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OPTIMAL MORREY ESTIMATE FOR PARABOLIC EQUATIONS IN DIVERGENCE FORM VIA GREEN'S FUNCTIONS

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ABSTRACT. This paper presents a local Morrey regularity with the optimal exponents for linear parabolic equations in divergence form under the assumption that the leading coefficient is independent of t and not necessarily symmetric based on a rather different approach. Here, we achieve it by applying natural growth properties of Green's functions through the use of parabolic operators and the hole-filling technique.

1. Introduction. Let $Q = \Omega \times [0,T] \subset \mathbb{R}^{n+1}$ be a cylindrical domain with an open connected set $\Omega \subset \mathbb{R}^n$ for $n \geq 1$ and $0 < T < \infty$, and let $u(x,t) : Q \to \mathbb{R}$ be a Sobolev function in $V_2^{1,0}(Q)$ (see Definition 2.1 below). The main purpose of this paper is to consider the following parabolic operator based on a rather different argument:

$$\mathcal{L}u := u_t - D_j(a_{ij}(x)D_i u), \quad i,j = 1,\ldots,n. \tag{1.1}$$

Here, we suppose that the coefficient $A(x) = (a_{ij}(x))_{i,j=1}^n$ is an $n \times n$ matrix whose entries are real-valued measurable functions satisfying the uniform boundedness condition and the strong ellipticity:

$$a_{ij}(x) \in L^\infty(\Omega) \quad \text{and} \quad \|a_{ij}\|_{L^\infty(\Omega)} \leq \Lambda, \tag{1.2}$$

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } x \in \Omega \subset \mathbb{R}^n \text{ for all } \xi \in \mathbb{R}^n \tag{1.3}$$

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with some positive constants $0 < \lambda \le \Lambda < \infty$. It is worth noting that the coefficients $a_{ij}(x)$ are assumed only to be time-independent due to our main proof which includes an estimate of u_t.

It is well known that fundamental solutions and Green's functions play important roles in studying the qualitative theory of classical partial differential equations. There is much literature on Green's functions of uniformly elliptic and parabolic equations of second order. For example, it has been established that the Harnack inequality, the existence of solution, the Wiener criterion of the regular boundary and the representation formula point to the classical Laplacian and heat operators defined in a bounded domain by way of using the properties of Green's functions, for details, see [**3, 14**]. To a uniformly elliptic operator with bounded measurable symmetrical coefficients, various estimates of Green's functions compared with that of Laplacian and its application to Wiener's criterion on the boundary point have been studied by Littman, et al. [**20**]. Later, Grüter and Widman [**15**] generalized these estimates of Green's functions to the uniformly elliptic operators with non-symmetrical coefficients and Mazzoni [**21**] further obtained local estimates of Green's functions for X-elliptic operators with non-regular coefficients. Recently, Hofmann, et al. [**12, 16, 24**] gave a unified approach for studying Green's functions for both scalar equations and systems of elliptic type. Later, Choi and Kim [**7**] also obtained similar properties of Green's functions on the Neumann boundary condition of second order divergence elliptic systems with bounded measurable coefficients in a bounded Lipschitz domain or a Lipschitz graph domain, which enjoys the assumption that weak solutions of the system satisfy an interior Hölder continuity.

As for the parabolic settings, in 1967, Aronson [**2**] proved Gaussian upper and lower bounds for the fundamental solutions of parabolic equations in divergence form with bounded measurable coefficients. In fact, to establish the Gaussian lower bound, Aronson made use of the Harnack inequality for nonnegative solutions, which was proven by Nash in [**22**]. From then on, much research has been conducted on this subject, see e.g., [**5, 6, 8, 12, 16, 18, 19**], and the references therein. Compared to the investigation of Green's functions for parabolic equations, there has been relatively little study on Green's matrices for parabolic systems. We observe that Cho, Dong and Kim [**5**] established global estimates for Green's matrix of second order divergence parabolic systems in a cylindrical domain, under the assumption that weak solutions vanish on a portion of the boundary and satisfy a certain local boundedness estimate as well as a local Hölder continuity estimate. Recently, Dong and Kim [**12**] improved the results in [**5**] by constructing Green's functions of similar parabolic systems in non smooth time-varying domains under the assumption that weak solutions satisfy an interior Hölder continuity estimate.

In this paper, we attempt to utilize these estimates of Green's functions from Cho, Dong and Kim's papers [**5, 6, 12**] to present a local Morrey regularity. It is our main aim of this paper to give a new approach for attaining the local Morrey estimate and Hölder continuity of the weak solution with the sharp regularity index instead of the classical argument of the De Giorgi, Moser, Nash iteration. Before stating the main result, we recall the definition of a VMO space. We say that a measurable function $a_{ij}(x)$ belongs to a VMO space if, for any $\rho > 0$,

$$\omega_\rho(a_{ij}) := \sup_{\substack{x\in\mathbb{R}^n \\ 0<r<\rho}} ⨍_{B_r(x)} |a_{ij}(y) - \overline{a}_{ij}|\, dy \longrightarrow 0 \quad \text{as } \rho \to 0,$$

where $\overline{a}_{ij} = ⨍_{B_r(x)} a_{ij}(y)\, dy$.

Theorem 1.1. *Let* $(q, s, n) \in ((n+2)/2, \infty) \times (n+2, \infty) \times \mathbb{N}$ *and* $u \in L^\infty(Q) \cap V_2^{1,0}(Q)$ *be any weak solution of linear parabolic equations*

$$u_t - \sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u) = g(x,t) - \sum_{i=1}^{n} D_i f^i \tag{1.4}$$

with the coefficients $a_{ij}(x)$ *satisfying* (1.2), (1.3) *and belonging to the* VMO *space. Assume that* $g(x,t) \in L^q(Q, \mathbb{R})$ *and* $f(x,t) \in L^s(Q, \mathbb{R}^n)$. *Then, we have*

$$Du \in L^{2,\lambda}_{\mathrm{loc}}(Q, \mathbb{R})$$

for every $0 < \lambda \le n + \alpha_0$ *with*

$$\alpha_0 = \min\left\{2 - \frac{2(n+2)}{s}, 4 - \frac{2(n+2)}{q}\right\} \in (0, 2).$$

Indeed, our argument for obtaining the local optimal Morrey estimate of (1.4) is inspired by some applications of Green's functions to elliptic problems from the recent papers **[13, 25]**. In addition, an important idea comes from Huang and Wang's paper **[17]**, in which they applied the Riesz potential with the parabolic metric to prove the $C^{1,\alpha}$-regularity of heat flow of harmonic maps.

As an immediate consequence, by the Morrey lemma, we obtain a local Hölder continuity with an optimal Hölder exponent. As is well known, it does not reach the optimal Hölder index, based upon the argument of the De Giorgi, Moser, Nash iteration.

Corollary 1.2. *Let $u \in L^\infty(Q) \cap V_2^{1,0}(Q)$ be any weak solution of linear parabolic equations* (1.4) *with coefficients $a_{ij}(x)$ and data $f(x,t)$ and $g(x,t)$ satisfying the same assumptions as Theorem* 1.1. *Then, we have*

$$u \in C_{x,t}^{\gamma,\gamma/2}(Q, \mathrm{loc})$$

with an optimal Hölder index $\gamma = \alpha_0/2$, where α_0 is as shown in Theorem 1.1.

Remark 1.3. As a local estimate of the weak solutions, we do not require the base Ω of the cylinder $Q = \Omega \times [0,T]$ to be bounded or to have a regular boundary.

