ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 8, 2018

JENSEN TYPE INEQUALITIES AND THEIR APPLICATIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. The present paper is devoted to the study of Jensen type inequalities for fractional integration on finite subintervals of the real axis. The complete form of Jensen's inequality and the generalized Jensen's inequality are investigated by using the Chebyshev inequality. As applications, some new integral inequalities, including Hölder's and Minkowski's inequalities, are obtained by using Jensen's inequality via Riemann-Liouville fractional integrals.

1. Introduction. The theory of fractional integro-differentiation was initiated by Liouville **[10]** in the 1830s. The suggested definition by Liouville was based on differentiating an exponential function f, which may be shown as

$$f(x)=\sum_{k=1}^{\infty} c_k e^{a_k x} \Longrightarrow D^p f(x)=\sum_{k=1}^{\infty} c_k a_k^p e^{a_k x}$$

for any $p \in \mathbb{C}$. Continuing this concept, the differentiation formula of a power function was derived by Liouville as

$$D^{-p} f(x)=\frac{1}{(-1)^p \Gamma(p)} \int_0^{\infty} \varphi(x+t) t^{p-1} \mathrm{d}t,$$

where $x \in \mathbb{R}$ and $\mathrm{Real}(p)>0$. This formula is called the Liouville form of fractional integration. There are many applications of fractional integration in geometry, physics, mechanics, etc., see **[3, 9]**.

The classical Jensen inequality is one of the interesting inequalities in the theory of differential and difference equations, as well as other

2010 AMS *Mathematics subject classification.* Primary 26A33, 26D15, 28A25.
Keywords and phrases. Riemann-Liouville fractional integrals, Jensen's inequality, Hölder's inequality, Minkowski's inequality.

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Received by the editors on February 23, 2018, and in revised form on May 14, 2018.

DOI:10.1216/RMJ-2018-48-8-2459

areas of mathematics. The well-known Jensen inequality for convex functions is given as follows: Let (X, Σ, μ) be a measure space and f a real valued μ-measurable and μ-integrable function on a set $D \in \Sigma$ with $\mu(D) \in]0, \infty[$. If φ is a convex function on an open interval I in $\mathbb{R}$, and, if $f(D) \subset I$, then

$$\varphi\left(\frac{1}{\mu(D)} \int_D f \mathrm{d}\mu\right) \leq \frac{1}{\mu(D)} \int_D \varphi \circ f \, \mathrm{d}\mu.$$

In recent years, there have been many extensions, refinements and similar results of the classical Jensen inequality, see **[1, 2, 6, 7, 16, 19, 21]**. For $1 \leq p \leq \infty$ and a μ-measurable set of real numbers E, $L^p(E)$ is a normed space of functions f with $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_E |f|^p \mathrm{d}\mu\right)^{1/p}.$$

The triangular inequality for $L^p(E)$, i.e.,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

is called Minkowski's inequality. The number q is called the Hölder conjugate number of p, if $1/p + 1/q = 1$ with $1 < p$. For $f \in L^p(E)$ and $g \in L^q(E)$, the fact that fg belongs to $L^p(E)$ is shown by Hölder's inequality as

$$\int_E |fg| \mathrm{d}\mu \leq \|f\|_p \cdot \|g\|_q,$$

where q is the Hölder conjugate number of p.

The continuous form of Jensen's inequality (integral version) and its extensions are important consequences of convexity. The aim of this paper is to obtain a generalization of Jensen-type inequalities for fractional integrals on a finite interval $[a, b]$. The complete form of Jensen's inequality **[15]** and the generalized Jensen's inequality **[17]** are investigated by using Chebyshev's integral inequality. As applications, Hölder's and Minkowski's inequalities via Riemann-Liouville fractional integrals are established.

This paper is organized as follows. The definition of the Riemann-Liouville fractional integral along with a series of their desirable properties are given in Section 2. Jensen's inequality, the complete form of Jensen's inequality and the generalized Jensen's inequality for convex functions via fractional integrals are presented in Section 3. Some applications and results related to other renowned integral inequalities are given in Section 4. Finally, a conclusion is given in Section 5.

2. Definitions and basic properties. Throughout this paper, $[a, b]$, $-\infty < a < b < \infty$, will denote a finite interval on the real axis $\mathbb{R}$. We use standard notation such as $\mathbb{R} :=]-\infty, \infty[$, $\mathbb{R}^+ := [0, \infty[$ and $\mathbb{R}_+ :=]0, \infty[$ for the entire real line, the closed half-line and the open half-line, respectively.

A function $f : I \to \mathbb{R}$ is called *convex* (on an interval I of the real axis $\mathbb{R}$), if:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all points x and y in I and all $\lambda \in [0, 1]$. It is called *strictly convex* if the above inequality strictly holds whenever x and y are distinct points and $\lambda \in]0, 1[$. If $-f$ is convex (respectively, strictly convex), then we say that f is concave (respectively, strictly concave). For an arbitrary function $f : I \to \mathbb{R}$, the following properties are well known **[17]**:

(1) f is convex if and only if there is at least one line of support for f at each $x_0 \in \operatorname{int} I$, i.e.,

$$f(x) \geq f(x_0) + \lambda(x - x_0), \quad \text{for all } x \in \operatorname{int} I,$$

where λ depends on x_0 and is given by $\lambda = f'(x_0)$, when f' exists, and $\lambda \in [f'_-(x_0), f'_+(x_0)]$, when $f'_-(x_0) \neq f'_+(x_0)$. We call the set $\partial f(x_0)$ of all such λ the *subdifferential* of f at x_0.

(2) Every function $\varphi : I \to \mathbb{R}$ for which $\varphi(x) \in \partial f(x)$ whenever $x \in \operatorname{int} I$ verifies the double inequality

$$f'_-(x) \leq \varphi(x) \leq f'_+(x),$$

and thus, it is nondecreasing on $\operatorname{int} I$.

The continuous form of Jensen's inequality (integral version) and its extensions are important consequences of the convexity. We will point out these concepts in the following.

Theorem 2.1 (**[15]**). *Let (X, Σ, μ) be a finite measure space, and let f be a real valued μ-measurable and μ-integrable function on X. If φ is a convex function given on an interval I that includes the image of f,* *then*

$$\frac{1}{\mu(X)}\int_X f\,\mathrm{d}\mu \in I,$$

and

$$\varphi\biggl(\frac{1}{\mu(X)}\int_X f\,\mathrm{d}\mu\biggr) \le \frac{1}{\mu(X)}\int_X \varphi\circ f\,\mathrm{d}\mu,$$

provided that $\varphi\circ f$ is μ-integrable. If φ is strictly convex on I, then the above inequality becomes an equality if and only if f is constant almost everywhere on X.

Theorem 2.2 (The complete form of Jensen's inequality [**15**])**.** *Let (X,Σ,μ) be a finite measure space, and let f be a real valued μ-measurable and μ-integrable function on a set X. If φ is a convex function given on an interval I that includes the image of f, and $g : I \to \mathbb{R}$ is a function such that*

(i) *$g(x) \in \partial\varphi(x)$ for every $x \in I$, and*

(ii) *$g\circ f$ and $f\cdot(g\circ f)$ are μ-integrable functions,*

then the following inequalities hold:

$$\begin{aligned}0 &\le \frac{1}{\mu(X)}\int_X \varphi\circ f\,\mathrm{d}\mu - \varphi\biggl(\frac{1}{\mu(X)}\int_X f\,\mathrm{d}\mu\biggr)\\ &\le \frac{1}{\mu(X)}\int_X f\cdot(g\circ f)\,\mathrm{d}\mu - \frac{1}{(\mu(X))^2}\biggl(\int_X f\,\mathrm{d}\mu\biggr)\biggl(\int_X g\circ f\,\mathrm{d}\mu\biggr).\end{aligned}$$

Theorem 2.3 (Generalized Jensen's inequality [**17**])**.** *Let (X,Σ,μ) be a finite measure space, and let f be a real valued μ-measurable function on a set X from $L^\infty(\mu)$. If φ is a convex function given on an interval I that includes the image of f, and $p : X \to \mathbb{R}$ is a nonnegative function from $L^1(\mu)$ such that $\int_X p\,\mathrm{d}\mu \neq 0$, then*

$$\varphi\biggl(\frac{1}{\int_X p\,\mathrm{d}\mu}\int_X p\cdot f\,\mathrm{d}\mu\biggr) \le \frac{1}{\int_X p\,\mathrm{d}\mu}\int_X p\cdot(\varphi\circ f)\,\mathrm{d}\mu.$$

If φ is strictly convex on I, then the above inequality becomes an equality if and only if f is constant almost everywhere on X.

