ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 8, 2018

FINITE ATOMIC LATTICES AND THEIR MONOMIAL IDEALS

PENG HE AND XUE-PING WANG

ABSTRACT. This paper primarily studies monomial ideals by their associated lcm-lattices. It first introduces notions of weak coordinatizations of finite atomic lattices which have weaker hypotheses than coordinatizations and shows the characterizations of all such weak coordinatizations. It then defines a finite super-atomic lattice in $\mathcal{L}(n)$, investigates the structures of $\mathcal{L}(n)$ by their super-atomic lattices and proposes an algorithm to calculate all of the super-atomic lattices in $\mathcal{L}(n)$. It finally presents a specific labeling of finite atomic lattice and obtains the conditions that the specific labelings of finite atomic lattices are the weak coordinatizations or the coordinatizations by using the terminology of super-atomic lattices.

1. Introduction. Let M be a monomial ideal in a polynomial ring $R = K[x_1, x_2, \ldots, x_n]$ where K is a field. We are interested in studying a minimal free resolution of R/M and, specifically, understanding the maps in this resolution (see [**1, 4, 6, 13, 14**]). For a monomial ideal M, a minimal resolution is completely dependent on the information in the lcm-lattice of M, or LCM(M), which is the lattice of least common multiples of the minimal generators of M partially ordered by divisibility. In 1999, Gasharov, Peeva, and Welker [**7**] expressed the multigraded Betti numbers of R/M using the homology groups of certain open intervals in LCM(M). They further showed that the combinatorial type of minimal resolutions of a monomial ideal is determined by its LCM lattice. In 2006, Phan [**12**] proved that all finite atomic lattices can be realized as the LCM lattice of some monomial ideal M. He gave a construction which is motivated by the observation

---

2010 AMS *Mathematics subject classification.* Primary 06D05, 13D02.

Keywords and phrases. Monomial ideal, finite atomic lattice, coordinatization, weak coordinatization, super-atomic lattice, labeling.

This research was supported by the National Natural Science Foundation of China, grant No. 11171242. The second author is the corresponding author.

Received by the editors on January 3, 2017, and in revised form on April 18, 2018.

DOI:10.1216/RMJ-2018-48-8-2503

that, for any coordinatization of an atomic lattice as a monomial ideal, the set of lattice elements for which a given variable has a given degree bound is an order ideal. Essentially, he identified which order ideals are necessary and labeled them with variables. In 2009, Mapes [**9**] gave a generalization of the main construction in [**12**] to describe all monomial ideals with a given LCM lattice, i.e., she proved a statement as below (see [**9, 10**]).

Any labeling $\mathcal{M}$ of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of the lattice P.

(A1) If $p \in \mathrm{mi}(P)$, then $m_p \neq 1$, i.e., all meet-irreducibles are labeled.

(A2) If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$, then p and q must be comparable, i.e., each variable only appears in monomials along one chain in P.

Mapes thought that it would be interesting to give an explicit formulation for when two coordinatizations are equivalent in this sense or to prove a version of the above result which has weaker hypotheses. This question was inadvertently answered by Katthän [**8**] and independently by Mapes and Piechnik [**11**] using different techniques. However, they do not give a general construction of the labeling $\mathcal{M}$, which does not satisfy conditions (A1) and (A2), although, in fact, $\mathcal{M}$ is a coordinatization.

On the other hand, the fact that the set of finite atomic lattices on n ordered atoms, denoted by $\mathcal{L}(n)$, is itself a finite atomic lattice leads us to the question: what is the relationship between minimal resolutions of coordinatizations of lattices in $\mathcal{L}(n)$? The answer, due to a result in [**7**], is that the total Betti numbers are weakly monotonic along chains in $\mathcal{L}(n)$. This inspires us to understand the structure of $\mathcal{L}(n)$. In 2013, Mapes [**10**] proved that, for any relation $P > Q$ in $\mathcal{L}(n)$, there exists a coordinatization of Q producing a monomial ideal M_Q and a deformation of exponents of M_Q such that the lcm-lattice of the deformed ideal is P.

This paper furthers the topics on describing all monomial ideals by their LCM lattices and understanding the structure of $\mathcal{L}(n)$, and is organized as follows. In Section 2, we give some preliminaries for convenience. In Section 3, we introduce notions of weak coordinatizations of finite atomic lattices and show their characterizations. In Section 4, we define a finite super-atomic lattice in $\mathcal{L}(n)$, investigate the structures of $\mathcal{L}(n)$ by their super-atomic lattices and propose an algorithm to calculate all the super-atomic lattices in $\mathcal{L}(n)$. At the end, we present a specific labeling of finite atomic lattice and obtain the conditions which are used to determine whether the specific labelings are weak coordinatizations or coordinatizations by terminology of super-atomic lattices.

2. Preliminaries. A poset is a structure $(P, \leq)$ where P is a nonempty set and $\leq$ an ordering (reflexive, antisymmetric and transitive) relation on P. We write $x\|y$ if $x \not\geq y$ and $y \not\geq x$, and we say that x and y are not comparable. Conversely, we write $x \nparallel y$ if $x \geq y$ or $y \geq x$, and we say that x and y are comparable. In addition, if $x < y$ and there is no element $z \in P$ such that $x < z < y$, then we say that x is covered by y (or y covers x), and we write $x \prec y$ (or $y \succ x$), see **[5]**.

Definition 2.1 (**[10]**). A lattice is a *poset* $(P, \leq)$ satisfying the following properties:

(1) P has a maximum element denoted by 1.

(2) P has a minimum element denoted by 0.

(3) Every pair of elements a and b in P has a join $a \vee b$ which is the least upper bound of the two elements.

(4) Every pair of elements a and b in P has a meet $a \wedge b$ which is the greatest lower bound of the two elements.

If P satisfies only conditions (2) and (4), then it is a *meet-semilattice*, and if P satisfies only conditions (1) and (3), then it is a *join-semilattice*. Furthermore, if P is a meet-semilattice with a unique maximal element, then it is a lattice. Equivalently, if P is a join-semilattice with a unique minimal element, then it is a lattice.

We define an atom of a lattice P to be an element $x \in P$ such that x covers 0. We denote the set of atoms in P by atoms(P), see **[5, 10]**. Let A and B be two sets. Then, we denote that $A \setminus B = \{x \in A : x \notin B\}$; for convenience, if $B = \{b\}$, then we write $A \setminus B$ as $A \setminus b$.

Definition 2.2 ([**10**]). If P is a lattice and every element in $P \setminus 0$ is the join of atoms, then P is an *atomic lattice*. Furthermore, if P is finite, then it is a *finite atomic lattice*.

If P is a lattice, then we define an element $x \in P$ to be meet-irreducible if $x \neq a \wedge b$ for any $a > x$, $b > x$. We denote the set of meet-irreducible elements in P by $\mathrm{mi}(P)$. Given an element $x \in P$, an order ideal of x is defined to be the set $\lfloor x \rfloor = \{a \in P : a \leq x\}$. Similarly, we define an order filter of x to be $\lceil x \rceil = \{a \in P : x \leq a\}$, see [**5, 10**].

Lemma 2.3 ([**10,** Lemma 2.3]). *Let P be a finite atomic lattice. Every element $p \in P$ is the meet of all the meet-irreducible elements l such that $l \geq p$.*

It will be convenient to consider finite atomic lattices as sets of sets in the following way. Let $\mathcal{S}$ be a set of subsets of $\{1, \ldots, n\}$ with no duplicates, closed under intersections, and containing the entire set, the empty set and the sets $\{i\}$ for all $1 \leq i \leq n$. Then, it is easy to see that $\mathcal{S}$ is a finite atomic lattice by ordering the sets in $\mathcal{S}$ by inclusion. Conversely, it is clear that any finite atomic lattice P can be expressed in this way, simply by letting

$$\mathcal{S}_P = \{\sigma : \sigma = \mathrm{supp}(p),\ p \in P\},$$

where $\mathrm{supp}(p) = \{a_i : a_i \leq p, a_i \in \mathrm{atoms}(P)\}$, see [**2, 3, 10**].

Definition 2.4 ([**7**]). The LCM *lattice*, $\mathrm{LCM}(M)$, of a monomial ideal M is the set of least common multiples of minimal generators of M, partially ordered by divisibility.

Example 2.5. For the monomial ideal $M = (a^2cd, abd, abc) \subseteq k[a, b, c, d]$, the Hasse diagram of the LCM lattice of M is shown in Figure 1 (note the minimal element of the lattice has been eliminated, as will often be the case).

One result in [**7**] is that, for monomial ideals, all minimal resolutions are completely dependent on the information in the LCM lattice. Specifically, we can compute multigraded Betti numbers using the LCM lattice $\mathrm{LCM}(M)$, and all ideals with a given LCM lattice have isomorphic minimal free resolutions.

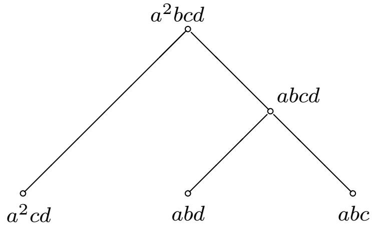

FIGURE 1. The lattice LCM(M).

Definition 2.6 ([**9**])**.** Define a *labeling* of a finite atomic lattice P to be any assignment of non-trivial monomials $\mathcal{M} = \{m_{p_1}, \ldots, m_{p_t}\}$ to some set of elements $p_i \in P$. It will be convenient to think of unlabeled elements as having the label 1. Define a *monomial ideal* $M_{P,\mathcal{M}}$ to be the ideal generated by monomials

$$x(a) = \prod_{p \in \lceil a \rceil^c} m_p \tag{2.1}$$

for each $a \in \text{atoms}(P)$, where $\lceil a \rceil^c$ means taking the complement of $\lceil a \rceil$ in P. We say that the labeling $\mathcal{M}$ is a coordinatization if the lcm-lattice of $M_{P,\mathcal{M}}$ is isomorphic to P.

Lemma 2.7 ([**9,** Proposition 3.2.1], [**10,** Theorem 3.2])**.** *Any labeling $\mathcal{M}$ of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of the lattice P.*

(A1) *If $p \in \text{mi}(P)$, then $m_p \neq 1$, i.e., all meet-irreducibles are labeled.*

(A2) *If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$, then $p \not\parallel q$, i.e., each variable only appears in monomials along one chain in P.*

Let $\mathcal{M}$ be a labeling with conditions (A1) and (A2), and let

$$f : P \longrightarrow \text{LCM}(M_{P,\mathcal{M}})$$

be denoted by

$$f(p) = \prod_{q \in \lceil p \rceil^c} m_q \tag{2.2}$$

for each $p \in P$. Then, f is an isomorphism from P to LCM($M_{P,\mathcal{M}}$).

Lemma 2.8 ([**10,** Lemma 3.3]). *If $p \in \lceil q \rceil^c$ for some $p, q \in P$, where P is a finite atomic lattice, then $\lfloor p \rfloor \subseteq \lceil q \rceil^c$.*

Let M be a monomial ideal with n generators, and let P_M be its lcm-lattice. For notational purposes, denote P_M as the set consisting of elements $\overline{p}$, which represent the monomials occurring in P_M. Now, define an abstract finite atomic lattice P, where the elements in P are formal symbols p satisfying the relations $p < p'$ if and only if $\overline{p} < \overline{p}'$ in P_M, in other words, P is the finite atomic lattice isomorphic to P_M obtained by simply forgetting the data of the monomials in P_M. Define a labeling of P in the following manner: let $\mathcal{D}$ be the set consisting of monomials m_p for each $p \in P$, defined by

$$m_p = \frac{\gcd\{\overline{t} : t > p\}}{\overline{p}}, \tag{2.3}$$

where, by convention, $\gcd\{\overline{t} : t > p\}$ for $p = 1$ is defined to be $\overline{1}$. Note that m_p is a monomial since, clearly, $\overline{p}$ divides $\overline{t}$ for all $t > p$.

Lemma 2.9 ([**10,** Proposition 3.6]). *Given M a monomial ideal with* lcm*-lattice P_M, if P is an abstract finite atomic lattice where P is isomorphic to P_M as lattices, then the labeling $\mathcal{D}$ of P, defined by* (2.3)*, is a coordinatization, and the resulting monomial ideal is $M_{P,\mathcal{D}} = M$.*

Although Lemma 2.9 shows that the labeling $\mathcal{D}$ of P, defined by (2.3), is a coordinatization, the following theorem will further verify that the labeling $\mathcal{D}$ induced by (2.3) is the same as $\mathcal{M}$ if $\mathcal{M}$ satisfies the conditions of Lemma 2.7. As is standard, we denote $\operatorname{lcm} \emptyset = 1$ and $\gcd \emptyset = 1$.

