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A NEW BLOW-UP CRITERION FOR NON-NEWTON FILTRATION EQUATIONS WITH SPECIAL MEDIUM VOID

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ABSTRACT. This paper deals with the finite time blow-up of solutions to the initial boundary value problem of a non-Newton filtration equation with special medium void. A new criterion for the solutions to blow up in finite time is established by using the Hardy inequality. Moreover, the upper and lower bounds for the blow-up time are also estimated. The results solve an open problem proposed by Liu in 2016 [**8**].

1. Introduction. In this paper, we investigate the blow-up properties of solutions to the following non-Newton filtration equation with special medium void:

$$\begin{cases} \dfrac{u_t}{|x|^2} - \Delta_p u = u^q & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary $\partial\Omega$, $0 \in \Omega$, $2 < p < n$, $p < q+1 < p^* = np/(n-p)$, $0 \leq u_0 \in W_0^{1,p}(\Omega)$ and $u_0(x) \not\equiv 0$. Moreover, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

By the conservation law, the motion of a fluid in a rigid porous medium can be described, under certain assumptions, by the following equation

$$a(x)u_t - \operatorname{div}(u\overrightarrow{V}) = f(u),$$

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where $a(x)$ is the void of the medium, $u(x,t)$ is the density of fluid, $\overrightarrow{V}$ is the velocity of filtration of fluid and $f(u)$ is the source, [**2, 12, 13**]. For the non-Newtonian filtration fluid, one obtains the p-Laplacian equation

$$a(x)u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u).$$

During the past few decades, much effort has been devoted to solving the above problems, mainly, for the case $a(x) = 1$ (see [**2**]). In 2004, Tan [**12**] considered the existence and asymptotic estimates of global solutions and finite time blow-up of local solutions to problem (1.1). By using the potential well method proposed by Sattinger and Payne [**9, 10**] and Hardy's inequality, he gave some sufficient conditions for the solutions to exist globally or to blow up in finite time, when the initial energy is subcritical, i.e., initial energy smaller than the mountain pass level. These results were extended to porous medium equations with a special medium void by Zhou [**13**]. In 2016, by defining new potential wells and their corresponding sets, Liu [**8**] showed the existence of global or finite time blow-up solutions to problem (1.1) when the initial energy is critical. He also proposed an open problem as to whether or not problem (1.1) admits blow-up solutions when the initial energy is supercritical.

In this short paper, we will answer this question. Our main results contain two aspects. Firstly, inspired by some ideas from [**3, 4, 11**] and with the help of Hardy's inequality, we will give a new criterion for the solutions to problem (1.1) to blow up in finite time. More precisely, we will show that, for any $M > 0$, there exists a $u_0 \in W_0^{1,p}(\Omega)$ satisfying $J(u_0) > M$, and the solutions to problem (1.1) with u_0 as initial value blow up in finite time. Secondly, the blow-up time is estimated from both above and below. In this process, Gagliardo-Nirenberg's inequality will play an important role.

The paper is organized as follows. Some preliminaries and the main results are introduced in Section 2, such as the definition of weak solutions to problem (1.1) and some functionals, as well as some auxiliary lemmas to be used later. The main results are proven in Section 3.

2. Preliminaries and main results. Throughout this paper, we denote the norm of $L^r(\Omega)$, $1 \leq r \leq \infty$, by $\|\cdot\|_r$, and denote by $(\ ,\ )$ the inner product in $L^2(\Omega)$. By $W_0^{1,p}(\Omega)$, we denote the Sobolev space such that both u and $|\nabla u|$ belong to $L^p(\Omega)$ for any $u \in W_0^{1,p}(\Omega)$. $W_0^{1,p}(\Omega)$ will be endowed with the equivalent norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$. In this paper, we consider weak solutions to problem (1.1).