Remark 2.4. If f is concave, then the inequalities in Theorems 2.1, 2.2 and 2.3 hold in the reverse direction.

Let $C(X,Y)$ be the set of all continuous functions from X to Y. The following corollaries are true if μ is the standard Lebesgue measure and $X=[a,b]$.

Corollary 2.5 ([**14**]). *If $f\in C\big([a,b],]c,d[\big)$ and $\varphi\in C(]c,d[,\mathbb{R})$ is convex, then*

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\le\frac{1}{b-a}\int_a^b\varphi(f(x))\,\mathrm{d}x. \tag{2.1}$$

Corollary 2.6 ([**15**]). *If $f\in C([a,b],]c,d[)$ and $\varphi\in C(]c,d[,\mathbb{R})$ is convex and also $g\in C(]c,d[,\mathbb{R})$ such that $g(s)\in\partial\varphi(s)$ for every $s\in\,]c,d[$, then*

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b\varphi(f(x))\,\mathrm{d}x-\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\\&\le\frac{1}{b-a}\int_a^b f\cdot(g\circ f)\,\mathrm{d}\mu-\frac{1}{(b-a)^2}\left(\int_a^b f\,\mathrm{d}\mu\right)\left(\int_a^b g\circ f\,\mathrm{d}\mu\right).\end{aligned} \tag{2.2}$$

Corollary 2.7 ([**11**]). *If $f\in C([a,b],]c,d[)$ and $\varphi\in C(]c,d[,\mathbb{R})$ is convex and also $p\in C([a,b],\mathbb{R}^+)$ such that $\int_a^b p(x)\,\mathrm{d}x\neq 0$, then*

$$\varphi\left(\frac{1}{\int_a^b p(x)\,\mathrm{d}x}\int_a^b p(x)f(x)\,\mathrm{d}x\right)\le\frac{1}{\int_a^b p(x)\,\mathrm{d}x}\int_a^b p(x)\varphi(f(x))\,\mathrm{d}x. \tag{2.3}$$

Let $L^1[a,b]$ be the space of all Riemann integrable functions on $[a,b]$. In the following, we will give the definition of Riemann-Liouville fractional integrals on $[a,b]$ and present some of their properties in $L^1[a,b]$. For more details, one can refer to [**8, 12, 18, 20**].

Definition 2.8 ([**8**]). Let $f\in L^1[a,b]$. The Riemann-Liouville fractional integrals j_{a+}^{α} and j_{b-}^{α} of order $\alpha\in\mathbb{R}$, $\alpha>0$, are defined by

$$j_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,\mathrm{d}t,\quad a<x\le b,$$

and

$$j_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t, \quad a \le x < b,$$

where $\Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} \mathrm{d}t$ is the Gamma function and $j_{a+}^0 f(x) = j_{b-}^0 f(x) = f(x)$. These integrals are called the *left-side* and the *right-side fractional integrals*, respectively.

In the case of $\alpha = 1$, the fractional integrals reduce to the classical integral.

Theorem 2.9 ([**13**]). *Let $f(x)$ and $g(x)$ be such that both $j_{a+}^{\alpha} f(x)$ and $j_{b-}^{\alpha} f(x)$ exist. Then, the following basic properties of the Riemann-Liouville fractional integrals hold.*

(i) *Interpolation* (*continuity*):

$$\lim_{\alpha \to n} j_{a+}^{\alpha} f(x) = j_{a+}^{n} f(x) \quad \textit{and} \quad \lim_{\alpha \to n} j_{b-}^{\alpha} f(x) = j_{b-}^{n} f(x),$$

where j_{a+}^n and j_{b-}^n, $n \in \mathbb{N}$, are the classical operators for n-fold integration.

(ii) *Linearity*:

$$j_{a+}^{\alpha} (\lambda f(x) + g(x)) = \lambda j_{a+}^{\alpha} f(x) + j_{a+}^{\alpha} g(x)$$

and

$$j_{b-}^{\alpha} (\lambda f(x) + g(x)) = \lambda j_{b-}^{\alpha} f(x) + j_{b-}^{\alpha} g(x), \quad \lambda \in \mathbb{R}.$$

(iii) *Semi-group property* (*law of exponents*):

$$j_{a+}^{\alpha} (j_{a+}^{\beta}) f(x) = j_{a^+}^{\alpha+\beta} f(x) \quad \textit{and} \quad j_{b-}^{\alpha} (j_{b-}^{\beta}) f(x) = j_{b^-}^{\alpha+\beta} f(x)$$

for each $\alpha > 0$ and $\beta > 0$.

(iv) *Commutativity*:

$$j_{a^+}^{\alpha+\beta} f(x) = j_{a^+}^{\beta+\alpha} f(x) \quad \textit{and} \quad j_{b^-}^{\alpha+\beta} f(x) = j_{b^-}^{\beta+\alpha} f(x)$$

for each $\alpha > 0$ and $\beta > 0$.

The following Chebyshev's inequality for fractional integrals has been proved by Belarbi and Dahamani [**4**].

Theorem 2.10. *If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable and synchronous (in the sense that $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all x and y), then:*

(i) *for each $x \in \,]a, b]$ and $\alpha > 0$, we have*

$$\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j^{\alpha}_{a^+} f(x)\ j^{\alpha}_{a^+} g(x) \leq j^{\alpha}_{a^+} (fg)(x).$$

(ii) *For each $x \in [a, b[$ and $\alpha > 0$, we have*

$$\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j^{\alpha}_{b^-} f(x)\ j^{\alpha}_{b^-} g(x) \leq j^{\alpha}_{b^-} (fg)(x).$$

3. The main results.

The aim of this section is to show the Jensen-type inequalities for convex functions via fractional integrals.

Theorem 3.1. *If $f \in C\big([a, b],]c, d[\big)$ and $\varphi \in C\big(]c, d[, \mathbb{R}\big)$ is convex, then*

(i)

$$\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j^{\alpha}_{a^+} f(x) \in \,]c, d[\quad \text{for all } x \in \,]a, b],$$

and the following inequality for fractional integrals holds:

$$\varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j^{\alpha}_{a^+} f(x)\right) \leq \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\right) j^{\alpha}_{a^+} \varphi(f(x)), \tag{3.1}$$

for all $\alpha > 0$ and $x \in \,]a, b]$.

(ii)

$$\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j^{\alpha}_{b^-} f(x) \in \,]c, d[,$$

for all $x \in [a, b[$, and the following inequality for fractional integrals holds:

$$\varphi\left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j^{\alpha}_{b^-} f(x)\right) \leq \left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\right) j^{\alpha}_{b^-} \varphi(f(x)), \tag{3.2}$$

for $\alpha > 0$ and $x \in [a, b[$.

Proof.

(i) By the assumption,

$$c < f(t) < d \tag{3.3}$$

for all $t \in [a,b]$. Multiplying inequality (3.3) by $(x-t)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[a,x]$, we obtain

$$\begin{aligned}\frac{c}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\mathrm{d}t &< \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t\\ &< \frac{d}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\mathrm{d}t\end{aligned}$$

for all $x \in \,]a,b]$. Consequently,

$$c < \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x) < d. \tag{3.4}$$

Choose a function $h : \,]c,d[\, \to \mathbb{R}$ such that $h(s) \in \partial\varphi(x)$ for every $s \in \,]c,d[$. Since

$$\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x) \in (c,b),$$

then

$$\begin{aligned}\varphi(f(t)) \geq \varphi&\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right)\\ &+\left(f(t)-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right)\cdot h\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right)\end{aligned}$$

for all $t \in [a,b]$. Multiplying the previous inequality by $(x-t)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[a,x]$, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi\big(f(t)\big)\,\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right)\mathrm{d}t\\ &\geq \left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\,\mathrm{d}t\right)\\ &\quad\cdot h\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right).\end{aligned}$$

Consequently,

$$j_{a+}^{\alpha}\varphi(f(x)) - \varphi\bigg(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}j_{a+}^{\alpha}f(x)\bigg)\bigg(\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg)$$
$$\geq \bigg(j_{a+}^{\alpha}f(x) - \bigg(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}j_{a+}^{\alpha}f(x)\bigg)\bigg(\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg)\bigg)h\bigg(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}j_{a+}^{\alpha}f(x)\bigg),$$

for each $x \in\,]a,b]$. Thus,

$$\varphi\bigg(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}j_{a+}^{\alpha}f(x)\bigg) \leq \bigg(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\bigg)j_{a+}^{\alpha}\varphi(f(x)).$$

The proof of (i) is now complete.

