\sigma$, we get

$$\begin{cases} \dot{u} = u(N + \sigma - u - pv), \\ \dot{v} = v(-N + 2 + u + v). \end{cases} \tag{2.6}$$

In the sequel, unless otherwise stated, $\phi(r)$ always represents a solution of (2.1) in (R_0, R) with $0 \le R_0 < R \le \infty$, and $\varphi = (u, v)$ is the associated solution of (2.6) in (T_0, T) with $-\infty \le T_0 = \ln R_0 < T = \ln R \le \infty$. In addition, it is always assumed that $p > 1$, $\sigma > -2$ and $N \ge 3$.

2.3. Linearization of autonomous system (2.6). Clearly, system (2.6) has the following stationary points: $P_1 = (0, 0)$, $P_2 = (0, N - 2)$, $P_3 = (N + \sigma, 0)$; for $p > (N + \sigma)/(N - 2)$,

$$P_4 = \left(\frac{(N-2)p - N - \sigma}{p - 1}, \frac{\sigma + 2}{p - 1}\right),$$

where $\sigma > -2$ and $N \ge 3$. Denote the stationary point by $P = (u^*, v^*)$. Then, the Jacobian matrix of system (2.6) is

$$\mathrm{A} := \begin{pmatrix} N + \sigma - 2u^* - pv^* & -pu^* \\ v^* & u^* + 2v^* - N + 2 \end{pmatrix}. \tag{2.7}$$

For P_1, we have eigenvalues $\lambda_1 = N+\sigma > 0$, $\lambda_2 = -N+2 < 0$ and corresponding eigenvectors $\xi_1 = (1,0)$, $\xi_2 = (0,1)$, and P_1 is a saddle.

For P_2, we have $\lambda_1 = N+\sigma-(N-2)p$, $\lambda_2 = N-2$ and $\xi_1 = (N+\sigma-(N-2)(p+1), N-2)$, $\xi_2 = (0,1)$. If $p < (N+\sigma)/(N-2)$, then $\lambda_1 > 0$, P_2 is an unstable improper node; if $p < (\sigma+2)/(N-2)$, then $0 < \lambda_2 < \lambda_1$, P_2 is a 2-tangential improper node; if $p = (\sigma+2)/(N-2)$, then $0 < \lambda_2 = \lambda_1$, P_2 is a 1-tangential node; if $(\sigma+2)/(N-2) < p < (N+\sigma)/(N-2)$, then $0 < \lambda_1 < \lambda_2$, P_2 is a 2-tangential improper node; if $p = (N+\sigma)/(N-2)$, then $\lambda_1 = 0$, P_2 is an unstable 2-tangential node; if $p > (N+\sigma)/(N-2)$, then $\lambda_1 < 0$, P_2 is a saddle.

For P_3, we have $\lambda_1 = -(N+\sigma) < 0$, $\lambda_2 = \sigma+2 > 0$ and $\xi_1 = (1,0)$, $\xi_2 = (-(N+\sigma)p/(N+2+2\sigma), 1)$, and P_3 is a saddle.

If $p > (N+\sigma)/(N-2)$,

$$P_4 = \left(\frac{(N-2)p-N-\sigma}{p-1}, \frac{\sigma+2}{p-1}\right) := (u_4^*, v_4^*),$$

we have $\lambda_{1,2} = v_4^* - (N-2)/2 \pm 1/2\sqrt{\Lambda(v_4^*)}$, where $\Lambda(\kappa) = 4p\kappa^2 - 4(N-2)p\kappa + (N-2)^2$ with two distinct roots $\kappa_{1,2} = (N-2)/2(1 \pm \sqrt{1-1/p})$. If

$$p < \frac{N+2+2\sigma}{N-2} \Longrightarrow v_4^* > \frac{N-2}{2},$$

then $\mathrm{Re}\lambda_j > 0$, $j = 1,2$, and P_4 is unstable. In particular, if $\kappa_1 \le v_4^* < N-2$, P_4 is an improper node, and $(N-2)/2 < v_4^* < \kappa_1$ is a spiral point. If

$$p = \frac{N+2+2\sigma}{N-2} \Longrightarrow v_4^* = \frac{N-2}{2},$$

then $\lambda_{1,2} = \pm(N-2)/2\sqrt{p-1}i$, and P_4 is a center. If

$$p > \frac{N+2+2\sigma}{N-2} \Longrightarrow 0 < v_4^* < \frac{N-2}{2},$$

then $\mathrm{Re}\lambda_j < 0$, $j = 1,2$, and P_4 is stable. In particular, for $\kappa_2 \le v_4^* < (N-2)/2$, P_4 is a spiral point, and, if $0 < v_4^* < \kappa_2, P_4$ is an improper node.

3. E-solutions. The following theorem characterizes E-solutions, which also imply their existence.

Theorem 3.1. *Assume that $\sigma > -2$ and $p > 1$. Then the following conclusions are equivalent*:

(i) *$\phi(r)$ is an* E-*solution.*

(ii) *There exists some constant $\alpha > 0$ such that*

$$\begin{cases} \phi(r) = \alpha - \frac{1}{N-2}\int_0^r \left[1 - \left(\frac{s}{r}\right)^{N-2}\right] s^{\sigma+1}\phi^p(s)\,ds, \\ \phi'(r) = -\frac{1}{r^{N-1}}\int_0^r s^{N+\sigma-1}\phi^p(s)\,ds \quad \text{for any } r > 0. \end{cases} \tag{3.1}$$

(iii) *There exists some constant $\alpha > 0$ such that*

$$\begin{cases} \phi(r) = \alpha - \frac{\alpha^p}{(N+\sigma)(\sigma+2)} r^{\sigma+2} \\ \qquad + \frac{p\alpha^{2p-1}}{2(N+\sigma)(\sigma+2)^2(N+2+2\sigma)} r^{2\sigma+4} + o(r^{2\sigma+4}), \\ \phi'(r) = -\frac{\alpha^p}{N+\sigma} r^{\sigma+1} + \frac{p\alpha^{2p-1}}{(N+\sigma)(\sigma+2)(N+2+2\sigma)} r^{2\sigma+3} \\ \qquad + o(r^{2\sigma+3}) \qquad r \to 0. \end{cases}$$

(iv) *There exists some constant $\alpha > 0$ such that*

$$\begin{cases} u(t) = (N+\sigma)\left[1 - \frac{p\alpha^{p-1}}{(N+\sigma)(N+2+2\sigma)} e^{(\sigma+2)t} \right. \\ \qquad \left. + o(e^{(\sigma+2)t})\right], \\ v(t) = \frac{\alpha^{p-1}e^{(\sigma+2)t}}{N+\sigma}\left[1 + \frac{[N+2+2\sigma-(N+\sigma)p]\alpha^{p-1}}{(N+\sigma)(\sigma+2)(N+2+2\sigma)} e^{(\sigma+2)t} \right. \\ \qquad \left. + o(e^{(\sigma+2)t})\right] \qquad t \to -\infty. \end{cases}$$

(v) *$\varphi(t) \to P_3$, $t \to -\infty$.*

Proof.

(i) $\to$ (ii). Follows from [**22,** Proposition 4.1].

(ii) $\to$ (iii). The existence and uniqueness of E-solutions can be established by application of Lemma 2.1. From the second equality of

(3.1), we obtain

$$\begin{aligned}\phi'(r) &= -\frac{1}{r^{N-1}}\int_0^r s^{N+\sigma-1}\phi^p(s)\,ds\\ &= -\frac{1}{r^{N-1}}\int_0^r s^{N+\sigma-1}\alpha^p[1+o(1)]\,ds\\ &= -\frac{\alpha^p}{N+\sigma}r^{\sigma+1}[1+o(1)].\end{aligned}$$

Integration yields

$$\phi(r) = \alpha - \frac{\alpha^p}{(N+\sigma)(\sigma+2)}r^{\sigma+2} + o(r^{\sigma+2}), \quad r \to 0.$$

