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INVARIANT SETS FOR QMF FUNCTIONS

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ABSTRACT. A quadrature mirror filter (QMF) function can be considered as the transition function for a Markov process on the unit interval. The QMF functions that generate scaling functions for multiresolution analyses are then distinguished by properties of their invariant sets. By characterizing these sets, we answer in the affirmative a question raised by Gundy **[8]**.

1. Introduction. The motivation for this paper comes from the study of a class of Markov processes that appear in the construction of scaling functions for multiresolution analyses (MRA). For definitions and background, see **[2, 3, 4, 5, 6, 12, 14, 16]** and, in particular, **[8]**. One way to construct a scaling function is to start with a 1-periodic function $p(\xi)$, $\xi \in \mathbb{R}$, that satisfies

$$p(\xi/2) + p(\xi/2 + 1/2) = 1 \quad \text{for every } \xi \in [0,1],\ p(0) = 1. \tag{1.1}$$

This condition is known as the *quadrature mirror filter* (QMF) condition. We reserve the symbol p for nonnegative, continuous 1-periodic functions that satisfy (1.1). We call them QMF *functions.*

To each p, we associate a Markov process $\xi_0, \xi_1, \xi_2, \dots$ on the interval $[0,1]$. Given $\xi_0 \in [0,1]$, the process evolves according to

$$\begin{aligned} &\xi_{t+1} = \xi_t/2 \quad \text{or} \quad \xi_t/2 + 1/2, \\ &\mathbb{P}_p(\xi_{t+1} = \xi_t/2 + j/2 \mid \xi_t) = p(\xi_t/2 + j/2), \quad j = 0, 1. \end{aligned} \tag{1.2}$$

If p generates a scaling function, then (see (4.4) below)

$$\mathbb{P}_p(\xi_t \to 0 \text{ or } 1 \mid \xi_0) = 1 \text{ for Lebesgue almost every } \xi_0 \in [0,1]. \tag{1.3}$$

For Hölder continuous p, the left-hand side of the equality in (1.3) is a continuous function of ξ_0. In this case, the equality in (1.3) must hold

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for every $\xi_0 \in [0,1]$ if p generates a scaling function. However, for some p which generate scaling functions, the equality in (1.3) fails on a set of Lebesgue measure zero.

An example of such a p is constructed in **[8]** starting from $p(\xi) = \cos^2(3\pi\xi)$, a QMF function with $p(1/3) = p(2/3) = 1$. That p takes the value one at $\xi = 1/3$ and $\xi = 2/3$ means that $B = \{1/3, 2/3\}$ is invariant (i.e., $\mathbb{P}_p(\xi_1 \in B \mid \xi_0) = 1$ for every $\xi_0 \in B$), so $\mathbb{P}_p(\xi_t \to 0 \text{ or } 1 \mid \xi_0) = 0$ if $\xi_0 \in B$. The sets $\{0\}$ and $\{1\}$ are invariant since $p(0) = p(1) = 1$, so the equality in (1.3) holds if $\xi_0 \in \{0,1\}$. To allow sample paths from initial points in the complement of B to converge to $\{0,1\}$, the function p is given sharp cusps at $1/3$ and $2/3$, with corresponding modifications near $1/6$ and $5/6$ to retain the QMF condition. Paths from points in the vicinity of B are still attracted to B, but B is "inaccessible": for every $\xi_0 \in B^c$, the sequence $\xi_0, \xi_1, \xi_2, \ldots$ converges to 0 or 1 with probability one. Hence, the equality in (1.3) holds for almost every $\xi_0 \in [0,1]$.

The set B in the above example is closed and invariant under multiplication by 2 (mod 1). Since every subset of $(0,1)$ with these properties has (Lebesgue) measure zero by the ergodicity of the doubling map, we may ask (cf., [**8,** page 1103]): given a closed set $B \subset (0,1)$, invariant under multiplication by 2 (mod 1), is there a p for which B is invariant, where

$$\mathbb{P}_p(\xi_t \to 0 \text{ or } 1 \mid \xi_0) = 1$$

for almost every $\xi_0 \in [0,1]$?

Our objective is to answer this question by establishing the following result, whose proof provides a characterization of those subsets of $(0,1)$ that are invariant with respect to some p (see (3.5) below).