(ii) Multiplying the inequality (3.3) by $(t-x)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[x,b]$, we obtain

$$\frac{c}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\mathrm{d}t < \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,\mathrm{d}t < \frac{d}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\,\mathrm{d}t$$

for each $x \in [a,b[$. Thus,

$$c < \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}j_{b-}^{\alpha}f(x) < d. \tag{3.5}$$

Since

$$\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}j_{b-}^{\alpha}f(x) \in (c,b),$$

then

$$\varphi(f(t)) \geq \varphi\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}j_{b-}^{\alpha}f(x)\bigg)$$
$$+ \bigg(f(t) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}j_{b-}^{\alpha}f(x)\bigg)h\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}j_{b-}^{\alpha}f(x)\bigg)$$

for all $t \in [a,b]$. Multiplying the above inequality by $(t-x)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[x,b]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi\big(f(t)\big)\,\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\bigg)\,\mathrm{d}t\\
&\quad\geq \bigg(\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\,\mathrm{d}t\bigg)\\
&\qquad\cdot h\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\bigg).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&j_{b^-}^\alpha \varphi(f(x)) - \varphi\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\bigg)\bigg(\frac{(b-x)^\alpha}{\Gamma(\alpha+1)}\bigg)\\
&\quad\geq \bigg(j_{b^-}^\alpha f(x) - \bigg(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\bigg)\bigg(\frac{(b-x)^\alpha}{\Gamma(\alpha+1)}\bigg)\bigg) h\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\bigg)
\end{aligned}$$

for each $x \in [a, b[$. Thus,

$$\varphi\bigg(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b^-}^\alpha f(x)\bigg) \leq \bigg(\frac{\Gamma(\alpha+1}{(b-x)^\alpha}\bigg) j_{b^-}^\alpha \varphi(f(x)),$$

and the proof of (ii) is complete. $\square$

Remark 3.2. If we let $\alpha = 1$ and $x = b$ in Theorem 3.1 (i) or $\alpha = 1$ and $x = a$ in Theorem 3.1 (ii), then the inequalities (3.1) and (3.2) become the inequality (2.1). We also note that, if the function φ is concave in Theorem 3.1, then the inequalities (3.1) and (3.2) hold in the reverse direction.

Combining (3.1) and (3.2), we get the following theorem.

Theorem 3.3. *If $f \in C\big([a,b],]c,d[\big)$ and $\varphi \in C\big(]c,d[, \mathbb{R}\big)$ is convex, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big(j_{a^+}^\alpha f(b) + j_{b^-}^\alpha f(a)\big) \in]c,d[,$$

and the following inequality for fractional integrals holds:

(3.6)
$$\varphi\bigg(\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big(j_{a^+}^\alpha f(b) + j_{b^-}^\alpha f(a)\big)\bigg) \leq \bigg(\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigg)\big(j_{a^+}^\alpha \varphi(f(b)) + j_{b^-}^\alpha \varphi(f(a))\big),$$

with $\alpha > 0$.

Proof. Combining (3.4) and (3.5) in the two previous theorems, we get

$$c < \left(\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \big(j^{\alpha}_{a+} f(b) + j^{\alpha}_{b-} f(a) \big) \right) < d.$$

Since

$$\alpha_0 := \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \big(j^{\alpha}_{a+} f(b) + j^{\alpha}_{b-} f(a) \big) \in \,]c, d[\,,$$

then

$$\varphi(f(t)) \geq \varphi(\alpha_0) + (f(t) - \alpha_0) h(\alpha_0) \tag{3.7}$$

for all $t \in [a, b]$. Multiplying the inequality (3.7) by $(x-t)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[a, x]$, we get

$$j^{\alpha}_{a+} \varphi(f(x)) - \varphi(\alpha_0) \left(\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \geq \left(j^{\alpha}_{a+} f(x) - \alpha_0 \left(\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \right) h(\alpha_0),$$

for each $x \in \,]a, b]$. In particular, taking $x = b$, we have

$$j^{\alpha}_{a+} \varphi\big(f(b)\big) - \varphi(\alpha_0) \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \geq \left(j^{\alpha}_{a+} f(b) - \alpha_0 \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \right) h(\alpha_0). \tag{3.8}$$

On the other hand, multiplying the inequality (3.7) by $(t-x)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[x, b]$, we get

$$j^{\alpha}_{b-} \varphi(f(x)) - \varphi(\alpha_0) \left(\frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \right) \geq \left(j^{\alpha}_{b-} f(x) - \alpha_0 \left(\frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \right) \right) h(\alpha_0),$$

for each $x \in [a, b[$. In particular, taking $x = a$, we have

$$j^{\alpha}_{b-} \varphi\big(f(a)\big) - \varphi(\alpha_0) \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \geq \left(j^{\alpha}_{b-} f(a) - \alpha_0 \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \right) h(\alpha_0). \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\begin{aligned} j^{\alpha}_{a+} \varphi\big(f(b)\big) + j^{\alpha}_{b-} \varphi\big(f(a)\big) - 2\varphi(\alpha_0) \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \\ \geq \left(j^{\alpha}_{a+} f(b) + j^{\alpha}_{b-} f(a) - 2\alpha_0 \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \right) h(\alpha_0). \end{aligned}$$

Thus,

$$\varphi(\alpha_0) \le \left(\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\right)\Big(j^\alpha_{a^+}\varphi(f(b)) + j^\alpha_{b^-}\varphi\big(f(a)\big)\Big),$$

i.e.,

$$\varphi\left(\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big(j^\alpha_{a^+}f(b) + j^\alpha_{b^-}f(a)\big)\right) \le \left(\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\right)\Big(j^\alpha_{a^+}\varphi(f(b)) + j^\alpha_{b^-}\varphi\big(f(a)\big)\Big).$$

Thus, the proof is finished. □

Remark 3.4. If we let $\alpha = 1$ in Theorem 3.3, then inequality (3.6) becomes inequality (2.1). We also note that, if the function φ is concave in Theorem 3.3, then inequality (3.6) holds in the reverse direction.

The complement of Jensen's inequality by Chebyshev's inequality is given in the next result.

Theorem 3.5 (The complete form of Jensen's inequality)**.** *If $f \in C([a,b],]c,d[)$, $\varphi \in C\big(]c,d[\,,\mathbb{R}\big)$ is convex and also $g \in C\big(]c,d[\,,\mathbb{R}\big)$ such that $g(s) \in \partial\varphi(x)$ for every $s \in\,]c,d[$, then*:

(i) *the following inequality for each $x \in\,]a,b]$ and $\alpha > 0$ holds*

$$0 \le \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\right)j^\alpha_{a^+}\varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}j^\alpha_{a^+}f(x)\right) \le \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\right)j^\alpha_{a^+}\big(f\cdot(g\circ f)\big)(x) - \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\right)^2 j^\alpha_{a^+}f(x)j^\alpha_{a^+}(g\circ f)(x). \tag{3.10}$$

(ii) *The following inequality for each $x \in [a,b[$ and $\alpha > 0$ holds*

$$0 \le \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\right)j^\alpha_{b^-}\varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}j^\alpha_{b^-}f(x)\right) \le \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\right)j^\alpha_{b^-}\big(f\cdot(g\circ f)\big)(x) - \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\right)^2 j^\alpha_{b^-}f(x)j^\alpha_{b^-}(g\circ f)(x). \tag{3.11}$$

Proof.