We will use a stepwise method to improve the asymptotic expansions. By virtue of (3.1), again, we have

$$\begin{aligned}r^{N-1}\phi'(r) &= -\int_0^r s^{N+\sigma-1}\phi^p(s)\,ds\\ &= -\int_0^r s^{N+\sigma-1}\left[\alpha - \frac{\alpha^p}{(N+\sigma)(\sigma+2)}s^{\sigma+2} + o(s^{\sigma+2})\right]^p ds\\ &= -\alpha^p\int_0^r s^{N+\sigma-1}\left[1 - \frac{p\alpha^{p-1}}{(N+\sigma)(\sigma+2)}s^{\sigma+2} + o(s^{\sigma+2})\right] ds\\ &= -\alpha^p\left[\frac{1}{N+\sigma}r^{N+\sigma} - \frac{p\alpha^{p-1}}{(N+\sigma)(\sigma+2)(N+2+2\sigma)}r^{N+2+2\sigma}\right.\\ &\qquad \left. + o(r^{N+2+2\sigma})\right]\\ \phi'(r) &= -\frac{\alpha^p}{N+\sigma}r^{\sigma+1} + \frac{p\alpha^{2p-1}}{(N+\sigma)(\sigma+2)(N+2+2\sigma)}r^{2\sigma+3} + o(r^{2\sigma+3}).\end{aligned}$$

Therefore,

$$\begin{aligned}\phi(r) = \alpha &- \frac{\alpha^p}{(N+\sigma)(\sigma+2)}r^{\sigma+2}\\ &+ \frac{p\alpha^{2p-1}}{2(N+\sigma)(\sigma+2)^2(N+2+2\sigma)}r^{2\sigma+4} + o(r^{2\sigma+4}).\end{aligned}$$

(iii) $\to$ (iv). Recalling the previous transformation (2.3), we get

$$
\begin{aligned}
u(t) &= r^{\sigma+1}\tfrac{\phi^p(r)}{-\phi'(r)}\\
&= r^{\sigma+1}\tfrac{\alpha^p[1-\alpha^{p-1}/((N+\sigma)(\sigma+2))r^{\sigma+2}}{[\alpha^p/(N+\sigma)]r^{\sigma+1}[1-p\alpha^{p-1}/[(\sigma+2)(N+2+2\sigma)]r^{\sigma+2}+o(r^{\sigma+2})]}\\
&\quad + \tfrac{p\alpha^{2p-2}/(2(N+\sigma)(\sigma+2)^2(N+2+2\sigma))r^{2\sigma+4}+o(r^{2\sigma+4})]^p}{[\alpha^p/(N+\sigma)]r^{\sigma+1}[1-p\alpha^{p-1}/[(\sigma+2)(N+2+2\sigma)]r^{\sigma+2}+o(r^{\sigma+2})]}\\
&= (N+\sigma)\Big[1-\tfrac{p\alpha^{p-1}r^{\sigma+2}}{(N+\sigma)(\sigma+2)}\\
&\qquad + \tfrac{[p^2(2N+2+3\sigma)-p(N+2+2\sigma)]\alpha^{2p-2}r^{2\sigma+4}}{2(N+\sigma)^2(\sigma+2)^2(N+2+2\sigma)}+o\big(r^{2\sigma+4}\big)\Big]\\
&\cdot\Big[1+\tfrac{p\alpha^{p-1}}{(\sigma+2)(N+2+2\sigma)}r^{\sigma+2}+o(r^{\sigma+2})\Big]\\
&= (N+\sigma)\Big[1-\tfrac{p\alpha^{p-1}}{(N+\sigma)(N+2+2\sigma)}r^{\sigma+2}+o(r^{\sigma+2})\Big].
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
v(t) &= r\tfrac{-\phi'(r)}{\phi(r)}\\
&= r\tfrac{\alpha^p r^{\sigma+1}/(N+\sigma)}{\alpha[1-\alpha^{p-1}/((N+\sigma)(\sigma+2))r^{\sigma+2}+p\alpha^{2p-2}/[2(N+\sigma)(\sigma+2)^2(N+2+2\sigma)]r^{2\sigma+4}+o(r^{2\sigma+4})]}\\
&\cdot\tfrac{[1-p\alpha^{p-1}/[(\sigma+2)(N+2+2\sigma)]r^{\sigma+2}+o(r^{\sigma+2})]}{\alpha[1-\alpha^{p-1}/((N+\sigma)(\sigma+2))r^{\sigma+2}+p\alpha^{2p-2}/[2(N+\sigma)(\sigma+2)^2(N+2+2\sigma)]r^{2\sigma+4}+o(r^{2\sigma+4})]}\\
&= \tfrac{\alpha^{p-1}}{N+\sigma}r^{\sigma+2}[1-p\alpha^{p-1}/[(\sigma+2)(N+2+2\sigma)]r^{\sigma+2}+o(r^{\sigma+2})]\\
&\cdot[1+\alpha^{p-1}/[(N+\sigma)(\sigma+2)]r^{\sigma+2}+o(r^{\sigma+2})]\\
&= \tfrac{\alpha^{p-1}}{N+\sigma}r^{\sigma+2}[1+[N+2+2\sigma-(N+\sigma)p]\alpha^{p-1}/\\
&\qquad [(N+\sigma)(\sigma+2)(N+2+2\sigma)]r^{\sigma+2}+o(r^{\sigma+2})].
\end{aligned}
$$

(iv) $\to$ (v) is trivial.

(v) $\to$ (i). Rewrite the equation of v in (2.6), we have

$$\dot{v} = v(-N+2+u)+v^2.$$

Let $y=v^{-1}$. Then, the above equation reduces to

$$\dot{y} = y(N-2-y)-1.$$

By some simple computations, we find

$$y(t) = y(t_0)e^{\Gamma(t)} - e^{\Gamma(t)}\int_{t_0}^{t} e^{-\Gamma(s)}ds$$

with some $t_0 \in I_\varphi$ and

$$\Gamma(t) = \int_{t_0}^{t} (N - 2 - u)\, ds.$$

Since $u(t) \to N + \sigma$ as $t \to -\infty$, we see that $\Gamma(t) = O(-(\sigma + 2)t)$, which also implies that $y(t) = O(e^{-(\sigma+2)t})$ as $t \to -\infty$, in other words, we have $v(t) = O(e^{(\sigma+2)t})$.

By the inverse transformation (2.5), we obtain

$$\begin{aligned}\phi^{p-1}(r) = \frac{u(\ln r)v(\ln r)}{r^2 K(r)} &= \frac{u(\ln r)[\widetilde{C} r^{\sigma+2} + o(r^{\sigma+2})]}{r^{\sigma+2}} \\ &\longrightarrow (N+\sigma)\widetilde{C} := \alpha^{p-1}, \quad r \to 0,\end{aligned}$$

where $\widetilde{C}$ is some positive constant independent of t. The proof is complete. □

4. M-solutions. In this section, let $p > 1$, $\sigma > -2$, $N \geq 3$. We consider solutions ϕ in $(0, R)$ with $0 < R \leq \infty$, and corresponding φ in $(-\infty, T)$, $T = \ln R$. In order to obtain more precise asymptotic expansions of the M-solutions, we investigate by distinguishing among the four different cases:

(i) $1 < p < (N + \sigma)/(N - 2)$ (three subcases with $\sigma > N - 4$: $1 < p < (\sigma + 2)/(N - 2)$, $p = (\sigma + 2)/(N - 2)$, $(\sigma + 2)/(N - 2) < p < (N+\sigma)/(N-2)$), the case $-2 < \sigma \leq N-4$ can be found in Remark 4.1;

(ii) $p = (N + \sigma)/(N - 2)$;

(iii) $p > (N + \sigma)/(N - 2)$; however, $p \neq (N + 2 + 2\sigma)/(N - 2)$;

(iv) $p = (N + 2 + 2\sigma)/(N - 2)$.

4.1. The case $1 < p < (N + \sigma)/(N - 2)$.

Theorem 4.1. *Suppose that* $1 < p < (N + \sigma)/(N - 2)$. *Then, the following conclusions are equivalent*:

(i) $\phi(r)$ *is an* M-*solution.*

(ii) *There exists some constant* $c > 0$ *such that*

$$\begin{cases} \phi(r) = \frac{c}{r^{N-2}}[1+o(1)], \\ \phi'(r) = -\frac{(N-2)c}{r^{N-1}}[1+o(1)] \quad r \to 0. \end{cases}$$

(iii) *There exists some constant* $c > 0$ *such that*

$$\begin{cases} u(t) = (N-2)c^{p-1}e^{[N+\sigma-(N-2)p]t}[1+o(1)], \\ v(t) = (N-2)[1+o(1)] \qquad\qquad t \to -\infty. \end{cases}$$

(iv) $\varphi(t) \to P_2$, $t \to -\infty$.

In addition, $\phi(r)$ *satisfies*

$$r^{N-1}\phi'(r) = -(N-2)c - \int_0^r s^{N-1}K(s)\phi^p(s)\,ds, \tag{4.1}$$

where $c > 0$ *is uniquely determined.*

Proof.

(i) $\to$ (ii). This conclusion can be established by the method of **[2, 3]**.