Remark 1.4. If the coefficient a_{ij} is a bounded, measurable function in x and t, then we can only obtain the Hölder continuity of Green's function with respect to the time variable due to De Girogi, Moser and Nash's iteration, as follows.

$$|G(X,Y) - G(X',Y)| \le C\delta(X,X')^{\mu_0}\delta(X,Y)^{-n-\mu_0}, \quad X, X', Y \in Q,$$

where μ_0 is a constant in $(0,1)$, and $\delta(\cdot,\cdot)$ is the parabolic distance (see below). Therefore, the pointwise estimate of D_tG is absent. However, if the coefficient a_{ij} is time-independent, Green's function of parabolic operators (1.1) is often called the heat kernel, studied by many authors, see Davies **[9, 10]** or Alexander and Andras **[1]**. In this case, the pointwise estimate of D_tG is present (see Lemma 2.6 below).

The remainder of this paper is organized as follows. In Section 2, we recall some related notation and basic facts, as well as some natural growth properties of Green's function. In Section 3, we provide a proof of Theorem 1.1 by the hole-filling technique based on Green's function as a part of test functions. Finally, we provide a brief conclusion.

2. Preliminaries. We denote by $X = (x, t)$ any point in $Q \subset \mathbb{R}^{n+1}$ with $x = (x_1, \ldots, x_n)$ in $\Omega \subset \mathbb{R}^n$. Similarly, we write $Y = (y, s)$, $X_0 = (x_0, t_0), \ldots$. We denote the parabolic distance between the points $X = (x, t)$ and $Y = (y, s)$ by

$$\delta(X, Y) := \max\{|x - y|, \sqrt{|t - s|}\},$$

where $|\cdot|$ denotes the usual Euclidean norm. Hence, we can easily see that there are positive constants C_1 and C_2 such that the double inequality

$$C_1 R \le \delta(X, X_0) \le C_2 R \quad \text{for all } X \in P(X_0, 2R) \setminus P(X_0, (1/2)R)$$

holds. For a cylinder $Q = \Omega \times [0, T]$, we set

$$SQ = \partial\Omega \times [0, T],$$
$$\partial_p Q = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}),$$
$$\widetilde{\partial}_p Q = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = T\}),$$

and define a distance function to the parabolic boundary $\partial_p Q$ by

$$\mathrm{dist}(X, \partial_p Q) := \inf\Big\{\delta(X, Y) : \text{for all } Y \in \partial_p Q\Big\}.$$

Set

$$P_R(X) = P(X, R) = B_R^{\delta}(X) = B_R(x) \times [t - R^2, t + R^2] \subset\subset Q,$$
$$P_-(X, R) = B_R(x) \times [t - R^2, t],$$
$$P_+(X, R) = B_R(x) \times [t, t + R^2],$$

and, if no confusion arises in the context, we will simply write $P_R = P_R(X_0)$. By $C(n, \lambda, \Lambda, \ldots)$, we denote a universal constant depending only upon prescribed quantities and possibly varying from line to line.

Let $\widehat{n}$ denote the Hausdorff dimension of $\mathbb{R}^{n+1}$ with respect to the parabolic distance δ. Then, we have $\widehat{n} = n + 2$. Throughout this paper, we denote the time derivative of u by $u_t = D_t u = \partial u/\partial t$, the spatial gradient of u by $Du = D_x u = (D_1 u, \ldots, D_n u)$, where $D_i u = D_{x_i} u = \partial u/\partial x_i$ for $i = 1, \ldots, n$.

The Sobolev space $W_p^{1,0}(Q)$ is the class of all functions $u \in L^p(Q)$ with its weak derivative $Du \in L^p(Q)$ obeying

$$\|u\|_{W_p^{1,0}(Q)} := \|u\|_{L^p(Q)} + \|Du\|_{L^p(Q)} < \infty.$$

Let $W_2^{1,1}(Q)$ denote the Hilbert space with the inner product

$$\langle u, v\rangle_{W_2^{1,1}(Q)} := \int_Q uv\, dX + \sum_{i=1}^n \int_Q D_i u D_i v\, dX + \int_Q u_t v_t\, dX,$$

and let $V_2(Q)$ denote the set of all $u \in W_2^{1,0}(Q)$ satisfying

$$\|u\|_{V_2(Q)} := \left\{\|Du\|^2_{L^2(Q)} + \operatorname*{ess\,sup}_{0\le t\le T} \|u(\cdot,t)\|^2_{L^2(\Omega)}\right\}^{1/2} < \infty.$$

Furthermore, $V_2^{1,0}(Q)$ stands for the set of all functions $u \in V_2(Q)$ such that

$$\lim_{h\to 0}\|u(\cdot, t+h) - u(\cdot,t)\|_{L^2(\Omega)} = 0, \quad t, t+h \in [0,T],$$

with the norm $\|u\|_{V_2(Q)}$. Clearly, $W_2^{1,1}(Q)$, $V_2(Q)$ and $V_2^{1,0}(Q)$ are all Banach spaces, and they have the relations:

$$W_2^{1,1}(Q) \subset V_2(Q) \subset V_2^{1,0}(Q).$$

In fact, $V_2^{1,0}(Q)$ is obtained by completing the set of $W_2^{1,1}(Q)$ in the norm of $\|u\|_{V_2(Q)}$. In any case, we also define $\mathring{W}_2^{1,1}(Q), \mathring{V}_2(Q)$ and $\mathring{V}_2^{1,0}(Q)$, respectively, to be the sets of all functions in $W_2^{1,1}(Q)$, $V_2(Q)$ and $V_2^{1,0}(Q)$ with $u(\cdot,t)|_{\partial\Omega} = 0$ for almost every $t \in [0,T]$.

Now, we understand the weak solution of equation (1.4) in the following distributional sense:

Definition 2.1. Let $(q,s) \in ((n+2)/2, \infty) \times (n+2, \infty)$ and $g(X) \in L^q(Q,\mathbb{R})$, $f(X) \in L^s(Q,\mathbb{R}^n)$. A real-valued function $u(X)$ is called a *bounded weak solution* of (1.4) if $u \in L^\infty(Q) \cap V_2^{1,0}(Q)$ such that

$$-\int_Q u\phi_t\, dX + \int_Q a_{ij} D_i u D_j \phi\, dX = \int_Q g\phi\, dX + \int_Q f^i D_i\phi\, dX \tag{2.1}$$

for any $\phi \in \mathring{V}_2^{1,0}(Q,\mathbb{R})$.

For the parabolic operator $\mathcal{L}$ of (1.1), its adjoint operator ${}^t\mathcal{L}$ is introduced by

$$ {}^t\mathcal{L} = -u_t - \sum_{i,j=1}^{n} D_j \left(\widetilde{a}_{ij}(x) D_i u\right), $$

where $(\widetilde{a}_{ij})_{i,j=1}^n$ is the transpose of $(a_{ij})_{i,j=1}^n$ with $\widetilde{a}_{ij} = a_{ji}$. It is obvious that the coefficients $\widetilde{a}_{ij}$ satisfy (1.2) and (1.3) with the same constants λ, Λ.

Next, we recall the definitions of Green's functions associated with $\mathcal{L}$ and ${}^t\mathcal{L}$, cf., [**5, 6, 12**].