Theorem 2.10. *Let $\mathcal{M} = \{m_p : p \in P\}$ be a labeling of a finite atomic lattice P satisfying the conditions of Lemma* 2.7*, and let $M = M_{P,\mathcal{M}}$ for each $p \in P$, $\overline{p} = f(p)$, where $f(p)$ is defined by* (2.2)*. Then, the labeling*

$$\mathcal{D} = \{m'_p : p \in P\}$$

of P, defined by (2.3)*, satisfies $m'_p = m_p$ for each $p \in P$.*

Proof. Suppose that P has n atoms. First, note that

$$\overline{p} = f(p) = \prod_{q \in \lceil p \rceil^c} m_q \quad \text{for all } p \in P.$$

Thus, formula (2.3) implies that

$$\begin{aligned} m'_p &= \frac{\gcd\{\prod_{q \in \lceil t \rceil^c} m_q : t > p\}}{\prod_{q \in \lceil p \rceil^c} m_q} \\ &= \frac{\prod_{q \in \lceil p \rceil^c} m_q * \gcd\{\prod_{q \in \lceil t \rceil^c \setminus \lceil p \rceil^c} m_q : t > p\}}{\prod_{q \in \lceil p \rceil^c} m_q} \\ &= \gcd\bigg\{ \prod_{q \in \lceil t \rceil^c \setminus \lceil p \rceil^c} m_q : t > p \bigg\}. \end{aligned}$$

Second, note that, if $a \geq b$, then $\lceil a \rceil^c \supseteq \lceil b \rceil^c$, which means

$$\prod_{q \in \lceil b \rceil^c \setminus \lceil p \rceil^c} m_q \mid \prod_{q \in \lceil a \rceil^c \setminus \lceil p \rceil^c} m_q,$$

thus,

$$m'_p = \gcd\bigg\{ \prod_{q \in \lceil t \rceil^c \setminus \lceil p \rceil^c} m_q : t > p \bigg\} = \gcd\bigg\{ \prod_{q \in \lceil t \rceil^c \setminus \lceil p \rceil^c} m_q : t \succ p \bigg\}.$$

It follows that

$$m'_p = m_p * \gcd\bigg\{ \prod_{q \in (\lceil t \rceil^c \setminus \lceil p \rceil^c) \setminus p} m_q : t \succ p \bigg\}$$

since $p \in \lceil t \rceil^c \setminus \lceil p \rceil^c$ for any $t \succ p$. Therefore, in order to prove $m_p = m'_p$ for all $p \in P$, we only need show

$$\gcd\bigg\{ \prod_{q \in (\lceil t \rceil^c \setminus \lceil p \rceil^c) \setminus p} m_q : t \succ p \bigg\} = 1,$$

as follows.

(a) If there is only one element $t \in P$ satisfying $t \succ p$, then $(\lceil t \rceil^c \setminus \lceil p \rceil^c) \setminus p = \emptyset$. Otherwise, there exists an element $d \in P$ such that $d > p$ and $d \ngeq t$, where $d \ngeq t$ implies that $d < t$ or $d \| t$. If $d < t$, then $t > d > p$, contrary to $t \succ p$. If $d \| t$, then we have an element $c \in P$ such that $d \geq c \succ p$ since $d > p$. Thus, $c = t$, and then, $d \geq t$, a contradiction. Therefore,

$$\gcd\left\{\prod_{q\in(\lceil t\rceil^c\setminus\lceil p\rceil^c)\setminus p} m_q : t \succ p\right\} = \gcd\,\emptyset = 1.$$

(b) Suppose that there are k elements $t_1, t_2, \ldots, t_k$ in P such that $t_i \succ p$ for any $1 \le i \le k$ where $k \ge 2$. If

$$\gcd\left\{\prod_{q\in(\lceil t\rceil^c\setminus\lceil p\rceil^c)\setminus p} m_q : t \succ p\right\} \neq 1,$$

then there exists a variable x_p such that

$$x_p \mid \gcd\left\{\prod_{q\in(\lceil t\rceil^c\setminus\lceil p\rceil^c)\setminus p} m_q : t \succ p\right\}.$$

Therefore, we have elements $q_i > p$ and $q_i \not\geq t_i$ such that $x_p \mid m_{q_i}$ for each $1 \le i \le k$. From Lemma 2.7 (A2), $\{q_1, q_2, \ldots, q_k\}$ lies in a chain in P. Hence, there exists an element $1 \le r \le k$ such that $\{q_1, q_2, \ldots, q_k, t_r\}$ is a chain, and then, for all $1 \le j \le k$, we have $q_j \ge t_r$ since $q_j > p$ and $t_r \succ p$. Thus, $q_r \ge t_r$, a contradiction. Therefore,

$$\gcd\left\{\prod_{q\in(\lceil t\rceil^c\setminus\lceil p\rceil^c)\setminus p} m_q : t \succ p\right\} = 1. \qquad \square$$

3. Weak coordinatizations. One of the main results in [**12**] is that every finite atomic lattice is, in fact, the lcm-lattice of a monomial ideal. In 2009, Mapes [**9**] introduced a definition of coordinatization. Moreover, she proved that there are some specific constructions which produce a monomial ideal whose lcm-lattice has a given lattice structure, i.e., Lemma 2.7 (also see [**10**]). Mapes thought that it would be interesting to give an explicit formulation for when two coordinatizations are equivalent in this sense, or to prove a version of Lemma 2.7 which has weaker hypotheses.

In this section, we shall introduce the notion of a weak coordinatization which has weaker hypotheses than Definition 2.6 and show a sufficient condition which yields a weak coordinatization.

Let P be a finite atomic lattice and $p \in P$. Define

$$B_p = \left\{ T \subseteq \mathrm{supp}(p) : \bigvee_{b \in T} b = p \right\}.$$

Definition 3.1. Let $\mathcal{M}$ be a labeling of a finite atomic lattice P. Define a *monomial ideal* $I_{P,\mathcal{M}}$ to be the ideal generated by monomials

$$\triangle(a) = \gcd\left\{ \mathrm{lcm}\{x(b) : b \in T\} : T \in \bigcup_{p \geq a} B_p \right\} \tag{3.1}$$

for each $a \in \mathrm{atoms}(P)$. We say that the labeling $\mathcal{M}$ is a weak coordinatization if the lcm-lattice of $I_{P,\mathcal{M}}$ is isomorphic to P.

We first have the following lemma.

Lemma 3.2. *A labeling $\mathcal{M}$ is a coordinatization of a finite atomic lattice P if and only if it is a weak coordinatization and $\triangle(a) = x(a)$ for all $a \in \mathrm{atoms}(P)$.*

Proof. By Definition 3.1, the sufficiency is clear. Now, we prove the necessity.

Firstly, for all $a \in \mathrm{atoms}(P)$, as $\{a\} \in \bigcup_{p \geq a} B_p$, equation (3.1) implies $\triangle(a) \mid x(a)$. Secondly, since $\mathcal{M}$ is a coordinatization, the map

$$g : P \longrightarrow \mathrm{LCM}(M_{P,\mathcal{M}}) \quad \text{with } g(a) = x(a)$$

for all $a \in \mathrm{atoms}(P)$ is an isomorphism. Thus, for any $p \in P$ and any $T \in B_p$,

$$g(p) = \mathrm{lcm}\{x(b) : b \in \mathrm{supp}(p)\} = \mathrm{lcm}\{x(b) : b \in T\}.$$

Finally, suppose that $a \in \mathrm{atoms}(P)$. Let $p \in P$ and $a \leq p$. Clearly, $a \in \mathrm{supp}(p)$, and then,

$$g(a) = x(a) \mid g(p) = \mathrm{lcm}\{x(b) : b \in T\} \quad \text{for any } T \in B_p,$$

so that $x(a) \mid \mathrm{lcm}\{x(b) : b \in T\}$ for any $T \in \bigcup_{p \geq a} B_p$. Furthermore, by (3.1), $x(a) \mid \triangle(a)$. Therefore, $\triangle(a) = x(a)$ for all $a \in \mathrm{atoms}(P)$, which, together with the fact that $\mathcal{M}$ is a coordinatization of P, yields that $\mathcal{M}$ is a weak coordinatization of P. $\square$

Note that a weak coordinatization of a finite atomic lattice P need not be a coordinatizaton. For instance, let P be the finite atomic lattice with labeling as in Figure 2. Then, by Definitions 2.6 and 3.1,

$$M_{P,\mathcal{M}} = (b^2c^2d^2e^2, acd^2e^2, a^2b^2d^2e^2, a^2b^3c^2e, a^2b^3c^2d),$$
$$I_{P,\mathcal{M}} = (b^2c^2d^2e^2, acd^2e^2, a^2b^2d^2e^2, a^2b^2c^2e, a^2b^2c^2d).$$

Then, it is obvious that the lattice $\mathrm{LCM}(I_{P,\mathcal{M}})$, shown as Figure 3, is isomorphic to P. Furthermore, the labeling $\mathcal{M}$ is a weak coordinatizaton of P. On the other hand, the lattice $\mathrm{LCM}(M_{P,\mathcal{M}})$ shown in Figure 4 is not isomorphic to P. It follows that $\mathcal{M}$ is not a coordinatizaton of P.

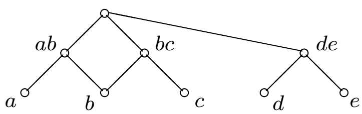

FIGURE 2. P with a labeling.

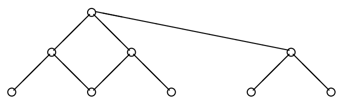

FIGURE 3. $\mathrm{LCM}(I_{P,\mathcal{M}})$.

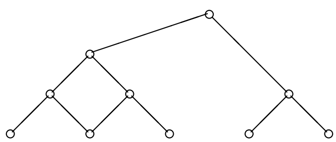

FIGURE 4. $\mathrm{LCM}(M_{P,\mathcal{M}})$.

Lemma 3.3. *Let $\mathcal{M}$ be a labeling of a finite atomic lattice P and $p \in P$. For each $R \in B_p$, if $b \in \mathrm{supp}(p) \setminus R$, then $\triangle(b) \mid \mathrm{lcm}\{\triangle(r) : r \in R\}$.*

Proof. Suppose that $\triangle(b) \nmid \mathrm{lcm}\{\triangle(r) : r \in R\}$. Then, there is a monomial x^{u_b} such that $x^{u_b} \nmid \mathrm{lcm}\{\triangle(r) : r \in R\}$, where x^{u_b} is the highest power of x dividing $\triangle(b)$. Let

$$S = \{a \in R : x^{u_b} \mid x(a)\}$$

and x^{u_a} be the highest power of x dividing $\triangle(a)$ for each $a \in S$. Then, $u_a < u_b$ since $x^{u_b} \nmid \mathrm{lcm}\{\triangle(r) : r \in R\}$. Moreover, it follows from formula (3.1) that, for any $a \in S$, there exist an element $q_a \in P$ with $q_a \geq a$ and a set $T_a \in B_{q_a}$ such that x^{u_a} is the highest power of x

dividing $\operatorname{lcm}\{x(t) : t \in T_a\}$. Thus,

$$x^{u_b} \nmid \operatorname{lcm}\{x(t) : t \in T_a\} \tag{3.2}$$

for each $a \in S$ since $u_a < u_b$.

Next, let $C = \bigcup_{a\in S} T_a \bigcup (R \setminus S)$. Clearly, we have

$$\begin{aligned}\bigvee_{c\in C} c &= \bigvee_{a\in S}\left(\bigvee T_a\right) \vee \bigvee (R \setminus S)\\ &= \bigvee_{a\in S} q_a \vee \bigvee (R\setminus S)\\ &\geq \bigvee_{a\in S} a \vee \bigvee (R \setminus S) = p \geq b\end{aligned}$$

and $C \in B_{\bigvee_{c\in C} c}$. Using (3.1), we have $\triangle(b) \mid \operatorname{lcm}\{x(c) : c \in C\}$. Thus,

$$x^{u_b} \mid \operatorname{lcm}\{x(c) : c \in C\}. \tag{3.3}$$

However, from (3.2), we know that, if $c \in \bigcup_{a\in S} T_a$, then $x^{u_b} \nmid x(c)$. Moreover, if $c \in R \setminus S$, then $x^{u_b} \nmid x(c)$ by the construction of S. Hence, $x^{u_b} \nmid \operatorname{lcm}\{x(c) : c \in C\}$, contrary to (3.3). Therefore, $\triangle(b) \mid \operatorname{lcm}\{\triangle(r) : r \in R\}$. □

Lemma 3.4. *Let $\mathcal{M}$ be a labeling of a finite atomic lattice P. For all $p, q \in P$, if $x_0 \mid m_p$ and $x_0 \mid m_q$ imply $p \not\parallel q$, then*

$$x_0 \nmid \frac{x(a)}{\gcd(\triangle(a), x(a))}$$

for any $a \in$ atoms(P).

Proof. Let $S = \{s \in P : x_0 \nmid m_s\}$ and $R = P \setminus S$. Suppose that $\overline{m_s} = x_s$ with $s \in S$ and $\overline{m_r} = x_0^r$, where x_0^r is the highest power of x_0 dividing m_r with $r \in R$. Then, from the hypotheses of Lemma 3.4, the labeling $\overline{\mathcal{M}} = \{\overline{m_p} : p \in P\}$ satisfies the conditions of Lemma 2.7. Thus, $\overline{\mathcal{M}}$ is a coordinatization of P. Hence, by Lemma 3.2, $\overline{\mathcal{M}}$ is a weak coordinatization of P, and

$$\overline{x(a)} = \overline{\triangle(a)} \tag{3.4}$$

for any atom $a \in$ atoms(P), where $\overline{x(a)} \in \overline{M}_{P,\overline{\mathcal{M}}}$ and $\overline{\triangle(a)} \in I_{P,\overline{\mathcal{M}}}$.

Now, assume that $x_0^{a_1}$ and $x_0^{\overline{a_1}}$ are the highest powers of x_0 dividing $x(a)$ and $\overline{x(a)}$, respectively, and $x_0^{a_2}$ and $x_0^{\overline{a_2}}$ are the highest powers of x_0 dividing $\triangle(a)$ and $\overline{\triangle(a)}$, respectively. By Definition 2.6, we have $a_1 = \overline{a_1}$, which, together with equation (3.1), implies that $a_2 = \overline{a_2}$. Using (3.4), we have $\overline{a_1} = \overline{a_2}$. Therefore, $a_1 = a_2$, which means that

$$x_0 \nmid \frac{x(a)}{\gcd(\triangle(a), x(a))}. \qquad \square$$

Theorem 3.5. *Any labeling $\mathcal{M}$ of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a weak coordinatization of the lattice P.*

(C1) *If $p \in \mathrm{mi}(P)$, then $m_p \neq 1$.*

(C2) *If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$, then either $p \nparallel q$, or*

$$r_q(p) = \frac{m_p}{\gcd(m_p, m_q)} \neq 1, \qquad r_p(q) = \frac{m_q}{\gcd(m_p, m_q)} \neq 1,$$

and, if $x, y \in \{s \in P : \gcd(r_q(p), m_s) \neq 1\}$ or $x, y \in \{s \in P : \gcd(r_p(q), m_s) \neq 1\}$, then $x \nparallel y$.