Theorem 1.1. *If $B \subset (0,1)$ is closed, invariant under multiplication by* 2 (mod 1)*, and invariant for some p, then there is a $\widetilde{p}$ for which B is invariant, where $\mathbb{P}_{\widetilde{p}}(\xi_t \to 0$ or $1 \mid \xi_0) = 1$ for almost every $\xi_0 \in [0,1]$.*

This paper is organized as follows. The next section restates the main question on a space of binary sequences. Having seen the role played by invariant subsets of the sequence space, we return to the unit interval in Section 3, where Theorem 1.1 is proven. Section 4 concludes our study.

2. The dynamics of sample paths. The process (1.2) is conveniently studied on the set of all sequences $\boldsymbol{\xi} = (\dots, x_{-1}, x_0)$ of zeros and ones (see **[3, 7, 8, 9]**), viewed as binary representations of points of $[0,1]$. We denote this set by $\mathscr{X}$. The correspondence between $\mathscr{X}$ and $[0,1]$ is given by $\tau\colon \mathscr{X} \to [0,1]$, where $\tau(\boldsymbol{\xi}) = \sum_{j=0}^{\infty} x_{-j} 2^{-(j+1)}$. With the topology induced by the metric

$$\rho(\boldsymbol{\xi}, \boldsymbol{\xi}') = \begin{cases} 0 & \text{if } \boldsymbol{\xi} = \boldsymbol{\xi}', \\ 2^{-\min\{|j| : x_j \neq x'_j\}} & \text{if } \boldsymbol{\xi} \neq \boldsymbol{\xi}', \end{cases} \tag{2.1}$$

$\mathscr{X}$ becomes a compact space.

After composition with τ, a QMF function defines a continuous $g\colon \mathscr{X} \to [0,1]$ that satisfies

$$g((\boldsymbol{\xi}, 0)) + g((\boldsymbol{\xi}, 1)) = 1 \quad \text{for all } \boldsymbol{\xi} \in X, \quad g(\mathbf{0}) = g(\mathbf{1}) = 1. \tag{2.2}$$

Here $\mathbf{0} = (\dots, 0, 0)$ and $\mathbf{1} = (\dots, 1, 1)$. If we define

$$(\dots, x_{-1}, x_0)^* = (\dots, x_{-1}, 1 - x_0), \tag{2.3}$$

we can write the first condition in (2.2) as the requirement that

$$g(\boldsymbol{\xi}) + g(\boldsymbol{\xi}^*) = 1 \quad \text{for all } \boldsymbol{\xi} \in X. \tag{2.4}$$

Let $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ be the Markov process on $\mathscr{X}$ that goes from $\boldsymbol{\xi}_t$ to $(\boldsymbol{\xi}_t, j)$ with probability $g((\boldsymbol{\xi}_t, j))$, $j = 0, 1$, and let $d\boldsymbol{\xi}$ denote the infinite product of normalized counting measure on $\{0,1\}$. Then, (1.3) is equivalent to the condition that

$$\mathbb{P}_g(\boldsymbol{\xi}_t \to \mathbf{0} \text{ or } \mathbf{1} \mid \boldsymbol{\xi}_0) = 1 \quad \text{for } d\boldsymbol{\xi}\text{-almost every } \boldsymbol{\xi}_0 \in X. \tag{2.5}$$

Before we describe the structure of what Gundy **[8]** refers to as inaccessible invariant sets, we discuss the subshifts of finite type studied in [**7,** Section 13]. For $n \geq 2$, let $K(n)$ be the set of all $\boldsymbol{\xi} \in \mathscr{X}$ that do not contain a string (or word) of n consecutive zeros, or a string of n consecutive ones. Then,

$$K(2) = \{(\dots, 1, 0, 1, 0) \quad \text{and} \quad (\dots, 0, 1, 0, 1)\} = \tau^{-1}(B), \tag{2.6}$$

where $B = \{1/3, 2/3\}$ is the set discussed in the introduction. For every $n \geq 2$, we have that $K(n)$ is a closed shift-invariant proper subset of $\mathscr{X}$ (a subshift). Such sets have measure zero by the ergodicity of the shift with respect to $d\boldsymbol{\xi}$.

