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ZEROS OF RANDOM ORTHOGONAL POLYNOMIALS WITH COMPLEX GAUSSIAN COEFFICIENTS

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ABSTRACT. Let $\{f_j\}_{j=0}^n$ be a sequence of orthonormal polynomials where the orthogonality relation is satisfied on either the real line or on the unit circle. We study zero distribution of random linear combinations of the form

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z),$$

where $\eta_0, \dots, \eta_n$ are complex-valued iid standard Gaussian random variables. Using the Christoffel-Darboux formula, the density function for the expected number of zeros of P_n in these cases takes a very simple shape. From these expressions, under the mere assumption that the orthogonal polynomials are from the Nevai class, we give the limiting value of the density function away from their respective sets where the orthogonality holds. In the case when $\{f_j\}$ are orthogonal polynomials on the unit circle, the density function shows that the expected number of zeros of P_n are clustering near the unit circle. To quantify this phenomenon, we give a result that estimates the expected number of complex zeros of P_n in shrinking neighborhoods of compact subsets of the unit circle.

1. Introduction. The study of the expected number of real zeros of polynomials $P_n(z) = \sum_{j=0}^n \eta_j z^j$ with random coefficients, called random algebraic polynomials, dates back to the 1930s. In 1932, Bloch and Pólya [**3**] showed that, when $\{\eta_j\}$ are independent and identically distributed (iid) random variables that take values from the set $\{-1, 0, 1\}$ with equal probabilities, the expected number of real zeros

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is $O(\sqrt{n})$. Other early advancements in the subject were later made by Littlewood and Offord [**25**], Kac [**21, 22**], Rice [**30**], Erdős and Offord [**11**], and many others. For a relevant history of the early progress in this topic, we refer the reader to the books by Bharucha-Reid and Sambandham [**2**] and by Farahmand [**14**].

It is common to refer to the density function for the expected number of zeros of a random polynomial as the intensity function or the first correlation function. In 1943, Kac [**21**] gave a formula for the intensity function of the expected number of real zeros of $P_n(z)$ when $\{\eta_j\}$ are real-valued iid normal Gaussian coefficients. Using that formula, he was able to show that the expected number of real roots of the random algebraic polynomial is asymptotic to $2\pi^{-1}\log n$ as $n \to \infty$. The error term in his asymptotic was further sharpened by Hammersley [**18**], Wang [**39**], Edelman and Kostlan [**10**] and Wilkins [**40**].

Remaining with the case when $\{\eta_j\}$ are real-valued iid normal Gaussian random variables, Shepp and Vanderbei gave a formula for the intensity function for the expected number of complex zeros of the random algebraic polynomial P_n in 1995. They were also able to obtain a limit of the intensity function as $n \to \infty$. Generalizations to other types of real-valued random variables and to other random polynomials with basis functions different than the monomials were made by Ibragimov and Zeitouni [**20**], Feildheim [**17**] and Vanderbei [**38**].

In 1996, Farahmand [**13**] produced a formula for the intensity function for a random algebraic polynomial when the random coefficients are complex-valued iid standard Gaussian random variables. As an application, Farahmand considered the spanning functions of the random polynomial to be cosine functions. For extensions of Faramand's result, we refer the reader to the works by Farahmand [**12**], Farahmand and Grigorash [**16**] and Farahmand and Jahangiri [**15**].

We will study a case of the expectation of the number zeros of random polynomials of the form

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z), \quad z \in \mathbb{C}, \tag{1.1}$$

where n is a fixed integer, $\{f_j\}_{j=0}^n$ are entire functions real-valued on the real line, $\eta_j = \alpha_j + i\beta_j$, $j = 0, 1, \ldots, n$, with $\{\alpha_j\}_{j=0}^n$ and $\{\beta_j\}_{j=0}^n$ being sequences of iid standard Gaussian random variables. The formula for the intensity function associated to P_n is expressed in terms of the kernels

$$K_n(z,w) = \sum_{j=0}^{n} f_j(z)\overline{f_j(w)}, \tag{1.2}$$

$$K_n^{(0,1)}(z,w) = \sum_{j=0}^{n} f_j(z)\overline{f_j'(w)},$$

and

$$K_n^{(1,1)}(z,w) = \sum_{j=0}^{n} f_j'(z)\overline{f_j'(w)}. \tag{1.3}$$

We note that, since the functions $f_j(z)$ are entire functions that are real-valued on the real line, by the Schwarz reflection principle, we have $\overline{f_j(z)} = f_j(\overline{z})$ for all $j = 0, 1, \dots, n$, and all $z \in \mathbb{C}$.

Let $N_n(\Omega)$ denote the (random) number of zeros of $P_n(z)$ as defined by (1.1) in a Jordan region Ω of the complex plane, and $\mathbb{E}$ denote the mathematical expectation. Due to Edelman and Kostlan **[10]** (with different proofs later given by Hough, et al., in **[19]**, Feldheim **[17]**, the author **[42]**, and Ledoan **[23]**) it is known that, for each Jordan region $\Omega \subset \{z \in \mathbb{C} : K_n(z,z) \neq 0\}$, we have that the intensity function ρ_n associated to P_n satisfies:

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_n(x,y)\,dx\,dy,$$

with

$$\rho_n(x,y) = \rho_n(z) = \frac{K_n^{(1,1)}(z,z)K_n(z,z) - |K_n^{(0,1)}(z,z)|^2}{\pi(K_n(z,z))^2}, \tag{1.4}$$

where the kernels $K_n(z,z)$, $K_n^{(0,1)}(z,z)$ and $K_n^{(1,1)}(z,z)$ are defined in (1.2) and (1.3). We note that, since all of the functions that make up ρ_n are real valued, the function ρ_n is real valued. The function ρ_n is, also, in fact nonnegative. Furthermore, for $(a,b) \subset \mathbb{R}$, it is known that $\mathbb{E}[N_n(a,b)] = 0$, so that ρ_n has no mass on the real line.