Proof. The proof of Theorem 3.5 is comprised of several steps. Let P' be the lcm-lattice of $I_{P,\mathcal{M}}$. For $b \in P$, define $g : P \to P'$ to be the map such that

$$g(b) = \mathrm{lcm}\{\triangle(a_i) : a_i \in \mathrm{supp}(b)\}. \tag{3.5}$$

Next, we shall show that g is an isomorphism from P to P'. Note that g is well defined.

(A) $\triangle(a) \nmid \triangle(b)$ and $\triangle(b) \nmid \triangle(a)$ for any $a, b \in \mathrm{atoms}(P)$ with $a \neq b$. By Lemma 2.3, the condition $a \neq b$ yields that $\mathrm{mi}(P) \cap \lceil a \rceil \neq \mathrm{mi}(P) \cap \lceil b \rceil$. Moreover, $a \| b$ since $a \neq b$ and $a, b \in \mathrm{atoms}(P)$. Thus, by Lemma 2.3,

$$\mathrm{mi}(P) \cap \lceil a \rceil \nsubseteq \mathrm{mi}(P) \cap \lceil b \rceil \quad \text{and} \quad \mathrm{mi}(P) \cap \lceil b \rceil \nsubseteq \mathrm{mi}(P) \cap \lceil a \rceil.$$

Hence,

$$\mathrm{mi}(P) \cap \lceil a \rceil^c \nsubseteq \mathrm{mi}(P) \cap \lceil b \rceil^c \quad \text{and} \quad \mathrm{mi}(P) \cap \lceil b \rceil^c \nsubseteq \mathrm{mi}(P) \cap \lceil a \rceil^c.$$

Therefore, there exists at least one element

$$q \in \mathrm{mi}(P) \cap \lceil a \rceil^c \quad \text{but } q \notin \mathrm{mi}(P) \cap \lceil b \rceil^c. \tag{3.6}$$

We shall prove the following statement.

(3.7) There exists a variable $x_q \mid m_q$ such that,

$$\text{for all } r \in P,\ x_q \mid m_r \text{ implies that } q \nparallel r.$$

Indeed, since q is meet-irreducible, condition (C1) of Theorem 3.5 yields that $m_q \neq 1$. Let y_q be a variable satisfying $y_q \mid m_q$. Then, there are two cases.

Case (1). If, for all $r \in P$, $y_q \mid m_r$ implies $q \nparallel r$, then, clearly, (3.7) is true.

Case (2). If there is a $t \in P$ such that $y_q \mid m_t$ but $q \| t$, then $\gcd(m_t, m_q) \neq 1$. Thus, $r_t(q) \neq 1$ by condition (C2) of Theorem 3.5. Let

$$x_q \mid r_t(q) \quad \text{and} \quad C_q = \{u \in P : x_q \mid m_u\}.$$

Then, $q \in C_q$ and

$$C_q \subseteq \{s \in P : \gcd(r_t(q), m_s) \neq 1\}.$$

Again, by Theorem 3.5 (C_2), $x \nparallel y$ for any $x, y \in \{s \in P : \gcd(r_t(q), m_s) \neq 1\}$, i.e., $\{s \in P : \gcd(r_t(q), m_s) \neq 1\}$ is a chain in P. Thus, the condition $C_q \subseteq \{s \in P : \gcd(r_t(q), m_s) \neq 1\}$ means that C_q is a chain in P. Note that $x_q \mid m_q$. Therefore, by the construction of C_q, we have that, for all $r \in P$, $x_q \mid m_r$ implies that $q \nparallel r$, i.e., (3.7) is true. In view of Cases (1) and (2), (3.7) holds.

Now, let x_q be a variable of m_q such that (3.7) holds, and let $D_q = \{v \in P : x_q \mid m_v\}$. Then, $q \in D_q$. Suppose that $p \in \lceil b \rceil^c$ satisfies $x_q \mid m_p$. Then, $p \nparallel q$ by (3.7). Note that $p \neq q$. Thus, either $q < p$ or $p < q$. If $q < p$, then $q \in \lfloor p \rfloor \subseteq \lceil b \rceil^c$ by $p \in \lceil b \rceil^c$ and Lemma 2.8, contrary to (3.6) such that $p < q$. Therefore, for all $p \in \lceil b \rceil^c$, if $x_q \mid m_p$, then $p < q$. Furthermore, from the construction of D_q, we know that, if $z \in D_q \cap \lceil b \rceil^c$, then $z < q$. Note that $q \in \lceil a \rceil^c$ by (3.6). Thus, $z < q \in \lceil a \rceil^c$, and it follows from Lemma 2.8 that $z \in \lceil a \rceil^c$. Thus, $D_q \cap \lceil b \rceil^c \subseteq D_q \cap \lceil a \rceil^c$. Note that $q \in \lceil a \rceil^c, q \in D_q$ and $q \notin D_q \cap \lceil b \rceil^c$. Therefore,

$$D_q \cap \lceil b \rceil^c \subsetneqq D_q \cap \lceil a \rceil^c. \tag{3.8}$$

Finally, let $x_q^{s_a}$ be the highest power of x_q dividing $x(a)$. Then, by the construction of D_q and formulae (2.1) and (3.8), we know that

$x_q^{s_a} \nmid x(b)$. Note that $\triangle(b) \mid x(b)$. Thus,

$$x_q^{s_a} \nmid \triangle(b). \tag{3.9}$$

On the other hand, by statement (3.7), x_q fulfills the conditions of Lemma 3.4. Thus,

$$x_q^{s_a} \text{ is the highest power of } x_q \text{ dividing } \triangle(a). \tag{3.10}$$

Therefore, $\triangle(a) \nmid \triangle(b)$ by (3.9). Similarly, we can prove that $\triangle(b) \nmid \triangle(a)$.

(B) Obviously, the map g is meet-preserving.

(C) The map g is join-preserving. Let $p, q \in P$. Obviously, $\operatorname{supp}(p) \cup \operatorname{supp}(q) \subseteq \operatorname{supp}(p \vee q)$. Now, let

$$T_{p\vee q} = \operatorname{supp}(p \vee q) \setminus (\operatorname{supp}(p) \cup \operatorname{supp}(q)).$$

Then,

$$g(p \vee q) = g(p) \vee g(q) \vee \operatorname{lcm}\{\triangle(a_v) : a_v \in T_{p\vee q}\}.$$

If $T_{p\vee q} = \emptyset$, then $g(p \vee q) = g(p) \vee g(q) \vee \operatorname{lcm} \emptyset = g(p) \vee g(q)$. Next, suppose that $T_{p\vee q} \neq \emptyset$. Then, by Lemma 3.3,

$$\operatorname{lcm}\{\triangle(a_v) : a_v \in T_{p\vee q}\} \mid \operatorname{lcm}\{\triangle(a_v) : a_v \in \operatorname{supp}(p) \cup \operatorname{supp}(q)\}$$

since $\operatorname{supp}(p) \cup \operatorname{supp}(q) \in B_{p\vee q}$. Therefore, $g(p \vee q) = g(p) \vee g(q)$, i.e., the map g is join-preserving.

(D) The map g is surjective. Assume that $p' \in P'$. Then, $p' = \operatorname{lcm}\{\triangle(a_i) : i \in I\}$ with $a_i \in \operatorname{atoms}(P)$ for each $i \in I$. Let $b = \bigvee_{i\in I} a_i \in P$. Then, $\{a_i : i \in I\} \in B_b$. Thus, by Lemma 3.3, $\triangle(a_j) \mid \operatorname{lcm}\{\triangle(a_i) : i \in I\}$ for all $a_j \in \operatorname{supp}(b) \setminus \{a_i : i \in I\}$. Therefore,

$$g(b) = \operatorname{lcm}\{\triangle(a_i) : a_i \in \operatorname{supp}(b)\} = \operatorname{lcm}\{\triangle(a_i) : i \in I\} = p',$$

which means that g is surjective.

(E) The map g is injective. Equivalently, we only need prove that $a = b$ when $g(a) = g(b)$. For any $a, b \in P$, distinguishing two situations, we can have either $0 \in \{a, b\}$ or $a, b \in P \setminus 0$. In the first case, we have $g(a) = g(b) = g(0) = 1$. Obviously, $a = 0 = b$ by (3.5) and statement (A). In the second case, the proof will be completed by two parts.

(i) Suppose that $b \not\leq a$. In this case, we easily see that

$$g(b) = \text{lcm}\{\triangle(a_i) : a_i \in \text{supp}(b) \cap \text{supp}(a)\} \vee \text{lcm}\{\triangle(a_j) : a_j \in \text{supp}(b) \setminus \text{supp}(a)\}. \tag{3.11}$$

From $b \not\leq a$, $\text{supp}(b) \setminus \text{supp}(a) \neq \emptyset$. Now, let $a_r \in \text{supp}(b) \setminus \text{supp}(a)$. Then, $a_r \leq b$; however, $a_r \not\leq a$, which, together with $a_r \in \text{atoms}(P)$, yields that $a_r \| a$. Thus, by Lemma 2.3, we have that

$$\text{mi}(P) \cap \lceil a_r \rceil \nsubseteq \text{mi}(P) \cap \lceil a \rceil \quad \text{and} \quad \text{mi}(P) \cap \lceil a \rceil \nsubseteq \text{mi}(P) \cap \lceil a_r \rceil,$$

and consequently,

$$\text{mi}(P) \cap \lceil a \rceil^c \nsubseteq \text{mi}(P) \cap \lceil a_r \rceil^c \quad \text{and} \quad \text{mi}(P) \cap \lceil a_r \rceil^c \nsubseteq \text{mi}(P) \cap \lceil a \rceil^c.$$

Hence, there exists an element q such that $q \in \text{mi}(P) \cap \lceil a_r \rceil^c$, but $q \notin \text{mi}(P) \cap \lceil a \rceil^c$. Let $a_m \in \text{supp}(a)$. Then, $a_m \leq a$, which implies that $\lceil a_m \rceil^c \cap \text{mi}(P) \subseteq \lceil a \rceil^c \cap \text{mi}(P)$. Thus, $q \notin \text{mi}(P) \cap \lceil a_m \rceil^c$. Therefore,

$$q \notin \text{mi}(P) \cap \lceil a_i \rceil^c \tag{3.12}$$

for all $a_i \in \text{supp}(a)$.

By statement (3.7), there exists a variable x_q in m_q such that, for all $r \in P$, $x_q \mid m_r$ implies that $q \nparallel r$. Let $x_q^{s_{a_r}}$ be the highest power of x_q dividing $x(a_r)$. Then, similarly to the proof of formula (3.10), we have $x_q^{s_{a_r}} \mid \triangle(a_r)$. Thus,

$$x_q^{s_{a_r}} \mid \text{lcm}\{\triangle(a_j) : a_j \in \text{supp}(b) \setminus \text{supp}(a)\}$$

since $a_r \in \text{supp}(b) \setminus \text{supp}(a)$. Therefore, $x_q^{s_{a_r}} \mid g(b)$ by (3.11). Furthermore, similarly to the proof of formula (3.9), from (3.12) we have that, for all $a_i \in \text{supp}(a)$, $x_q^{s_{a_r}} \nmid \triangle(a_i)$. Thus, $x_q^{s_{a_r}} \nmid g(a)$. Consequently, $g(b) \nmid g(a)$, contrary to $g(a) = g(b)$. This yields $b \leq a$.

(ii) Similarly to the proof of (i), the condition $a \not\leq b$ will show a contradiction. With (i) and (ii), we know that $a = b$ if $g(a) = g(b)$ in the case where $a, b \in P \setminus 0$. Therefore, the map g is injective.

From (B), (C), (D) and (E), g is an isomorphism from P to P'. Furthermore, by (3.5), $\mathcal{M}$ is a weak coordinatization of P. □

The following two examples will illustrate Theorem 3.5.

Example 3.6. Let P be a finite atomic lattice with a labeling as in Figure 5. It is easy to see that the labeling of P satisfies the conditions of Theorem 3.5 and does not satisfy the conditions of Lemma 2.7.

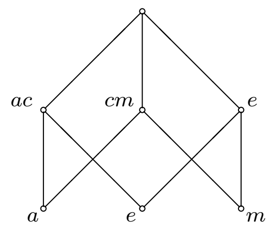

FIGURE 5. The lattice P with labeling $\mathcal{M}$.

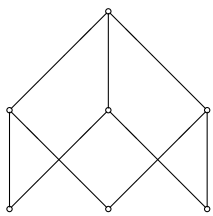

FIGURE 6. $\mathrm{LCM}(I_{P,\mathcal{M}})$.

We can clarify that $I_{P,\mathcal{M}} = (e^2m, acm^2, a^2ce)$, and $\mathrm{LCM}(I_{P,\mathcal{M}})$ is isomorphic to P (see Figures 5 and 6). Moreover, we can verify that $\mathcal{M}$ is a weak coordinatization and $I_{P,\mathcal{M}} = M_{P,\mathcal{M}}$.

Example 3.7. We consider the finite atomic lattice P, again with labeling as in Figure 2. We can verify that the labeling of P satisfies the conditions of Theorem 3.5 and does not satisfy the conditions of Lemma 2.7. Moreover, the labeling $\mathcal{M}$ is a weak coordinatization, and $I_{P,\mathcal{M}} \neq M_{P,\mathcal{M}}$.

Remark 3.8. From Theorem 2.10, if the monomial ideal $M = M_{P,\mathcal{M}}$ with the labeling $\mathcal{M}$ satisfies the conditions of Lemma 2.7, then $\mathcal{D} = \mathcal{M}$. On the other hand, by Lemma 2.9, we know that, if the monomial ideal $M = I_{P,\mathcal{M}}$ with the labeling $\mathcal{M}$ satisfies the conditions of Theorem 3.5 and does not satisfy the conditions of Lemma 2.7, then M must induce a new labeling $\mathcal{D}$ which is different from $\mathcal{M}$ and $D_{P,\mathcal{D}} = I_{P,\mathcal{M}} = M$.