(i) The left hand side inequality is that of Jensen (fractional integral version). Since

$$\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x) \in (c,b) \quad \text{for all } x \in \left]a,b\right],$$

then the right hand side inequality can be obtained from

$$\varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x)\right) - \varphi(f(t)) \geq \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x) - f(t)\right) g(f(t))$$

for all $t \in [a,b]$. Multiplying the previous inequality by $(x-t)^{\alpha-1}/\Gamma(\alpha)$ and integrating the resulting inequality with respect to t over $[a,x]$, we obtain

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x)\right) \mathrm{d}t - \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(f(t))\, \mathrm{d}t \\
\geq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x) g(f(t))\, \mathrm{d}t \\
- \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \cdot g(f(t))\, \mathrm{d}t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x)\right)\left(\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}\right) - j^\alpha_{a^+} \varphi(f(x)) \\
\geq \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x) j^\alpha_{a^+} g(f(x)) - j^\alpha_{a^+} f(x) \cdot g\big(f(x)\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\right) j^\alpha_{a^+} \varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j^\alpha_{a^+} f(x)\right) \\
&\leq \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\right) j^\alpha_{a^+} \big(f \cdot (g \circ f)\big)(x) - \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\right)^2 j^\alpha_{a^+} f(x) j^\alpha_{a^+} (g \circ f)(x).
\end{aligned}$$

Thus, the proof of (i) is finished.

(ii) The proof of (ii) is similar to that of (i). $\square$

Remark 3.6. If we let $\alpha = 1$ and $x = b$ in Theorem 3.5 (i) or $\alpha = 1$ and $x = a$ in Theorem 3.5 (ii), then inequalities (3.10) and (3.11) become inequality (2.2). We also note that, if the function φ is concave in Theorem 3.1, then the inequalities (3.10) and (3.11) hold in the reverse direction.

The next results are related to the generalized Jensen's inequality, see Theorem 2.3.

Theorem 3.7. *If $f \in C\big([a,b],]c,d[\big)$, and $\varphi \in C\big(]c,d[\,,\mathbb{R}\big)$ is convex as well as $p \in C\big([a,b],\mathbb{R}^+\big)$, then:*

(i) *the following inequality for each $\alpha > 0$ and $x \in\,]a,b]$ with $j^{\alpha}_{a+}p(x) > 0$ holds*

$$\varphi\left(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}\left(p(x)f(x)\right)\right) \leq \frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}\big(p(x)\varphi(f(x))\big). \tag{3.12}$$

(ii) *The following inequality for each $\alpha > 0$ and $x \in [a,b[$ with $j^{\alpha}_{b-}p(x) > 0$ holds*

$$\varphi\left(\frac{1}{j^{\alpha}_{b-}p(x)} j^{\alpha}_{b-}\left(p(x)f(x)\right)\right) \leq \frac{1}{j^{\alpha}_{b-}p(x)} j^{\alpha}_{b-}\big(p(x)\varphi(f(x))\big). \tag{3.13}$$

Proof.

(i) Under the above-mentioned assumptions,

$$c < f(t) < d \tag{3.14}$$

for all $t \in [a,b]$. Multiplying inequality (3.14) by $((x-t)^{\alpha-1}/\Gamma(\alpha))p(t)$ and integrating the resulting inequality with respect to t over $[a,x]$, we get

$$\begin{aligned} \frac{c}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} p(t)\,\mathrm{d}t &< \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} p(t) f(t)\,\mathrm{d}t \\ &< \frac{d}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} p(t)\,\mathrm{d}t \end{aligned}$$

for each $x \in\,]a,b]$. Consequently,

$$c < \frac{j^{\alpha}_{a+}\left(p(x)f(x)\right)}{j^{\alpha}_{a+}p(x)} < d.$$

Choose a function $h :]c,d[\to \mathbb{R}$ such that $h(s) \in \partial\varphi(x)$ for every $s \in]c,d[$. Since $j^{\alpha}_{a+}(p(x)f(x))/(j^{\alpha}_{a+}p(x)) \in (c,b)$, then

$$\varphi(f(t)) \geq \varphi\left(\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right) + \left(f(t) - \frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right) h\left(\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right)$$

for all $t \in [a,b]$. Multiplying the previous inequality by $((x-t)^{\alpha-1}/\Gamma(\alpha))p(t)$ and integrating the resulting inequality with respect to t over $[a,x]$, we get

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}&\int_a^x (x-t)^{\alpha-1}p(t)\varphi(f(t))\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi\left(\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right)p(t)\mathrm{d}t \\ &\geq \left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}p(t)f(t)\,\mathrm{d}t\right. \\ &\qquad \left. - \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}p(t)\,\mathrm{d}t\right) h\left(\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}j^{\alpha}_{a+}p(x)\varphi(f(x)) &- \varphi\left(\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right)j^{\alpha}_{a+}p(x) \\ &\geq \left(j^{\alpha}_{a+}p(x)f(x) - \frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}j^{\alpha}_{a+}p(x)\right) h\left(\frac{j^{\alpha}_{a+}(p(x)f(x))}{j^{\alpha}_{a+}p(x)}\right),\end{aligned}$$

i.e.,

$$\varphi\left(\frac{1}{j^{\alpha}_{a+}p(x)}j^{\alpha}_{a+}(p(x)f(x))\right) \leq \frac{1}{j^{\alpha}_{a+}p(x)}j^{\alpha}_{a+}\Big(p(x)\varphi(f(x))),$$

and the proof of (i) is finished.

(ii) The proof of (ii) is similar to that of (i). $\square$

Remark 3.8. If we let $\alpha = 1$ and $x = b$ in Theorem 3.7 (i) or $\alpha = 1$ and $x = a$ in Theorem 3.7 (ii), then inequalities (3.12) and (3.13) become inequality (2.3). We also note that, if the function φ is concave in Theorem 3.7, then the inequalities (3.12) and (3.13) hold in the reverse direction.

Corollary 3.9. *If $f \in C\big([a,b],]c,d[\big)$ and $\varphi \in C\big(]c,d[, \mathbb{R}\big)$ is convex and, in addition, $p \in C([a,b],\mathbb{R})$, then:*

(i) *the following inequality for each $\alpha > 0$ and $x \in]a,b]$ with $j^{\alpha}_{a^+}|p(x)| > 0$ holds*

$$\varphi\left(\frac{1}{j^{\alpha}_{a^+}|p(x)|} j^{\alpha}_{a^+}(|p(x)|f(x))\right) \le \frac{1}{j^{\alpha}_{a^+}|p(x)|} j^{\alpha}_{a^+}(|p(x)|\varphi(f(x))). \tag{3.15}$$

(ii) *The following inequality for each $\alpha > 0$ and $x \in [a,b[$ with $j^{\alpha}_{b^-}|p(x)| > 0$ holds*

$$\varphi\left(\frac{1}{j^{\alpha}_{b^-}|p(x)|} j^{\alpha}_{b^-}(|p(x)|f(x))\right) \le \frac{1}{j^{\alpha}_{b^-}|p(x)|} j^{\alpha}_{b^-}(|p(x)|\varphi(f(x))). \tag{3.16}$$

Remark 3.10. In particular, for case $p = 1$, Corollary 3.9 reduces to Theorem 3.1.

Corollary 3.11. *If $p, f\ [a,b] \to \mathbb{R}$ are integrable and $\varphi \in C\big(]c,d[, \mathbb{R}\big)$ is convex, then:*

(i) *the following inequality for each $\alpha > 0$ and $x \in]a,b]$ with $j^{\alpha}_{a^+}|p(x)| > 0$ holds*

$$\varphi\left(\frac{1}{j^{\alpha}_{a^+}|p(x)|} j^{\alpha}_{a^+}(|p(x)|f(x))\right) \le \frac{1}{j^{\alpha}_{a^+}|p(x)|} j^{\alpha}_{a^+}(|p(x)|\varphi(f(x))) \tag{3.17}$$

where $f([a,b]) \subseteq]c,d[$.