(ii) $\to$ (iii), (iii) $\to$ (iv). Trivial.

(iv) $\to$ (i). $\phi(r)$ must be an M- or E-solution. For the latter case, $\varphi(t) \to P_3$ by Theorem 3.1, which contradicts (iv).

Finally, (4.1) follows from integration by parts. □

In the sequel, we will show that the M-solution $\phi(r)$ has a splitting form:

$$\phi = S + \Theta, \tag{4.2}$$

where S is a singular term of the form $S = (c/r^{N-2})P(r)$ with an elementary, explicitly given function P of r with $P(r) = 1 + o(1)$, $r \to 0$, whereas Θ is a regular solution of the initial value problem

(4.3)
$$\begin{cases} (r^{N-1}\Theta')'/r^{N-1} = -K(r)(\Theta+S)^p - (r^{N-1}S')'/r^{N-1} \quad 0 < r < R, \\ \Theta(0) = \beta \in \mathbb{R}. \end{cases}$$

In terms of P, we have
(4.4)

$$\begin{aligned}\frac{1}{r^{N-1}}(r^{N-1}\Theta')' &= -K(r)\left(\frac{c}{r^{N-2}}P+\Theta\right)^p - \frac{c}{r^{N-2}}P'' + \frac{(N-3)c}{r^{N-1}}P' \\ &= -K(r)\frac{c^p}{r^{(N-2)p}}P^p\left[\left(1+\frac{r^{N-2}}{cP}\Theta\right)^p - 1\right] \\ &\quad + \left[-\frac{c}{r^{N-2}}P'' + \frac{(N-3)c}{r^{N-1}}P' - c^p r^{\sigma-(N-2)p}P^p\right] \\ &=: f_1(r,\Theta) + f_2(r).\end{aligned}$$

Since

$$K(r)\frac{c^p}{r^{(N-2)p}}P^p(r) = O(r^{\sigma-(N-2)p})$$

and

$$\left(1+\frac{r^{N-2}}{cP}\Theta\right)^p - 1 = O(r^{N-2}), \quad r \to 0,$$

we see that $f_1(r,\Theta)$ satisfies the hypotheses of Lemma 2.1. Hence, it suffices to verify that $f_2(r)$ also fulfills the assumptions of Lemma 2.1. Then, we can conclude that (4.3) has a unique solution. In the following three subsections, we shall divide into three subcases: $1 < p < (\sigma + 2)/(N-2), p = (\sigma+2)/(N-2)$ and $(\sigma+2)/(N-2) < p < (N+\sigma)/(N-2)$ with $\sigma > N-4$ to derive the different forms of S and asymptotic expansions for Θ.

4.1.1. The case $1 < p < (\sigma + 2)/(N - 2)$.

Theorem 4.2. *Assume that* $1 < p < (\sigma + 2)/(N - 2)$. *Then*:

(i) *every* M-*solution* $\phi(r)$ *has the form* $\phi = S + \Theta$, *where*

$$S(r) = \frac{c}{r^{N-2}},$$

and Θ *solves the initial problem* (4.3). *Moreover,*

$$\begin{aligned}\Theta(r) = \beta &- \frac{c^p r^{\sigma+2-(N-2)p}}{[N+\sigma-(N-2)p][\sigma+2-(N-2)p]} \\ &- \frac{p\beta c^{p-1} r^{N+\sigma-(N-2)p}}{[2N+\sigma-(N-2)p-2][N+\sigma-(N-2)p]}\end{aligned}$$

$$+ o(r^{N+\sigma-(N-2)p}), \quad r \to 0,$$

for some uniquely defined constants $c > 0$, $\beta \in \mathbb{R}$.

(ii) *Conversely, given any* $c > 0$ *and* $\beta \in \mathbb{R}$, *there exists a unique solution* Θ *of* (4.3) *with* $S(r) = c/r^{N-2}$, *and* $\phi = S + \Theta$ *is an* M-*solution. In addition,*

(4.5)

$$\begin{cases} \Theta(r) = \beta - \frac{1}{N-2}\int_0^r \left[1 - \left(\frac{s}{r}\right)^{N-2}\right] sK(s)[\frac{c}{s^{N-2}} + \Theta(s)]^p\, ds \quad 0 < r < R, \\ u(t) = \frac{c^{p-1}e^{[N+\sigma-(N-2)p]t}}{N-2}\Big[1 + \frac{p\beta}{c}e^{(N-2)t} - \frac{(\sigma+2)c^{p-1}e^{[N+\sigma-(N-2)p]t}}{(N-2)[\sigma+2-(N-2)p][N+\sigma-(N-2)p]} \\ \qquad + \frac{p(p-1)\beta^2}{2c^2}e^{(2N-4)t} + o(e^{\max\{N+\sigma-(N-2)p, 2N-4\}t})\Big], \\ v(t) = N - 2 - \frac{(N-2)\beta}{c}e^{(N-2)t} + \frac{c^{p-1}}{\sigma+2-(N-2)p}e^{[N+\sigma-(N-2)p]t} \\ \qquad + \frac{(N-2)\beta^2}{c^2}e^{(2N-4)t} + o\left(e^{\max\{N+\sigma-(N-2)p, 2N-4\}t}\right) \quad t \to -\infty. \end{cases}$$

Proof.

(i) Let $c > 0$ be determined by Theorem 4.1. Since $1 < p < (\sigma+2)/(N-2)$, we see that

$$\frac{1}{r^{N-1}}\int_0^r s^{N-1}K(s)\phi^p(s)\, ds = O(r^{\sigma+1-(N-2)p})$$

is integrable at $r = 0$. It follows from (4.1) with $r_0 \in (0, R)$ that

$$\begin{aligned} \phi(r) &= \frac{c}{r^{N-2}} - \frac{c}{r_0^{N-2}} + \phi(r_0) - \int_{r_0}^r \frac{dt}{t^{N-1}}\int_0^t s^{N-1}K(s)\phi^p(s)\, ds, \\ &= \frac{c}{r^{N-2}} + \beta_0 - \int_{r_0}^r \frac{dt}{t^{N-1}}\int_0^t s^{N-1}K(s)\phi^p(s)\, ds, \\ &= \frac{c}{r^{N-2}} + \Theta(r), \quad 0 < r < R. \end{aligned}$$

Let

$$\beta = \beta_0 - \int_{r_0}^0 \frac{dt}{t^{N-1}}\int_0^t s^{N-1}K(s)\phi^p(s)\, ds$$

and $S(r) = c/r^{N-2}$. It is not difficult to see that Θ satisfies (4.2) and (4.3) with $(r^{N-1}S')' = 0$. Integrating by parts, we can obtain the first part of (4.5). Any equation

$$\frac{c_1}{r^{N-2}} + \Theta_1(r) = \frac{c_2}{r^{N-2}} + \Theta_2(r)$$

means $c_1 = c_2$ and $\Theta_1(r) = \Theta_2(r)$, which demonstrates the uniqueness of c, Θ and β. Recall that $\Theta = \beta + o(1)$. Then,

$$\begin{aligned}
\Theta'(r) &= -\frac{1}{r^{N-1}} \int_0^r s^{N-1} K(s) \left[\frac{c}{s^{N-2}} + \Theta(s)\right]^p ds \\
&= -\frac{c^p}{r^{N-1}} \int_0^r s^{N+\sigma-(N-2)p-1} \left[1 + \frac{s^{N-2}}{c}\Theta(s)\right]^p ds \\
&= -\frac{c^p}{r^{N-1}} \int_0^r s^{N+\sigma-(N-2)p-1} \left[1 + \frac{p\beta}{c} s^{N-2} + o(s^{N-2})\right] ds \\
&= -\frac{c^p r^{\sigma+1-(N-2)p}}{N+\sigma-(N-2)p} - \frac{p\beta c^{p-1} r^{N+\sigma-(N-2)p-1}}{2N+\sigma-(N-2)p-2} \\
&\quad + o(r^{N+\sigma-(N-2)p-1}).
\end{aligned}$$

The expansion for Θ follows by integration.

(ii) Given c, β, we define $S(r) = c/r^{N-2}$. Lemma 2.1 indicates that (4.3) (where $(r^{N-1}S')' = 0$) has a unique solution Θ, and $\phi = S + \Theta$ is an M-solution.