Definition 2.2. We say that a function $G(X, X_0) = G(x, t, x_0, t_0)$, defined on the set $\{(X, X_0) \in Q \times Q : X \neq X_0\}$, is a *Green's function* of $\mathcal{L}$ in Q, if it satisfies the following properties:

(i) $G(\cdot, X_0) \in V_2^{1,0}(Q \setminus P_R(X_0))$ for each fixed point $X_0 \in Q$, small $R > 0$, and $G(\cdot, X_0)$ vanishes on SQ.

(ii) $G(\cdot, X_0) \in W_{1,\mathrm{loc}}^{1,0}(Q)$, and $\mathcal{L}G(\cdot, X_0) = \delta_{X_0}$ for all $X_0 \in Q$ is understood in the weak sense

$$ -\int_Q G(\cdot, X_0)\phi_t \, dX + \int_Q a_{ij} D_i G(\cdot, X_0) D_j \phi \, dX = \phi(X_0) \tag{2.2} $$
$$ \text{for all } \phi \in \overset{\circ}{V}{}_2^{1,0}(Q, \mathbb{R}). $$

(iii) For any $h \in L_c^\infty(Q)$, the function u, given by

$$ u(X) = \int_Q G(X, X_0) h(X_0) \, dX_0, $$

belongs to $\overset{\circ}{V}{}_2^{1,0}(Q)$ and satisfies ${}^t\mathcal{L}u = h$ in the sense that

$$ \int_Q u\phi_t \, dX + \int_Q \widetilde{a}_{ij} D_i u D_j \phi \, dX = \int_Q h\phi \, dX \quad \text{for all } \phi \in \overset{\circ}{V}{}_2^{1,0}(Q, \mathbb{R}). $$

Definition 2.3. Similarly, we say that a function $\widetilde{G}(X, X_0) = \widetilde{G}(x, t, x_0, t_0)$ is a Green's function of ${}^t\mathcal{L}$, defined on the set $\{(X, X_0) \in Q \times Q : X \neq X_0\}$, if it satisfies the following properties:

(i) $\widetilde{G}(\cdot, X_0) \in V_2^{1,0}(Q \setminus P_R(X_0))$ for each fixed point $X_0 \in Q$, small $R > 0$, and $\widetilde{G}(\cdot, X_0)$ vanishes on SQ.

(ii) $\widetilde{G}(\cdot, X_0) \in W^{1,0}_{1,\text{loc}}(Q)$ and $\mathcal{L}\widetilde{G}(\cdot, X_0) = \delta_{X_0}$ for all $X_0 \in Q$ in the following sense:

$$\int_Q \widetilde{G}(\cdot, X_0)\phi_t\, dX + \int_Q \widetilde{a}_{ij} D_i \widetilde{G}(\cdot, X_0) D_j \phi\, dX = \phi(X_0) \tag{2.3}$$
$$\text{for all } \phi \in \overset{\circ}{V}{}^{1,0}_2(Q, \mathbb{R}).$$

(iii) For any $h \in L^\infty_c(Q)$, the function u, given by

$$u(X) = \int_Q \widetilde{G}(X, X_0) h(X_0)\, dX_0,$$

belongs to $\overset{\circ}{V}{}^{1,0}_2(Q)$ and satisfies $\mathcal{L}u = h$ in the sense of (2.1).

Remark 2.4. Definition 2.3 (iii), combined with the uniqueness of weak solutions of ${}^t\mathcal{L}u = h$ and $\mathcal{L}u = h$ in $\overset{\circ}{V}{}^{1,0}_2(Q)$ for any $h \in L^\infty_c(Q)$, implies that Green's functions $G(X, X_0)$ and $\widetilde{G}(X, X_0)$ are unique.

Let us recall that the weak-L^p spaces $L^p_*(P_R)$ comprise the class of all functions $f \in L^p(P_R)$ such that

$$\|f\|_{L^p_*(P_R)} := \inf\{C : \mu|\{X \in P_R : |f(X)| > \mu\}|^{1/p} \le C \text{ for all } \mu > 0\} < \infty$$

for all $p \ge 1$. In particular, for any $1 \le q < p$, the following hold:

$$\|f\|_{L^p_*(P_R)} \le \|f\|_{L^p(P_R)}$$

and

$$\|f\|_{L^q(P_R)} \le \left(\frac{p}{p-q}\right)^{1/q} |P_R|^{1/q-1/p} \|f\|_{L^p_*(P_R)}, \tag{2.4}$$

cf., [**14**]. According to [**12,** Corollary 4.9], we know that the assumption that coefficients $a_{ij}(x)$ belong to VMO implies that the weak solution of $\mathcal{L}u = 0$ enjoys interior Hölder continuity, which in turn guarantees that the Green's function of $\mathcal{L}$ exists and satisfies the following natural growth properties, cf., [**6,** Theorem 2.7], [**12,** Theorem 3.1], [**23,** Lemma 5].

Lemma 2.5. *For any fixed point $X_0 \in Q$, the Green's function $G(X, X_0)$ of $\mathcal{L}$ has the following properties*:

(i) $0 \le G(X, X_0) \le C\delta(X, X_0)^{-n}$, *whenever* $0 < \delta(X, X_0) < (1/2)\operatorname{dist}(X_0, \partial_p Q)$ *and* $X, X_0 \in Q$;

(ii) *there exist fixed constants* C_1 *and* C_2 *depending on* n, λ *and* Λ *such that*

$$G(X, X_0) \ge \frac{1}{C_1(t-t_0)^{n/2}}\, e^{-C_2|x-x_0|^2/t-t_0}.$$

(iii) $\|G(X, X_0)\|_{L^\nu(Q(X_0,R))} \le CR^{-n+(n+2)/\nu}$ *for any* $0 < R < \operatorname{dist}(X_0, \partial_p Q)$, *and* $\nu \in [1, (n+2)/n)$;

(iv) $G(X, X_0) \in L^\kappa_*(Q)$ *for* $\kappa = (n+2)/n$, *with*

$$\|G(X, X_0)\|_{L^\kappa_*(Q)} \le C(n, \lambda, \Lambda);$$

(v) $\|DG(X, X_0)\|_{L^p(Q(X_0,R))} \le CR^{-n-1+(n+2)/p}$ *for any* $0 < R < \operatorname{dist}(X_0, \partial_p Q)$, *and* $p \in [1, (n+2)/(n+1))$;

(vi) $DG(X, X_0) \in L^\tau_*(Q)$ *for* $\tau = (n+2)/(n+1)$, *with*

$$\|DG(X, X_0)\|_{L^\tau_*(Q)} \le C(n, \lambda, \Lambda).$$

The above natural growth properties are also valid for the Green's function $\widetilde{G}$ *of the adjoint operator* ${}^t\mathcal{L}$.

In what follows, we state the pointwise estimate of D_tG to parabolic equations, provided by Alexander and Andras [**1,** Corollary 5.7] via an argument for the heat semigroup. More precisely, we have

Lemma 2.6. *For any fixed point* $X_0 \in Q$, *the derivative of the Green's function* $G(X, X_0)$, *with respect to time variable* t, *satisfies*

$$|D_tG(X, X_0)| \le C\delta(X, X_0)^{-(n+2)}$$

for any $0 < \delta(X, X_0) < \infty$ *and some positive constant* $C = C(n)$. *The above estimate is also valid for the Green's function* $\widetilde{G}$ *of the adjoint operator* ${}^t\mathcal{L}$ *in* Q.

The next lemma states that the weak solution of (1.4) satisfies a Poincaré-type inequality, cf., [**6,** Lemma 2.4], [**23,** Lemma 3].