Definition 2.1 (**[12]**)**.** A function u is called a (weak) solution to problem (1.1) in $\Omega \times (0,T)$ if

$$u \in L^\infty(0,T; W_0^{1,p}(\Omega)), \qquad \int_0^T \left\| \frac{u_t}{|x|} \right\|_2^2 \mathrm{d}t < \infty,$$

and $u(x,t)$ satisfies $u(x,0) = u_0(x)$ and

$$\int_\Omega \left(\frac{u_t}{|x|^2} v + |\nabla u|^{p-2} \nabla u \cdot \nabla v \right) \mathrm{d}x = \int_\Omega u^q v \,\mathrm{d}x,$$
$$\text{for all } v \in W_0^{1,p}(\Omega), \quad t \in (0,T).$$

Definition 2.2. Let $u(x,t)$ be a weak solution to problem (1.1). We say that $u(x,t)$ blows up at a finite time T_0, provided that

$$\lim_{t \to T_0} \left\| \frac{u(t)}{|x|} \right\|_2^2 = +\infty.$$

For any $u \in W_0^{1,p}(\Omega)$, define the functionals

$$J(u) = \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{q+1} \|u\|_{q+1}^{q+1}$$

and

$$I(u) = \|\nabla u\|_p^p - \|u\|_{q+1}^{q+1}.$$

Since $q + 1 < p^*$, both functionals are well defined and continuous in $W_0^{1,p}(\Omega)$. Let $u(x,t)$ be a weak solution to problem (1.1). Then, standard arguments show that $J(u(x,t))$ is non-increasing for $t \in [0,T)$. Moreover, it has also been shown **[12]** that the solution to problem (1.1) blows up in finite time when $J(u_0) \le 0$. These results can be summarized into the following lemma.

Lemma 2.3 (**[8, 12]**)**.** *Let $u(x,t)$ be a weak solution to problem* (1.1)*. Then, $J(u(x,t))$ is non-increasing in t and it holds, for any $t \in (0,T)$,* *that*

$$\int_0^t \left\| \frac{u_\tau}{|x|} \right\|_2^2 \mathrm{d}\tau + J(u(x,t)) = J(u_0). \tag{2.1}$$

Moreover, if $J(u_0) \le 0$, then $u(x,t)$ blows up in finite time.

The next three lemmas are necessary for the proof of the main results. The first classical result is essentially due to Hardy **[5]**, the second is a special form of Gagliardo-Nirenberg's inequality **[1]** and the third can be viewed as a descendant of Levine's concavity method **[6]**.

Lemma 2.4. *Assume that $1 < p < n$ and $u \in W^{1,p}(\mathbb{R}^n)$. Then, $u/|x| \in L^p(\mathbb{R}^n)$, and*

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \mathrm{d}x \le C_{n,p} \int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x, \tag{2.2}$$

where $C_{n,p} = (p/(n-p))^p$. Denote $C_{n,p}$ by C_n when $p = 2$.

Remark 2.5. For any $u \in W_0^{1,p}(\Omega)$, extend $u(x)$ to be 0 for $x \in \mathbb{R}^n \backslash \Omega$. Then, $u \in W^{1,p}(\mathbb{R}^n)$, and therefore, (2.2) also holds for $u \in W_0^{1,p}(\Omega)$.

Lemma 2.6. *Let $2 \le p < q+1 < p^* = np/(n-p)$. Then, for any $u \in W_0^{1,p}(\Omega)$, we have*

$$\|u\|_{q+1}^{q+1} \le C_G \|\nabla u\|_p^{\alpha(q+1)} \|u\|_2^{(1-\alpha)(q+1)}, \tag{2.3}$$

where

$$\alpha = \frac{q-1}{2(q+1)} \left(\frac{1}{2} + \frac{1}{n} - \frac{1}{p} \right)^{-1} \in (0,1)$$

and $C_G > 0$ is a constant depending only on n, p and q.

Lemma 2.7 ([**7**]). *Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \ge 0,$$

where $\theta > 0$. *If* $\psi(0) > 0$ *and* $\psi'(0) > 0$, *then* $\psi(t) \to \infty$ *as*

$$t \longrightarrow t_* \leq t^* = \frac{\psi(0)}{\theta\psi'(0)}.$$

For simplicity, we denote by $u(t)$ the solution $u(x,t)$ to problem (1.1), and write $U_0 = \|u_0/|x|\|_2^2$,

$$U(t) = \left\| \frac{u(t)}{|x|} \right\|_2^2$$

in the sequel. The main results of this paper can be summarized in the following two theorems.