(ii) *The following inequality for each $\alpha > 0$ and $x \in [a,b[$ with $j^{\alpha}_{b^-}|p(x)| > 0$ holds*

$$\varphi\left(\frac{1}{j^{\alpha}_{b^-}|p(x)|} j^{\alpha}_{b^-}(|p(x)|f(x))\right) \le \frac{1}{j^{\alpha}_{b^-}|p(x)|} j^{\alpha}_{b^-}(|p(x)|\varphi(f(x))) \tag{3.18}$$

where $f([a,b]) \subseteq]c,d[$.

Remark 3.12. In particular, for cases $\alpha = 1$ and $x = b$ in Corollary 3.9 (i) or $x = a$ in Corollary 3.9 (ii), the inequalities (3.15) and (3.16) coincide with the inequality (2.3). Also, in particular, for case $\alpha = 1$ in Corollary 3.11, the inequalities (3.17) and (3.18) coincide with the inequality (7.1) in **[14]**.

Remark 3.13. Let $f \in C([a,b],\mathbb{R}_+)$, $p \in C([a,b],\mathbb{R})$ and $\varphi(x) = x^\beta$ on $]0,\infty[$. It is clear that φ is convex on $]0,\infty[$ for $\beta < 0$ or $\beta > 1$, and f is concave on $]0,\infty[$ for $\beta \in]0,1[$. Then:

(i) The following inequalities for each $\alpha > 0$ and $x \in]a,b]$ with $j^\alpha_{a+}|p(x)| > 0$ hold:

$$\left(\frac{1}{j^\alpha_{a+}|p(x)|} j^\alpha_{a+}(|p(x)|f(x))\right)^\beta \leq \frac{1}{j^\alpha_{a+}|p(x)|} j^\alpha_{a+}\big(|p(x)|f^\beta(x)\big),$$

if $\beta < 0$ or $\beta > 1$;

$$\left(\frac{1}{j^\alpha_{a+}|p(x)|} j^\alpha_{a+}(|p(x)|f(x))\right)^\beta \geq \frac{1}{j^\alpha_{a+}|p(x)|} j^\alpha_{a+}\big(|p(x)|f^\beta(x)\big),$$

if $\beta \in]0,1[$.

(ii) The following inequalities for each $\alpha > 0$ and $x \in [a,b[$ with $j^\alpha_{b-}|p(x)| > 0$ hold:

$$\left(\frac{1}{j^\alpha_{b-}|p(x)|} j^\alpha_{b-}(|p(x)|f(x))\right)^\beta \leq \frac{1}{j^\alpha_{b-}|p(x)|} j^\alpha_{b-}\big(|p(x)|f^\beta(x)\big),$$

if $\beta < 0$ or $\beta < 0$ or $\beta > 1$;

$$\left(\frac{1}{j^\alpha_{b-}|p(x)|} j^\alpha_{b-}(|p(x)|f(x))\right)^\beta \geq \frac{1}{j^\alpha_{b-}|p(x)|} j^\alpha_{b-}\big(|p(x)|f^\beta(x)\big),$$

if $\beta \in]0,1[$.

Remark 3.14. Let $f \in C([a,b],\mathbb{R}_+)$, $p \in C([a,b],\mathbb{R})$ and $\varphi(x) = \ln(x)$ on $]0,\infty[$. Obviously, φ is concave on $]0,\infty[$. Then

$$\ln\left(\frac{1}{j^\alpha_{a+}|p(x)|} j^\alpha_{a+}(|p(x)|f(x))\right) \geq \frac{1}{j^\alpha_{a+}|p(x)|} j^\alpha_{a+}(|p(x)|\ln(f(x))),$$

if $x \in]a,b]\,, 0 < \alpha$;

$$\ln\left(\frac{1}{j^\alpha_{b-}|p(x)|} j^\alpha_{b-}(|p(x)|f(x))\right) \geq \frac{1}{j^\alpha_{b-}|p(x)|} j^\alpha_{b-}(|p(x)|\ln(f(x))),$$

if $x \in [a,b[, 0 < \alpha$.

Remark 3.15. Let $f \in C([a,b], \mathbb{R}_+)$, $p = 1$ and $\varphi(x) = \ln(x)$ on $]0, \infty[$. Then:

(i) if $x \in \,]a,b]$, $0 < \alpha$,

$$\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^\alpha f(x) \geq \exp\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^\alpha \ln(f(x))\right); \tag{3.19}$$

(ii) if $x \in [a,b[$, $0 < \alpha$,

$$\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^\alpha f(x) \geq \exp\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^\alpha \ln(f(x))\right). \tag{3.20}$$

Remark 3.16. If we let $\alpha = 1$ and $x = b$ in Remark 3.15 (i) or $\alpha = 1$ and $x = a$ in Remark 3.15 (ii), then the inequalities (3.19) and (3.20) coincide with [**15,** Remark 1.8.2]. In the above remark, by choosing $a = 0$, $b = 1$, $f = x^\beta$, $\beta > 0$ and $\alpha > 0$, we conclude that

$$\frac{\Gamma(\alpha+1)}{x^\alpha} j_{0+}^\alpha x^\beta \geq \exp\left(\frac{\Gamma(\alpha+1)}{x^\alpha} j_{0+}^\alpha \ln(x^\beta)\right).$$

On the other hand, by [**12,** equations (2.3), (3.21)], we know that

$$j_{0+}^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}$$

and

$$j_{0+}^\alpha \ln(x) = \frac{x^\alpha}{\Gamma(\alpha+1)} \big(\ln(x) - \gamma - \psi(\alpha+1)\big),$$

where ψ is the digamma function and γ is Euler's constant. Thus,

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^\beta \geq \exp(\beta(\ln(x) - \gamma - \psi(\alpha+1))).$$

In the previous inequality, by choosing $\beta = 1$, $\alpha = 1/2$ and $x = 1/2$, we have

$$\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(5/2)} \frac{1}{2} \geq \exp\left(\ln\left(\frac{1}{2}\right) - \gamma - \psi\left(\frac{3}{2}\right)\right)$$

or

$$\frac{1}{3} \geq \exp\left(\ln\left(\frac{1}{2}\right) - \gamma - \psi\left(\frac{3}{2}\right)\right).$$

We note that $\psi(3/2) = 2 - \gamma - \ln 4$.

Remark 3.17. Let $f \in C([a,b],\mathbb{R})$, $p \in C([a,b],\mathbb{R}^+)$ and $\varphi(x) = \exp(x)$ on $\mathbb{R}$. Obviously, φ is convex on $\mathbb{R}$. Then:

(i) the following inequality for each $\alpha > 0$ and $x \in \,]a,b]$ holds

$$j^\alpha_{a+}p(x)\cdot \exp\left(\frac{j^\alpha_{a+}(p(x)f(x))}{j^\alpha_{a+}p(x)}\right) \le j^\alpha_{a+}(p(x)\exp(f(x)));$$

(ii) the following inequality for each $\alpha > 0$ and $x \in [a,b[$ holds

$$j^\alpha_{b-}p(x)\cdot \exp\left(\frac{j^\alpha_{b-}(p(x)f(x))}{j^\alpha_{b-}p(x)}\right) \le j^\alpha_{b-}(p(x)\exp(f(x))).$$

4. Applications of Jensen-type inequalities via fractional integrals. Using the generalized Jensen's inequality via fractional integrals, the following well-known inequalities are proven.

Theorem 4.1. *Let* $f,p \in C\big([a,b],\mathbb{R}_+\big)$. *Then:*

(i) *the following inequality for each* $\alpha > 0$ *and* $x \in \,]a,b]$ *with* $j^\alpha_{a+}(f(x)p(x)) > 0$, $j^\alpha_{a+}f(x) > 0$ *and* $j^\alpha_{a+}f(x)/p(x) > 0$ *holds*

$$\left(\frac{1}{j^\alpha_{a+}(f(x)/p(x))}j^\alpha_{a+}\left(\frac{f(x)}{p(x)}\ln p(x)\right)\right) < \frac{1}{j^\alpha_{a+}(f(x)p(x))}j^\alpha_{a+}\big(f(x)p(x)\ln p(x)\big).$$

(ii) *The following inequality for each* $\alpha > 0$ *and* $x \in [a,b[$ *with* $j^\alpha_{b-}(f(x)p(x)) > 0$, $j^\alpha_{b-}f(x) > 0$ *and* $j^\alpha_{b-}f(x)/p(x) > 0$ *holds*

$$\left(\frac{1}{j^\alpha_{b-}(f(x)/p(x))}j^\alpha_{b-}\left(\frac{f(x)}{p(x)}\ln p(x)\right)\right) < \frac{1}{j^\alpha_{b-}(f(x)p(x))}j^\alpha_{b-}\big(f(x)p(x)\ln p(x)\big).$$

Proof.