Lemma 2.7. *There exists a positive constant $C = C(n, \lambda, \Lambda)$ such that, if u is a weak solution of equation (1.4) in P_R, then*

$$\int_{P_R} |u - u_R|^2 dX \le CR^2 \int_{P_R} |Du|^2 dX + CR^{4+(n+2)(1-2/q)} \|g\|^2_{L^q(P_R)} + CR^{2+(n+2)(1-2/s)} \|f\|^2_{L^s(P_R)}, \tag{2.5}$$

where $u_R = \fint_{P_R} u \, dX$.

The following iteration lemma will be needed later; its proof may be found in [**14**].

Lemma 2.8. *Let ω be a non-decreasing function defined on the interval $(0, R]$, which satisfies inequality*

$$\omega(\tau r) \le \theta \omega(r) + K r^\alpha,$$

where $0 < \theta$, $\tau < 1$. Then, for $\delta \in (0, \alpha)$, we have

$$\omega(r) \le C \left(\frac{r}{R} \right)^\delta \Big(\omega(R) + K R^\alpha \Big),$$

where both $C = C(\tau, \theta)$ and $\delta = \delta(\tau, \theta, \alpha)$ are positive constants.

Finally, we introduce a version of Morrey space from [**3**], which is slightly stronger than the standard Morrey space.

Definition 2.9. Let $(p, \lambda) \in [1, \infty) \times (0, \widehat{n})$. A real-valued function $u(X) \in L^p(Q)$ belongs to the Morrey space $L^{p,\lambda}(Q)$ if and only if

$$\|u\|_{L^{p,\lambda}(Q)} := \sup_{\substack{X_0 \in Q \\ 0 < \varrho \le d}} \left(\int_{Q(X_0,\rho)} \frac{|u|^p}{\delta(X, X_0)^\lambda} \, dX \right)^{1/p} < \infty,$$

where $Q(X_0, \rho) = P(X_0, \rho) \cap Q$ and $d = \mathrm{diam}(Q)$.

3. Proof of the main theorem.

Proof of Theorem 1.1. For any given point $X_0 = (x_0, t_0) \in Q$ and a constant $0 < R_0 < (1/4)\,\mathrm{dist}(X_0, \partial_p Q)$, let $\eta(X) \in C_0^\infty(P_{2R})$ be a cut-off function such that $\eta(X) \equiv 1$ for $X \in P_{R/2}$ and

$$\begin{gathered} 0 \le \eta(X) \le 1, \\ |D\eta| \le \frac{K_1}{R}, \\ |\eta_t| \le \frac{K_2}{R^2} \quad \text{for all } X \in P_{2R}; \end{gathered} \tag{3.1}$$

where $0 < R < R_0$ and K_1 and K_2 are two positive constants. We denote u_R to be an integral average of u over $P_{2R} \setminus P_{R/2}$ with

$$u_R = \frac{1}{|P_{2R} \setminus P_{R/2}|} \int_{P_{2R}\setminus P_{R/2}} u(X)\, dX \quad \text{for all } P_{2R} \subset Q.$$

Since $\psi(X) = \eta^2 \widetilde{G}(X, X_0) \in \mathring{V}_2^{1,0}(P_{2R}, \mathbb{R})$ and $u \in L^\infty(Q) \cap V_2^{1,0}(Q)$, we derive that $\phi(X) = \psi(X)(u - u_R) \in V_2(P_{2R}, \mathbb{R})$ satisfying $\phi(X) = 0$ on $\partial_p P_{2R}$ and

$$\begin{aligned} &\lim_{h\to 0} \|\phi(\cdot, t+h) - \phi(\cdot, t)\|_{L^2(\Omega)} \\ &\qquad \le \|u\|_{L^\infty} \lim_{h\to 0} \|\psi(\cdot, t+h) - \psi(\cdot, t)\|_{L^2(\Omega)} = 0, \\ &\qquad\qquad t, t+h \in [0, T]. \end{aligned}$$

This implies that $\phi(X) \in \mathring{V}_2^{1,0}(P_{2R}, \mathbb{R})$; thus, we can take $\phi(X)$ as a test function of (2.1), yielding

$$\phi_t = \eta^2 D_t \widetilde{G}(u - u_R) + 2\eta \widetilde{G}(u - u_R)\eta_t + \eta^2 \widetilde{G} D_t(u - u_R)$$

and

$$D_i\phi = \psi D_i u + D_i \psi (u - u_R), \quad i = 1, \dots, n.$$

Substituting the above formula into (2.1), we deduce

$$\begin{aligned} &\int_Q u_t \phi\, dX + \int_Q a_{ij} D_i u D_j u \psi\, dX + \int_Q a_{ij} D_i u D_j \psi (u - u_R)\, dX \\ &= \int_Q g\psi(u - u_R)\, dX + \int_Q (f^i, \psi D_i u)\, dX + \int_Q (f^i, D_i\psi(u - u_R))\, dX, \end{aligned}$$

which can be rewritten as

$$\int_Q a_{ij} D_i u D_j u\psi\, dX = \mathrm{I} + \mathrm{II}, \tag{3.2}$$

with

$$\mathrm{I} = -\int\limits_Q u_t\phi\, dX - \int\limits_Q a_{ij} D_i u D_j \psi(u-u_R)\, dX =: \mathrm{I}_1 + \mathrm{I}_2$$

and

$$\mathrm{II} = \int\limits_Q g\psi(u-u_R)\, dX + \int\limits_Q (f^i, \psi D_i u)\, dX + \int\limits_Q (f^i, D_i\psi(u-u_R))\, dX.$$

In the sequel, we focus on the estimates of $|\mathrm{I}|$ and $|\mathrm{II}|$, respectively. In order to estimate $|\mathrm{I}|$, by employing integration by parts and substituting ϕ_t into I_1, we have

$$\begin{aligned} \mathrm{I}_1 &= -\int\limits_Q D_t(u-u_R)\phi\, dX = \int\limits_Q (u-u_R)\phi_t\, dX \\ &= \int\limits_Q \eta^2 D_t\widetilde{G}(u-u_R)^2 dX \\ &\quad + 2\int\limits_Q \eta\widetilde{G}(u-u_R)^2\eta_t\, dX + \int\limits_Q D_t(u-u_R)\phi\, dX \\ &= \frac{1}{2}\int\limits_Q \eta^2 D_t\widetilde{G}(u-u_R)^2\, dX + \int\limits_Q \eta\widetilde{G}(u-u_R)^2\eta_t\, dX. \end{aligned} \tag{3.3}$$

Note that

$$D_j\psi = 2\eta\widetilde{G}D_j\eta + \eta^2 D_j\widetilde{G},$$

and substituting into I_2, it follows that

$$\mathrm{I}_2 = -\int\limits_Q a_{ij} D_i u D_j\widetilde{G}(u-u_R)\eta^2 dX - 2\int\limits_Q a_{ij} D_i u\widetilde{G}(u-u_R) D_j\eta\eta\, dX. \tag{3.4}$$