Finally,

$$\begin{aligned}
u(t) &= r^{\sigma+1} \frac{\phi^p(r)}{-\phi'(r)} = r^{\sigma+1} \frac{[c/r^{N-2} + \Theta(r)]^p}{(N-2)c/r^{N-1} - \Theta'(r)} \\
&= \frac{c^{p-1}}{N-2} r^{N+\sigma-(N-2)p} \frac{[1 + (r^{N-2}/c)\Theta(r)]^p}{1 - (r^{N-1}/(N-2)c)\Theta'(r)} \\
&= \frac{c^{p-1}}{N-2} r^{N+\sigma-(N-2)p} \left[1 + \frac{p}{c} r^{N-2}\Theta(r) + O(r^{N-2}\Theta(r))^2\right] \\
&\cdot \left[1 + \frac{r^{N-1}}{(N-2)c}\Theta'(r) + O(r^{N-1}\Theta'(r))^2\right] \\
&= \frac{c^{p-1}}{N-2} r^{N+\sigma-(N-2)p} \left[1 + \frac{p\beta}{c} r^{N-2} \right. \\
&\qquad - \frac{pc^{p-1} r^{N+\sigma-(N-2)p}}{[N+\sigma-(N-2)p][\sigma+2-(N-2)p]} \\
&\qquad \left. + \frac{p(p-1)\beta^2}{2c^2} r^{2N-4} + o(r^{\max\{N+\sigma-(N-2)p, 2N-4\}})\right] \\
&\cdot \left[1 - \frac{c^{p-1} r^{N+\sigma-(N-2)p}}{(N-2)[N+\sigma-(N-2)p]} \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{p\beta c^{p-2} r^{2N+\sigma-(N-2)p-2}}{(N-2)[2N+\sigma-(N-2)p-2]} + o(r^{2N+\sigma-(N-2)p-2})\Bigg] \\
&= \frac{c^{p-1}}{N-2} r^{N+\sigma-(N-2)p} \Bigg[1 + \frac{p\beta}{c} r^{N-2} \\
&\qquad - \frac{(\sigma+2)c^{p-1} r^{N+\sigma-(N-2)p}}{(N-2)[N+\sigma-(N-2)p][\sigma+2-(N-2)p]} \\
&\qquad + \frac{p(p-1)\beta^2}{2c^2} r^{2N-4} + o(r^{\max\{N+\sigma-(N-2)p, 2N-4\}})\Bigg].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v(t) &= r\frac{-\phi'(r)}{\phi(r)} = r\frac{(N-2)c/r^{N-1} - \Theta'(r)}{c/r^{N-2} + \Theta(r)} = \frac{N-2-(r^{N-1}/c)\Theta'(r)}{1+(r^{N-2}/c)\Theta(r)} \\
&= \left[N-2-\frac{r^{N-1}}{c}\Theta'(r)\right] \cdot \left[1 - \frac{r^{N-2}}{c}\Theta(r) + O\left(\frac{r^{N-2}}{c}\Theta(r)\right)^2\right] \\
&= \left[N-2+\frac{c^{p-1} r^{N+\sigma-(N-2)p}}{N+\sigma-(N-2)p} + o(r^{N+\sigma-(N-2)p})\right] \\
&\cdot \Bigg[1 - \frac{\beta}{c} r^{N-2} + \frac{c^{p-1} r^{N+\sigma-(N-2)p}}{[N+\sigma-(N-2)p][\sigma+2-(N-2)p]} \\
&\qquad + \frac{\beta^2}{c^2} r^{2N-4} + o(r^{\max\{N+\sigma-(N-2)p, 2N-4\}})\Bigg] \\
&= N-2-\frac{(N-2)\beta r^{N-2}}{c} + \frac{c^{p-1} r^{N+\sigma-(N-2)p}}{\sigma+2-(N-2)p} + \frac{(N-2)\beta^2 r^{2N-4}}{c^2} \\
&\quad + o(r^{\max\{N+\sigma-(N-2)p, 2N-4\}}). \qquad \square
\end{aligned}
$$

4.1.2. The case $p = (\sigma+2)/(N-2)$.

Theorem 4.3. *Suppose that* $p = (\sigma+2)/(N-2)$. *Then*:

(i) *every* M-*solution* $\phi(r)$ *has the form* $\phi = S + \Theta$, *where*

$$
S(r) = \frac{c}{r^{N-2}} - \frac{c^p}{N-2}\ln r, \tag{4.6}
$$

and Θ *solves* (4.3) *with the following expansions*

$$\Theta(r) = \beta + \frac{pc^{2p-1}}{2(N-3)^3} r^{N-2} \ln r - \frac{3pc^{2p-1}}{4(N-2)^3} r^{N-2} + o(r^{N-2}), \quad r \to 0$$

for some uniquely defined constants $c > 0$, $\beta \in \mathbb{R}$.

(ii) *Conversely, given any* $c > 0$ *and* $\beta \in \mathbb{R}$, *there exists a unique solution* Θ *of* (4.3) *with* S *given by* (4.6), *and* $\phi = S + \Theta$ *is an* M*-solution. Moreover,* Θ *satisfies* (4.4) *with* $f_2(r) = (c^p/r^2) - c^p r^{-2} P^p(r) = O(r^{N-4} \ln r)$.

$$\begin{cases} u(t) = \frac{c^{p-1}e^{(N-2)t}}{N-2}\Big[1 - \frac{pc^{p-1}}{N-2} te^{(N-2)t} + \frac{p\beta e^{(N-2)t}}{c} \\ \qquad\qquad - \frac{c^{p-1}e^{(N-2)t}}{(N-2)^2} + o(e^{(N-2)t})\Big], \\ v(t) = N - 2 + c^{p-1}te^{(N-2)t} + \Big[\frac{c^{p-1}}{N-2} - \frac{(N-2)\beta}{c}\Big] te^{(N-2)t} \\ \qquad\qquad + o(e^{(N-2)t}) \qquad\qquad t \to -\infty. \end{cases}$$

Proof.

(i) Let $c > 0$ be determined by Theorem 4.1. Then,

$$s^{N-1}K(s)\phi^p(s) = s^{N+\sigma-1} \frac{c^p}{s^{(N-2)p}} \left(\frac{s^{N-2}}{c} \phi(s) \right)^p = c^p s^{N-3}[1 + o(1)].$$

By virtue of (4.1), we find

$$\begin{aligned} r^{N-1}\phi'(r) &= -(N-2)c - \int_0^r s^{N-1}K(s)\phi^p(s)\,ds, \\ \phi'(r) &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{(N-2)r} + o\left(\frac{1}{r}\right), \\ \phi(r) &= \frac{c}{r^{N-2}} \left[1 - \frac{c^{p-1}}{N-2} r^{N-2} \ln r + o(r^{N-2} \ln r) \right]. \end{aligned}$$

Clearly, the expansions for $\phi(r)$ are all singular. Hence, we have to apply the above iterative process once more. Again,

$$\begin{aligned} s^{N-1}K(s)\phi^p(s) &= c^p s^{N-3} \left[1 - \frac{c^{p-1}}{N-2} s^{N-2} \ln s + o(s^{N-2} \ln s) \right]^p \\ &= c^p s^{N-3} \left[1 - \frac{pc^{p-1}}{N-2} s^{N-2} \ln s + o(s^{N-2} \ln s) \right]. \end{aligned}$$

Therefore,

$$\phi'(r) = -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{(N-2)r} + \frac{pc^{2p-1}}{2(N-2)^2} r^{N-3} \cdot \ln r - \frac{pc^{2p-1}}{4(N-2)^3} r^{N-3} + o(r^{N-3}).$$

Set

$$S'(r) = -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{(N-2)r}$$

and $\Theta'(r) = \phi'(r) - S'(r)$. Since Θ' is integrable, Θ fulfills (4.3), and $\Theta(0) =: \beta$ exists. Hence,

$$\phi(r) = \frac{c}{r^{N-2}} - \frac{c^p}{N-2} \ln r + \beta + \frac{pc^{2p-1}}{2(N-3)^3} r^{N-2} \cdot \ln r - \frac{3pc^{2p-1}}{4(N-2)^3} r^{N-2} + o(r^{N-2}).$$

(ii) Given c and β, we define

$$S(r) := \frac{c}{r^{N-2}} - \frac{c^p}{N-2} \ln r.$$

Then,

$$\begin{aligned} f_2(r) &= -c^p r^{\sigma-(N-2)p} P^p(r) - \frac{1}{r^{N-1}} (r^{N-1} S')' \\ &= -c^p r^{-2} \left(1 - \frac{c^p}{N-2} r^{N-2} \ln r\right)^p + c^p r^{-2} \\ &= O(r^{N-4} \ln r). \end{aligned}$$

It is not difficult to verify that the assumptions of Lemma 2.1 are satisfied; thus, (4.3) admits a unique solution Θ. Then, $\phi = S + \Theta$ is an M-solution.