Theorem 2.8. *Assume that* $2 < p < n$, $p < q + 1 < p^*$, *and that* $u(x,t)$ *is a weak solution to problem* (1.1). *If*

$$0 < K_1 J(u_0) < \frac{1}{2} U_0 - K_2,$$

then $T < +\infty$, *which means that* $u(x,t)$ *blows up at some finite time* T. *Moreover, the upper bound for* T *has the following form*

$$T \leq \frac{4K_1 U_0}{(q-1)^2 H(0)},$$

where $H(0) = (1/2)U_0 - K_1 J(u_0) - K_2 > 0$,

$$K_1 = \frac{p(q+1)C_n}{2(q+1-p)}, \qquad K_2 = \frac{C_n}{2}|\Omega|,$$

C_n *is the positive constant given in Hardy inequality and* $|\Omega|$ *is the Lebesgue's measure of* Ω.

Theorem 2.9. *Let all of the assumptions in Theorem* 2.8 *hold, and assume, in addition, that* $q + 1 < p(1 + 2/n)$. *Then, the maximal existence time satisfies* $T \geq U_0^{1-\gamma}/C^*(\gamma - 1)$, *where* $\gamma > 1$ *and* $C^* > 0$ *are two constants that will be determined in the proof.*

3. Proofs of the main results.

Proof of Theorem 2.8. By using some ideas from **[11]** and an application of Hardy's inequality, we first show that the solutions to problem (1.1) will blow up in finite time when $0 < K_1 J(u_0) < (1/2)U_0 - K_2$.

Suppose, on the contrary, that $u(x,t)$ is global, i.e., $T = +\infty$. Then, for any $t \in (0,\infty)$, the following holds:

(3.1)
$$\int_0^t \left\| \frac{u_\tau(\tau)}{|x|} \right\|_2 \mathrm{d}\tau \geq \left\| \int_0^t \frac{u_\tau(\tau)}{|x|} \,\mathrm{d}\tau \right\|_2 = \left\| \frac{u(t)}{|x|} - \frac{u_0}{|x|} \right\|_2 \geq U^{1/2}(t) - U_0^{1/2}.$$

By using Hölder's inequality and recalling Lemma 2.3, we obtain

(3.2)
$$U^{1/2}(t) \leq U_0^{1/2} + \left(t \int_0^t \left\| \frac{u_\tau(\tau)}{|x|} \right\|_2^2 \mathrm{d}\tau \right)^{1/2} = U_0^{1/2} + \{t[J(u_0) - J(u(t))]\}^{1/2}.$$

Since $u(x,t)$ is global, from Lemma 2.3, it is known that $J(u(t)) > 0$ for any $t \in [0,\infty)$. Therefore, from (3.2), we see that

$$U^{1/2}(t) \leq U_0^{1/2} + [J(u_0) - J(u(t))]^{1/2} t^{1/2} < U_0^{1/2} + J^{1/2}(u_0) t^{1/2}. \tag{3.3}$$

On the other hand, since $p > 2$, by Hardy's inequality and direct computation, we have

(3.4)
$$\begin{aligned} \frac{1}{2}\frac{d}{dt}U(t) &= -I(u(t)) = \int_\Omega \frac{u u_t}{|x|^2} \,\mathrm{d}x = -\|\nabla u\|_p^p + \|u\|_{q+1}^{q+1} \\ &= \frac{q+1-p}{p}\|\nabla u\|_p^p - (q+1)J(u(t)) \\ &\geq \frac{q+1-p}{p}(\|\nabla u\|_2^2 - |\Omega|) - (q+1)J(u(t)) \\ &\geq \frac{2(q+1-p)}{pC_n}\left[\frac{1}{2}U(t) - \frac{p(q+1)C_n}{2(q+1-p)}J(u(t)) - \frac{C_n}{2}|\Omega|\right] \\ &\triangleq K_0\left(\frac{1}{2}U(t) - K_1 J(u(t)) - K_2\right), \end{aligned}$$

where

$$K_0 = \frac{2(q+1-p)}{pC_n}, \qquad K_1 = \frac{p(q+1)C_n}{2(q+1-p)}$$

and $K_2 = (C_n/2)|\Omega|$. Set $H(t) = (1/2)U(t) - K_1 J(u(t)) - K_2$. Noticing from Lemma 2.3 that $(d/dt)J(u(t)) \le 0$, we further obtain

$$\frac{d}{dt}H(t) \ge \frac{d}{dt}\left(\frac{1}{2}U(t)\right) \ge K_0 H(t). \tag{3.5}$$