4. Finite super-atomic lattices. Let $\mathcal{L}(n)$ be the set of all finite atomic lattices with n ordered atoms. $\mathcal{L}(n)$ has a partial order where $Q \leq P$ if and only if there exists a join-preserving map which is a bijection on atoms from P to Q (note that such a map will also be surjective) [**10**].

In this section, we shall discuss the structure of lattice $\mathcal{L}(n)$. We first define a finite super-atomic lattice and then give an algorithm to find all of the finite super-atomic lattices in $\mathcal{L}(n)$.

Definition 4.1. A finite atomic lattice P is called *super-atomic* if it satisfies that, for each $p \in (P \setminus \text{atoms}(P)) \setminus 0$, there exists a $T_0 = \{a_1, a_2\} \in B_p$ such that $T_0 \subseteq T$ for any $T \in B_p$.

For example, the finite atomic lattice P, shown in Figure 7, is super-atomic.

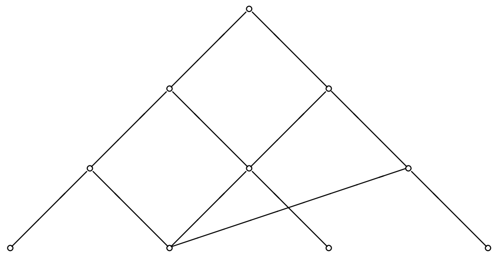

FIGURE 7. A finite super-atomic lattice.

Theorem 4.2. *A lattice P is super-atomic if and only if, for each $p \in (P \setminus \text{atoms}(P)) \setminus 0$, there exists a $\{a_1, a_2\} \in B_p$ such that* $\text{supp}(p) \setminus a_1 \in \mathcal{S}_P$ *and* $\text{supp}(p) \setminus a_2 \in \mathcal{S}_P$.

Proof. Suppose that P is super-atomic. Then, there exists a $\{a_1, a_2\} \in B_p$ for each $p \in (P \setminus \text{atoms}(P)) \setminus 0$. Now, assume that $\text{supp}(p) \setminus a_1 \notin \mathcal{S}_P$. Then,

$$\text{supp}(p) \supseteq \overline{\bigvee_{a \in \text{supp}(p) \setminus a_1}} \text{supp}(a) = \overline{\bigvee_{a \in \text{supp}(p) \setminus a_1}} \{a\} \supsetneq \text{supp}(p) \setminus a_1,$$

in which $\overline{\bigvee}$ is the join of $(\mathcal{S}_P, \subseteq)$. Thus,

$$\overline{\bigvee_{a \in \text{supp}(p) \setminus a_1}} \text{supp}(a) = \text{supp}(p). \tag{4.1}$$

From the definition of $\mathcal{S}_P$, $(\mathcal{S}_P, \subseteq)$ is the same as lattice P. Thus,

$$\text{supp}(q) \text{ corresponds to } q \text{ for each } q \in P, \tag{4.2}$$

and,

$$\text{for any } S \in \mathcal{S}_P, \text{ there exists a } q \in P \text{ such that } S = \text{supp}(q). \tag{4.3}$$

Therefore, by formulas (4.1) and (4.2), we have $\bigvee_{a\in \mathrm{supp}(p)\setminus a_1} a = p$, which means that $\mathrm{supp}(p) \setminus a_1 \in B_p$. Since P is super-atomic, there exists a $T_0 = \{b_1, b_2\} \in B_P$ such that $T_0 \subseteq \mathrm{supp}(p) \setminus a_1 \bigcap\{a_1, a_2\}$, a contradiction; therefore, $\mathrm{supp}(p) \setminus a_1 \in \mathcal{S}_P$. Similarly, we can prove that $\mathrm{supp}(p) \setminus a_2 \in \mathcal{S}_P$.

Conversely, let $p \in (P\setminus \mathrm{atoms}(P))\setminus 0$. Then, by the hypothesis, there exists a $\{a_1, a_2\} \in B_p$ such that $\mathrm{supp}(p)\setminus a_1 \in \mathcal{S}_P$ and $\mathrm{supp}(p)\setminus a_2 \in \mathcal{S}_P$. Note that $T \subseteq \mathrm{supp}(p)$ for all $T \in B_p$.

Next, we prove that $\{a_1, a_2\} \subseteq T$ for all $T \in B_p$. If there exists a $T \in B_p$ such that $\{a_1, a_2\} \nsubseteq T$, then either $\overline{\bigvee}_{a\in T}\{a\} = \overline{\bigvee}_{a\in T}\mathrm{supp}(a) \subseteq \mathrm{supp}(p) \setminus a_1 \in \mathcal{S}_P$ or $\overline{\bigvee}_{a\in T}\{a\} = \overline{\bigvee}_{a\in T}\mathrm{supp}(a) \subseteq \mathrm{supp}(p) \setminus a_2 \in \mathcal{S}_P$. From (4.2) and (4.3), in any case, we have that $\bigvee_{a\in T} a < p$ is contrary to $T \in B_p$. Hence,

$$\{a_1, a_2\} \subseteq T \quad \text{for all } T \in B_p. \tag{4.4}$$

Therefore, by Definition 4.1 and (4.4), P is a finite super-atomic lattice. □

From Definition 4.1 and Theorem 4.2, we obviously have the next lemma.

Lemma 4.3. *Let P be a super-atomic lattice in $\mathcal{L}(n)$ with* $\mathrm{atoms}(P) = \{1, 2, \ldots, n\}$ *and $n \geq 2$. Then, $(\mathcal{S}_P, \subseteq)$ satisfies the following statements*:

(D1) $\{\emptyset, \{1\}, \ldots, \{n\}, \{1, \ldots, n\}\} \subseteq \mathcal{S}_P$.

(D2) *If $S \in \mathcal{S}_P \setminus \{\emptyset, \{1\}, \ldots, \{n\}\}$, then there exist two different atoms $\{i\}, \{j\} \in \mathcal{S}_P$ such that $S = \{i\} \vee \{j\}$ and $S \setminus k \in \mathcal{S}_P$ for any $k \in \{i, j\}$.*

(D3) *Let $S_1, S_2 \in \mathcal{S}_P$. If $S_1 = \{u\} \vee \{v\}$, $S_2 = \{k\} \vee \{h\}$ and $S_1 \| S_2$, then $\{u, v\} \nsubseteq S_2$ and $\{k, h\} \nsubseteq S_1$.*

In what follows, we shall suggest an algorithm to construct all finite super-atomic lattices in $\mathcal{L}(n)$ with $n \geq 2$.

Algorithm 4.4.
Input: $X = \{1, 2, \ldots, n\}$.
Output: $\mathcal{S}^*$.
Step 1. *Take* $\mathcal{S}_0 = \{\emptyset\}, \mathcal{S}_1 = \{\{1\}, \ldots, \{n\}\}, \mathcal{S}_n = \{X\}$, $\mathcal{S}^* =: \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_n$ *and* $k := 0$.
Step 2. *If* $n - k = 2$, *then go to* Step 7.
Step 3. *For any* $S \in \mathcal{S}_{n-k}$, *take* $\delta(S) = \{i_S, j_S\} \subseteq S$ *satisfying* $\delta(S) \nsubseteq T$ *for all* $T \in \mathcal{S}_{n-k} \setminus S$.
Step 4. $\mathcal{S}_{n-k-1} = \bigcup_{S \in \mathcal{S}_{n-k}} \{S \setminus i_S, S \setminus j_S\}$.
Step 5. $k := k + 1$.
Step 6. $\mathcal{S}^* := \mathcal{S}^* \cup \mathcal{S}_{n-k}$, *and go to* Step 2.
Step 7. *Stop.*

Theorem 4.5. *Every output* $(\mathcal{S}^*, \subseteq)$ *in Algorithm* 4.4 *is a finite super-atomic lattice in* $\mathcal{L}(n)$. *Furthermore, every finite super-atomic lattice in* $\mathcal{L}(n)$ *can be constructed by Algorithm* 4.4.

Proof. Throughout the proof, let $\bigvee \delta(S) = \{i_S\} \vee \{j_S\}$ for any $S \in \mathcal{S}^* \setminus (\mathcal{S}_0 \bigcup \mathcal{S}_1)$. First, we shall prove that every output $(\mathcal{S}^*, \subseteq)$ in Algorithm 4.4 is a finite super-atomic lattice in four steps, below.

(B1) Obviously, $(\mathcal{S}^*, \subseteq)$ has a minimum element $\emptyset$ and a maximum element $\{1, \ldots, n\}$.

(B2) If $S \in \mathcal{S}^* \setminus (\mathcal{S}_1 \bigcup \mathcal{S}_0)$, then $S = \bigvee \delta(S)$.

Observe that there exists a $t \in \{2, \ldots, n\}$ such that $S \in \mathcal{S}_t$, and

$$\delta(S) \nsubseteq T \tag{4.5}$$

for all $T \in \mathcal{S}_t \setminus S$ by Algorithm 4.4. Set

$$\mathcal{D} = \{D \in \mathcal{S}^* : \delta(S) \subseteq D\} \tag{4.6}$$

and

$$\mathcal{D}_* = \{D : D \text{ is a minimal element of } \mathcal{D}\}.$$

Let $D \in \mathcal{D}_*$. We claim that $D \notin \mathcal{S}_u$ for any integer u with $0 \leq u < t$. Indeed, if $D \in \mathcal{S}_u$, then there exists a $G \in \mathcal{S}_t$ such that $D \subsetneq G$ by Algorithm 4.4. Thus, $\delta(S) \subseteq G$, which, together with (4.5), yields that $G = S$. Therefore, $D \subseteq S \setminus i_S$ or $D \subseteq S \setminus j_S$ by Algorithm 4.4, contrary to $\delta(S) \subseteq D$.

Below, assume that $D \in \mathcal{S}_v$ with $n \geq v \geq t$. Now, we shall prove $v = t$. Suppose that $n \geq v > t$. From Algorithm 4.4, there exists an $R \in \mathcal{S}_v$ such that $R \supsetneq S$. There are two cases.

Case (1). If $D = R$, then $D \supsetneq S$, contrary to $D \in \mathcal{D}_*$ since $S \in \mathcal{D}$.

Case (2). Let $D \neq R$. We first claim that $\delta(D) = \delta(S)$. Otherwise, either $\delta(S) \subseteq D \setminus i_D \subsetneq D$ or $\delta(S) \subseteq D \setminus j_D \subsetneq D$, contrary to $D \in \mathcal{D}_*$. Hence, $\delta(D) = \delta(S) \subseteq S \subsetneq R$, contrary to $\delta(D) \nsubseteq R$ since $R \neq D$ and both R and D in $\mathcal{S}_v$ (see (4.5)).

Cases (1) and (2) imply that $v = t$. Therefore, $D = S$ by formulas (4.5) and (4.6), which means that $\mathcal{D}_*$ contains exactly one element S and $S = \bigvee \delta(S)$.

(B3) If $S_1, S_2 \in \mathcal{S}^*$, then $S_1 \vee S_2$ exists in $\mathcal{S}^*$. Obviously, if $S_1 \nparallel S_2$, then $S_1 \vee S_2 = S_1$ or $S_1 \vee S_2 = S_2$.

Next, suppose that $S_1 \| S_2$. Observe that S_1 and S_2 are not in $\mathcal{S}_0$. There are three cases.

Case (i). If $S_1 = \{i\}, S_2 = \{j\}$ and $i \neq j$, then $S_1 \vee S_2$ exists in $\mathcal{S}^*$. In this case, set

$$M = \{S \in \mathcal{S}^* : \{i, j\} \subseteq S\}$$

and

$$M_* = \{S : S \text{ is a minimal element of } M\}.$$

Note that $M \neq \emptyset$. Hence, $M_* \neq \emptyset$. Assume that $S \in M_*$. Then, $S \in \mathcal{S}^* \setminus (\mathcal{S}_1 \bigcup \mathcal{S}_0)$. Thus, by (B2), $S = \bigvee \delta(S)$. If $\{i, j\} \neq \delta(S)$, then $\{i, j\} \subseteq S \setminus i_S \in \mathcal{S}^*$ or $\{i, j\} \subseteq S \setminus j_S \in \mathcal{S}^*$ by Algorithm 4.4, contrary to the fact that $S \in M_*$. Therefore, $\{i, j\} = \delta(S)$, which means that $S_1 \vee S_2 = S \in \mathcal{S}^*$.

Case (ii). If $S_1 = \{i\}$ and $S_2 \in \mathcal{S}^* \setminus (\mathcal{S}_1 \bigcup \mathcal{S}_0)$ with $i \notin S_2$, then $S_1 \vee S_2$ exists in $\mathcal{S}^*$. Indeed, by (B2), $S_2 = \bigvee \delta(S_2)$. Suppose that $S_1 \vee S_2$ does not exist in $\mathcal{S}^*$. Then, $\mathcal{S}^*$ contains two different minimal elements containing $S_1 \bigcup S_2$, say S_a, S_b. Clearly, $S_a \| S_b$.

We claim that

$$\delta(S_a) \subseteq \{i, i_{S_2}, j_{S_2}\} \quad \text{and} \quad \delta(S_a) \neq \delta(S_2). \tag{4.7}$$

Suppose that $\delta(S_a) \nsubseteq \{i, i_{S_2}, j_{S_2}\}$. From Algorithm 4.4, $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a \setminus i_{S_a} \in \mathcal{S}^*$ or $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a \setminus j_{S_a} \in \mathcal{S}^*$. From $S_2 = \bigvee \delta(S_2)$, if $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a \setminus i_{S_a}$, then

$$S_1 \bigcup S_2 \subseteq S_a \setminus i_{S_a} \subsetneq S_a,$$

a contradiction. Similarly, we can prove that $\{i, i_{S_2}, j_{S_2}\} \subseteq S_a \setminus j_{S_a} \in \mathcal{S}^*$ will show a contradiction. Therefore, $\delta(S_a) \subseteq \{i, i_{S_2}, j_{S_2}\}$. Now, assume that $\delta(S_a) = \delta(S_2)$. Then,

$$S_a = \bigvee \delta(S_a) = \bigvee \delta(S_2) = S_2,$$

which implies $i \in S_2$, a contradiction.