Finally,

$$\begin{aligned} u(t) = r^{\sigma+1} \frac{\phi^p(r)}{-\phi'(r)} &= r^{\sigma+1} \frac{c/r^{N-2} - c^p \ln r/(N-2) + \Theta(r)^p}{(N-2)c/r^{N-1} + c^p/(N-2)r - \Theta'(r)} \\ &= \frac{c^{p-1} r^{N-2}}{N-2} \frac{[1 - c^{p-1} r^{N-2} \ln r/(N-2) + (r^{N-2}/c)\Theta(r)]^p}{1 - (c^{p-1}/(N-2)^2) r^{N-2} - (r^{N-1}/(N-2)c)\Theta'(r)} \end{aligned}$$

$$= \frac{c^{p-1}r^{N-2}}{N-2}\bigg[1 - \frac{pc^{p-1}r^{N-2}\ln r}{N-2} + \frac{pr^{N-2}}{c}\Theta(r) + o\bigg(\frac{c^{p-1}r^{N-2}\ln r}{N-2} - \frac{r^{N-2}}{c}\Theta(r)\bigg)\bigg]$$
$$\cdot \bigg[1 - \frac{c^{p-1}r^{N-2}}{(N-2)^2} + \frac{r^{N-1}}{(N-2)c}\Theta'(r) + o\bigg(\frac{c^{p-1}r^{N-2}}{(N-2)^2} - \frac{r^{N-1}}{(N-2)c}\Theta'(r)\bigg)\bigg]$$
$$= \frac{c^{p-1}r^{N-2}}{N-2}\bigg[1 - \frac{pc^{p-1}r^{N-2}\ln r}{N-2} + \frac{p\beta r^{N-2}}{c} + o(r^{N-2})\bigg] \cdot \bigg[1 - \frac{c^{p-1}r^{N-2}}{(N-2)^2} + o(r^{N-2})\bigg]$$
$$= \frac{c^{p-1}r^{N-2}}{N-2}\bigg[1 - \frac{pc^{p-1}}{N-2}r^{N-2}\ln r + \frac{p\beta r^{N-2}}{c} - \frac{c^{p-1}r^{N-2}}{(N-2)^2} + o(r^{N-2})\bigg],$$

and

$$v(t) = r\frac{-\phi'(r)}{\phi(r)} = r\frac{(N-2)c/r^{N-1} + c^p/(N-2)r - \Theta'(r)}{c/r^{N-2} - c^p\ln r/(N-2) + \Theta(r)}$$
$$= \bigg[N - 2 + \frac{c^{p-1}}{N-2}r^{N-2} - \frac{r^{N-1}}{c}\Theta'(r)\bigg]$$
$$\cdot \bigg[1 + \frac{c^{p-1}}{N-2}r^{N-2}\ln r - \frac{r^{N-2}}{c}\Theta(r) + o\bigg(- \frac{c^{p-1}}{N-2}r^{N-2}\ln r + \frac{r^{N-2}}{c}\Theta(r)\bigg)\bigg]$$
$$= N-2+c^{p-1}r^{N-2}\ln r+\bigg[\frac{c^{p-1}}{N-2} - \frac{(N-2)\beta}{c}\bigg]r^{N-2}+o(r^{N-2}). \qquad \square$$

4.1.3. The case $(\sigma + 2)/(N - 2) < p < (N + \sigma)/(N - 2)$.

Theorem 4.4. *Let $(\sigma + 2)/(N - 2) < p < (N + \sigma)/(N - 2)$. Define $\mu := N + \sigma - (N - 2)p \in (0, N - 2)$, and choose $k_0 \in \mathbb{N}$ such that $k_0\mu < N - 2 \le (k_0 + 1)\mu$. Then, there exist some constants $\overline{a}_j$, $j = 1, 2, \dots, k_0 + 3$, depending on c, σ, p, N such that:*

(i) *every* M*-solution $\phi(r)$ has the form $\phi = S + \Theta$, where*

$$S(r) = \begin{cases} \frac{c}{r^{N-2}}\left(1+\sum_{j=1}^{k_0}\overline{a}_j r^{j\mu}\right) & N-2<(k_0+1)\mu, \\ \frac{c}{r^{(k_0+1)\mu}}\left(1+\sum_{j=1}^{k_0}\overline{a}_j r^{j\mu}+\overline{a}_{k_0+1}r^{(k_0+1)\mu}\ln r\right) & \\ & N-2=(k_0+1)\mu, \end{cases}$$

and Θ *is a solution of* (4.3) *with the expansions*

$$\Theta(r) = \begin{cases} \beta + c\overline{a}_{k_0+1}r^{(k_0+1)\mu-N+2}+o(r^{(k_0+1)\mu-N+2}) & \\ & N-2<(k_0+1)\mu, \\ \beta + c\overline{a}_{k_0+2}r^{\mu}\ln r + c\overline{a}_{k_0+3}r^{\mu}+o(r^{\mu}) & N-2=(k_0+1)\mu \end{cases}$$

for uniquely determined constants $c>0$, $\beta\in\mathbb{R}$.

(ii) *Conversely, given any* $c>0$, $\beta\in\mathbb{R}$, *there exists a unique solution* Θ *of* (4.3) *with* S *given by* (i), *and* $\phi=S+\Theta$ *is an* M*-solution. Furthermore,* Θ *satisfies* (4.4) *with*

$$f_2(r) = \begin{cases} O(r^{(k_0+1)\mu-N+2}) & N-2<(k_0+1)\mu, \\ O(r^{\mu-2}\ln r) & N-2=(k_0+1)\mu, \end{cases}$$

and

$$\begin{cases} u(t) = \frac{c^{p-1}}{N-2}e^{\mu t}\left[1-\frac{(\sigma+2)c^{p-1}}{(N-2)\mu(\mu-N+2)}e^{\mu t}+o(e^{\mu t})\right], & \\ v(t) = N-2+\frac{c^{p-1}}{\mu-N+2}e^{\mu t}+o(e^{\mu t}) & t\to-\infty. \end{cases}$$

Proof.

(i) Let c be determined by Theorem 4.1. We have

$$s^{N-1}K(s)\phi^p(s) = c^p s^{\mu-1}[1+o(1)].$$

By virtue of (4.1), we obtain

$$\begin{aligned} r^{N-1}\phi'(r) &= -(N-2)c - \int_0^r s^{N-1}K(s)\phi^p(s)\,ds, \\ \phi'(r) &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{\mu}r^{\mu-N+1}+o(r^{\mu-N+1}), \\ \phi(r) &= \frac{c}{r^{N-2}} - \frac{c^p}{\mu(\mu-N+2)}r^{\mu-N+2}+o(r^{\mu-N+2})+C_1 \\ &= \frac{c}{r^{N-2}}\left[1-\frac{c^{p-1}}{\mu(\mu-N+2)}r^{\mu}+o(r^{\mu})\right], \end{aligned}$$

where C_1 is some constant.

Repeating the above iterative process, we find

$$s^{N-1}K(s)\phi^p(s) = c^p s^{\mu-1}\left[1 - \frac{pc^{p-1}}{\mu(\mu-N+2)}s^\mu + o(s^\mu)\right],$$

$$\begin{aligned}\phi'(r) &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{r^{N-1}}\\ &\quad\cdot\int_0^r s^{\mu-1}\left[1-\frac{pc^{p-1}}{\mu(\mu-N+2)}s^\mu+o(s^\mu)\right]ds\\ &= -\frac{(N-2)c}{r^{N-1}}-\frac{c^p r^{\mu-N+1}}{\mu}+\frac{pc^{2p-1}r^{2\mu-N+1}}{2\mu^2(\mu-N+2)}+o(r^{2\mu-N+1}).\end{aligned}$$

If $k_0 = 1$, i.e., $\mu < N-2 < 2\mu$, there exists some constant $\beta \in \mathbb{R}$ such that, for $\mu < N-2 < 2\mu$,

$$\phi(r)=\beta+\frac{c}{r^{N-2}}\left[1-\frac{c^{p-1}r^\mu}{\mu(\mu-N+2)}+\frac{pc^{2p-2}r^{2\mu}}{2\mu^2(\mu-N+2)(2\mu-N+2)}+o(r^{2\mu})\right].$$

For the case $N-2 = 2\mu$, we obtain

$$\phi(r) = \frac{c}{r^{2\mu}}\left[1 + \frac{c^{p-1}r^\mu}{\mu^2} - \frac{pc^{2p-2}r^{2\mu}\ln r}{2\mu^3} + o(r^{2\mu}\ln r)\right].$$