Suppose that we have defined g so that g is continuous and such that $K := K(3)$ is invariant, i.e., $\mathbb{P}_g(\boldsymbol{\xi}_1 \in K \mid \boldsymbol{\xi}_0) = 1$ for every $\boldsymbol{\xi}_0 \in K$. The last condition is met if and only if $g(\boldsymbol{\xi}) = 0$ for every $\boldsymbol{\xi} \in K_e$, where

$$K_e := \{\boldsymbol{\xi} \in K^c \colon (\dots, x_{-2}, x_{-1}) \in K\}$$

is the set of *points of exit* from K **[11]**. It is possible to define g in such a way that g has no zeros outside K_e besides the zeros at $\mathbf{0}^*$ and $\mathbf{1}^*$, which are required for $g(\mathbf{0}) = g(\mathbf{1}) = 1$ **[13]**. To prevent sample paths from initial points in the complement of K from converging to K, we modify g so that

$$U_{\text{exit}} := \{\boldsymbol{\xi} \in \mathscr{X} \colon (x_{-2}, x_{-1}, x_0) = (0,0,0) \text{ or } (1,1,1)\} \tag{2.7}$$

is visited infinitely often. (If $\boldsymbol{\xi}_t \in U_{\text{exit}}$, then $\rho(\boldsymbol{\xi}_t, K) \geq 2^{-3}$, so paths that visit U_{exit} infinitely often do not converge to K.) By Levy's conditional Borel-Cantelli lemma (see **[1]** or **[7,** Lemma 4.1**]**), we have $\boldsymbol{\xi}_t \in U_{\text{exit}}$ for infinitely many values of $t \geq 1$, $\mathbb{P}_g(\,\cdot \mid \boldsymbol{\xi}_0)$-almost surely, if

$$\sum_{t=0}^{\infty} \mathbb{P}_g(\boldsymbol{\xi}_{t+1} \in U_{\text{exit}} \mid \boldsymbol{\xi}_t) = +\infty, \quad \mathbb{P}_g(\,\cdot \mid \boldsymbol{\xi}_0)\text{-almost surely.} \tag{2.8}$$

The words $(0,0)$ and $(1,1)$ are *critical* in the sense that, if one of these words appears as the initial word in $\boldsymbol{\xi}_t$, then U_{exit} can be reached in one step. By our assumptions on g, the probability to reach

$$U_{\text{crit}} := \{\boldsymbol{\xi} \in \mathscr{X} \colon (x_{-1}, x_0) = (0,0) \text{ or } (1,1)\} \tag{2.9}$$

in at most two steps is positive for every $\boldsymbol{\xi}_0 \in X$. (If $\boldsymbol{\xi}_0 \in U_{\text{crit}}$, no steps have to be taken. If $\boldsymbol{\xi}_0 \in K$, then either $(\boldsymbol{\xi}_0, 0)$ and $(\boldsymbol{\xi}_0, 0, 0)$ are both in K, or $(\boldsymbol{\xi}_0, 1)$ and $(\boldsymbol{\xi}_0, 1, 1)$ are both in K. Since g is strictly positive on K, we can then reach U_{crit} in two steps. Finally, if $\boldsymbol{\xi}_0 \in (K \cup U_{\text{crit}})^c$, then neither $(\boldsymbol{\xi}_0, 0)$ nor $(\boldsymbol{\xi}_0, 1)$ is in $K_e \cup \{\mathbf{0}^*, \mathbf{1}^*\}$, so both transitions have positive probability. Since $(\boldsymbol{\xi}_0, 1) \in U_{\text{crit}}$ if $(\boldsymbol{\xi}_0, 0) \notin U_{\text{crit}}$, we can then reach U_{crit} in one step.) The strictly positive finite-step transition probability is a continuous function of $\boldsymbol{\xi}_0$; thus, it is bounded away from zero. By the Renewal theorem, we can find $\beta > 0$, not dependent on $\boldsymbol{\xi}_0 \in X$ such that the recurrence times $t_1, t_2, \dots$ for critical words (i.e., the times when $\boldsymbol{\xi}_t \in U_{\text{crit}}$) satisfy $t_j \leq \beta j$, $\mathbb{P}_g(\,\cdot \mid \boldsymbol{\xi}_0)$-almost surely. Setting $g = |\log_2 \rho(\boldsymbol{\xi}, K_e)|^{-1}$ on $U_{\text{exit}} \setminus K_e$, with a corresponding modification on ${U_{\text{exit}}}^* \setminus (K_e)^*$, we get

$$\mathbb{P}_g(\boldsymbol{\xi}_{t_j+1} \in U_{\text{exit}} \mid \boldsymbol{\xi}_{t_j}) \geq \frac{1}{l+t_j} \geq \frac{1}{l+\beta j},$$