Now, we insert (3.3) and (3.4) into the formula I. This yields

$$
\begin{aligned}
\mathrm{I} &= \frac{1}{2}\int_Q \eta^2 D_t\widetilde{G}(u-u_R)^2 dX + \int_Q \eta\widetilde{G}(u-u_R)^2\eta_t\,dX \\
&\quad - \int_Q a_{ij}D_iuD_j\widetilde{G}(u-u_R)\eta^2 dX - 2\int_Q a_{ij}D_iu\widetilde{G}(u-u_R)D_j\eta\eta\,dX \\
&= -\Big[-\frac{1}{2}\int_Q \eta^2 D_t\widetilde{G}(u-u_R)^2 dX + \int_Q a_{ij}D_iuD_j\widetilde{G}(u-u_R)\eta^2 dX\Big] \\
&\quad + \int_Q \eta\widetilde{G}(u-u_R)^2\eta_t\,dX - 2\int_Q a_{ij}D_iu\widetilde{G}(u-u_R)D_j\eta\eta\,dX \\
&= -\Big[-\int_Q D_t\widetilde{G}\Big(\frac{1}{2}(u-u_R)^2\eta^2\Big)\,dX + \int_Q \widetilde{a}_{ji}D_j\widetilde{G}D_i\Big(\frac{1}{2}(u-u_R)^2\eta^2\Big)dX\Big] \\
&\quad + \int_Q \widetilde{a}_{ji}D_j\widetilde{G}D_i\eta(u-u_R)^2\eta\,dX + \int_Q \eta\widetilde{G}(u-u_R)^2\eta_t\,dX \\
&\quad - 2\int_Q a_{ij}D_iu\widetilde{G}(u-u_R)D_j\eta\eta\,dX \\
&= -\frac{1}{2}(u(X_0)-u_R)^2\eta^2(X_0) + \int_Q \widetilde{a}_{ji}D_j\widetilde{G}D_i\eta(u-u_R)^2\eta\,dX \\
&\quad + \int_Q \eta\widetilde{G}(u-u_R)^2\eta_t\,dX - 2\int_Q a_{ij}D_iu\widetilde{G}(u-u_R)D_j\eta\eta\,dX \\
&\le \int_Q \widetilde{a}_{ji}D_j\widetilde{G}D_i\eta(u-u_R)^2\eta\,dX + \int_Q \eta\widetilde{G}(u-u_R)^2\eta_t\,dX \\
&\quad - 2\int_Q a_{ij}D_iu\widetilde{G}(u-u_R)D_j\eta\eta\,dX.
\end{aligned}
$$

In the fourth equality above, we use $a_{ij} = \widetilde{a}_{ji}$ as well as equality (2.3), which yield the Green's function of the adjoint operator ${}^t\mathcal{L}$. Therefore,

$$
\begin{aligned}
|\,\mathrm{I}\,| \le \Lambda \int_Q |D\widetilde{G}||D\eta||u-u_R|^2\eta\,dX \\
+ \int_Q \eta\widetilde{G}|u-u_R|^2|\eta_t|\,dX
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&+ 2\Lambda \int_Q |Du||\widetilde{G}|u-u_R||D\eta|\eta \, dX \\
&=: \mathsf{A}_1 + \mathsf{A}_2 + \mathsf{A}_3.
\end{aligned}
$$

Estimate of A_1. Using Young's inequality with an arbitrary $\varepsilon_1 > 0$, we find that

$$
\begin{aligned}
\mathsf{A}_1 &= \Lambda \int_Q |D\widetilde{G}||D\eta||u-u_R|^2 \eta \, dX \\
&\le \Lambda \int_{P_{2R}\setminus P_{R/2}} (|u-u_R||D\eta|\widetilde{G}^{1/2}) \left(\eta |u-u_R| \frac{|D\widetilde{G}|}{\widetilde{G}^{1/2}} \right) dX \\
&\le \varepsilon_1 \int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 |D\eta|^2 \widetilde{G} \, dX + C(\Lambda, \varepsilon_1) \\
&\cdot \int_{P_{2R}\setminus P_{R/2}} \eta^2 |u-u_R|^2 \frac{|D\widetilde{G}|^2}{\widetilde{G}} \, dX \\
&=: \mathsf{A}_{11} + \mathsf{A}_{12}
\end{aligned}
$$

By virtue of Lemma 2.5 (i), (3.1) and Lemma 2.7, we deduce

$$
\begin{aligned}
\mathsf{A}_{11} &= \varepsilon_1 \int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 |D\eta|^2 \widetilde{G} \, dX \le C\varepsilon_1 \int_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2 \delta(X, X_0)^n} \, dX \\
&\le \frac{C\varepsilon_1}{R^{n+2}} \int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 dX \le \frac{C\varepsilon_1}{R^n} \int_{P_{2R}\setminus P_{R/2}} |Du|^2 dX \\
&\quad + C\varepsilon_1 R^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} + C\varepsilon_1 R^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}.
\end{aligned}
$$

Then, it remains to estimate A_{12}. We introduce a new, smooth cut-off function, satisfying:

$$
\xi(X) = \begin{cases} 0 & \text{for } X \in P_{R/2}, \\ \eta(X) & \text{for } X \in \mathbb{R}^{n+1} \setminus P_{R/2}. \end{cases}
$$

For the Green's function $\widetilde{G}$, defined by (2.3), we take $\phi = \widetilde{G}^{-1/2}(u - u_R)^2 \xi^2 \in C_0^\infty(P_{2R} \setminus P_{R/2}, \mathbb{R})$ as the test function. Note that $\phi(X_0) \equiv 0$, and

$$D_j\phi = -\frac{1}{2}\widetilde{G}^{-3/2}D_j\widetilde{G}(u-u_R)^2\xi^2 + 2\widetilde{G}^{-1/2}[\xi(u-u_R)][D_j(\xi(u-u_R))].$$

By substituting $\phi(X_0)$ and $D_j\phi$ into (2.3), we find that

$$\begin{aligned}
&\frac{1}{2}\int_{P_{2R}\setminus P_{R/2}} \widetilde{a}_{ij}D_i\widetilde{G}D_j\widetilde{G}\widetilde{G}^{-3/2}(u-u_R)^2\xi^2 dX \\
&= -\int_{P_{2R}\setminus P_{R/2}} D_t\widetilde{G}\widetilde{G}^{-1/2}(u-u_R)^2\xi^2 dX \\
&\quad + 2\int_{P_{2R}\setminus P_{R/2}} \widetilde{a}_{ij}D_i\widetilde{G}\widetilde{G}^{-1/2}[\xi(u-u_R)][D_j(\xi(u-u_R))]\, dX.
\end{aligned} \tag{3.6}$$

Now, combining (1.2), (3.6) and (1.3), we have

$$\begin{aligned}
&\frac{\lambda}{2}\int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2\widetilde{G}^{-3/2}(u-u_R)^2\xi^2 dX \le \int_{P_{2R}\setminus P_{R/2}} |D_t\widetilde{G}|\widetilde{G}^{-1/2}|u-u_R|^2\xi^2 dX \\
&\quad + 2\Lambda\int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|\widetilde{G}^{-1/2}\xi|u-u_R||D_j(\xi(u-u_R))|\, dX.
\end{aligned} \tag{3.7}$$

Applying Lemma 2.5 (i), Lemma 2.6, (3.1) and Lemma 2.7, the first term on the right-hand side of (3.7) satisfies

$$\begin{aligned}
&\int_{P_{2R}\setminus P_{R/2}} |D_t\widetilde{G}|\widetilde{G}^{-1/2}|u-u_R|^2\xi^2 dX \le C\int_{P_{2R}\setminus P_{R/2}} \frac{\delta(X,X_0)^{n/2}}{\delta(X,X_0)^{n+2}}|u-u_R|^2 dX \\
&\le C\int_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{\delta(X,X_0)^{n/2+2}}\, dX \\
&\le CR^{-n/2-2}\int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 dX \\
&\le CR^{-n/2}\int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4+n/2-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} \\
&\quad + CR^{2+n/2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.
\end{aligned} \tag{3.8}$$