(i) Since $\varphi(x) = -\ln(x)$ is strictly convex, it follows from Theorem 3.7 that

$$\begin{aligned}\ln\left(\frac{1}{j_{a+}^{\alpha}(f(x)/p(x))}j_{a+}^{\alpha}f(x)\right) &= -\ln\left(\frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}f(x)\frac{1}{p(x)}\right)\\ &< -\frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}f(x)\ln\left(\frac{1}{p(x)}\right)\\ &= \frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}f(x)\ln p(x).\end{aligned}$$

Thus,

$$\left(\frac{1}{j_{a+}^{\alpha}(f(x)/p(x))}j_{a+}^{\alpha}f(x)\right) < \exp\left(\frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}f(x)\ln p(x)\right). \tag{4.1}$$

Similarly,

$$\begin{aligned}\frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}f(x)p(x) &= \frac{1}{j_{a+}^{\alpha}(f(x)p(x))/p(x)}j_{a+}^{\alpha}f(x)p(x)\\ &< \exp\left(\frac{1}{j_{a+}^{\alpha}f(x)p(x)}j_{a+}^{\alpha}f(x)p(x)\ln p(x)\right).\end{aligned} \tag{4.2}$$

Since exp is a strictly convex function, it follows from (4.1), (4.2) and Theorem 3.7 that

$$\begin{aligned}&\exp\left(\frac{1}{j_{a+}^{\alpha}(f(x)/p(x))}j_{a+}^{\alpha}\left(\frac{f(x)}{p(x)}\ln p(x)\right)\right)\\ &< \frac{1}{j_{a+}^{\alpha}\frac{f(x)}{p(x)}}j_{a+}^{\alpha}\left(\frac{f(x)}{p(x)}\exp(\ln p(x))\right) \quad \text{by Theorem 3.7}\\ &\quad = \frac{1}{j_{a+}^{\alpha}(f(x)/p(x))}j_{a+}^{\alpha}\left(\frac{f(x)}{p(x)}p(x)\right) = \frac{1}{j_{a+}^{\alpha}(f(x)/p(x))}j_{a+}^{\alpha}f(x)\\ &\quad < \exp\left(\frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}(f(x)\ln p(x))\right) \quad \text{by (4.1)}\\ &\quad < \frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}(f(x)\exp(\ln p(x))) \quad \text{by Theorem 3.7}\\ &\quad = \frac{1}{j_{a+}^{\alpha}f(x)}j_{a+}^{\alpha}(f(x)p(x))\\ &\quad < \exp\left(\frac{1}{j_{a+}^{\alpha}(f(x)p(x))}j_{a+}^{\alpha}\big(f(x)p(x)\ln p(x)\big)\right) \quad \text{by (4.2)},\end{aligned}$$

which completes the proof.

(ii) The proof of (ii) is similar to that of (i). □

In the standard proof of Hölder's inequality, the basic Young inequality $x^{1/p}y^{1/q} \le x/p + y/q$ for nonnegative x and y is used. Here, we present a proof based on the application of the generalized Jensen's inequality via fractional integrals.

Theorem 4.2 (Generalized Hölder's inequality via fractional integrals)**.** *Let $f, g, h \in C\big([a,b], \mathbb{R}^+\big)$ and q be the Hölder conjugate numbers of p. Then*:

(i) *the following inequality for each $\alpha > 0$ and $x \in \,]a,b]$ with $j^{\alpha}_{a^+}\big(h(x)g^q(x)\big) > 0$ and $j^{\alpha}_{a^+}h(x) > 0$ holds*

$$j^{\alpha}_{a^+}(h(x)f(x)g(x)) \le (j^{\alpha}_{a^+}(h(x)f^p(x)))^{1/p}(j^{\alpha}_{a^+}(h(x)g^q(x)))^{1/q}. \tag{4.3}$$

(ii) *The following inequality for each $\alpha > 0$ and $x \in [a,b[$ with $j^{\alpha}_{b^-}(h(x)g^q(x)) > 0$ and $j^{\alpha}_{b^-}h(x) > 0$ holds*

$$j^{\alpha}_{b^-}(h(x)f(x)g(x)) \le (j^{\alpha}_{b^-}(h(x)f^p(x)))^{1/p}(j^{\alpha}_{b^-}\big(h(x)g^q(x)))^{1/q}. \tag{4.4}$$

Proof. Since $\varphi(x) = x^p$ is convex for $1 < p$, it follows from Theorem 3.7 that

$$\left(\frac{1}{j^{\alpha}_{a^+}h(x)} j^{\alpha}_{a^+}(h(x)g(x))\right)^p \le \frac{1}{j^{\alpha}_{a^+}h(x)} j^{\alpha}_{a^+}(h(x)g^p(x)). \tag{4.5}$$

Replacing g by $fg^{-q/p}$ and h by hg^q in inequality (4.5), we get

$$\begin{aligned}\left(\frac{1}{j^{\alpha}_{a^+}\big(h(x)g^q(x)\big)} j^{\alpha}_{a^+}(h(x)g^q(x)f(x)g^{-q/p}(x))\right)^p \\ \le \frac{1}{j^{\alpha}_{a^+}(h(x)g^q(x))} j^{\alpha}_{a^+}(h(x)g^q(x)(f(x)g^{-q/p}(x))^p).\end{aligned}$$

Using the fact that $1/p + 1/q = 1$, we deduce

$$j^{\alpha}_{a^+}(h(x)f(x)g(x)) \le (j^{\alpha}_{a^+}(h(x)f^p(x)))^{1/p}(j^{\alpha}_{a^+}(h(x)g^q(x)))^{1/q}.$$

(ii) The proof of (ii) is similar to that of (i). □

Remark 4.3. In particular, for case $h = 1$, Theorem 4.2 gives Hölder's inequality via fractional integrals:

$$j^{\alpha}_{a^+}(f(x)g(x)) \le (j^{\alpha}_{a^+}f^p(x))^{1/p}(j^{\alpha}_{a^+}g^q(x))^{1/q}$$

if $x \in \,]a,b]$, $0 < \alpha$, $j^{\alpha}_{a+} g^q(x) > 0$,

$$j^{\alpha}_{b-}(f(x)g(x)) \leq (j^{\alpha}_{b-} f^p(x))^{1/p} (j^{\alpha}_{b-} g^q(x))^{1/q}$$

if $x \in [a,b[$, $0 < \alpha$, $j^{\alpha}_{a+} g^q(x) > 0$.

Remark 4.4. In particular, for cases $\alpha = 1$, $h = 1$ and $x = b$ in Theorem 4.2 (i) or $\alpha = 1$, $h = 1$ and $x = a$ in Theorem 4.2 (ii), the inequalities (4.3) and (4.4) coincide with the classical Hölder's inequality:

$$\int_a^b f(x)g(x)\,\mathrm{d}x \leq \left(\int_a^b f^p(x)\,\mathrm{d}x \right)^{1/p} \left(\int_a^b g^q(x)\mathrm{d}x \right)^{1/q},$$

where $1/p + 1/q = 1$ and $1 < p$.