where l is the integer with $\rho(\boldsymbol{\xi}_0, K) = 2^{-l}$. (Here we have used the fact that $\rho(\boldsymbol{\xi}_0, K) = 2^{-l}$ implies $\rho(\boldsymbol{\xi}_t, K_e) \geq 2^{-(t+l)}$: the initial word in $\boldsymbol{\xi}_t$ of length $t+l$ cannot be the initial word of a point of K_e if the initial word in $\boldsymbol{\xi}_0$ of length l does not appear in a point of K.) Since (2.9) holds, U_{exit} is visited infinitely often. If we set $g \equiv 1$ on a neighborhood of $\{\mathbf{0}, \mathbf{1}\}$, then $\mathbb{P}_g(\boldsymbol{\xi}_t \to \mathbf{0} \text{ or } \mathbf{1} \mid \boldsymbol{\xi}_0)$ is positive for every $\boldsymbol{\xi}_0 \in U_{\text{exit}}$. We then obtain that the equality in (2.5) holds for all $\boldsymbol{\xi}_0 \in K^c$, hence almost everywhere.

The above construction relies (only) upon the assumption that K is a subshift of finite type [**15,** Definition 2.1.1]. If K is a g-invariant subshift that is not of finite type, then g must take the value zero at some point of K [**13**]. (The frontier of K_e is a non-empty subset of K if K is not of finite type [**11**]. Since g must vanish on $\overline{K_e}$ if g is continuous and K is invariant, we must then have $g(\boldsymbol{\xi}) = 0$ for certain $\boldsymbol{\xi} \in K$.) This may leave us without a lower bound on the probability to encounter a critical word in any number of steps. However, as long as the zeros of g are contained in $\overline{K_e} \cup (U_{\text{exit}})^*$, the set $\{\mathbf{0}, \mathbf{1}\}$ remains accessible from any $\boldsymbol{\xi}_0 \in K^c$ in the sense that $\mathbb{P}_g(\rho(\boldsymbol{\xi}_k, \{\mathbf{0}, \mathbf{1}\}) \leq 2^{-k} \mid \boldsymbol{\xi}_0) > 0$ for every $k \geq 1$. Consider, therefore, a sequence of (dependent) trials, where trial $n \geq 0$ consists of the attempt to reach

$$U_{0,1} := \left\{ \boldsymbol{\xi} \in X \colon (x_{-k}, \ldots, x_0) = (\underbrace{0, 0, \ldots, 0}_{k+1 \text{ zeros}}) \text{ or } (\underbrace{1, 1, \ldots, 1}_{k+1 \text{ ones}}) \right\} \tag{2.10}$$

by k consecutive steps towards either $\mathbf{0}$ or $\mathbf{1}$, depending on whether the initial symbol in $\boldsymbol{\xi}_{nk}$ is 0 or 1. For k so large that $U_{0,1}$ is disjoint from K and with $g(\boldsymbol{\xi}) = |\log_2 \rho(\boldsymbol{\xi}, \overline{K_e})|^{-1/k}$ on a neighborhood of $\overline{K_e}$, we obtain (below), for some $\lambda' > 0$ and all $n \geq 1$, that

$$\mathbb{P}_g(\boldsymbol{\xi}_{nk+k} \in U_{0,1} | \boldsymbol{\xi}_{nk}) \geq \frac{\lambda'}{l+nk+k}, \quad \text{where } l = |\log_2 \rho(\boldsymbol{\xi}_0, K)|. \tag{2.11}$$

Setting $g \equiv 1$ on $U_{0,1}$ achieves (2.5) since $U_{0,1}$ is visited infinitely often if $\boldsymbol{\xi}_0 \in K^c$, again by Borel-Cantelli.

A construction of the second type is possible whenever $K \subset X \setminus \{\mathbf{0}, \mathbf{1}\}$ satisfies

$$\overline{K_e} \cap (\overline{K_e})^* = \emptyset. \tag{2.12}$$

This condition is necessary if we require that g be continuous, for g-invariance then implies that $g(\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi} \in \overline{K_e}$ (the closure of K_e), and hence, that $g(\boldsymbol{\xi}) = 1$ for all $\boldsymbol{\xi} \in (\overline{K_e})^*$ (cf., **[11, 13]**). The construction does not answer the question from the introduction, however, as it does not provide a continuous $p(\xi)$, $\xi \in \mathbb{R}$. To answer the question that we began with, we return to the unit interval.

3. Proving Theorem 1.1.

3.1. Definitions. When we say that $B \subset (0,1)$ is invariant under multiplication by 2 (mod 1), we mean that, if B is considered as a subset of the circle $[0,1)$, then $B = \theta(B)$, where $\theta(\xi) := 2\xi \pmod 1$.