Similarly, we must apply the above process once more; thus,

$$s^{N-1}K(s)\phi^p(s) = c^p s^{\mu-1}\left[1 + \frac{pc^{p-1}s^\mu}{\mu^2} - \frac{p^2c^{2p-2}s^{2\mu}\ln s}{2\mu^3} + o(s^{2\mu}\ln s)\right]$$

and

$$\begin{aligned}\phi'(r) &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{r^{N-1}}\\ &\quad\cdot\int_0^r s^{\mu-1}\left[1 + \frac{pc^{p-1}s^\mu}{\mu^2} - \frac{p^2c^{2p-2}s^{2\mu}\ln s}{2\mu^3} + o(s^{2\mu}\ln s)\right]ds\\ &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p r^{\mu-N+1}}{\mu} - \frac{pc^{2p-1}r^{-1}}{2\mu^3} + \frac{p^2c^{3p-2}r^{3\mu-N+1}\ln r}{6\mu^4}\\ &\quad - \frac{p^2c^{3p-2}r^{3\mu-N+1}}{18\mu^5} + o(r^{3\mu-N+1}).\end{aligned}$$

Integration gives

$$\phi(r) = \beta + \frac{c}{r^{2\mu}}\left[1 + \frac{c^{p-1}r^{\mu}}{\mu^2} - \frac{pc^{2p-2}r^{2\mu}\ln r}{2\mu^3} + \frac{p^2c^{3p-3}r^{3\mu}\ln r}{6\mu^5} - \frac{2p^2c^{3p-3}r^{3\mu}}{9\mu^6} + o(r^{3\mu})\right].$$

Now, the singular term S and the regular term Θ can be read off exactly.

By induction, we may assume that, for $k_0\mu < N-2 < (k_0+1)\mu$,

$$\phi(r) = \frac{c}{r^{N-2}}\left[1 + \sum_{j=1}^{k_0} \overline{a}_j r^{j\mu} + o(r^{k_0\mu})\right],$$

where $\overline{a}_j$, $j = 1, 2, \ldots, k_0$, are some constants depending on c, σ, p and N. It follows from similar arguments as before that

$$\begin{aligned} s^{N-1}K(s)\phi^p(s) &= c^p s^{\mu-1}\left[1 + \sum_{j=1}^{k_0} \overline{a}_j s^{j\mu} + o(s^{k_0\mu})\right]^p \\ &= c^p s^{\mu-1}\left[1 + \sum_{j=1}^{k_0} \widehat{a}_j s^{j\mu} + o(s^{k_0\mu})\right] \end{aligned}$$

for some appropriate constants $\widehat{a}_j$ depending upon c, σ, p, N and k_0. Hence,

$$\begin{aligned} \phi'(r) &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p}{r^{N-1}}\int_0^r s^{\mu-1}\left[1 + \sum_{j=1}^{k_0} \widehat{a}_j s^{j\mu} + o(s^{k_0\mu})\right] ds \\ &= -\frac{(N-2)c}{r^{N-1}} - \frac{c^p r^{\mu-N+1}}{\mu} \\ &\quad - \sum_{j=1}^{k_0} \frac{c^p\widehat{a}_j r^{(j+1)\mu-N+1}}{(j+1)\mu} + o(r^{(k_0+1)\mu-N+1}), \end{aligned}$$

$$\begin{aligned} \phi(r) &= \beta + \frac{c}{r^{N-2}}\left[1 - \frac{c^{p-1}r^{\mu}}{\mu(\mu-N+2)} - \sum_{j=1}^{k_0} \frac{c^{p-1}\widehat{a}_j r^{(j+1)\mu}}{(j+1)\mu[(j+1)\mu-N+2]} + o(r^{(k_0+1)\mu})\right] \\ &= \beta + \frac{c}{r^{N-2}}\left[1 + \sum_{j=1}^{k_0+1} \overline{a}_j r^{j\mu} + o(r^{(k_0+1)\mu})\right], \end{aligned}$$

where $\overline{a}_1 = -c^{p-1}/(\mu(\mu - N + 2))$, $\overline{a}_{j+1} = -c^{p-1}\widehat{a}_j/[(j+1)\mu[(j+1)\mu - N + 2]]$, $j = 1, 2, \ldots, k_0$. Now, the singular term S and regular term Θ can be precisely obtained again. We can handle the case $(k_0 + 1)\mu = N - 2$ similarly.

(ii) To show that (4.3) has a unique solution Θ, it suffices to show that $f_2(r)$ satisfies the hypotheses of Lemma 2.1. For the case $N - 2 < (k_0 + 1)\mu$, we have

$$
\begin{aligned}
f_2(r) &= -c^p r^{\sigma-(N-2)p} P^p(r) - \frac{1}{r^{N-1}}(r^{N-1}S')' \\
&= -c^p r^{\mu-N}\left(1 + \sum_{j=1}^{k_0} \overline{a}_j r^{j\mu}\right)^p - \sum_{j=1}^{k_0} j\mu(j\mu - N + 2)c\overline{a}_j r^{j\mu-N} \\
&= -c^p r^{\mu-N}\left[1 + \sum_{j=1}^{k_0} \widehat{a}_j r^{j\mu} + o(r^{k_0\mu})\right] \\
&\quad - \sum_{j=1}^{k_0} j\mu(j\mu - N + 2)c\overline{a}_j r^{j\mu-N} = O(r^{(k_0+1)\mu-N}).
\end{aligned}
$$

Here, we have used the relations between $\overline{a}_j$ and $\widehat{a}_j$. The other case can be investigated similarly.

Finally,

$$
\begin{aligned}
u(t) &= r^{\sigma+1}\frac{\phi^p(r)}{-\phi'(r)} \\
&= r^{\sigma+1}\left[(c/r^{N-2})\left(1 + \sum_{j=1}^{k_0} \overline{a}_j r^{j\mu}\right) + \Theta(r)\right]^p \Big/ \\
&\qquad \Big[(N-2)c/r^{N-1} + c^p r^{\mu-N+1}/\mu \\
&\qquad + \sum_{j=1}^{k_0} c^p \widehat{a}_j r^{(j+1)\mu-N+1}/[(j+1)\mu] + o(r^{(k_0+1)\mu-N+1})\Big] \\
&= \frac{c^{p-1}}{N-2} r^\mu \left[1 - \frac{pc^{p-1}}{\mu(\mu - N + 2)} r^\mu + o(r^\mu)\right] \cdot \left[1 - \frac{c^{p-1}}{(N-2)\mu} r^\mu + o(r^\mu)\right] \\
&= \frac{c^{p-1}}{N-2} r^\mu \left[1 - \frac{(\sigma + 2)c^{p-1}}{(N-2)\mu(\mu - N + 2)} r^\mu + o(r^\mu)\right],
\end{aligned}
$$

and

$$\begin{aligned}
v(t) &= r\frac{-\phi'(r)}{\phi(r)}\\
&= r\frac{(N-2)c/r^{N-1}+c^p r^{\mu-N+1}/\mu}{(c/r^{N-2})(1+\sum_{j=1}^{k_0}\overline{a}_j r^{j\mu})+\Theta}\\
&\quad+\frac{\sum_{j=1}^{k_0} c^p\widehat{a}_j r^{(j+1)\mu-N+1}/[(j+1)\mu]+o(r^{(k_0+1)\mu-N+1})}{(c/r^{N-2})(1+\sum_{j=1}^{k_0}\overline{a}_j r^{j\mu})+\Theta}\\
&= \left[N-2+\frac{c^{p-1}}{\mu}r^{\mu}+o(r^{\mu})\right]\cdot\left[1+\frac{c^{p-1}}{\mu(\mu-N+2)}r^{\mu}+o(r^{\mu})\right]\\
&= N-2+\frac{c^{p-1}}{\mu-N+2}r^{\mu}+o(r^{\mu}). \qquad \square
\end{aligned}$$

Remark 4.5. For the case $-2<\sigma\le N-4$, since $(\sigma+2)/(N-2)\le 1$, we have one case only: $1<p<(N+\sigma)/(N-2)$. Here, the a priori estimate (Theorem 4.1) of the M-solutions still holds. If the hypotheses $(\sigma+2)/(N-2)<p<(N+\sigma)/(N-2)$ in Theorem 4.4 is replaced by $1<p<(N+\sigma)/(N-2)$, then, we can also obtain similar conclusions as in Theorem 4.4.