Arguing as in formula (4.7), we have

$$\delta(S_b) \subseteq \{i, i_{S_2}, j_{S_2}\} \quad \text{and} \quad \delta(S_b) \neq \delta(S_2). \tag{4.8}$$

Formulas (4.7) and (4.8) imply that both $\delta(S_a)$ and $\delta(S_b)$ are equal to $\{i, i_{S_2}\}$ or $\{i, j_{S_2}\}$. We claim that

$$\delta(S_a) \neq \delta(S_b). \tag{4.9}$$

Indeed, if $\delta(S_a) = \delta(S_b)$, then

$$S_a = \bigvee \delta(S_a) = \bigvee \delta(S_b) = S_b,$$

contrary to $S_a \| S_b$. Thus, if $\delta(S_a) = \{i, i_{S_2}\}$, then $\delta(S_b) = \{i, j_{S_2}\}$. Clearly, $\{i, j_{S_2}\} \subseteq S_a \setminus i_{S_2} \in \mathcal{S}^*$. Thus,

$$S_b = \bigvee \delta(S_b) = \{i\} \vee \{j_{S_2}\} \subseteq S_a \setminus i_{S_2} \subsetneq S_a,$$

contrary to $S_a \| S_b$. Similarly, we can prove that $\delta(S_a) = \{i, j_{S_2}\}$ will show a contradiction. Therefore, $S_1 \vee S_2$ exists in $\mathcal{S}^*$.

Case (iii). If $S_1, S_2 \in \mathcal{S}^* \setminus (\mathcal{S}_1 \bigcup \mathcal{S}_0)$ and $S_1 \| S_2$, then $S_1 \vee S_2$ exists in $\mathcal{S}^*$. First, if $\delta(S_1) \subseteq S_2$, then $\bigvee \delta(S_1) = S_1 \subseteq S_2$, a contradiction. Thus, $\delta(S_1) \nsubseteq S_2$. Similarly, we can prove $\delta(S_2) \nsubseteq S_1$.

Assume that $S_1 \vee S_2$ does not exist in $\mathcal{S}^*$. Then, $\mathcal{S}^*$ contains two different minimal elements containing $S_1 \bigcup S_2$, say C_1, C_2. Clearly, $C_1 \| C_2$. Similarly to the proof of formula (4.7) in Case (ii), we can prove that

$$\delta(C_1) \subseteq \delta(S_1) \cup \delta(S_2), \delta(C_1) \neq \delta(S_1) \quad \text{and} \quad \delta(C_1) \neq \delta(S_2). \tag{4.10}$$

Using (4.10), we know that $\delta(C_1)$ equals one of the four sets $\{i_{S_1} i_{S_2}\}$, $\{i_{S_1}, j_{S_2}\}, \{j_{S_1}, i_{S_2}\}$, $\{j_{S_1}, j_{S_2}\}$.

Similarly, we can prove that $\delta(C_2)$ also equals one of the four sets $\{i_{S_1}, i_{S_2}\}$, $\{i_{S_1}, j_{S_2}\}$, $\{j_{S_1}, i_{S_2}\}$, $\{j_{S_1}, j_{S_2}\}$.

Similarly to the proof of formula (4.9) in Case (ii), we can prove $\delta(C_1) \neq \delta(C_2)$. Now, suppose that $\delta(C_1) = \{i_{S_1}, i_{S_2}\}$. Then, $C_1 \setminus i_{S_1}$, $C_1 \setminus i_{S_2} \in \mathcal{S}^*$ by Algorithm 4.4. If $\delta(C_2) = \{i_{S_1}, j_{S_2}\}$, then

$$C_2 = \{i_{S_1}\} \vee \{j_{S_2}\} \subseteq C_1 \setminus i_{S_2},$$

contrary to $C_1 \| C_2$. All of the other cases may be similarly proven, yielding a contradiction. Therefore, $S_1 \vee S_2$ exists in $\mathcal{S}^*$.

(B4) $(\mathcal{S}^*, \subseteq)$ is super-atomic. From (B1), (B2), (B3) and Definitions 2.1 and 2.2, $(\mathcal{S}^*, \subseteq)$ is a finite atomic lattice.

Next, we shall prove that $(\mathcal{S}^*, \subseteq)$ is super-atomic. Suppose that $S \in \mathcal{S}^* \setminus (\mathcal{S}_1 \bigcup \mathcal{S}_0)$ and $T \in B_S$. Note that $\bigvee T = S$. If $\{i_S\} \notin T$, then $\bigcup T \subseteq S \setminus i_S \in \mathcal{S}^*$, which implies $\bigvee T \subseteq S \setminus i_S$, contrary to $\bigvee T = S$. Thus, $\{i_S\} \in T$.

Similarly, we have $\{j_S\} \in T$. Hence, $\{\{i_S\}, \{j_S\}\} \subseteq T$. Again, by (B2), $\{i_S\} \vee \{j_S\} = \bigvee \delta(S) = S$, which means that $\{\{i_S\}, \{j_S\}\} \in B_S$. Thus, by Definition 4.1, the lattice $(\mathcal{S}^*, \subseteq)$ is super-atomic.

We shall finally prove that every super-atomic lattice in $\mathcal{L}(n)$ can be constructed by Algorithm 4.4. Let $(\mathcal{S}, \subseteq)$ be a super-atomic lattice in $\mathcal{L}(n)$. For each $0 \leq i \leq n$, define

$$\mathcal{T}_i = \{S \in \mathcal{S} : |S| = i\}.$$

Then,

$$\mathcal{S} = \mathcal{T}_0 \bigcup \mathcal{T}_1 \bigcup \cdots \bigcup \mathcal{T}_n.$$

In what follows, we prove that there is an output $\mathcal{S}^*$ by Algorithm 4.4 such that $\mathcal{S}^* = \mathcal{S}$. In fact, from Algorithm 4.4, we know that $\mathcal{S}^* = \mathcal{S}_0 \bigcup \mathcal{S}_1 \bigcup \cdots \bigcup \mathcal{S}_n$. Therefore, in order to construct $\mathcal{S}^*$ by Algorithm 4.4 such that $\mathcal{S}^* = \mathcal{S}$, we merely need to construct $\mathcal{S}_i$ such that $\mathcal{T}_i = \mathcal{S}_i$ for all $0 \leq i \leq n$.

First, by Algorithm 4.4 and (D1) in Lemma 4.3, we have

$$\mathcal{T}_i = \mathcal{S}_i \quad \text{for all } i \in \{0, 1, n\}. \tag{4.11}$$

Then, by (D2), there exist $\{i^S\}, \{j^S\} \in \mathcal{T}_1$ such that $\{i^S\} \vee \{j^S\} = S$ for each $S \in \mathcal{T}_n$ with $n \geq 2$. Since $\mathcal{T}_n = \mathcal{S}_n$, by (4.11), we can take $\delta(S) = \{i^S, j^S\}$ in Step 3 of Algorithm 4.4 for all $S \in \mathcal{S}_n$. Thus, $\mathcal{T}_{n-1} \supseteq \mathcal{S}_{n-1}$, by Step 4 and (D2).

We claim that $\mathcal{T}_{n-1} = \mathcal{S}_{n-1}$. Otherwise, there exists a $W \in \mathcal{T}_{n-1}$ such that $W \notin \mathcal{S}_{n-1}$. Let $K \in \mathcal{S}$ with $K \succ W$. Then, by (D2), there exist $\{i^K\}, \{j^K\} \in \mathcal{T}_1$ such that $\{i^K\} \vee \{j^K\} = K$. If $\{i^K, j^K\} \subseteq W$, then

$$K = \{i^K\} \vee \{j^K\} \subseteq W \prec K,$$

a contradiction. Thus, $\{i^K, j^K\} \nsubseteq W$. It follows from (D2) that

$$W \subseteq K \setminus i^K \prec K \quad \text{or} \quad W \subseteq K \setminus j^K \prec K,$$

which means that $W = K \setminus i^K$ or $W = K \setminus j^K$. Therefore, $K \in \mathcal{T}_n$, which, together with $\mathcal{T}_n = \mathcal{S}_n$, yields that $W \in \mathcal{S}_{n-1}$ since $\delta(K) = \{i^K, j^K\}$, a contradiction.

Similarly, we can construct $\mathcal{T}_h = \mathcal{S}_h$ by taking $\delta(T) = \{i^T, j^T\}$ for any $T \in \mathcal{T}_{h+1}$, in which $\{i^T\}, \{j^T\} \in \mathcal{T}_1$ and $\{i^T\} \vee \{j^T\} = T$ for all $2 \leq h \leq n-2$. Consequently, $\mathcal{T}_i = \mathcal{S}_i$ for all $0 \leq i \leq n$. □

The next example will illustrate Algorithm 4.4.

Example 4.6. Let $n = 3$. Then, by Algorithm 4.4, we have three super-atomic lattices in $\mathcal{L}(n)$, as follows:

$$\mathcal{Q}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}, \{1,2\}, \{1,3\}\},$$
$$\mathcal{Q}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}, \{1,2\}, \{2,3\}\},$$
$$\mathcal{Q}_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}, \{1,3\}, \{2,3\}\}.$$

It can easily be verified that $(\mathcal{Q}_1, \subseteq)$, $(\mathcal{Q}_2, \subseteq)$ and $(\mathcal{Q}_3, \subseteq)$ are all the super-atomic lattices in $\mathcal{L}(n)$.

5. Specific labelings. In [**9**], there are three specific coordinatizations, i.e., minimal squarefree, minimal depolarized and greedy; we can see that all of them are based on the labeling described as in Lemma 2.7.

In this section, we shall give a type of labeling on a lattice P which does not satisfy the conditions of Lemma 2.7 and show the conditions that our labeling is either a coordinatization or a weak coordinatization.

Let $P \in \mathcal{L}(n)$ with $\text{atoms}(P) = \{a_1, a_2, \ldots, a_n\}$. We define a labeling $\mathcal{C}$ of P as $\mathcal{C} = \{m_p : p \in P \setminus 0\}$, where

$$m_p = \prod_{a_i \in \text{supp}(p)} a_i, \tag{5.1}$$

in which every a_i means both an atom in P and a variable in labeling $\mathcal{C}$.

In what follows, let

$$[a, b] = \{p \in P : a \leq p \leq b\} \quad \text{and} \quad N([a, b]) = |[a, b]|$$

for the purposes of convenience.

Theorem 5.1. *Let $P \in \mathcal{L}(n)$. For each $p \in (P \setminus \text{atoms}(P)) \setminus 0$, if there exist $a_i, a_j \in \text{supp}(p)$ such that $p = a_i \vee a_j$ and $N([a_r \vee a_k, 1]) < N([p, 1])$ for a fixed number $r \in \{i, j\}$ and all $a_k \in \text{atoms}(P) \setminus \text{supp}(p)$, then the labeling $\mathcal{C}$ of P, as defined by (5.1), is a weak coordinatization.*

Proof. For $b \in P$, define $g : P \to \text{LCM}(I_{P,\mathcal{C}})$ to be a map such that

$$g(b) = \text{lcm}\{\triangle(u) : u \in \text{supp}(b)\}.$$

The main part is to show that g is an isomorphism of lattices. Similarly to (B), (C) and (D) in the proof of Theorem 3.5, we can prove that the map g is meet-preserving, join-preserving and surjection. Thus, we only need show that g is injective. The proof will be split into two parts.

$(*)$ Let $a_u, a_v \in \text{atoms}(P)$. Then, $a_u \mid \triangle(a_v)$ if and only if $a_u \neq a_v$. Suppose that $a_u \mid \triangle(a_v)$. From formula (5.1), $a_u \mid m_p$ if and only if $p \geq a_u$. Thus, $a_u \nmid x(a_u)$ by (2.1). This means that $a_u \nmid \triangle(a_u)$ since $\triangle(a_u) \mid x(a_u)$. Therefore, $a_u \neq a_v$.

Conversely, assume that $a_w \in \text{atoms}(P) \setminus a_u$. Then, $a_u \in \lceil a_w \rceil^c$. Thus, $a_u \mid x(a_w)$ by equations (2.1) and (5.1). On the other hand, let

$$F \in \bigcup_{p \geq a_v} B_p.$$

Then, $\bigvee F \geq a_v$ such that $a_u \neq \bigvee F$ since $a_u \neq a_v$. Thus, there exists an $a_z \in F$ such that $a_z \neq a_u$. Hence, $a_u \mid \text{lcm}\{x(a_j) : a_j \in F\}$. This, together with equation (3.1), implies that $a_u \mid \triangle(a_v)$.

(∗∗) The map g is injective. Clearly, if $0 \in \{a, b\}$ and $g(a) = g(b)$, then $g(a) = g(b) = g(0) = 1$, which implies that $a = 0 = b$. Next, let $a, b \in P \setminus 0$ and $g(a) = g(b)$.

Now, we shall prove $a = b$. Suppose that $b \not\leq a$. Then, we have either $a \in \text{atoms}(P)$ or $a \in (P \setminus \text{atoms}(P)) \setminus 0$. In the first case, we have $\text{supp}(b) \setminus \text{supp}(a) \neq \emptyset$. Thus, there exists a $c \in \text{supp}(b) \setminus \text{supp}(a)$. By statement (∗), $a \nmid \triangle(a)$ and $a \mid \triangle(c)$. Therefore, $a \mid g(b)$ and $a \nmid g(a)$, a contradiction.