The map $\xi \mapsto \xi^* = \xi + 1/2 \pmod 1$, which is unambiguously defined on $[0,1)$, corresponds to the map in (2.3). We define ξ^* for all $\xi \in [0,1]$ by

$$\xi^* = \begin{cases} \xi + 1/2 & \text{if } \xi \in [0,1/2], \\ \xi - 1/2 & \text{if } \xi \in (1/2,1]. \end{cases} \tag{3.1}$$

The first condition in (1.1) then says that

$$p(\xi) + p(\xi^*) = 1 \quad \text{for every } \xi \in [0,1]. \tag{3.2}$$

For $E \subset [0,1]$, we let $E^* = \{\xi^* : \xi \in E\}$. Finally, the distance between $\xi \in [0,1]$ and E is given by

$$d_E(\xi) = \inf_{\xi' \in E} |\xi - \xi'|. \tag{3.3}$$

3.2. The structure of invariant sets. A set $B \subset [0,1]$ is invariant for the process (1.2) if and only if $p(\xi) = 0$ for every $\xi \in B_e$, where

$$B_e = \{\xi \in [0,1] \setminus B : \xi = \xi'/2 + j/2 \text{ for some } \xi' \in B, j \in \{0,1\}\}. \tag{3.4}$$

(If we identify 0 and 1, we can write $B_e = \{\xi \in [0,1) \setminus B : \theta(\xi) \in B\}$.) If this condition holds, then we have $p(\xi) = 0$ for every $\xi \in \overline{B_e}$, and consequently, $p(\xi) = 1$ for every $\xi \in (\overline{B_e})^*$, so that

$$\overline{B_e} \cap \overline{(B_e)}^* = \emptyset. \tag{3.5}$$

The proof of Theorem 1.1 shows that every closed θ-invariant $B \subset (0,1)$ which satisfies (3.5) is invariant for some p.

3.3. Proof of Theorem 1.1. The proof of Theorem 1.1 proceeds in two steps. Given a closed θ-invariant $B \subset (0,1)$ that satisfies (3.5), we first construct p such that B is invariant. We then verify that p satisfies (1.3).

Step 1: Construction. Our construction relies on the following result.

Lemma 3.1. *Suppose that $B \subset (0,1)$ is closed and θ-invariant.*

(a) $\{0, 1/2\} \cap (B \cup \overline{B_e} \cup (\overline{B_e})^*) = \emptyset$.

(b) *If B satisfies* (3.5), *there is a closed $N_e \subset [0,1]$ such that*

(i) *N_e contains $\overline{B_e}$ is in its interior*;
(ii) $N_e \cap N_e^* = \emptyset$;
(iii) $\{0, 1/2, 1\} \cap (N_e \cup N_e^*) = \emptyset$.

Proof.

(a) That B is θ-invariant means that $1/2 \notin B$. Thus, we can find $\varepsilon > 0$ such that

$$B \subset C_\varepsilon := (\varepsilon, 1/2 - \varepsilon) \cup (1/2 + \varepsilon, 1 - \varepsilon).$$

Then, we have $B_e \subset C_\varepsilon$. It follows that

$$\{0, 1/2\} \cap (B \cup \overline{B_e} \cup (\overline{B_e})^*) = \emptyset.$$

(b) That B satisfies (3.5) means that we can take $\delta > 0$ so that $|\xi - \xi'| > \delta$ if $\xi \in \overline{B_e}$ and $\xi' \in (\overline{B_e})^*$. We can cover the (compact) set $\overline{B_e}$ by a finite union of closed intervals whose lengths do not exceed $\delta/3$ and that each contain a point of $\overline{B_e}$ in its interior. If we let N_e be such a union, then N_e contains $\overline{B_e}$ in its interior. The set N_e^* is a finite union of closed intervals whose lengths do not exceed $\delta/3$ and that each contain a point of $(\overline{B_e})^*$ in its interior. Since

$$\{0,1/2,1\} \cap (\overline{B_e} \cup (\overline{B_e})^*) = \emptyset,$$

we have

$$\{0,1/2,1\} \cap (N_e \cup N_e^*) = \emptyset,$$

if we take the intervals that define N_e sufficiently short. Our choice of δ gives $N_e \cap N_e^* = \emptyset$. □