Using Young's inequality with arbitrary $\varepsilon_2 > 0$, Lemma 2.5 (i) and Lemma 2.7, the second term on the right-hand side of (3.7) satisfies

(3.9)
$$
\begin{aligned}
2\Lambda & \int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|\widetilde{G}^{-1/2}\xi|u-u_R||D_j(\xi(u-u_R))|\,dX \\
&= 2\Lambda \int_{P_{2R}\setminus P_{R/2}} [|D\widetilde{G}|\widetilde{G}^{-3/4}\xi|u-u_R|][\widetilde{G}^{1/4}|D_j(\xi(u-u_R))|]\,dX \\
&\le \varepsilon_2 \int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2\widetilde{G}^{-3/2}|u-u_R|^2\xi^2 dX + C(\Lambda,\varepsilon_2) \\
&\cdot \int_{P_{2R}\setminus P_{R/2}} \widetilde{G}^{1/2}(|D\xi|^2|u-u_R|^2+\xi^2|Du|^2)\,dX \\
&\le \varepsilon_2 \int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2\widetilde{G}^{-3/2}|u-u_R|^2\xi^2 dX + CR^{-n/2} \\
&\cdot \int_{P_{2R}\setminus P_{R/2}} (|D\xi|^2|u-u_R|^2+\xi^2|Du|^2)\,dX \\
&\le \varepsilon_2 \int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2\widetilde{G}^{-3/2}|u-u_R|^2\xi^2 dX + CR^{-n/2} \\
&\cdot \int_{P_{2R}\setminus P_{R/2}} \left(\frac{|u-u_R|^2}{R^2}+|Du|^2\right)dX \\
&\le \varepsilon_2 \int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2\widetilde{G}^{-3/2}|u-u_R|^2\xi^2 dX + CR^{-n/2}\int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX \\
&\quad + CR^{4+n/2-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} + CR^{2+n/2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.
\end{aligned}
$$

Letting $\varepsilon_2 < \lambda/2$, from (3.8) and (3.9), inequality (3.7) becomes

$$
\begin{aligned}
(3.10)\qquad & \int_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2\widetilde{G}^{-3/2}|u-u_R|^2\xi^2 dX \\
&\le CR^{-n/2}\int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4+n/2-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} \\
&\quad + CR^{2+n/2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.
\end{aligned}
$$

Again using Lemma 2.5 (i), together with (3.10), we deduce

$$\begin{aligned}
\mathsf{A}_{12} &= C \int\limits_{P_{2R}\setminus P_{R/2}} \eta^2 |u-u_R|^2 \frac{|D\widetilde{G}|^2}{\widetilde{G}}\, dX \\
&= C \int\limits_{P_{2R}\setminus P_{R/2}} \widetilde{G}^{1/2} |D\widetilde{G}|^2 \widetilde{G}^{-3/2} |u-u_R|^2 \xi^2 dX \\
&\le CR^{-n/2} \int\limits_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^2 \widetilde{G}^{-3/2} |u-u_R|^2 \xi^2 dX \\
&\le CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} \\
&\quad + CR^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}.
\end{aligned}$$

Thus, the estimates of A_{11} and A_{12} imply

$$\begin{aligned}
\mathsf{A}_1 &\le CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} \\
&\quad + CR^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}.
\end{aligned}$$

Estimate of A_2. We use (3.1), Lemma 2.5 (i) and Lemma 2.7 to deduce

$$\begin{aligned}
\mathsf{A}_2 &= \int\limits_{Q} \eta \widetilde{G} |u-u_R|^2 |\eta_t|\, dX \\
&\le \frac{K_2}{R^2} \int\limits_{P_{2R}\setminus P_{R/2}} \widetilde{G} |u-u_R|^2 dX \\
&\le \frac{C}{R^2} \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{\delta(X,X_0)^n}\, dX \\
&\le \frac{C}{R^{n+2}} \int\limits_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 dX
\end{aligned}$$

$$\leq \frac{C}{R^n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4-(2n+4)/q}\|g\|^2_{L^q(P_{2R})}$$
$$+ CR^{2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.$$

Estimate of A_3. By (3.1), Young's inequality with arbitrary $\varepsilon_3 > 0$ and Lemma 2.5 (i), we can derive

(3.11)
$$\begin{aligned}
\mathsf{A}_3 &= 2\Lambda \int\limits_Q |Du|\widetilde{G}|u-u_R||D\eta|\eta\, dX \\
&\leq 2K_1\Lambda \int\limits_{P_{2R}\setminus P_{R/2}} (\eta|Du|\widetilde{G}^{1/2})\left(\frac{|u-u_R|}{R}\widetilde{G}^{1/2}\right) dX \\
&\leq \varepsilon_3 \int\limits_{P_{2R}\setminus P_{R/2}} \eta^2|Du|^2\widetilde{G}\, dX + C(\Lambda,\varepsilon_3) \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2}\widetilde{G}\, dX \\
&\leq C\varepsilon_3 \int\limits_{P_{3R}\setminus P_{R/2}} \frac{|Du|^2}{\delta(X,X_0)^n}\, dX \\
&\quad + C \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2}\widetilde{G}\, dX,
\end{aligned}$$

and the second term in (3.11) satisfies

$$\begin{aligned}
C \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2}\widetilde{G}\, dX &\leq C \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2\delta(X,X_0)^n}\, dX \\
&\leq \frac{C}{R^{n+2}} \int\limits_{P_{2R}\setminus P_{R/2}} |u-u_R|^2\, dX \\
&\leq \frac{C}{R^n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} \\
&\quad + CR^{2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})},
\end{aligned}$$

whence

$$\mathsf{A}_3 \le CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} + CR^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}.$$

Now, placing the estimations of A_1, A_2 and A_3 into (3.5), we obtain

$$|\,\mathrm{I}\,| \le CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} + CR^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}. \tag{3.12}$$

Next, we are ready to derive

$$\begin{aligned} |\mathrm{II}| &\le \int\limits_Q |g||u-u_R||\psi|\, dX + \int_Q |f||Du||\psi|\, dX \\ &\quad + \int\limits_Q |f||u-u_R||D\psi|\, dX \\ &=: \mathsf{B}_1 + \mathsf{B}_2 + \mathsf{B}_3. \end{aligned} \tag{3.13}$$

Estimate of B_1. Recalling $\psi = \eta^2 \widetilde{G}$ and using Young's inequality with arbitrary $\varepsilon_4 > 0$, we have

$$\begin{aligned} \mathsf{B}_1 &= \int\limits_Q |g||u-u_R|\widetilde{G}\eta^2 dX \\ &\le \int\limits_{P_{2R}\setminus P_{R/2}} (R|g|\widetilde{G}^{1/2}) \Big(\frac{|u-u_R|\widetilde{G}^{1/2}}{R} \Big) dX \\ &\le C(\varepsilon_4) R^2 \int_{P_{2R}\setminus P_{R/2}} |g|^2 \widetilde{G}\, dX + \varepsilon_4 \int_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2} \widetilde{G}\, dX \\ &=: \mathsf{B}_{11} + \mathsf{B}_{12}. \end{aligned}$$

For B_{11}, Lemma 2.5 (iv) tells us that $\|\widetilde{G}\|_{L^\kappa_*} \le C(n,\lambda,\Lambda)$ with

$\kappa = (n+2)/n$. Hence, by the Hölder inequality and (2.4), we have

$$\begin{aligned}
\mathsf{B}_{11} &= R^2 \int_{P_{2R}\setminus P_{R/2}} |g|^2 \widetilde{G}\, dX \\
&\le R^2 \Big(\int_{P_{2R}\setminus P_{R/2}} \widetilde{G}^{\nu} dX \Big)^{1/\nu} \Big(\int_{P_{2R}\setminus P_{R/2}} |g|^{2\nu'} dX \Big)^{1/\nu'} \\
&\le CR^2 |P_{2R}|^{(1/\nu - 1/\kappa)} \|\widetilde{G}\|_{L^{\kappa}_{*}} |P_{2R}|^{(1/\nu' - 2/q)} \|g\|^2_{L^q} \\
&\le CR^2 |P_{2R}|^{(1 - 1/\kappa - 2/q)} \|\widetilde{G}\|_{L^{\kappa}_{*}} \|g\|^2_{L^q} \\
&\le CR^{2+(n+2)(1-1/\kappa-2/q)},
\end{aligned}$$

where the exponent of R is positive due to $q > (n+2)/2$, and ν' is the Hölder conjugate number of ν with $1 \le \nu < (n+2)/n$.