In the second case, let $a_k \in \text{atoms}(P) \setminus \text{supp}(a)$. Then, by the hypothesis of the theorem, there exist two elements $a_i, a_j \in \text{supp}(a)$ such that $a = a_i \vee a_j$ and $N([a_j \vee a_k, 1]) < N([a, 1])$ (set $r = j$). For convenience, let $a_j^{n_y}$ be the highest power of a_j dividing $x(a_y)$ for each $a_y \in \text{atoms}(P)$. Clearly, by (2.1)

$$x(a_k) = \prod_{q \in \lceil a_k \rceil^c} m_q = \prod_{q_1 \in \lceil a_k \rceil^c \cap \lceil a_j \rceil} m_{q_1} * \prod_{q_2 \in \lceil a_k \rceil^c \cap \lceil a_j \rceil^c} m_{q_2}.$$

Thus, by (5.1), $n_k = |\lceil a_k \rceil^c \cap \lceil a_j \rceil|$. On the other hand, $\lceil a_k \rceil^c \cap \lceil a_j \rceil = [a_j, 1] - [a_j \vee a_k, 1]$ such that $n_k = N([a_j, 1]) - N([a_j \vee a_k, 1])$. Similarly, $n_i = N([a_j, 1]) - N([a_j \vee a_i, 1])$. Therefore,

$$\begin{aligned} n_k - n_i &= N([a_j \vee a_i, 1]) - N([a_j \vee a_k, 1]) \\ &= N([a, 1]) - N([a_j \vee a_k, 1]) \geq 1. \end{aligned} \tag{5.2}$$

Let $r \geq a_k$. Suppose that $T \in B_r$. We claim that there exists an $a_t \in T$ such that $a_t \in \text{atoms}(P) \setminus \text{supp}(a)$. Otherwise, $T \subseteq \text{supp}(a)$, which means that

$$a_k \leq r = \bigvee T \leq \bigvee \text{supp}(a) = a,$$

contrary to $a_k \notin \text{supp}(a)$. Hence, $n_t - n_i \geq 1$ by (5.2). Thus,

$$a_j^{n_i+1} \mid \text{lcm}\{x(a_w) : a_w \in T\}. \tag{5.3}$$

Below, let $a_j^{m_y}$ be the highest power of a_j dividing $\triangle(a_y)$ for each $a_y \in \text{atoms}(P)$. Thus, $m_k \geq n_i + 1$ by formulas (3.1) and (5.3). Clearly, $m_i \leq n_i$ since $\triangle(a_i) \mid x(a_i)$. Therefore,

$$m_k > n_i \geq m_i. \tag{5.4}$$

Clearly, there exists an $a_s \in \text{supp}(b) \setminus \text{supp}(a) \subseteq \text{atoms}(P) \setminus \text{supp}(a)$ such that $a_s \vee a_e = b$ for some $a_e \in \text{supp}(b)$. It follows that $g(b) = \text{lcm}\{\triangle(a_s), \triangle(a_e)\}$ since g is join-preserving. Now, let a_j^m be the highest power of a_j dividing $g(b)$. Then, $m \geq m_s$. Using formula (5.4), $m_s > n_i \geq m_i$. Thus, $m \geq m_s > m_i$.

On the other hand, $g(a) = \text{lcm}\{\triangle(a_i), \triangle(a_j)\}$. By statement $(*)$, we have $a_j \nmid \triangle(a_j)$. Thus, $a_j^{m_i}$ is the highest power of a_j dividing $g(a)$. Since $m \geq m_s > m_i$, we finally have that $g(b) \nmid g(a)$, a contradiction. Therefore, the assumption of $b \nleq a$ yields a contradiction. Consequently, $b \leq a$.

Similarly, we can prove that $a \leq b$; it follows from $b \leq a$ that $a = b$ finally. □

Remark 5.2. The labeling $\mathcal{C}$ as defined by (5.1) need not satisfy condition (C2) generally. For example, consider the lattice shown in Figure 8.

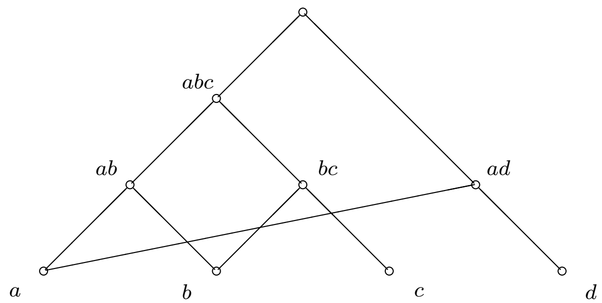

FIGURE 8. The lattice P with a labeling $\mathcal{C}$.

Clearly, the lattice P satisfies the conditions of Theorem 5.1, and its labeling $\mathcal{C}$ yields that $I_{P,\mathcal{C}} = \{b^2c^2d, a^2cd^2, a^3b^2d^2, a^3b^4c^3\}$. It can be verified that $\text{LCM}(I_{P,\mathcal{C}}) \cong P$. Obviously, the labeling $\mathcal{C}$ is a weak coordinatization, and it does not satisfy condition (C2).

Theorem 5.3. *Let P be a super-atomic lattice. Then, the labeling $\mathcal{C}$ of P as defined by* (5.1) *is a coordinatization if and only if, for each $p \in (P \setminus \text{atoms}(P)) \setminus 0$, either*

$$N([a_i \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$$

or

$$N([a_j \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$$

for any $a_k, a_r \in \mathrm{supp}(p)$*, where* $\{a_i, a_j\} \in B_p$*.*

Proof. Let $\mathcal{C}$ be a coordinatization. Then, there exists an isomorphism

$$g : P \longrightarrow \mathrm{LCM}(C_{P,\mathcal{C}}),$$

with $g(a) = x(a)$ for each $a \in \mathrm{atoms}(P)$. Suppose that $p \in (P \setminus \mathrm{atoms}(P)) \setminus 0$ and there exist $a_k, a_r \in \mathrm{supp}(p)$ such that

$$N([a_i \vee a_k, 1]) > N([a_r \vee a_k, 1])$$

and

$$N([a_j \vee a_k, 1]) > N([a_r \vee a_k, 1]),$$

where $\{a_i, a_j\} \in B_p$. Let $a_k^{n_y}$ be the highest power of a_k dividing $x(a_y)$ for any $a_y \in \mathrm{atoms}(P)$. Then, similarly to the proof of (5.2), we have $n_r > n_i$ and $n_r > n_j$. Thus, $a_k^{n_r} \nmid \mathrm{lcm}\{x(a_i), x(a_j)\}$, i.e., $x(a_r) \nmid \mathrm{lcm}\{x(a_i), x(a_j)\}$. Note that $g(p) = \mathrm{lcm}\{x(a_i), x(a_j)\}$. Hence, $x(a_r) \nmid g(p)$. However, $a_r \in \mathrm{supp}(p)$ yields that $g(a_r) = x(a_r) \mid g(p)$, a contradiction.

Conversely, suppose that, for all $p \in (P \setminus \mathrm{atoms}(P)) \setminus 0$, either

$$N([a_i \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$$

or

$$N([a_j \vee a_k, 1]) \leq N([a_r \vee a_k, 1])$$

for any $a_k, a_r \in \mathrm{supp}(p)$, where $\{a_i, a_j\} \in B_p$.

In what follows, we first prove that $\triangle(a) = x(a)$ for all $a \in \mathrm{atoms}(P)$. The proof will be completed in two parts.

(E1) Let $p \in (P \setminus \mathrm{atoms}(P)) \setminus 0$ and $\{a_i, a_j\} \in B_p$. Now, we prove that

$$x(a_s) \mid \mathrm{lcm}\{x(a_i), x(a_j)\} \quad \text{if } a_s \in \mathrm{supp}(p) \setminus \{a_i, a_j\}. \tag{5.5}$$

Since P is super-atomic,

$$a_i \vee a_s < p \quad \text{and} \quad a_j \vee a_s < p. \tag{5.6}$$

Let $a_t \in \text{atoms}(P)$ and $a_t^{n_y}$ be the highest power of a_t dividing $x(a_y)$ for any $a_y \in \text{atoms}(P)$. We claim

$$a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}. \tag{5.7}$$

If $a_t = a_s$, then, clearly, $n_s = 0$. It follows that (5.7) holds. If $a_t \neq a_s$, then there are two cases.

Case (1*). Suppose that $a_t \notin \text{supp}(p)$. Then, $a_i \vee a_j \vee a_t \vee a_s = p \vee a_t > p$. Thus,

$$a_i \vee a_j \neq a_i \vee a_j \vee a_t \vee a_s,$$
$$a_i \vee a_s \neq a_i \vee a_j \vee a_t \vee a_s$$

and

$$a_j \vee a_s \neq a_i \vee a_j \vee a_t \vee a_s$$

by (5.6). We claim that $a_s \vee a_t \neq a_i \vee a_j \vee a_t \vee a_s$. Otherwise, $a_s \vee a_t = a_i \vee a_j \vee a_t$ since $a_i \vee a_j \vee a_t = a_i \vee a_j \vee a_t \vee a_s$, which, together with P is super-atomic, yields $s = i$ or $s = j$, a contradiction. Therefore, either $a_i \vee a_j \vee a_t \vee a_s = a_i \vee a_t$ or $a_i \vee a_j \vee a_t \vee a_s = a_j \vee a_t$.

Obviously, $a_i \vee a_j \vee a_t \vee a_s = a_i \vee a_t$ implies that

$$a_s \vee a_t < a_i \vee a_j \vee a_t \vee a_s = a_i \vee a_t.$$

Thus, $N([a_s \vee a_t, 1]) > N([a_i \vee a_t, 1])$. Similarly to the proof of (5.2), we have $n_s < n_i$. It follows that

$$a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}.$$

Similarly, we can prove that $a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$ when $a_i \vee a_j \vee a_t \vee a_s = a_j \vee a_t$. Therefore,

$$a_t^{n_s} \mid \text{lcm}\{x(a_i), x(a_j)\}$$

in the case of $a_t \notin \text{supp}(p)$.

Case (2*). Suppose that $a_t \in \text{supp}(p)$. From the hypotheses, either

$$N([a_i \vee a_t, 1]) \leq N([a_s \vee a_t, 1])$$

or

$$N([a_j \vee a_t, 1]) \leq N([a_s \vee a_t, 1]).$$

In the first case, similarly to the proof of (5.2), we have $n_s \leq n_i$. Thus, $a_t^{n_s} \mid \mathrm{lcm}\{x(a_i), x(a_j)\}$. Similarly, we can prove $a_t^{n_s} \mid \mathrm{lcm}\{x(a_i), x(a_j)\}$ when $N([a_j \vee a_t, 1]) \leq N([a_s \vee a_t, 1])$. Hence,

$$a_t^{n_s} \mid \mathrm{lcm}\{x(a_i), x(a_j)\}$$

in the case of $a_t \in \mathrm{supp}(p)$. Therefore, from Case (1*) and Case (2*), we know that (5.7) holds if $a_t \neq a_s$.

From the definition of $\mathcal{C}$, we have that, if x is a variable of $x(a_s)$, then $x \in \mathrm{atoms}(P)$. Thus, by formula (5.7)

$$x(a_s) \mid \mathrm{lcm}\{x(a_i), x(a_j)\} \quad \text{if } a_s \in \mathrm{supp}(p) \setminus \{a_i, a_j\},$$

i.e., (5.5) is true.

(E2) We shall prove that

$$\triangle(a) = x(a) \tag{5.8}$$

for each $a \in \mathrm{atoms}(P)$. Indeed, let $q \in P$ and $q \geq a$. We claim that

$$x(a) \mid \mathrm{lcm}\{x(r) : r \in T\} \tag{5.9}$$

for any $T \in B_q$. If $q = a$, then, clearly, (5.9) holds.

If $q > a$, then there exist $a_u, a_v \in \mathrm{supp}(q)$ such that $a_u \vee a_v = q$. Since P is super-atomic, a_u, $a_v \in T$ for any $T \in B_q$. Using (5.5), we have that $x(c) \mid \mathrm{lcm}\{x(a_u), x(a_v)\}$ for all $c \in \mathrm{supp}(q)$. Note that $a \in \mathrm{supp}(q)$. Thus, $x(a) \mid \mathrm{lcm}\{x(a_u), x(a_v)\}$. Therefore, (5.9) is true.

Formula (5.9) implies that

$$x(a) \mid \mathrm{lcm}\{x(r) : r \in T\}$$

for any $T \in B_q$ if $q \geq a$. Thus, $x(a) \mid \triangle(a)$ by (3.1). Note that $\triangle(a) \mid x(a)$. Therefore, $\triangle(a) = x(a)$, i.e., (5.8) holds.

In order to prove that $\mathcal{C}$ is a coordinatization, by Lemma 3.2, it suffices to prove that $\mathcal{C}$ is a weak coordinatization. For $q \in P$, define

$$g : P \longrightarrow \mathrm{LCM}(I_{P,\mathcal{C}})$$

to be a map such that

$$g(q) = \mathrm{lcm}\{\triangle(w) : w \in \mathrm{supp}(q)\}. \tag{5.10}$$

Obviously, g is meet-preserving, join-preserving and a surjection by (B), (C) and (D) in the proof of Theorem 3.5. Thus, we only need prove that g is injective. Clearly, if $g(u) = g(v)$ and $0 \in \{u, v\}$, then $u = 0 = v$.

Next, suppose that $g(u) = g(v)$ and $u, v \in P \setminus 0$. We shall prove $u = v$. Indeed, if $v \not\leq u$, then $\text{supp}(v) \setminus \text{supp}(u) \neq \emptyset$. Let $a_t \in \text{supp}(v) \setminus \text{supp}(u)$. There are two cases.

Case (k1). If $u \in \text{atoms}(P)$, then by statement $(*)$ in the proof of Theorem 5.1, $u \nmid \triangle(u)$ and $u \mid \triangle(a_t)$. Hence $u \mid g(v)$ and $u \nmid g(u)$, contrary to $g(u) = g(v)$.