Remark 4.5. In particular, for cases $h = 1$ and $p = q = 2$, Theorem 4.2 gives the Schwarz inequality via the fractional integrals:

$$j^{\alpha}_{a+}(f(x)g(x)) \leq \sqrt{(j^{\alpha}_{a+} f^2(x))(j^{\alpha}_{a+} g^2(x))} \quad \text{if } x \in \,]a,b]\,, \quad 0 < \alpha,$$
$$j^{\alpha}_{b-}(f(x)g(x)) \leq \sqrt{(j^{\alpha}_{b-} f^2(x))(j^{\alpha}_{b-} g^2(x))} \quad \text{if } x \in [a,b[\,, \quad 0 < \alpha.$$

Remark 4.6. In particular, for cases $\alpha = 1$, $h = 1$, $p = q = 2$ and $x = b$ in Theorem 4.2 (i) or $\alpha = 1$, $h = 1$ and $x = a$ in Theorem 4.2 (ii), inequalities (4.3) and (4.4) coincide with the classical Schwarz's inequality

$$\int_a^b f(x)g(x)\,\mathrm{d}x \leq \sqrt{\left(\int_a^b f^2(x)\,\mathrm{d}x \right) \left(\int_a^b g^2(x)\,\mathrm{d}x \right)}.$$

We are now in a position to prove the Minkowski inequality using Hölder's inequality.

Theorem 4.7 (Minkowski's inequality via fractional integrals)**.** *If* $f, g, h \in C\big([a,b], \mathbb{R}^+\big)$ *and* $p > 1$*, then:*

(i) *the following inequality for each* $\alpha > 0$ *and* $x \in \,]a,b]$ *with* $j^{\alpha}_{a+}(f(x) + g(x))^p > 0$ *holds*

$$j^{\alpha}_{a+}(f(x) + g(x))^{1/p} \leq \big(j^{\alpha}_{a+} f^p(x)\big)^{1/p} + \big(j^{\alpha}_{a+} g^p(x)\big)^{1/p}.$$

(ii) *The following inequality for each* $\alpha > 0$ *and* $x \in [a, b[$ *with* $j_{b^-}^{\alpha}(f(x)+g(x))^p > 0$ *holds*

$$j_{b^-}^{\alpha}(f(x)+g(x))^{1/p} \leq \left(j_{b^-}^{\alpha} f^p(x)\right)^{1/p} + \left(j_{b^-}^{\alpha} g^p(x)\right)^{1/p}.$$

Proof.

(i) We have

$$\begin{aligned} j_{a^+}^{\alpha}(f(x)+g(x))^p &= j_{a^+}^{\alpha}((f(x)+g(x))^{p-1}(f(x)+g(x))) \\ &= j_{a^+}^{\alpha}(f(x)(f(x)+g(x))^{p-1}+g(x)(f(x)+g(x))^{p-1}). \end{aligned}$$

Now, applying Hölder's inequality with $q = p/(p-1)$, we obtain

$$\begin{aligned} &j_{a^+}^{\alpha}(f(x)+g(x))^p \\ &= \left(j_{a^+}^{\alpha} f^p(x)\right)^{1/p}(j_{a^+}^{\alpha}(f(x)+g(x))^{q(p-1)})^{1/q} \\ &\quad + (j_{a^+}^{\alpha} g^p(x))^{1/p}(j_{a^+}^{\alpha}(f(x)+g(x))^{q(p-1)})^{1/q} \\ &= ((j_{a^+}^{\alpha} f^p(x))^{1/p} + \left(j_{a^+}^{\alpha} g^p(x)\right)^{1/p})(j_{a^+}^{\alpha}(f(x)+g(x))^p)^{1/q}. \end{aligned}$$

Dividing both sides of the last inequality by $(j_{a^+}^{\alpha}(f(x)+g(x))^p)^{1/q}$, we get the desired conclusion.

(ii) The proof of (ii) is similar to that of (i). $\square$

As another application of Hölder's inequality via fractional integrals, we have the following theorem.

Theorem 4.8. *If* $f, g, h \in C\big([a,b], \mathbb{R}^+\big)$, *then*

(i) *the following inequality for each* $\alpha > 0$ *and* $x \in]a, b]$ *with* $j_{a^+}^{\alpha} h(x) > 0$ *and* $j_{a^+}^{\alpha} h(x) f(x) > 0$ *holds*

$$((j_{a^+}^{\alpha} h(x)f(x))^p + (j_{a^+}^{\alpha} h(x)g(x))^p)^{1/p} \leq j_{a^+}^{\alpha} h(x)(f^p(x)+g^p(x))^{1/p},$$

if $p > 1$;

$$((j_{a^+}^{\alpha} h(x)f(x))^p + (j_{a^+}^{\alpha} h(x)g(x))^p)^{1/p} \geq j_{a^+}^{\alpha} h(x)(f^p(x)+g^p(x))^{1/p},$$

if $0 < p < 1$.

(ii) *The following inequality for each* $\alpha > 0$ *and* $x \in [a,b[$ *with* $j^{\alpha}_{b^-}h(x) > 0$ *and* $j^{\alpha}_{b^-}h(x)f(x) > 0$ *holds*

$$((j^{\alpha}_{b^-}h(x)f(x))^p + (j^{\alpha}_{b^-}h(x)g(x))^p)^{1/p} \leq j^{\alpha}_{b^-}h(x)(f^p(x) + g^p(x))^{1/p},$$

if $p > 1$;

$$((j^{\alpha}_{b^-}h(x)f(x))^p + (j^{\alpha}_{b^-}h(x)g(x))^p)^{1/p} \geq j^{\alpha}_{b^-}h(x)(f^p(x) + g^p(x))^{1/p},$$

if $0 < p < 1$.

Proof.

(i) Clearly, $\varphi(x) = (1 + x^p)^{1/p}$ is convex on $]0, \infty[$ for $p > 1$. Hence, by Theorem 3.7, we have

$$\left(1 + \left(\frac{j^{\alpha}_{a^+}h(x)f(x)}{j^{\alpha}_{a^+}h(x)}\right)^p\right)^{1/p} \leq \frac{j^{\alpha}_{a^+}h(x)\big(1 + f^p(x)\big)^{1/p}}{j^{\alpha}_{a^+}h(x)}. \tag{4.6}$$

Replacing h by hf and f by g/f in inequality (4.6), we get our desired result. If $0 < p < 1$, then φ is concave, and hence, the inequality (4.6) holds in the reverse direction.

(ii) Similarly, we can prove case (ii). $\square$

Finally, some applications of Jensen-type inequalities via fractional integrals for the continuous function φ, which is twice differentiable on $]c, d[$ and there exists $m = \inf_{c<x<d} \varphi''(x)$ or $M = \sup_{c<x<d} \varphi''(x)$, will be presented.

Theorem 4.9. *If* $f \in C\big([a,b],]c,d[\big)$, $p \in C\big([a,b], \mathbb{R}^+\big)$ *and* $\varphi \in C\big(]c,d[, \mathbb{R}\big)$ *is twice differentiable and, in addition, there exists an* $m = \inf_{c<x<d} \varphi''(x)$, *then:*

(i) *the following inequality for each* $\alpha > 0$ *and* $x \in]a,b]$ *with* $j^{\alpha}_{a^+}p(x) > 0$ *holds*

$$\frac{1}{j^{\alpha}_{a^+}p(x)} j^{\alpha}_{a^+}(p(x)\varphi(f(x))) - \varphi\left(\frac{1}{j^{\alpha}_{a^+}p(x)} j^{\alpha}_{a^+}(p(x)f(x))\right)$$
$$\geq \frac{m}{2}\left(\frac{1}{j^{\alpha}_{a^+}p(x)} j^{\alpha}_{a^+}(p(x)(f(x))^2) - \left(\frac{1}{j^{\alpha}_{a^+}p(x)} j^{\alpha}_{a^+}(p(x)f(x))\right)^2\right).$$

(ii) *The following inequality for each* $\alpha > 0$ *and* $x \in [a, b[$ *with* $j_{b-}^{\alpha} p(x) > 0$ *holds*

$$\frac{1}{j_{b-}^{\alpha} p(x)} j_{b-}^{\alpha} \left(p(x)\varphi(f(x))\right) - \varphi\left(\frac{1}{j_{b-}^{\alpha} p(x)} j_{b-}^{\alpha} \left(p(x)f(x)\right)\right)$$
$$\geq \frac{m}{2}\left(\frac{1}{j_{b-}^{\alpha} p(x)} j_{b-}^{\alpha} \left(p(x)(f(x))^2\right) - \left(\frac{1}{j_{b-}^{\alpha} p(x)} j_{b-}^{\alpha} \left(p(x)f(x)\right)\right)^2\right).$$

Proof.