Since $H(0) = (1/2)U_0 - K_1 J(u_0) - K_2 > 0$, $H(t) > 0$ for all $t \ge 0$. Integration of (3.5) over $[0, t]$ yields

$$H(t) \ge H(0)e^{K_0 t}. \tag{3.6}$$

Since $0 < J(u(t)) \le J(u_0)$ for all $t \in [0, \infty)$, from (3.6), we have

$$U(t) \ge 2H(t) \ge 2H(0)e^{K_0 t}, \quad t \in [0, \infty),$$

which contradicts with (3.3) for sufficiently large t. Therefore, $T < +\infty$, and $u(x, t)$ blows up in finite time.

Next, we derive the upper bound for T. We first observe from the definitions of $I(u)$, $J(u)$, the initial condition $K_1 J(u_0) + K_2 < (1/2)U_0$ and Hardy's inequality that

$$\begin{aligned} I(u_0) &= (q+1)J(u_0) - \frac{q+1-p}{p}\|\nabla u_0\|_p^p \\ &= \frac{2(q+1-p)}{pC_n}\left[K_1 J(u_0) - \frac{1}{2}U_0 + K_2\right] \\ &\quad - \frac{q+1-p}{p}\left[\|\nabla u_0\|_p^p + |\Omega| - \frac{1}{C_n}U_0\right] \\ &\le \frac{2(q+1-p)}{pC_n}\left[K_1 J(u_0) - \frac{1}{2}U_0 + K_2\right] \\ &\quad - \frac{q+1-p}{p}\left[\|\nabla u_0\|_2^2 - \frac{1}{C_n}U_0\right] \\ &< 0. \end{aligned}$$

We claim that $I(u(t)) < 0$ for all $t \in [0, T)$. If not, there would exist a $t_0 \in (0, T)$ such that $I(u(t)) < 0$ for all $t \in [0, t_0)$ and $I(u(t_0)) = 0$. By (3.4), we know that $U(t)$ is strictly increasing in t for $t \in [0, t_0)$, and therefore,

$$K_1 J(u_0) + K_2 < \frac{1}{2}U_0 < \frac{1}{2}U(t_0). \tag{3.7}$$

On the other hand, from the monotonicity of $J(u(t))$ and Hardy's inequality, we obtain

$$\begin{aligned}K_1 J(u_0) + K_2 &\geq K_1 J(u(t_0)) + K_2 \\ &= \frac{C_n}{2}(\|\nabla u(t_0)\|_p^p + |\Omega|) + \frac{pC_n}{2(q+1-p)} I(u(t_0)) \\ &\geq \frac{C_n}{2}\|\nabla u(t_0)\|_2^2 \geq \frac{1}{2}U(t_0),\end{aligned}$$

which contradicts with (3.7). Therefore, $I(u(t)) < 0$ for all $t \in [0, T)$, as claimed, and $U(t)$ is strictly increasing on $[0, T)$.

For any $T^* \in (0, T)$, $\beta > 0$ and $\sigma > 0$, define

$$F(t) = \int_0^t U(\tau)\mathrm{d}\tau + (T - t)U_0 + \beta(t + \sigma)^2, \quad t \in [0, T^*]. \tag{3.8}$$

By direct computations,

$$\begin{aligned}F'(t) &= U(t) - U_0 + 2\beta(t + \sigma) \\ &= \int_0^t \frac{d}{d\tau}U(\tau)\,\mathrm{d}\tau + 2\beta(t + \sigma) \\ &= 2\int_0^t \left(u(\tau), \frac{u_\tau(\tau)}{|x|^2}\right)\mathrm{d}\tau + 2\beta(t + \sigma),\end{aligned} \tag{3.9}$$

$$\begin{aligned}F''(t) &= 2\left(u(t), \frac{u_t(t)}{|x|^2}\right) + 2\beta = -2I(u(t)) + 2\beta \\ &= -2(q+1)J(u(t)) + \frac{2(q+1-p)}{p}\|\nabla u(t)\|_p^p + 2\beta \\ &= -2(q+1)J(u_0) + 2(q+1)\int_0^t \left\|\frac{u_\tau}{|x|}\right\|_2^2 \mathrm{d}\tau \\ &\quad + \frac{2(q+1-p)}{p}\|\nabla u(t)\|_p^p + 2\beta.\end{aligned} \tag{3.10}$$