Case (k2). If $u \in (P \setminus \text{atoms}(P)) \setminus 0$, then, there exists an $\{a_i, a_j\} \in B_u$. Thus,

$$g(u) = \text{lcm}\{\triangle(a_i), \triangle(a_j)\}. \tag{5.11}$$

Obviously, $u = a_i \vee a_j \neq a_t \vee a_i \vee a_j$ since $a_t \not\leq u$. Thus, either $a_t \vee a_i = a_t \vee a_i \vee a_j$ or $a_t \vee a_j = a_t \vee a_i \vee a_j$. In the first case, note that $a_t \vee a_i > a_j \vee a_i$. Then,

$$N([a_j \vee a_i, 1]) > N([a_t \vee a_i, 1]).$$

Let $a_i^{n_j}$ be the highest power of a_i dividing $x(a_j)$ and $a_i^{n_t}$ the highest power of a_i dividing $x(a_t)$. Similarly to the proof of (5.2), $n_t > n_j$. Thus, $x(a_t) \nmid x(a_j)$. Again, by statement $(*)$ in the proof of Theorem 5.1, $a_i \nmid x(a_i)$ since $x(a_i) = \triangle(a_i)$, and this means that $x(a_t) \nmid x(a_i)$. Therefore, $x(a_t) \nmid \text{lcm}\{x(a_i), x(a_j)\}$. As $\triangle(a) = x(a)$ for any $a \in \text{atoms}(P)$, we have

$$\triangle(a_t) \nmid \text{lcm}\{\triangle(a_i), \triangle(a_j)\}.$$

From formulas (5.10) and (5.11), we have $\triangle(a_t) \nmid g(u)$, but $\triangle(a_t) \mid g(v)$ since $a_t \in \text{supp}(v)$, contrary to $g(u) = g(v)$. In the second case, with an analogous proof to the first case of $a_t \vee a_i = a_t \vee a_i \vee a_j$, we can deduce a contradiction.

Cases (k1) and (k2) tell us that the assumption of $v \not\leq u$ will yield a contradiction. Hence, $v \leq u$. Arguing as above, we can prove that $u \leq v$. Therefore, $u = v$. Consequently, g is injective. $\square$

Using Theorem 5.3, we can determine whether the labeling, defined by (5.1), of a super-atomic lattice is a coordinatization.

As a conclusion of this section, we shall consider when the labeling, defined by (5.1), of a non-super-atomic lattice is also a coordinatization.

Lemma 5.4. *Let* $P, Q \in \mathcal{L}(n)$ *with* $\operatorname{atoms}(P) = \operatorname{atoms}(Q) = \{1, 2, \ldots, n\}$*. If* $\mathcal{S}_P \setminus \mathcal{S}_Q = \{S\}$*, then* S *is meet-irreducible in* $(\mathcal{S}_P, \subseteq)$*.*

Proof. If S is not meet-irreducible in $(\mathcal{S}_P, \subseteq)$, then there exist two different elements $S_1, S_2 \in \mathcal{S}_P$ such that $S_1 \succ S$ and $S_2 \succ S$ in lattice $(\mathcal{S}_P, \subseteq)$. Note that $S_1, S_2 \in \mathcal{S}_Q$. We claim that $\bigvee_{t \in S}\{t\} = S_1$ in lattice $(\mathcal{S}_Q, \subseteq)$. Otherwise, we have

$$\bigvee_{t \in S}\{t\} = R \subsetneq S_1$$

for some $R \in \mathcal{S}_Q$ in lattice $(\mathcal{S}_Q, \subseteq)$. Clearly, $S \subseteq R \subsetneq S_1$. Since $S \notin \mathcal{S}_Q$, $S \neq R$, which means that $S \subsetneq R$. Therefore, $S \subsetneq R \subsetneq S_1$, which, together with $S, R,\ S_1 \in \mathcal{S}_P$ yields that $S_1 \not\succ S$ in lattice $(\mathcal{S}_P, \subseteq)$, a contradiction. Consequently,

$$\bigvee_{t \in S}\{t\} = S_1 \quad \text{in } (\mathcal{S}_Q, \subseteq).$$

Similarly, we also have

$$\bigvee_{t \in S}\{t\} = S_2 \quad \text{in } (\mathcal{S}_Q, \subseteq).$$

Therefore, $S_1 = S_2$, contrary to $S_1 \neq S_2$. $\square$

Let $P \in \mathcal{L}(n)$ with $\operatorname{atoms}(P) = \{a_1, a_2, \ldots, a_n\}$. Next, we denote by $\mathcal{C}_P$ the labeling of P defined by (5.1), that is,

$$m_c = \prod_{a_i \in \operatorname{supp}(c)} a_i$$

for any $c \in P \setminus 0$. Note that $(\mathcal{S}_P, \subseteq)$ is the lattice corresponding to P, see Section 2. Then, for any $C \in \mathcal{S}_P \setminus \emptyset$, we have that

$$m_C = \prod_{a_i \in C} a_i,$$

where C corresponds to c. Again, we denote by $x_P(\{a_i\})$ the monomials corresponding to $(\mathcal{S}_P, \subseteq)$ defined by (2.1). Then, we define $C_{\mathcal{S}_P, \mathcal{C}_P}$ as the ideal generated by monomials $x_P(\{a_i\})$ for each $i \in \{1, 2, \ldots, n\}$. We denote by $\triangle_P(\{a_i\})$ the monomials corresponding to $(\mathcal{S}_P, \subseteq)$ defined by (3.1), and define $I_{\mathcal{S}_P, \mathcal{C}_P}$ as the ideal generated by monomials

$\triangle_P(\{a_i\})$ for each $i \in \{1, 2, \ldots, n\}$. Then, we have the following theorem.

Theorem 5.5. *Let* $(\mathcal{S}_Q, \subseteq), (\mathcal{S}_P, \subseteq), (\mathcal{S}_R, \subseteq) \in \mathcal{L}(n)$ *and* $(\mathcal{S}_R, \subseteq)$ *be a super-atomic lattice. If* $\mathcal{S}_P \subseteq \mathcal{S}_R$, $\mathcal{S}_P \setminus \mathcal{S}_Q = \{S\}$ *and* $\mathcal{C}_P$ *are a coordinatization, then* $\mathcal{C}_Q$ *is a coordinatization if and only if* $\triangle_Q(\{a_k\}) = x_Q(\{a_k\})$ *for any* $k \in \{1, 2, \ldots, n\}$.

Proof. We only need show the sufficiency of the theorem since the necessity is obvious. First note that $I_{\mathcal{S}_Q,\mathcal{C}_Q} = C_{\mathcal{S}_Q,\mathcal{C}_Q}$ since $\triangle_Q(\{a_k\}) = x_Q(\{a_k\})$ for any $k \in \{1, 2, \ldots, n\}$. Define a map

$$h : (\mathcal{S}_Q, \subseteq) \longrightarrow \mathrm{LCM}(I_{\mathcal{S}_Q,\mathcal{C}_Q}) = \mathrm{LCM}(C_{\mathcal{S}_Q,\mathcal{C}_Q})$$

as

$$h(C) = \mathrm{lcm}\{\triangle_Q(\{a_i\}) : a_i \in C\} = \mathrm{lcm}\{x_Q(\{a_i\}) : a_i \in C\}$$

for any $C \in \mathcal{S}_Q$. According to Lemma 3.2, we merely need to prove that $\mathcal{C}_Q$ is a weak coordinatization, i.e., we only need prove that h is an isomorphism. By (B), (C) and (D) in the proof of Theorem 3.5, we can verify that h is meet-preserving, join-preserving and surjective.

Now, we shall prove that h is injective. For $C \in \mathcal{S}_P$, we define a map

$$g : (\mathcal{S}_P, \subseteq) \longrightarrow \mathrm{LCM}(C_{\mathcal{S}_P,\mathcal{C}_P})$$

such that

$$g(C) = \mathrm{lcm}\{x_P(\{a_i\}) : a_i \in C\}.$$

Obviously, g is an isomorphism from $(\mathcal{S}_P, \subseteq)$ to $\mathrm{LCM}(C_{\mathcal{S}_P,\mathcal{C}_P})$ since $\mathcal{C}_P$ is a coordinatization.

By Lemma 5.4 there exists exactly one element $T \in \mathcal{S}_P$ such that $T \succ S$ in lattice $(\mathcal{S}_P, \subseteq)$. Clearly, $S \notin \mathrm{atoms}(\mathcal{S}_P) \bigcup \{\emptyset\}$. If $a_j \in \{a_1, a_2, \ldots, a_n\} \setminus S$, then $S \notin \lceil \{a_j\} \rceil_P$ since $\{a_j\} \nsubseteq S$. Thus, $\lceil \{a_j\} \rceil_P = \lceil \{a_j\} \rceil_Q$, which implies that

$$x_Q(\{a_j\}) = \prod_{C \in \lceil \{a_j\} \rceil_Q^c} m_C = \prod_{C \in \mathcal{S}_Q \setminus \lceil \{a_j\} \rceil_Q} m_C = \frac{\prod_{C \in \mathcal{S}_P \setminus \lceil \{a_j\} \rceil_P} m_C}{m_S} = \frac{x_P(\{a_j\})}{\prod_{a_i \in S} a_i}. \tag{5.12}$$

If $a_j \in S$, then $S \in \lceil \{a_j\} \rceil_P$ since $\{a_j\} \subseteq S$. Thus, $\lceil \{a_j\} \rceil_P = \lceil \{a_j\} \rceil_Q \cup \{S\}$, which implies that

$$x_Q(\{a_j\}) = \prod_{C \in \lceil \{a_j\} \rceil_Q^c} m_C = \prod_{C \in \mathcal{S}_Q \setminus \lceil \{a_j\} \rceil_Q} m_C = \prod_{C \in \mathcal{S}_P \setminus \lceil \{a_j\} \rceil_P} m_C = x_P(\{a_j\}). \tag{5.13}$$

The proof is completed using three parts.

(I) Let $C_1, D_1 \in \mathcal{S}_Q$. If $h(C_1) = h(D_1)$ and $C_1 \subseteq D_1$, then $C_1 = D_1$. Suppose that $C_1 \neq D_1$. Then, $C_1 \subsetneq D_1$. Thus, there exists a $C_2 \in \mathcal{S}_Q$ such that

$$C_1 \prec C_2 \subseteq D_1 \quad \text{in } (\mathcal{S}_Q, \subseteq), \tag{5.14}$$

and

$$h(C_1) = h(C_2) \tag{5.15}$$

since h is meet-preserving. Clearly, if $C_1 = \emptyset$, then $h(C_1) = 1 = h(D_1)$, which implies that $C_1 = D_1$.

Next, we suppose that $C_1 \in \mathcal{S}_Q \setminus \emptyset$. If $C_1 \in \text{atoms}((\mathcal{S}_Q, \subseteq))$, then let $C_1 = \{a_u\}$. Clearly, there exists an $\{a_v\} \subseteq C_2$ such that $\{a_v\} \neq \{a_u\}$ by (5.14). From statement ($*$), we know that $a_u \nmid \triangle_Q(\{a_u\})$ and $a_u \mid \triangle_Q(\{a_v\})$. Hence, $a_u \mid h(C_2)$ and $a_u \nmid h(C_1)$, contrary to formula (5.15). If $C_1 \in (\mathcal{S}_Q \setminus \text{atoms}((\mathcal{S}_Q, \subseteq))) \setminus \emptyset$, then there exist $\{a_i\}, \{a_j\} \in \text{atoms}((\mathcal{S}_Q, \subseteq))$ such that

$$C_1 = \{a_i\} \vee \{a_j\} \tag{5.16}$$

in $(\mathcal{S}_Q, \subseteq)$ since $(\mathcal{S}_R, \subseteq)$ is super-atomic and $\mathcal{S}_Q \subseteq \mathcal{S}_R$. Furthermore, by (5.14), there exists an $\{a_k\} \in \text{atoms}((\mathcal{S}_Q, \subseteq))$ such that

$$C_2 = \{a_i\} \vee \{a_j\} \vee \{a_k\} \tag{5.17}$$

in $(\mathcal{S}_Q, \subseteq)$. Using formulas (5.15), (5.16) and (5.17), we have

$$h(C_1) = \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} = \text{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\}), x_Q(\{a_k\})\} = h(C_2). \tag{5.18}$$

Thus, we shall distinguish the six types, as follows. For convenience, let $a_y^{m_{xy}}$ be the highest power of a_y dividing $x_P(\{a_x\})$ and $a_y^{n_{xy}}$ the highest power of a_y dividing $x_Q(\{a_x\})$ for any $x, y \in \{1, 2, \ldots, n\}$.

Type 1. $a_i, a_j, a_k \in S$. We first claim that $C_1 \neq T$. If $C_1 = T$, then $\{a_i\} \vee \{a_j\} = S$ in $(\mathcal{S}_P, \subseteq)$ since $a_i, a_j \in S$. Thus, $\{a_i\} \vee \{a_j\} \vee \{a_k\} = S$ in $(\mathcal{S}_P, \subseteq)$ since $a_k \in S$. From formula (5.17), $C_2 = T$ such that $C_2 = C_1$, a contradiction. Hence, $C_1 \neq T$ and $C_1 \subsetneq T$ since $\{a_i, a_j\} \subseteq T$. Therefore,

$$\{a_i\} \vee \{a_j\} = C_1 \subsetneq S \tag{5.19}$$

in $(\mathcal{S}_P, \subseteq)$.

Using formula (5.13), we have $x_Q(\{a_t\}) = x_P(\{a_t\})$ for any $t \in \{i, j, k\}$. Then,

$$\text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \text{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_P(\{a_k\})\} \tag{5.20}$$

by formula (5.18). There are two subcases.

Subcase (1) (i). If $C_2 = T$, then $\{a_i\} \vee \{a_j\} \vee \{a_k\} = S$ in $(\mathcal{S}_P, \subseteq)$ since $a_i, a_j, a_k \in S$, which, together with formulas (5.19) and (5.20), implies that $g(C_1) = g(S)$. However, $g(C_1) < g(S)$ since $C_1 \subsetneq S$, and g is isomorphic, a contradiction.

Subcase (2) (i). If $C_2 \neq T$, then $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$. From formulas (5.19) and (5.20), $g(C_1) = g(C_2)$, contrary to $g(C_1) < g(C_2)$.