(i) We choose

$$\psi(x) = \varphi(x) - \frac{m}{2}x^2.$$

Differentiating two times on both sides of ψ, we get

$$\psi''(x) = \varphi''(x) - m \geq 0.$$

Thus, ψ is a convex function on $]c, d[$. By Theorem 3.7, we have

$$\psi\left(\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \left(p(x)f(x)\right)\right) \leq \frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \big(p(x)\psi(f(x))\big).$$

Now, by the definition of ψ and Proposition 2.9 (ii), we conclude that

$$\varphi\left(\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \left(p(x)f(x)\right)\right) - \frac{m}{2}\left(\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \left(p(x)f(x)\right)\right)^2$$
$$\leq \frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \big(p(x)\varphi(f(x))\big) - \frac{m}{2}\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \big(p(x)(f(x))^2\big). \tag{4.7}$$

Then, by (4.7), we can write

$$\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \big(p(x)\varphi(f(x))\big) - \varphi\left(\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \left(p(x)f(x)\right)\right)$$
$$\geq \frac{m}{2}\left(\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \big(p(x)(f(x))^2\big) - \left(\frac{1}{j_{a+}^{\alpha} p(x)} j_{a+}^{\alpha} \left(p(x)f(x)\right)\right)^2\right).$$

(ii) The proof of (ii) is similar to that of (i). □

Theorem 4.10. *If* $f \in C\big([a, b],]c, d[\big)$, $p \in C\big([a, b], \mathbb{R}^+\big)$ *and* $\varphi \in C\big(]c, d[, \mathbb{R}\big)$ *is twice differentiable and, in addition, there exists an* $M = \sup_{c<x<d} \varphi''(x)$, *then*:

(i) *the following inequality for each* $\alpha > 0$ *and* $x \in]a,b]$ *with* $j^{\alpha}_{a+}p(x) > 0$ *holds*

$$\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}\big(p(x)\varphi(f(x))\big) - \varphi\bigg(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}(p(x)f(x))\bigg) \\ \leq \frac{M}{2}\bigg(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}(p(x)(f(x))^2) - \bigg(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}(p(x)f(x))\bigg)^2\bigg).$$

(ii) *The following inequality for each* $\alpha > 0$ *and* $x \in [a,b[$ *with* $j^{\alpha}_{b-}p(x) > 0$ *holds*

$$\frac{1}{j^{\alpha}_{b-}p(x)} j^{\alpha}_{b-}\big(p(x)\varphi(f(x))\big) - \varphi\bigg(\frac{1}{j^{\alpha}_{b-}p(x)} j^{\alpha}_{b-}(p(x)f(x))\bigg) \\ \leq \frac{M}{2}\bigg(\frac{1}{j^{\alpha}_{b-}p(x)} j^{\alpha}_{b-}(p(x)(f(x))^2) - \bigg(\frac{1}{j^{\alpha}_{b-}p(x)} j^{\alpha}_{b-}(p(x)f(x))\bigg)^2\bigg).$$

Proof.

(i) We choose $\psi(x) = \varphi(x) - (M/2)x^2$. Differentiating twice on both sides of ψ, we get $\psi''(x) = \varphi''(x) - M \leq 0$. Then, ψ is a concave function on $]c,d[$ and, by a similar proof as that of Theorem 4.9, we obtain

$$\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}\big(p(x)\varphi(f(x))\big) - \varphi\bigg(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}(p(x)f(x))\bigg) \\ \leq \frac{M}{2}\bigg(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}(p(x)(f(x))^2) - \bigg(\frac{1}{j^{\alpha}_{a+}p(x)} j^{\alpha}_{a+}(p(x)f(x))\bigg)^2\bigg).$$

(ii) Similarly, we can prove case (ii). $\square$

Remark 4.11. In particular, for cases $\alpha = 1$ and $x = b$ in Theorem 4.9 (i) and 4.10(i) or $\alpha = 1$ and $x = a$ in Theorem 4.9 (ii) and 4.10 (ii), Theorems 4.9 and 4.10 coincide with [**5,** Theorem 1.4].

By choosing $p = 1$ in Theorems 4.9 and 4.10, we have the following corollary.

Corollary 4.12. *If $f \in C\big([a,b],]c,d[\big)$ and $\varphi \in C\big(]c,d[, \mathbb{R}\big)$ is twice differentiable and, in addition, there exists an $m = \inf_{c<x<d} \varphi''(x)$, then*:

(i) *the following inequality for each $\alpha > 0$ and $x \in]a,b]$ holds*

$$\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} \varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right) \geq \frac{m}{2}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} (f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right)^2\right).$$

(ii) *The following inequality for each $\alpha > 0$ and $x \in [a,b[$ holds*

$$\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} \varphi(f(x)) - \varphi \leq \left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} f(x)\right) \geq \frac{m}{2}\left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} (f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} f(x)\right)^2\right).$$

Corollary 4.13. *If $f \in C\big([a,b],]c,d[\big)$ and $\varphi \in C\big(]c,d[, \mathbb{R}\big)$ is twice differentiable and, in addition, there exists an $M = \sup_{c<x<d} \varphi''(x)$, then*:

(i) *the following inequality for each $\alpha > 0$ and $x \in]a,b]$ holds*

$$\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} \varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right) \leq \frac{M}{2}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} (f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} j_{a^+}^{\alpha} f(x)\right)^2\right).$$

(ii) *The following inequality for each $\alpha > 0$ and $x \in [a,b[$ holds*

$$\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} \varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} f(x)\right) \leq \frac{M}{2}\left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} (f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}} j_{b^-}^{\alpha} f(x)\right)^2\right).$$

Remark 4.14. In particular, for cases $\alpha = 1$ and $x = b$ in Corollaries 4.12 (i) and 4.13 (i) or $\alpha = 1$ and $x = a$ in Corollaries 4.12 (ii) and 4.13 (ii), Corollaries 4.12 and 4.13 coincide with [**5,** Corollary 1.5].

Corollary 4.15. *If $f \in C\big([a,b],]c,d[\big)$ and $\varphi \in C\big(]c,d[,\mathbb{R}\big)$ is twice differentiable and, in addition, there exist an $m = \inf_{c<x<d} \varphi''(x)$ and $M = \sup_{c<x<d} \varphi''(x)$, then:*

(i) *the following inequalities for each $\alpha > 0$ and $x \in]a,b]$ hold*

$$\begin{aligned}
&\frac{m}{2}\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^{\alpha}(f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^{\alpha} f(x)\right)^2\right)\\
&\quad\le \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^{\alpha} \varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^{\alpha} f(x)\right)\\
&\quad\le \frac{M}{2}\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^{\alpha}(f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} j_{a+}^{\alpha} f(x)\right)^2\right).
\end{aligned}$$

(ii) *The following inequalities for each $\alpha > 0$ and $x \in [a,b[$ hold*

$$\begin{aligned}
&\frac{m}{2}\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^{\alpha}(f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^{\alpha} f(x)\right)^2\right)\\
&\quad\le \frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^{\alpha} \varphi(f(x)) - \varphi\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^{\alpha} f(x)\right)\\
&\quad\le \frac{M}{2}\left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^{\alpha}(f(x))^2 - \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} j_{b-}^{\alpha} f(x)\right)^2\right).
\end{aligned}$$

Remark 4.16. In particular, for cases $\alpha = 1$, $a = 0$ and $x = b = 1$ in Corollaries 4.15 (i) or $\alpha = 1$, $b = 1$ and $x = a = 0$ in Corollary 4.15 (ii), Corollary 4.15 coincide with [**5,** Corollary 1.7].

5. Conclusion.

The concept of convexity has a great impact on our everyday lives, and there are numerous applications of this concept in industry, business, medicine, art, etc. The applications of the convexity in equilibrium of non-cooperative games and the problems of optimum allocation of resources are significant. Jensen's inequality is the first important result for convex (concave) functions defined on an interval. The classical Jensen's inequality, the complete form of Jensen's inequality and the generalized Jensen's inequality for convex functions are important results in theoretical and applied mathematics, and in this paper we studied these inequalities via fractional integrals.

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