For $t \in [0, T^*]$, set

$$\begin{aligned}f(t) = &\left(\int_0^t U(\tau)\mathrm{d}\tau + \beta(t+\sigma)^2\right)\left(\int_0^t \left\|\frac{u_\tau}{|x|}\right\|_2^2 \mathrm{d}\tau + \beta\right) \\ &- \left(\int_0^t \left(u, \frac{u_\tau}{|x|^2}\right)\mathrm{d}\tau + \beta(t+\sigma)\right)^2.\end{aligned}$$

Then it is easy to verify, by using the Cauchy-Schwarz inequality and Hölder's inequality, that $f(t)$ is nonnegative on $[0,T^*]$. Therefore, in view of (3.8)–(3.10) and noting the monotonicity of $U(t)$, we have

(3.11)

$$\begin{aligned}
&F(t)F''(t)-\frac{q+1}{2}(F'(t))^2\\
&\quad=F(t)F''(t)-2(q+1)\bigg(\int_0^t\Big(u,\frac{u_\tau}{|x|^2}\Big)\mathrm{d}\tau+\beta(t+\sigma)\bigg)^2\\
&\quad=F(t)F''(t)+2(q+1)\bigg[f(t)-(F-(T-t)U_0)\bigg(\int_0^t\Big\|\frac{u_\tau}{|x|}\Big\|_2^2\mathrm{d}\tau\mathrm{d}\tau+\beta\bigg)\bigg]\\
&\quad\geq F(t)F''(t)-2(q+1)F(t)\bigg(\int_0^t\Big\|\frac{u_\tau}{|x|}\Big\|_2^2\mathrm{d}\tau+\beta\bigg)\\
&\quad=F(t)\bigg[-2(q+1)J(u_0)+2(q+1)\int_0^t\Big\|\frac{u_\tau}{|x|}\Big\|_2^2\mathrm{d}\tau+\frac{2(q+1-p)}{p}\|\nabla u(t)\|_p^p\\
&\qquad\qquad+2\beta-2(q+1)\int_0^t\Big\|\frac{u_\tau}{|x|}\Big\|_2^2\mathrm{d}\tau-2(q+1)\beta\bigg]\\
&\quad>F(t)\bigg[-2(q+1)J(u_0)+\frac{2(q+1-p)}{p}\|\nabla u(t)\|_p^p-2(q+1)\beta\bigg]\\
&\quad\geq F(t)\bigg[-2(q+1)J(u_0)+\frac{2(q+1-p)}{p}(\|\nabla u(t)\|_2^2-|\Omega|)-2(q+1)\beta\bigg]\\
&\quad\geq F(t)\bigg[-2(q+1)J(u_0)+\frac{2(q+1-p)}{pC_n}U(t)-\frac{2(q+1-p)|\Omega|}{p}-2(q+1)\beta\bigg]\\
&\quad\geq F(t)\bigg[-2(q+1)J(u_0)+\frac{2(q+1-p)}{pC_n}U_0-\frac{2(q+1-p)|\Omega|}{p}-2(q+1)\beta\bigg]\\
&\quad=2(q+1)F(t)\bigg\{\frac{1}{K_1}\Big(\frac{1}{2}U_0-K_1J(u_0)-K_2\Big)-\beta\bigg\}\\
&\quad=2(q+1)F(t)\Big(\frac{H(0)}{K_1}-\beta\Big)\geq 0,
\end{aligned}$$

for any $t\in[0,T^*]$ and $\beta\in(0,H(0)/K_1]$. In view of Lemma 2.7, we can see that

$$T^*\leq\frac{2F(0)}{(q-1)F'(0)}=\frac{U_0}{(q-1)\beta\sigma}T+\frac{\sigma}{q-1}.$$

From the arbitrariness of $T^* < T$, it follows that

$$T \leq \frac{U_0}{(q-1)\beta\sigma} T + \frac{\sigma}{q-1}, \tag{3.12}$$

for any $\beta \in (0, H(0)/K_1]$ and $\sigma > 0$.