For B_{12}, by Lemma 2.7, it follows that

$$\begin{aligned}
\mathsf{B}_{12} &= \varepsilon_4 \int_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2}{R^2} \widetilde{G}\, dX \\
&\le \frac{C\varepsilon_4}{R^{n+2}} \int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 dX \\
&\le \frac{C\varepsilon_4}{R^{n}} \int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + C\varepsilon_4 R^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} \\
&\quad + C\varepsilon_4 R^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}.
\end{aligned}$$

Combining the estimates of B_{11} and B_{12}, we obtain

$$\begin{aligned}
\mathsf{B}_{1} \le{}& \frac{C\varepsilon_4}{R^{n}} \int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{2+(n+2)(1-1/\kappa-2/q)} \\
&+ C\varepsilon_4 R^{4-(2n+4)/q} \|g\|^2_{L^q(P_{2R})} + C\varepsilon_4 R^{2-(2n+4)/s} \|f\|^2_{L^s(P_{2R})}.
\end{aligned}$$

Estimate of B_2. Note that $\psi = \eta^2 \widetilde{G}$ and $\mathrm{supp}(\eta) = P_{2R}$. From Lemma 2.5 (iv), it follows that

(3.14)

$$\begin{aligned}
\int_{P_{2R}\setminus P_{R/2}} |f|^2 \widetilde{G}\, dX &\le \Big(\int_{P_{2R}\setminus P_{R/2}} \widetilde{G}^{\nu} dX \Big)^{1/\nu} \Big(\int_{P_{2R}\setminus P_{R/2}} |f|^{2\nu'} dX \Big)^{1/\nu'} \\
&\le C|P_{2R}|^{(1/\nu - 1/\kappa)} \|\widetilde{G}\|_{L^{\kappa}_{*}} |P_{2R}|^{(1/\nu' - 2/s)} \|f\|^2_{L^s}
\end{aligned}$$

$$\le C|P_{2R}|^{(1-1/\kappa-2/s)}\|\widetilde{G}\|_{L_*^\kappa}\|f\|_{L^s}^2$$
$$\le CR^{(n+2)(1-1/\kappa-2/s)},$$

where the exponent of R is positive due to $s > n+2$. Then, using (3.14) and Young's inequality with arbitrary $\varepsilon_5 > 0$, we have

$$\begin{aligned}\mathsf{B}_2 &= \int_Q |f||Du|\widetilde{G}\eta^2 dX \\ &\le C(\varepsilon_5)\int_{P_{2R}\setminus P_{R/2}} |f|^2\widetilde{G}\,dX + \varepsilon_5\int_{P_{2R}\setminus P_{R/2}} |Du|^2\eta^4\widetilde{G}\,dX \\ &\le CR^{(n+2)(1-1/\kappa-2/s)} + \frac{C\varepsilon_5}{R^n}\int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX.\end{aligned}$$

Estimate of B_3. Since

$$\mathsf{B}_3 \le 2\int_Q |f||u-u_R|\widetilde{G}|D\eta|\eta\,dX + \int_Q |f||u-u_R|\eta^2|D\widetilde{G}|\,dX =: \mathsf{B}_{31}+\mathsf{B}_{32},$$

it suffices to estimate B_{31} and B_{32}, respectively. Similar to the estimate of B_2, we find that

$$\begin{aligned}\mathsf{B}_{31} &= 2\int_Q |f||u-u_R|\widetilde{G}|D\eta|\eta\,dX \\ &\le C\int_{P_{2R}\setminus P_{R/2}} |f|^2\widetilde{G}\eta^2 dX + C\int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2|D\eta|^2\widetilde{G}\,dX \\ &\le C\int_{P_{2R}\setminus P_{R/2}} |f|^2\widetilde{G}\,dX + CR^{-n-2}\int_{P_{2R}\setminus P_{R/2}} |u-u_R|^2 dX \\ &\le CR^{(n+2)(1-1/\kappa-2/s)} + CR^{-n}\int_{P_{3R}\setminus P_{R/2}} |Du|^2 dX \\ &\quad + CR^{4-(2n+4)/q}\|g\|_{L^q(P_{2R})}^2 + CR^{2-(2n+4)/s}\|f\|_{L^s(P_{2R})}^2.\end{aligned}$$

We use Young's inequality with arbitrary $\varepsilon_6 > 0$ to see that

$$\mathsf{B}_{32} = \int_Q |f||u-u_R|\eta^2|D\widetilde{G}|\,dX \tag{3.15}$$

$$\leq \int\limits_{P_{2R}\setminus P_{R/2}} \left(\frac{|f||D\widetilde{G}|R}{\widetilde{G}^{1/2}}\right)\left(\frac{|u-u_R|\widetilde{G}^{1/2}}{R}\right)dX$$
$$\leq C(\varepsilon_6)R^2 \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|f|^2|D\widetilde{G}|^2}{\widetilde{G}}\,dX + \varepsilon_6 \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2\widetilde{G}}{R^2}\,dX.$$

Looking at the first term in (3.15), from Lemma 2.5 (vi), we know that $\|D\widetilde{G}\|_{L^\tau_*} \leq C(n,\lambda,\Lambda)$ with $\tau = (n+2)/(n+1)$.

$$\begin{aligned}
(3.16)\quad & R^2 \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|f|^2|D\widetilde{G}|^2}{\widetilde{G}}\,dX \\
&\leq CR^2 \int\limits_{P_{2R}\setminus P_{R/2}} \delta(X,X_0)^n|f|^2|D\widetilde{G}|^2 dX \\
&\leq CR^{n+2} \int\limits_{P_{2R}\setminus P_{R/2}} |f|^2|D\widetilde{G}|^2 dX \\
&\leq CR^{n+2}\left(\int\limits_{P_{2R}\setminus P_{R/2}} |D\widetilde{G}|^{2\delta}dX\right)^{1/\delta}\left(\int\limits_{P_{2R}\setminus P_{R/2}} |f|^{2\delta'}dX\right)^{1/\delta'} \\
&\leq CR^{n+2}|P_{2R}|^{(1/\delta-2/\tau)}\|D\widetilde{G}\|_{L^\tau_*}|P_{2R}|^{(1/\delta'-2/s)}\|f\|^2_{L^s} \\
&\leq CR^{n+2}|P_{2R}|^{(1-2/\tau-2/s)}\|D\widetilde{G}\|_{L^\tau_*}\|f\|^2_{L^s} \\
&\leq CR^{n+2+(n+2)(1-2/\tau-2/s)},
\end{aligned}$$

where δ' is the Hölder conjugate number of δ with $1 \leq \delta < (n+2)/(n+1)$. Since $s > n+2$, direct calculation gives

$$n+2+(n+2)\left(1-\frac{2}{\tau}-\frac{2}{s}\right) > 0.$$

With the same argument as for B_{12}, we have

$$\begin{aligned}
(3.17)\quad & \varepsilon_6 \int\limits_{P_{2R}\setminus P_{R/2}} \frac{|u-u_R|^2\widetilde{G}}{R^2}\,dX \leq \frac{C\varepsilon_6}{R^n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX \\
&+ C\varepsilon_6 R^{4-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} + C\varepsilon_6 R^{2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.
\end{aligned}$$