Now, given a closed θ-invariant $B \subset (0,1)$ that satisfies (3.5), let N_e be as in Lemma 3.1. Choose $\varepsilon > 0$ so small that $N_e \cup (N_e)^*$ is disjoint from

$$N_{0,1}{:=}[0,\varepsilon] \cup [1-\varepsilon, 1].$$

Then, $N_e \cup (N_e)^*$ is also disjoint from

$$N_{1/2}{:=}[1/2-\varepsilon, 1/2+\varepsilon] = N_{0,1}^*.$$

Fix a positive integer k with $2^{-k} < \varepsilon$. (This choice of k will ensure that, starting from any $\xi_0 \in [0,1]$ and using the transitions (1.2), we can reach $N_{0,1}$ by k consecutive steps towards 0, or by k consecutive steps towards 1.) Define

$$p(\xi) = \begin{cases} |\log_2(d_{B_e}(\xi))|^{-1/k} & \text{if } \xi \in N_e \setminus \overline{B_e}, \\ 0 & \text{if } \xi \in \overline{B_e}, \\ 0 & \text{if } \xi \in N_{1/2}. \end{cases} \tag{3.6}$$

For $\xi \in (N_e)^* \cup N_{0,1}$, let $p(\xi) = 1 - p(\xi^*)$. Now, p is defined on

$$N{:=}N_e \cup (N_e)^* \cup N_{1/2} \cup N_{0,1},$$

and the equality in (3.2) holds if $\xi \in N$. Extend p to $[0,1/2] \setminus N$ continuously in such a way that $0 < p(\xi) < 1$ for all $\xi \in [0,1/2] \setminus N$. If we set $p(\xi) = 1 - p(\xi^*)$ for $\xi \in [1/2,1] \setminus N$ and extend periodically, then p is a QMF function with $\{\xi \in [0,1] \colon p(\xi) = 0\} = \overline{B_e} \cup N_{1/2}$. In particular, B is invariant for p.

Step 2: Condition (1.3). The verification of (1.3) uses (3.9) below, which gives an estimate on the speed at which sample paths can approach B_e. We take the sample space for the process (1.2) to be the set $\{0,1\}^{\mathbb{N}}$ of all binary sequences $x^+ = (x_1, x_2, \dots)$, each $x_i \in \{0,1\}$, and define the sample path $\xi_t(x^+)$ from a fixed $\xi_0 \in [0,1]$ recursively, via

$$\xi_t = \xi_{t-1}/2 + x_t/2, \quad t \geq 1. \tag{3.7}$$

Note that the estimates in (3.8) and (3.9) below do not involve p.

Lemma 3.2. *Let $B \subset (0,1)$ be closed and θ-invariant, and let $\xi_0 \in B^c$. There is a constant $\alpha = \alpha(\xi_0) > 0$ such that, for any sample path $\xi_t = \xi_t(x^+), t \geq 0$, from ξ_0,*

$$d_B(\xi_t) \geq \alpha 2^{-t} \quad \textit{for all } t \geq 0, \tag{3.8}$$

$$d_{B_e}(\xi_t) \geq \alpha 2^{-t} \quad \textit{for all } t \geq 1. \tag{3.9}$$

Proof. Since $1/2 \notin B \cup \overline{B_e}$ (Lemma 3.1 (a)), we can pick $\delta \in (0,1)$ such that $|\xi - 1/2| > \delta$ for all $\xi \in B \cup \overline{B_e}$. Let $\xi_0 \in B^c$ be given, and consider the sample path $\xi_t = \xi_t(x^+)$ defined by $x^+ \in \{0,1\}^{\mathbb{N}}$ and the recursion (3.7). We show that (3.8) holds with

$$\alpha := \min(\delta, d_B(\xi_0)).$$

This choice of $\alpha > 0$ gives $d_B(\xi_t) \geq \alpha 2^{-t}$ if $t = 0$. Thus, it is sufficient to show that $d_B(\xi_{t-1}) \geq \alpha 2^{-(t-1)}$ implies $d_B(\xi_t) \geq \alpha 2^{-t}$ for all $t \geq 1$. Suppose, therefore, that $d_B(\xi_{t-1}) \geq \alpha 2^{-(t-1)}$, where $t \geq 1$. To estimate $d_B(\xi_t)$, fix an arbitrary $\xi \in B$. Since B is θ-invariant, we can write $\xi = \xi'/2 + j/2$ with $\xi' \in B$ and $j \in \{0,1\}$. If $j \neq x_t^+$, then $|\xi_t - \xi| \geq \delta$. (This is since $B \subset C_\delta := (0, 1/2 - \delta) \cup (1/2 + \delta, 1)$.) In this case, we immediately obtain $|\xi_t - \xi| \geq \alpha 2^{-t}$ from the definition of α. If $x_t^+ = j$, then

$$|\xi_t - \xi| = \left| \frac{\xi_{t-1}}{2} + \frac{x_t^+}{2} - \left(\frac{\xi'}{2} + \frac{j}{2} \right) \right| = |\xi_{t-1} - \xi'|/2 \geq d_B(\xi_{t-1})/2.$$

Using that $d_B(\xi_{t-1}) \geq \alpha 2^{-(t-1)}$, we get $|\xi_t - \xi| \geq \alpha 2^{-t}$. Since $\xi \in B$ was arbitrary, $d_B(\xi_t) \geq \alpha 2^{-t}$.