Fix a $\beta_0 \in (0, H(0)/K_1]$. Then, for any $\sigma \in ((U_0/(q-1)\beta_0), +\infty)$, we have $0 < (U_0/(q-1)\beta_0\sigma) < 1$, which, together with (3.12), implies that

$$T \leq \frac{\sigma}{q-1}\left(1 - \frac{U_0}{(q-1)\beta_0\sigma}\right)^{-1} = \frac{\beta_0\sigma^2}{(q-1)\beta_0\sigma - U_0}. \tag{3.13}$$

Minimizing the right hand side in (3.13) for $\sigma \in ((U_0/(q-1)\beta_0), +\infty)$ yields

$$T \leq \frac{4U_0}{(q-1)^2\beta_0} \quad \text{for all } \beta_0 \in (0, H(0)/K_1]. \tag{3.14}$$

Then, minimizing the right hand side of (3.14) with respect to $\beta_0 \in (0, H(0)/K_1]$, we finally obtain

$$T \leq \min_{\beta_0 \in (0, H(0)/K_1]} \frac{4U_0}{(q-1)^2\beta_0} = \frac{4K_1U_0}{(q-1)^2H(0)}.$$

The proof is complete. □

Corollary 3.1. *Let all of the assumptions in Theorem* 2.8 *hold. Then, for any $M > 0$, there exists a u_0 such that $J(u_0) = M$, while the corresponding solution $u(x,t)$ to problem* (1.1) *with u_0 as initial datum blows up in finite time.*

Proof. First, we recall a well-known result that, for any bounded smooth domain Ω in $\mathbb{R}^n$, $n \geq 3$, and $q \in (p-1, p^*-1)$, the functional $J(u)$ defined on $W_0^{1,p}(\Omega)$ has a sequence of critical points $\{w_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega)$ such that

$$J(w_k) = \frac{1}{p}\|\nabla w_k\|_p^p - \frac{1}{q+1}\|w_k\|_{q+1}^{q+1} \longrightarrow +\infty, \quad k \to +\infty. \tag{3.15}$$

For any $M > 0$, set $R = 2K_1M + 2K_2$. Let Ω_1 and Ω_2 be two disjoint smooth subdomains of Ω. Using the remark above, there exists

a sequence $\{w_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega_2)$ such that
(3.16)
$$J(w_k) = \frac{1}{p}\int_{\Omega_2} |\nabla w_k|^p \mathrm{d}x - \frac{1}{q+1}\int_{\Omega_2} |w_k|^{q+1}\mathrm{d}x \longrightarrow +\infty, \quad k \to +\infty.$$

Let $v \in W_0^{1,p}(\Omega_1)$ be an arbitrary nontrivial function. Then, there exists an $r_1 > 0$ such that

$$\int_{\Omega_1} \left|\frac{r_1 v}{|x|}\right|^2 \mathrm{d}x = r_1^2 \int_{\Omega_1} \left|\frac{v}{|x|}\right|^2 \mathrm{d}x > R. \tag{3.17}$$

On the other hand, since $q+1 > p$, the following holds:

$$M - \frac{r^p}{p}\int_{\Omega_1} |\nabla v|^p \mathrm{d}x + \frac{r^{q+1}}{q+1}\int_{\Omega_1} |v|^{q+1}\mathrm{d}x \longrightarrow +\infty, \quad r \to +\infty. \tag{3.18}$$

From (3.16) and (3.18), it follows that there exist $k_0 \in \mathbb{N}$ and $r > r_1$, both sufficiently large such that

$$\begin{aligned} M - \frac{r^p}{p}\int_{\Omega_1} |\nabla v|^p \mathrm{d}x + \frac{r^{q+1}}{q+1}\int_{\Omega_1} |v|^{q+1}\mathrm{d}x \\ = \frac{1}{p}\int_{\Omega_2} |\nabla w_{k_0}|^p \mathrm{d}x - \frac{1}{q+1}\int_{\Omega_2} |w_{k_0}|^{q+1}\mathrm{d}x. \end{aligned} \tag{3.19}$$