Hence, combining (3.15), (3.16) and (3.17), we obtain

$$\mathsf{B}_{32} \le C\varepsilon_6 R^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{n+2+(n+2)(1-(2/\tau)-(2/s))} + C\varepsilon_6 R^{4-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} + C\varepsilon_6 R^{2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.$$

This estimate, together with the estimate of B_{31}, implies

$$\begin{aligned}\mathsf{B}_3 \le{}& CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX \\ &+ CR^{(n+2)(1-1/\kappa-2/s)} + CR^{n+2+(n+2)(1-2/\tau-2/s)} \\ &+ CR^{4-(2n+4)/q}\|g\|^2_{L^q(P_{2R})} + CR^{2-(2n+4)/s}\|f\|^2_{L^s(P_{2R})}.\end{aligned}$$

Now, combining the estimates of B_1, B_2 and B_3, we obtain

$$|\mathrm{II}| \le CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{\alpha_0}, \tag{3.18}$$

where

$$\begin{aligned}\alpha_0 = \min\Bigg\{&2 + (n+2)\left(1 - \frac{1}{\kappa} - \frac{2}{q}\right), (n+2)\left(1 - \frac{1}{\kappa} - \frac{2}{s}\right), \\ &n+2+(n+2)\left(1 - \frac{2}{\tau} - \frac{2}{s}\right), 4 - \frac{2n+4}{q}, 2 - \frac{2n+4}{s}\Bigg\} \\ &= \min\left\{2 - \frac{2(n+2)}{s}, 4 - \frac{2(n+2)}{q}\right\} \in (0,2).\end{aligned}$$

Combining (3.2), (3.12) and (3.18), it follows that

$$\lambda \int\limits_Q \widetilde{G}\eta^2 |Du|^2 dX \le CR^{-n} \int\limits_{P_{3R}\setminus P_{R/2}} |Du|^2 dX + CR^{\alpha_0}.$$

In accordance with Lemma 2.5 (ii), we have

$$\begin{aligned}&\lambda \int\limits_Q \widetilde{G}\eta^2 |Du|^2 dX \\ &\quad\ge \lambda \int\limits_{P_{R/2}} |Du|^2 \frac{1}{C_1(t-t_0)^{n/2}} e^{-C_2|X-X_0|^2/t-t_0} dX\end{aligned}$$

$$
\begin{aligned}
(3.19) \qquad &= \lambda \sum_{j=1}^{\infty} \int_{P_{R/2^j} \setminus P_{R/2^{j+1}}} |Du|^2 \frac{1}{C_1 (t-t_0)^{n/2}} e^{-C_2|X-X_0|^2/t-t_0} dX \\
&\geq \lambda \sum_{j=1}^{\infty} \int_{P_{R/2^j} \setminus P_{R/2^{j+1}}} |Du|^2 \frac{2^{nj}}{C_1 R^n} e^{-C_2(R/2^j)^2/(R/2^{j+1})^2} dX \\
&= \frac{\lambda e^{-4C_2}}{C_1 R^n} \sum_{j=1}^{\infty} 2^{nj} \int_{P_{R/2^j} \setminus P_{R/2^{j+1}}} |Du|^2 dX \\
&\geq \frac{\lambda e^{-4C_2}}{C_1 R^n} \sum_{j=1}^{\infty} \int_{P_{R/2^j} \setminus P_{R/2^{j+1}}} |Du|^2 dX \\
&= \frac{\lambda e^{-4C_2}}{C_1 R^n} \int_{P_{R/2}} |Du|^2 dX.
\end{aligned}
$$

Thus, there is a basic estimate for positive constants K_0 and C, which only depends upon n, λ and Λ such that

$$
\begin{aligned}
R^{-n} \int_{P_{R/2}} |Du|^2 dX &\leq K_0 R^{-n} \int_{P_{3R} \setminus P_{R/2}} |Du|^2 dX + CR^{\alpha_0} \\
&\leq K_0 R^{-n} \int_{P_{3R}} |Du|^2 dX - K_0 R^{-n} \int_{P_{R/2}} |Du|^2 dX + CR^{\alpha_0},
\end{aligned}
$$

that is,

$$
R^{-n} \int_{P_{R/2}} |Du|^2 dX \leq \left(\frac{K_0}{K_0+1} \right) R^{-n} \int_{P_{3R}} |Du|^2 dX + CR^{\alpha_0}, \quad \alpha_0 > 0.
$$

Since $K_0/(K_0+1) < 1$, Lemma 2.8 implies

$$
(3.20) \qquad \int_{P_R} |Du|^2 dX \leq CR^{n+\alpha_0}.
$$

By virtue of the hole-filling technique **[11]**, we conclude that $Du \in L^{2,\lambda}_{\mathrm{loc}}(Q, \mathbb{R}^n)$ for every $0 < \lambda \leq n + \alpha_0$, and this completes the proof of Theorem 1.1. □

The following is a parabolic version of the Morrey lemma, and Corollary 1.2 follows as a direct consequence of Theorem 1.1.

Lemma 3.1 (Morrey lemma **[14]**). *Suppose that* $u \in W^{1,1}_{p,\mathrm{loc}}(Q)$ *with* $Q \subset \mathbb{R}^{n+1}$ *satisfies the following inequality. There exist a constant* $M > 0$ *and some* $\beta \in (0,1)$ *such that*

$$\int_{P_R} |Du|^p dx \leq M^p R^{n+2-p+p\beta},$$

for any $P_R \subset Q$. *Then,* $u \in C^{\beta,\beta/2}_{x,t}(Q, \mathrm{loc})$. *Moreover, for any* $Q' \subset\subset Q$ *the following holds*

$$\sup_{Q'} |u| + \sup_{\substack{X \\ Y \in Q' \\ X \neq Y}} \frac{|u(X) - u(Y)|}{\delta(X,Y)^n} \leq C(M + \|u\|_{L^p(Q)}),$$

where $M = \sup_Q |u|$ *and* $C = C(n, \beta, Q', Q) > 0$.

Proof of Corollary 1.2. Since (3.20) holds, we can easily derive that

$$\int_{P_R} |Du|^2 dX \leq CR^{n+\alpha_0}.$$

Therefore, following from Lemma 3.1, the proof of Corollary 1.2 is complete. □

4. Conclusions. In this paper, we first reviewed some natural growth properties of Green's functions to linear parabolic operator (1.1), including the estimate of the derivative with respect to the time variation t. Then, as an application of these estimates, we derived a local regularity in Morrey spaces for the weak solution of equation (1.4) by employing Green's functions as a part of test functions and the hole-filling technique. We also gave an alternative proof of a locally Hölder continuity with optimal Hölder exponent to the weak solution of linear parabolic equations with time-independent coefficients.

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