Now, we prove (3.9). Let $t \geq 1$. To estimate $d_{B_e}(\xi_t)$, let $\xi \in B_e$, so that (by the definition of B_e) $\xi = \xi'/2 + j/2$ for some $\xi' \in B$ and $j \in \{0,1\}$. If $j \neq x_t^+$, then $|\xi_t - \xi| \geq \delta$. (This follows from $B_e \subset C_\delta$.) Thus, $|\xi_t - \xi| \geq \alpha 2^{-t}$ if $j \neq x_t^+$. If $x_t^+ = j$, then

$$|\xi_t - \xi| = \left|\frac{\xi_{t-1}}{2} + \frac{x_t^+}{2} - \left(\frac{\xi'}{2} + \frac{j}{2}\right)\right| = |\xi_{t-1} - \xi'|/2 \geq d_B(\xi_{t-1})/2,$$

so $|\xi_t - \xi| \geq \alpha 2^{-t}$. Since $\xi \in B_e$ was arbitrary, $d_{B_e}(\xi_t) \geq \alpha 2^{-t}$. □

Let B, N_e, $N_{0,1}$, $N_{1/2}$ and p be as in Step 1. Since B has measure zero, we are done if we can show that

$$\mathbb{P}_p(\xi_t \longrightarrow 0 \text{ or } 1 \mid \xi_0) = 1$$

for every $\xi_0 \in B^c$. Let ξ_0 be any point of B^c. That $p \equiv 1$ on $N_{0,1}$ means that, if a sample path from ξ_0 reaches $N_{0,1}$, it goes to 0 (if it reaches $[0, \varepsilon]$) or 1 (if it reaches $[1-\varepsilon, 1]$). Thus, it suffices to show that $\xi_t \in N_{0,1}$ for some t, $\mathbb{P}_p(\,\cdot \mid \xi_0)$-almost surely. By Borel-Cantelli, we will have $\xi_t \in N_{0,1}$ for some (in fact, infinitely many) t, if

$$\sum_{n=0}^{\infty} \mathbb{P}_p(\xi_{nk+k} \in N_{0,1} \mid \xi_{nk}) = +\infty, \quad \mathbb{P}_p(\cdot \mid \xi_0)\text{-almost surely.} \tag{3.10}$$

We verify (3.10) by showing that there is a constant $\lambda \in (0,1)$ and $a > 0$ such that, for all $n \geq 1$,

$$\mathbb{P}_p(\xi_{nk+k} \in N_{0,1} \mid \xi_{nk}) \geq \frac{\lambda}{a + nk + k}, \tag{3.11}$$

$\mathbb{P}_p(\,\cdot \mid \xi_0)$-almost surely.

Case 1: $\xi_{nk} \leq 1/2$. Then, $\xi_{nk}/2^k \in N_{0,1}$ by our choice of k; thus,

$$\mathbb{P}_p(\xi_{nk+k} \in N_{0,1} \mid \xi_{nk}) \geq \mathbb{P}_p(\xi_{nk+k} = \xi_{nk}/2^k \mid \xi_{nk}) = \prod_{i=1}^{k} p(\xi_{nk}/2^i).$$

That $\xi_{nk} \leq 1/2$ implies that $\xi_{nk}/2^i \leq 1/4$ for all $i \geq 1$. Since $\{\xi \colon p(\xi) = 0\} = \overline{B_e} \cup N_{1/2}$ and $\overline{B_e}$ is in the interior of N_e, we can find $c \in (0,1)$ such that $p(\xi) \geq c$ for all

$$\xi \in ([0,1] \setminus N_e) \cap ([0,1/4] \cup [3/4,1]).$$

Then, we have

$$\prod_{i=1}^{k} p(\xi_{nk}/2^i) \geq c^k \quad \text{if } \xi_{nk}/2^i \notin N_e \text{ for } i = 1, \ldots, k.$$