Denote $w = w_{k_0}$. Extend v and w to be 0 in $\Omega \setminus \Omega_1$ and $\Omega \setminus \Omega_2$, respectively, and still denote them, respectively, by v and w. Then, $v,\ w \in W_0^{1,p}(\Omega)$. Set $u_0 = rv + w$. It can be directly verified that $J(u_0) = J(rv) + J(w) = M$, and

$$\left\|\frac{u_0}{|x|}\right\|_2^2 \geq \left\|\frac{rv}{|x|}\right\|_2^2 > R = 2K_1 M + 2K_2 = 2K_1 J(u_0) + 2K_2. \tag{3.20}$$

According to Theorem 2.8, the solutions to problem (1.1) with initial data u_0 blow up in finite time, and the proof is complete. □

Proof of Theorem 2.9. From the proof of Theorem 2.8, we know that $I(u(t)) < 0$ for all $t \in [0, T)$, which then implies

$$\|\nabla u(t)\|_p^p < \|u(t)\|_{q+1}^{q+1}, \quad t \in [0, T). \tag{3.21}$$

Combining (3.21) with Lemma 2.6, we obtain

$$\begin{aligned}\|u(t)\|_{q+1}^{q+1} &\le C_G\|\nabla u(t)\|_p^{\alpha(q+1)}\|u(t)\|_2^{(1-\alpha)(q+1)}\\ &< C_G(\|u(t)\|_{q+1}^{q+1})^{\alpha(q+1)/p}(\|u(t)\|_2^2)^{(1-\alpha)(q+1)/2}\\ &\le C_G(\|u(t)\|_{q+1}^{q+1})^{\alpha(q+1)/p}[(\mathrm{diam}(\Omega))^2U(t)]^{(1-\alpha)(q+1)/2}\\ &\triangleq C_*(\|u(t)\|_{q+1}^{q+1})^{\alpha(q+1)/p}[U(t)]^{(1-\alpha)(q+1)/2},\end{aligned}$$

which implies

$$(\|u(t)\|_{q+1}^{q+1})^{1-\alpha(q+1)/p} < C_*[U(t)]^{(1-\alpha)(q+1)/2}, \tag{3.22}$$

where $C_* = C_G(\mathrm{diam}(\Omega))^{(1-\alpha)(q+1)}$,

$$\alpha = \left(\frac{1}{2} - \frac{1}{q+1}\right)\left(\frac{1}{2} + \frac{1}{n} - \frac{1}{p}\right)^{-1},$$

and $\mathrm{diam}(\Omega) > 0$ is the diameter of Ω. Since $q+1 < p(1+(2/n))$, it can be directly checked that $1-\alpha(q+1)/p > 0$ and

$$\gamma \triangleq \frac{(1-\alpha)(q+1)/2}{1-\alpha(q+1)/p} > 1.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}U(t) &= -2I(u(t)) < 2\|u(t)\|_{q+1}^{q+1}\\ &< 2C_*^{1/[1-\alpha(q+1)/p]}U^\gamma(t) \triangleq C^*U^\gamma(t), \quad t\in[0,T),\end{aligned} \tag{3.23}$$

where $C^* = 2C_*^{1/[1-\alpha(q+1)/p]}$. In view of the negativity of $I(u(t))$ on $[0,T)$, we know that $U(t) > 0$ for $t \in [0,T)$. Thus, dividing both sides of (3.23) by $U^\gamma(t)$ and integrating the resulting inequality over $[0,t)$, we obtain

$$\frac{1}{1-\gamma}\{U^{1-\gamma}(t) - U^{1-\gamma}(0)\} \le C^*t.$$

From Theorem 2.8, we know that $\lim_{t\to T} U(t) = +\infty$. Therefore, letting $t \to T$ in the above inequality and recalling that $\gamma > 1$, we obtain

$$T \ge \frac{U_0^{1-\gamma}}{C^*(\gamma-1)}.$$

The proof is complete. $\square$

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