4.2. The case $p=(N+\sigma)/(N-2)$.

Theorem 4.6. *Suppose that* $p=(N+\sigma)/(N-2)$. *Then, the following conclusions are equivalent*:

(i) $\phi(r)$ *is an* M-*solution.*

(ii) $\phi(r)$ *fulfills*

$$\begin{cases}\phi(r)=\left(\frac{N-2}{p-1}\right)^{1/(p-1)}(-\ln r)^{1/(p-1)}\cdot\frac{1}{r^{N-2}}[1+o(1)],\\ \phi'(r)=\left(\frac{N-2}{p-1}\right)^{(1/p-1)}(-\ln r)^{1/(p-1)}\cdot\frac{-(N-2)}{r^{N-1}}[1+o(1)] \quad r\to 0.\end{cases}$$

(iii) $u(t)$ *and* $v(t)$ *satisfy*

$$u(t)=-\frac{1}{(p-1)t}[1+o(1)],$$

$$v(t) = N - 2 + \frac{1}{(p-1)t}[1 + o(1)], \quad t \to -\infty.$$

(iv) $\varphi(t) \to P_2$, $t \to -\infty$.

Proof.

(i) $\to$ (iv). By [**1,** Theorem 3.1], we see that $C^-(\varphi)$ is bounded. By the Poincaré-Bendixson theorem for autonomous systems, $\varphi(t)$ must converge to a stationary point of (2.6). Moreover, the convergence to P_3 is impossible by Theorem 3.1, and the convergence to P_1 is impossible, too. Therefore, $\varphi(t) \to P_2$ as $t \to -\infty$.

(iv) $\to$ (iii). Set $z = (N - 2 - v)/u$, and differentiate it with respect to t. We have

$$\begin{aligned}\dot{z} &= \frac{-u\dot{v} - (N - 2 - v)\dot{u}}{u^2} \\ &= \frac{v(N - 2 - u - v)}{u} - \frac{N - 2 - v}{u}(N + \sigma - u - pv) \\ &= [(p + 1)v + u - N - \sigma]z - v.\end{aligned}$$

Define

$$\Gamma(t) := \int_{t_\delta}^{t} \gamma(s)\, ds$$

for some $t_\delta \in I_\varphi$, which indicates that

$$\begin{aligned}z(t) &= z(t_\delta)e^{\Gamma(t)} - e^{\Gamma(t)} \int_{t_\delta}^{t} v(s)e^{-\Gamma(s)} ds \\ &= z(t_\delta)e^{\Gamma(t)} + e^{\Gamma(t)} \int_{t_\delta}^{t} [\gamma(s) - v(s)]e^{-\Gamma(s)} ds + 1 - e^{\Gamma(t)} \\ &=: I_1 + I_2 + 1 + I_3.\end{aligned}$$

Since $\gamma(t) \to N - 2$ as $t \to -\infty$, we see that $\Gamma(t) \to -\infty$ as $t \to -\infty$, which implies that $I_1, I_3 \to 0$. Clearly, $v(t) \to N - 2$ as $t \to -\infty$. Hence, for any $\delta > 0$, we can select $t_\delta \in I_\varphi$ such that, if $s < t_\delta$,

$$|\gamma(s) - v(s)| < \delta\gamma(s).$$

It follows that $|I_2| < \delta$. Therefore, we have $z(t) \to 1$ as $t \to -\infty$.

Through some simple calculations, we obtain

$$\begin{aligned}\dot{\left(\frac{1}{u}\right)} &= -\frac{1}{u^2}\dot{u} = -\frac{N+\sigma-pv}{u} + 1 \\ &= -pz(t) + 1 \longrightarrow -p+1, \qquad \text{as } t \to -\infty.\end{aligned}$$

Then, the expansions for u and v follow.

(iii) $\to$ (ii). By the inverse transformation (2.5), we see that

$$\begin{aligned}\phi(r) &= \left[\frac{u(\ln r)v(\ln r)}{r^{\sigma+2}}\right]^{1/(p-1)} = \left(\frac{N-2}{p-1}\right)^{1/(p-1)} (-\ln r)^{1/(p-1)} \\ &\quad \cdot \frac{1}{r^{N-2}}[1+o(1)], \\ \phi'(r) &= \left(\frac{N-2}{p-1}\right)^{1/(p-1)} (-\ln r)^{1/(p-1)} \cdot \frac{-(N-2)}{r^{N-1}}[1+o(1)].\end{aligned}$$

(ii) $\to$ (i) is obvious. □

4.3. Cases $p > (N+\sigma)/(N-2)$ and $p \neq (N+2+2\sigma)/(N-2)$.

Theorem 4.7. *Suppose that $p > (N+\sigma)/(N-2)$ and $p \neq (N+2+2\sigma)/(N-2)$. Then, the following conclusions are equivalent*:

(i) $\phi(r)$ *is an* M-*solution.*

(ii) $\phi(r)$ *satisfies*

$$\begin{aligned}\phi(r) &= \widetilde{C} r^{-(\sigma+2)/(p-1)}[1+o(1)], \\ \phi'(r) &= -\widetilde{C}\frac{\sigma+2}{p-1} r^{-(\sigma+p+1)/(p-1)}[1+o(1)], \quad r \to 0,\end{aligned}$$

where
$$\widetilde{C}^{p-1} = \frac{[(N-2)p-N-\sigma](\sigma+2)}{(p-1)^2}.$$

(iii) u *and* v *fulfill*

$$\begin{aligned}u(t) &= \frac{(N-2)p-N-\sigma}{p-1}[1+o(1)], \\ v(t) &= \frac{\sigma+2}{p-1}[1+o(1)], \quad t \to -\infty.\end{aligned}$$

(iv) $\varphi(t) \to P_4$, $t \to -\infty$.

Proof. The proof follows by similar arguments to those of Theorem 4.5. □

4.4. The case $p = (N+2+2\sigma)/(N-2)$. For this case, we see that the real part of complex conjugate eigenvalues of (2.7) is zero. By means of the Hopf bifurcation theory **[10]**, we can derive more accurate forms of (u, v), respectively. Then, the form of $\phi(r)$ follows from an inverse transformation (2.5).

Theorem 4.8. *Suppose that $p = (N + 2 + 2\sigma)/(N - 2)$. Then, the following conclusions hold.*

(i) $\phi(r)$ *is an* M-*solution with the form*

$$\phi(r) = \frac{\psi(\ln r)}{r^{(N-2)/2}},$$

where $\psi(\ln r)$ is a strictly positive function with small oscillations about $\psi_0 = ((N-2)/2)^{(N-2)/(\sigma+2)}$.

(ii) *There exists some sufficiently small ϵ such that $u(t)$ and $v(t)$ can be described by the following forms, respectively*:

$$\begin{cases} u(t) = \frac{N-2}{2} + \epsilon\cos(2\pi t/T_\epsilon) + O(\epsilon^2), \\ v(t) = \frac{N-2}{2} + \dfrac{\epsilon}{p_*^2}[(p_* - 1)\sin(2\pi t/T_\epsilon) \\ \qquad -(p_* + \sqrt{p_* - 1})\cos(2\pi t/T_\epsilon)] + O(\epsilon^2), \end{cases}$$

where

$$p_* = \frac{N+2+2\sigma}{N-2}$$

and

$$T_\epsilon = \frac{4\pi}{(N-2)\sqrt{p_* - 1}}\left[1 + \frac{p_* + 3}{6(N-2)^2(p_* - 1)}\epsilon^2 + O(\epsilon^4)\right].$$

Proof. From subsection 2.3, we see that

$$\lambda_1(p) = v_4^* - \frac{N-2}{2} + \frac{1}{2}\sqrt{-\Lambda(v_4^*)}i := \alpha(p) + i\omega(p)$$

if $\kappa_2 < v_4^* < \kappa_1$. Obviously,

$$\omega_0 := \omega(p_*) = \frac{N-2}{2}\sqrt{p_* - 1} > 0$$

with $p_* = (N+2+2\sigma)/(N-2)$. By some simple computations, we obtain

$$\alpha'(p_*) = -\frac{N-2}{2(p_*-1)} < 0, \qquad \omega'(p_*) = \frac{(N-2)^3}{4}\sqrt{p_*-1} > 0.$$