Type 2. $a_i, a_j, a_k \notin S$. From formula (5.12),

$$x_P(\{a_t\}) = \left(\prod_{a_r \in S} a_r \right) * x_Q(\{a_t\}) \quad \text{for any } t \in \{i, j, k\}.$$

Then, $h(C_1) = h(C_2)$ implies that

$$h(C_1) * \prod_{a_r \in S} a_r = h(C_2) * \prod_{a_r \in S} a_r.$$

Furthermore, by formula (5.18),

$$\mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_P(\{a_k\})\}.$$

On the other hand, as $a_i, a_j, a_k \notin S$, $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, obviously. Therefore, $g(C_1) = g(C_2)$, contrary to $g(C_1) < g(C_2)$.

Type 3. $a_i, a_j \notin S$ and $a_k \in S$. By formulas (5.13) and (5.18), we have that

$$\mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} = \mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\}), x_P(\{a_k\})\}.$$

Thus, $x_P(\{a_k\}) \mid \mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\}$. Similarly to the proof of Type 2, we know that

$$\{a_i\} \vee \{a_j\} = C_1, \{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2 \quad \text{in } (\mathcal{S}_P, \subseteq)$$

and

$$x_P(\{a_t\}) = \left(\prod_{a_r \in S} a_r\right) * x_Q(\{a_t\}) \quad \text{for any } t \in \{i, j\}.$$

Thus, $x_P(\{a_k\}) \mid \mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$, which implies that

$$\mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_P(\{a_k\})\}.$$

Therefore, $g(C_1) = g(C_2)$, contrary to $g(C_1) < g(C_2)$.

Type 4. $a_i \in S$, $a_j \notin S$ and $a_k \in S$. Using (5.13) and (5.18), we have that

$$\mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\} = \mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\}), x_P(\{a_k\})\}.$$

Similarly to the proof of Type 3, we have that $x_P(\{a_k\}) \mid \mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$ and $g(C_1) = g(C_2)$ with $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, contrary to $g(C_1) < g(C_2)$.

Type 5. $a_i, a_j \in S$ and $a_k \notin S$. Using (5.13) and (5.18), we have that

$$\mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\} = \mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\}), x_Q(\{a_k\})\}.$$

Then,

$$x_Q(\{a_k\}) \mid \mathrm{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}. \tag{5.21}$$

Using (5.12), we have

$$x_P(\{a_k\}) = \left(\prod_{a_r\in S} a_r\right) * x_Q(\{a_k\}).$$

Thus, $n_{k_i}+1 = m_{k_i}$ since $a_i \in S$. We note that $\{a_i\}\vee\{a_j\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$ since $a_k \notin S$. Then,

$$\{a_i\}\vee\{a_k\} = C_2 \quad \text{or} \quad \{a_j\}\vee\{a_k\} = C_2$$

in $(\mathcal{S}_P, \subseteq)$ since $\mathcal{S}_P \subseteq \mathcal{S}_R$ and $(\mathcal{S}_R, \subseteq)$ is super-atomic. There are two subcases.

Subcase (1). If $C_1 = T$, then $\{a_i\}\vee\{a_j\} = S$ in $(\mathcal{S}_P, \subseteq)$. Thus, $S \subsetneq C_1 = T \subsetneq C_2$. Assume that $\{a_i\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$. Then, we have that

$$N([\{a_i\}\vee\{a_j\}, 1]) \geq N([\{a_i\}\vee\{a_k\}, 1]) + 2$$

in $(\mathcal{S}_P, \subseteq)$ since $S \subsetneq T \subsetneq C_2$. Similarly to the proof of (5.2), we have that $m_{k_i} \geq m_{j_i} + 2$. Thus, $n_{k_i} \geq m_{j_i} + 1$, which implies that $x_Q(\{a_k\}) \nmid x_P(\{a_j\})$. From Lemma 3.2, $\triangle_P(\{a_i\}) = x_P(\{a_i\})$ since $\mathcal{C}_P$ is a coordinatization. Furthermore, by statement $(*)$, we know that $a_i \nmid x_P(\{a_i\})$. Therefore, $x_Q(\{a_k\}) \nmid \operatorname{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\}$, contrary to (5.21).

If $\{a_j\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, then, with an analogous proof to the case of $\{a_i\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, we may obtain a contradiction.

Subcase (2). If $C_1 \neq T$, then $C_1 \subsetneq S$ and $\{a_i\}\vee\{a_j\} = C_1$ in $(\mathcal{S}_P, \subseteq)$ by the proof of Type 1. Suppose that $\{a_i\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$. Then,

$$N([\{a_i\}\vee\{a_j\}, 1]) > N([\{a_i\}\vee\{a_k\}, 1])$$

in $(\mathcal{S}_P, \subseteq)$ since $C_1 \subsetneq C_2$. Note that $C_2 \nsubseteq S$ since $a_k \notin S$. Thus,

$$N([\{a_i\}\vee\{a_j\}, 1]) \geq N([\{a_i\}\vee\{a_k\}, 1]) + 2$$

in $(\mathcal{S}_P, \subseteq)$ since $C_1 \subsetneq S$. Similarly to Subcase (1), we can prove that

$$x_Q(\{a_k\}) \nmid \operatorname{lcm}\{x_P(\{a_i\}), x_P(\{a_j\})\},$$

contrary to (5.21). If $\{a_j\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, then with an analogous proof to the case of $\{a_i\}\vee\{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, we may get a contradiction.

Type 6. $a_i \in S$ and $a_j, a_k \notin S$. By (5.13) and (5.18), we have that

$$\mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\} = \mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\}), x_Q(\{a_k\})\}.$$

Thus,

$$x_Q(\{a_k\}) \mid \mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\}. \tag{5.22}$$

Clearly, $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_i\} \vee \{a_j\} \vee \{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$ since $a_j, a_k \notin S$. From the proof of Type 5, we know that

$$\{a_i\} \vee \{a_k\} = C_2 \text{ or } \{a_j\} \vee \{a_k\} = C_2$$

in $(\mathcal{S}_P, \subseteq)$. There are two subcases.

Subcase (i). If $\{a_i\} \vee \{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, by the proof of Type 5, we have that

$$N([\{a_i\} \vee \{a_j\}, 1]) > N([\{a_i\} \vee \{a_k\}, 1])$$

in $(\mathcal{S}_P, \subseteq)$. Clearly, $m_{k_i} > m_{j_i}$, i.e., $x_P(\{a_k\}) \nmid x_P(\{a_j\})$. Using (5.12), we have

$$x_P(\{a_j\}) = \left(\prod_{a_r \in S} a_r\right) * x_Q(\{a_j\})$$

and

$$x_P(\{a_k\}) = \left(\prod_{a_r \in S} a_r\right) * x_Q(\{a_k\}).$$

Hence, $x_Q(\{a_k\}) \nmid x_Q(\{a_j\})$.

From Lemma 3.2, $\triangle_P(\{a_i\}) = x_P(\{a_i\})$ since $\mathcal{C}_P$ is a coordinatization. Furthermore, by statement $(*)$, $a_i \nmid x_P(\{a_i\})$. Thus,

$$x_Q(\{a_k\}) \nmid \mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\},$$

contrary to formula (5.22).

Subcase (ii). If $\{a_j\} \vee \{a_k\} = C_2$ in $(\mathcal{S}_P, \subseteq)$, then, we note that

$$N([\{a_i\} \vee \{a_j\}, 1]) > N([\{a_j\} \vee \{a_k\}, 1])$$

in $(\mathcal{S}_Q, \subseteq)$. Clearly, $n_{k_j} > n_{i_j}$. Again, we know that $n_{i_j} = m_{i_j}$ since $x_P(\{a_i\}) = x_Q(\{a_i\})$. Hence, $x_Q(\{a_k\}) \nmid x_P(\{a_i\})$. Since $\triangle_Q(\{a_j\})$

$= x_Q(\{a_j\})$, we have $a_j \nmid x_Q(\{a_j\})$, by statement $(*)$. Therefore,

$$x_Q(\{a_k\}) \nmid \mathrm{lcm}\{x_P(\{a_i\}), x_Q(\{a_j\})\},$$

contrary to formula (5.22).

Types 1–6 tell us that, if $h(C_1) = h(D_1)$ and $C_1 \subseteq D_1$, then $C_1 = D_1$.

Similarly to (I), we can prove that

(II) If $h(C_1) = h(D_1)$ and $C_1 \supseteq D_1$, then $C_1 = D_1$.

(III) If $h(C_1) = h(D_1)$, then $C_1 \subseteq D_1$ or $C_1 \supseteq D_1$.

Assume that $C_1 \| D_1$. Let $\{a_i\} \vee \{a_j\} = C_1$ and $\{a_k\} \vee \{a_e\} = D_1$ in $(\mathcal{S}_Q, \subseteq)$. Then,

$$C = C_1 \vee D_1 = \{a_i\} \vee \{a_j\} \vee \{a_k\} \vee \{a_e\} \supsetneq \{a_i\} \vee \{a_j\} = C_1 \quad \text{in } (\mathcal{S}_Q, \subseteq).$$

Thus, by (I), we have that $h(C_1) < h(C)$. It follows that

$$\mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} < \mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\}), x_Q(\{a_k\}), x_Q(\{a_e\})\}.$$

Therefore,

$$x_Q(\{a_k\}) \nmid \mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} \tag{5.23}$$

or

$$x_Q(\{a_e\}) \nmid \mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\},$$

and formula (5.23) implies that

$$h(D_1) = \mathrm{lcm}\{x_Q(\{a_k\}), x_Q(\{a_e\})\} \nmid \mathrm{lcm}\{x_Q(\{a_i\}), x_Q(\{a_j\})\} = h(C_1),$$

i.e., $h(C_1) \neq h(D_1)$, a contradiction. From (I), (II) and (III), we know that the map h is injective. $\square$

The following example will illustrate Theorem 5.5.

Example 5.6. Let $\mathcal{S}_P = \{\{a_1, a_2, a_3, a_4\}, \{a_2, a_3, a_4\}, \{a_1, a_3, a_4\}, \{a_3, a_4\}, \{a_2, a_3\}, \{a_1, a_4\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \emptyset\}$. It is easy to see that $(\mathcal{S}_P, \subseteq)$ is a super-atomic lattice in $\mathcal{L}(4)$. Denote $\mathcal{C}_P$ as a labeling of $\mathcal{S}_P$ defined by (5.1). Then, $C_{\mathcal{S}_P,\mathcal{C}_P} = \{a_2^3a_3^4a_4^3, a_1^3a_3^3a_4^4, a_1^2a_2a_4^2, a_1a_2^2a_3^2\}$. Clearly, the labeling $\mathcal{C}_P$ is a coordinatization. Let $\mathcal{S}_Q = \mathcal{S}_P \setminus \{\{a_2,$

$a_3, a_4\}\}$. Clearly, $x_Q(\{a_i\}) = \triangle_Q(\{a_i\})$ for any $i \in \{1, 2, 3, 4\}$. Then, by Theorem 5.5,

$$C_{\mathcal{S}_Q, \mathcal{C}_Q} = \{a_2^2 a_3^3 a_4^2, a_1^3 a_3^3 a_4^4, a_1^2 a_2 a_4^2, a_1 a_2^2 a_3^2\}.$$

Furthermore, it can be verified that $\mathrm{LCM}(C_{\mathcal{S}_Q, \mathcal{C}_Q}) \cong (\mathcal{S}_Q, \subseteq)$, i.e., $\mathcal{C}_Q$ is a coordinatization.

6. Conclusions.

In this paper, we studied monomial ideals by their associated lcm-lattices. First, we introduced notions of weak coordinatizations which have weaker hypotheses than coordinatizations, and next we showed the characterizations of all such weak coordinatizations which partly answer the problem given by Mapes in **[10]**. We then defined a finite super-atomic lattice in $\mathcal{L}(n)$, used to investigate the structures of $\mathcal{L}(n)$ and to identify that a specific labeling, given by us, of a finite atomic lattice is a weak coordinatization. It should be very interesting in the future to study a minimal free resolution of R/M by our results.

Acknowledgments. The authors thank the referees for their valuable comments and suggestions.

REFERENCES

1. D. Bayer, I. Peeva and B. Sturmfels, *Monomial resolutions*, Math. Res. Lett. **5** (1998), 31–46.

2. G. Birkhoff and O. Frink, *Representations of lattices by sets*, Trans. Amer. Math. Soc. **64** (1948), 299–316.

3. J.R. Büchi, *Representation of complete lattices by sets*, Portugal. Math. **11** (1952), 151–167.

4. Timothy B.P. Clark, *Poset resolutions and lattice-linear monomial ideals*, J. Algebra **323** (2010), 899–919.

5. P. Crawley and R.P. Dilworth, *Algebraic theory of lattices*, Prentice Hall, Englewood Cliffs, NJ, 1973.

6. S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129** (1990), 1–25.

7. V. Gasharov, I. Peeva and W. Volkmar, *The* lcm-*lattice in monomial resolutions*, Math. Res. Lett. **6** (1999), 521–532.

8. L. Katthän, *Stanley depth and simplicial spanning trees*, J. Alg. Combin. **42** (2015), 507–536.

9. S. Mapes, *Finite atomic lattices and their relationship to resolutions of monomial ideals*, Ph.D. dissertation, Columbia University, New York, 2009.

10. S. Mapes, *Finite atomic lattices and resolutions of monomial ideals*, J. Algebra **379** (2013), 259–276.

11. S. Mapes and L. Piechnik, *Constructing monomial ideals with a given minimal resolution*, Rocky Mountain J. Math. **47** (2017), 1963–1985.

12. J. Phan, *Order properties of monomial ideals and their free resolutions*, Ph.D. dissertation, Columbia University, New York, 2006.

13. A.B. Tchernev, *Representations of matroids and free resolutions for multigraded modules*, Adv. Math. **208** (2007), 75–134.

14. M. Velasco, *Minimal free resolutions that are not supported by a* CW-*complex*, J. Algebra **319** (2008), 102–114.

Sichuan Normal University, College of Mathematics and Software Science, Chengdu, Sichuan 610066, P.R. China
Email address: **443966297@qq.com**

Sichuan Normal University, College of Mathematics and Software Science, Chengdu, Sichuan 610066, P.R. China
Email address: **xpwang1@hotmail.com**