To verify that (3.11) holds, we need a lower bound on $\prod_{i=1}^{k} p(\xi_{nk}/2^i)$ for the case when $\xi_{nk}/2^i \in N_e$ for at least one $i \in \{1,\dots,k\}$. By Lemma 3.2, we can choose $\alpha_0 > 0$ so that $d_{B_e}(\xi_t(x^+)) \geq 2^{-t-\alpha_0}$ for every sample path $\xi_t(x^+)$ from ξ_0. For $i \in \{1,\dots,k\}$, take $x^+ \in \{0,1\}^{\mathbb{N}}$ so that $\xi_{nk+i}(x^+) = \xi_{nk}/2^i$. (The first nk entries of x^+ define the itinerary from ξ_0 to ξ_{nk}, and $x^+_{nk+j} = 0$ for $j = 1,\dots,i$.) If $\xi_{nk}/2^i \in N_e$, the definition (3.6) of p together with (3.9) gives

$$\begin{aligned} p(\xi_{nk}/2^i) = |\log_2(d_{B_e}(\xi_{nk}/2^i))|^{-1/k} &= |\log_2(d_{B_e}(\xi_{nk+i}(x^+)))|^{-1/k} \\ &\geq |\log_2(2^{-(nk+i+\alpha_0)})|^{-1/k} \\ &= \left(\frac{1}{\alpha_0 + nk + i}\right)^{1/k}. \end{aligned}$$

Letting $i_1,\dots,i_m$ be the m ($m \leq k$) integers i with $\xi_{nk}/2^i \in N_e$,

$$\prod_{i=1}^{k} p(\xi_{nk}/2^i) \geq c^{k-m} \prod_{j=1}^{m} \frac{1}{(\alpha_0 + nk + i_j)^{1/k}} \geq c^{k-m} \cdot \frac{1}{\alpha_0 + nk + k}.$$

This shows that (3.11) holds with $a = \alpha_0$ and $\lambda = c^k$.

Case 2: $\xi_{nk} > 1/2$. If $\xi_{nk} > 1/2$, then $N_{0,1}$ can be reached by k consecutive steps to the right: $\xi_{nk+i} = \xi_{nk+i-1}/2 + 1/2$ for $i = 1,\dots,k$. We then have $\xi_{nk+i} \geq 3/4$, and the above c bounds $p(\xi_{nk+i})$ when $\xi_{nk+i} \in ([0,1] \setminus N_e) \cap ([0,1/4] \cup [3/4,1])$. Lemma 3.2 and the argument in Case 1 give $p(\xi_{nk+i}) \geq (\alpha_0 + nk + i)^{-1/k}$ when $\xi_{nk+i} \in N_e$. This means that (3.11) again holds with $a = \alpha_0$ and $\lambda = c^k$.

Since $n \geq 1$ was arbitrary, (3.11) holds for all $n \geq 1$ with $a = \alpha_0(\xi_0)$ and $\lambda = c^k$. Hence, (3.10) is satisfied.

4. Concluding remarks.

A QMF function $p(\xi)$, $\xi \in \mathbb{R}$, generates a scaling function for an MRA if and only if the infinite product

$$\widehat{\Phi}_p(\xi) := \prod_{j=1}^{\infty} p(\xi/2^j), \quad \xi \in \mathbb{R}, \tag{4.1}$$

satisfies (see **[8, 10]**)

$$\sum_{k \in \mathbb{Z}} \widehat{\Phi}_p(\xi + k) = 1 \quad \text{for almost every } \xi \in [0,1], \tag{4.2}$$

$$\lim_{j\to\infty} \widehat{\Phi}_p(2^{-j}\xi) = 1 \quad \text{for almost every } \xi \in \mathbb{R}. \tag{4.3}$$

That (4.3) holds for the p that we constructed in the previous section follows from the fact that this $p \equiv 1$ is on an open interval containing 0. That the equality in (4.2) holds almost everywhere for this p follows from the fact that, for every p and *every* $\xi_0 \in [0, 1]$, we have (see **[8]**)

$$\sum_{k\in\mathbb{Z}} \widehat{\Phi}_p(\xi_0 + k) = \mathbb{P}_p(\xi_t \longrightarrow 0 \text{ or } 1 \mid \xi_0). \tag{4.4}$$

The discovery of continuous p for which $\sum_{k\in\mathbb{Z}} \widehat{\Phi}_p(\xi + k) = 1$ fails on a set of measure zero was made in **[5]**. The notion of an inaccessible invariant set comes from **[7]**, where the example from **[5]** is included in a class of such invariant sets obtained from subshifts of finite type. In this paper, we have described their structure completely.

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