Set

$$\mathrm{B} = (\mathrm{Re}\xi_1, -\mathrm{Im}\xi_1) = \begin{pmatrix} 1 & 0 \\ -1/p_* & \sqrt{p_*-1}/p_* \end{pmatrix},$$

where

$$\xi_1 = \left(1, -\frac{1}{p_*} - \frac{\sqrt{p_*-1}}{p_*}i\right)^T$$

is the eigenvector of the matrix corresponding to the eigenvalue $\lambda_1(p_*) = i\omega_0$. Applying a change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{N-2}{2} \\ \frac{N-2}{2} \end{pmatrix} + \mathrm{B}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

we have

$$\begin{cases} \dot{y}_1 = -\frac{(N-2)\sqrt{p_*-1}}{2}y_2 - \sqrt{p_*-1}y_1y_2 := F^1(y_1,y_2), \\ \dot{y}_2 = \frac{(N-2)\sqrt{p_*-1}}{2}y_1 - \frac{\sqrt{p_*-1}}{p_*}y_1^2 - \frac{2}{p_*}y_1y_2 \\ \qquad + \frac{\sqrt{p_*-1}}{p_*}y_2^2 := F^2(y_1,y_2). \end{cases} \tag{4.7}$$

The Jacobian matrix $\partial F^i/\partial y_j(0)$, $i, j = 1, 2, \ldots$, of (4.7) will have the real canonical form

$$\left.\begin{pmatrix} \dfrac{\partial F^1}{\partial y_1} & \dfrac{\partial F^1}{\partial y_2} \\ \dfrac{\partial F^2}{\partial y_1} & \dfrac{\partial F^2}{\partial y_2} \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 0 & -\omega_0 \\ -\omega_0 & 0 \end{pmatrix}.$$

In view of the formulae of [**10,** Chapter 2], we have

$$g_{11}(p_*) = \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial y_1^2} + \frac{\partial^2 F^1}{\partial y_2^2} + i\left(\frac{\partial^2 F^2}{\partial y_1^2} + \frac{\partial^2 F^2}{\partial y_2^2}\right)\right] = 0,$$

$$\begin{aligned} g_{02}(p_*) &= \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial y_1^2} - \frac{\partial^2 F^1}{\partial y_2^2} - 2\frac{\partial^2 F^2}{\partial y_1 \partial F_2} + i\left(\frac{\partial^2 F^2}{\partial y_1^2} - \frac{\partial^2 F^2}{\partial y_2^2} + 2\frac{\partial^2 F^1}{\partial y_1 \partial y_2}\right)\right] \\ &= \frac{1}{2p_*}[2 - (p_*+2)\sqrt{p_*-1}i], \end{aligned}$$

$$
\begin{aligned}
g_{20}(p_*) &= \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial y_1^2} - \frac{\partial^2 F^1}{\partial y_2^2} + 2\frac{\partial^2 F^2}{\partial y_1 \partial F_2} + i\left(\frac{\partial^2 F^2}{\partial y_1^2} - \frac{\partial^2 F^2}{\partial y_2^2} - 2\frac{\partial^2 F^1}{\partial y_1 \partial y_2}\right)\right] \\
&= \frac{1}{2p_*}[-2 + (p_* - 2)\sqrt{p_* - 1} i], \\
g_{21}(p_*) &= G_{21}(p_*) = 0,
\end{aligned}
$$

$$
\begin{aligned}
c_1(p_*) = \frac{i}{2\omega_0}&\left[g_{20}(p_*)g_{11}(p_*) - 2|g_{11}(p_*)|^2 - \frac{1}{3}|g_{02}(p_*)|^2\right] \\
&+ \frac{g_{21}(p_*)}{2} = -\frac{p_* + 3}{24\omega_0} i.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mu_2 &= -\mathrm{Re} c_1(p_*)/\alpha'(p_*) = 0, \qquad \beta_2 = 2\mathrm{Re} c_1(p_*) = 0, \\
\tau_2 &= -[\mathrm{Im} c_1(p_*) + \mu_2 \omega'(p_*)]/\omega_0 = \frac{p_* + 3}{24\omega_0^2} > 0.
\end{aligned}
$$

Unfortunately, the result $\mu_2 = 0$ does not imply the direction of bifurcation, and $\beta_2 = 0$ does not indicate the stability of the bifurcation periodic solution either. We do know that the periods of the oscillations increase as their amplitudes grow. For all sufficiently small ϵ, if we pose the initial conditions $y_1(0) = \epsilon, y_2(0) = 0$, the solution of (4.7) must exist for at least $2\pi/\omega_0$ units of time and must cross the line $y_2 = 0$ for some time near π/ω_0. Since the symmetry of the y_1, y_2 phase plane, the trajectory backwards in time from the same initial conditions is the reflection in the line $y_2 = 0$ of the forwards trajectory, and the trajectory meet at $y_2 = 0$, $y_1 = -\epsilon + O(\epsilon^2)$. Hence, there exists a family of periodic solution at $p = p_*$. Moreover, this family of periodic solutions is described by

$$
\begin{pmatrix} y_1(t;\epsilon) \\ y_2(t;\epsilon) \end{pmatrix} = \frac{\epsilon}{p_*}\begin{pmatrix} p_* \cos(2\pi t/T_\epsilon) \\ \sqrt{p_* - 1}\sin(2\pi t/T_\epsilon) - \cos(2\pi t/T_\epsilon) \end{pmatrix} + O(\epsilon^2),
$$

where the period is

$$
T_\epsilon = \frac{2\pi}{\omega_0}\left(1 + \frac{p_* + 3}{24\omega_0^2}\epsilon^2 + O(\epsilon^4)\right).
$$

In cylindrical coordinates, the family of bifurcating tori, belonging to the system (2.6), is characterized by

$$\begin{cases} u(t) = \frac{N-2}{2} + \epsilon \cos(2\pi t/T_\epsilon) + O(\epsilon^2), \\ v(t) = \frac{N-2}{2} + \frac{\epsilon}{p_*^2}[(p_* - 1)\sin(2\pi t/T_\epsilon) \\ \qquad\qquad -(p_* + \sqrt{p_* - 1})\cos(2\pi t/T_\epsilon)] + O(\epsilon^2). \end{cases}$$

With the aid of the inverse transformation (2.5), we have

$$\begin{aligned} \phi^{p_*-1}(r) &= \frac{u(\ln r)v(\ln r)}{r^{\sigma+2}} \\ &= \left[\frac{N-2}{2} + \epsilon\cos(2\pi t/T_\epsilon) + O(\epsilon^2)\right] \\ &\quad \cdot \left\{\frac{N-2}{2} + \frac{\epsilon}{p_*^2}\Big[(p_*-1)\sin(2\pi t/T_\epsilon)\right. \\ &\qquad \left. - (p_* + \sqrt{p_*-1})\cos(2\pi t/T_\epsilon)\Big] + O(\epsilon^2)\right\} r^{-\sigma-2} \\ &= \left\{\left(\frac{N-2}{2}\right)^2 + \frac{N-2}{2p_*^2}\epsilon\Big[(p_*-1)\sin(2\pi t/T_\epsilon)\right. \\ &\qquad \left. + (p_*^2 - p_* - \sqrt{p_*-1})\cos(2\pi t/T_\epsilon)\Big] + O(\epsilon^2)\right\} r^{-\sigma-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi(r) = &\left\{\left(\frac{N-2}{2}\right)^2 + \frac{N-2}{2p_*^2}\epsilon\Big[(p_*-1)\sin(2\pi t/T_\epsilon)\right. \\ &\left. + (p_*^2 - p_* - \sqrt{p_*-1})\cos(2\pi t/T_\epsilon)\Big] + O(\epsilon^2)\right\}^{1/(p_*-1)} r^{-(\sigma+2)/(p_*-1)} \\ := &\,\psi(\ln r) r^{-(N-2)/2}, \end{aligned}$$

where $\psi(\ln r)$ is strictly positive with small oscillations about $\psi_0 = (N-2/2)^{(N-2)/(\sigma+2)}$. The proof is complete. $\square$

5. F-solutions. By the definition of F-solutions, we see that the F-solution $\phi(r)$ fulfills

$$\text{(5.1)} \qquad \begin{cases} \phi''(r) + \frac{N-1}{r}\phi'(r) + r^\sigma \phi^p(r) = 0 \quad r > R_0 > 0, \\ \phi(R_0) = \alpha, \ \phi'(R_0) = 0. \end{cases}$$

The following results can be established by similar arguments to those of **[24, 27]**.

Theorem 5.1. *Assume that $\phi(r)$ is a solution of* (5.1). *Then, there exists a unique constant $\alpha^* > 0$ such that*:

(i) *if $\alpha > \alpha^*$, $\phi(r)$ has a finite zero and finite total mass*;

(ii) *if $\alpha = \alpha^*$, $\phi(r)$ has an infinite zero and finite total mass*;

(iii) *if $0 < \alpha < \alpha^*$, $\phi(r)$ has an infinite zero and infinite total mass.*

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