In the following results, we will be considering the case when the spanning functions $\{f_j\}$ of (1.1) are polynomials either orthogonal on the real line (OPRL), or polynomials orthogonal on the unit circle (OPUC). We say that a collection of polynomials $\{p_j\}_{j\geq 0}$ are *orthogonal* *on the real line* with respect to μ, with $\operatorname{supp}\mu \subseteq \mathbb{R}$, if

$$\int p_n(x)p_m(x)d\mu(x) = \delta_{nm} \quad \text{for all } n, m \in \mathbb{N} \cup \{0\}.$$

We note that, when polynomials are orthogonal on the real line, they have real coefficients, and thus, are real-valued on the real line.

As mentioned above, our other choice of the basis of the random sum P_n will be from OPUC. These are orthogonal polynomials $\{\varphi_j\}_{j\geq 0}$, defined by a probability Borel measure μ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\, d\mu(e^{i\theta}) = \delta_{nm} \quad \text{for all } n, m \in \mathbb{N} \cup \{0\}.$$

When we restrict μ to be symmetric with respect to conjugation, the sequence $\{\varphi_j\}$ of OPUC will have real coefficients and, consequently, be real-valued on the real line.

For analogs of our results concerning random linear combinations of OPRL or OPUC with the random coefficients $\{\eta_j\}$ of P_n being real-valued standard iid Gaussian, the reader is referred to the research of Das [**8**], Das and Bhatt [**9**], Lubinsky, Pritsker and Xie [**26**], [**27,** Theorems 2.2, 2.3], and Yattselev and the author [**41**].

We note that there has also been work done in the higher-dimensional analogs of the settings mentioned (cf., Shiffman and Zelditch [**32**]–[**34**], Bloom [**4, 5**], Bloom and Shiffman [**7**], Bloom and Levenberg [**6**] and Bayraktar [**1**]).

Using the Christoffel-Darboux formula, we show that the intensity function from (1.4) greatly simplifies when the spanning functions are orthogonal polynomials.

Theorem 1.1. *Let $P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z)$, where $\{\eta_j\}_{j=0}^{n}$ are complex-valued* iid *standard Gaussian random variables, and $\{f_j\}_{j=0}^{n}$ are orthogonal polynomials. Let ρ_n be defined as in* (1.4)*.*

(i) *When $f_j = p_j$, $j = 0, \ldots, n$, where the p_j's are* OPRL*, the intensity function ρ_n simplifies as*

$$\rho_n(z) = \frac{1 - h_n(z)^2}{4\pi(\operatorname{Im}(z))^2}, \qquad h_n(z) = \frac{\operatorname{Im}(z)|a_n'(z)|}{\operatorname{Im}(a_n(z))}, \qquad a_n(z) = \frac{p_{n+1}(z)}{p_n(z)}, \tag{1.5}$$

for $z \in \mathbb{C}$.

(ii) *Let* $f_j = \varphi_j$, $j = 0, \ldots, n$, *where the* φ_j*'s are* OPUC *associated to the conjugate-symmetric measure* μ. *When* $|z| \neq 1$, *the intensity function* ρ_n *reduces to*

$$\rho_n(z) = \frac{1-|k_n(z)|^2}{\pi(1-|z|^2)^2}, \qquad k_n(z) = \frac{(1-|z|^2)b_n'(z)}{1-|b_n(z)|^2}, \qquad b_n(z) = \frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)}, \tag{1.6}$$

where $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\overline{z})}$.

We note that

$$\mathrm{Im}(a_n(z)) = 0 \iff a_n(z) = \overline{a_n(z)} = a_n(\overline{z}) \iff z \in \mathbb{R}.$$

Thus, as written in the shape above (written as such for purposes of the computation of the limit as $n \to \infty$), the intensity function ρ_n in (1.5) has singularities on the real axis due to $\mathrm{Im}(z)$ and $\mathrm{Im}(a_n(z))$ in the denominators. However, these singularities exist only due to the way the intensity function is written. For, in the form of ρ_n at (1.4), the only potential singularity can come from $K_n(z,z) = 0$. In the case of $f_j = p_j$, $j = 0, 1, \ldots, n$, with $\{p_j\}$ being OPRL, it follows that $p_0(z) \neq 0$ gives $K_n(z,z) = \sum_{j=0}^n |p_j(z)|^2 > 0$. Thus, the intensity function (1.5) is well defined and continuous everywhere on $\mathbb{C}$.

The restriction $|z| \neq 1$ in (1.6) of Theorem 1.1 is present due to the use, and hence, assumptions of the Christoffel-Darboux formula for OPUC. This restriction is from the fact that only when $|z| = 1$ do we have $|\varphi_{n+1}^*(z)| = |\varphi_{n+1}(z)|$ (i.e., $|b_n(z)| = 1$). Furthermore, it is known that all of the zeros of $\varphi_{n+1}(z)$ lie in $\mathbb{D}$, and all the zeros of $\varphi_{n+1}^*(z)$ are outside of $\mathbb{D}$. Thus, these two polynomials cannot simultaneously vanish.

Our limiting results of ρ_n will be phrased in terms of assumptions on the recurrence coefficients of the orthogonal polynomials. For a sequence $\{p_n\}$ of OPRL, the three term recurrence relation [**36,** Theorem 3.2.1] states

$$xp_n(z) = a_n p_{n+1}(z) + b_n p_n(z) + a_{n-1}p_{n-1}(z), \quad n = 1, 2, \ldots, \tag{1.7}$$

where the recurrence coefficient sequences $\{a_n\}$ and $\{b_n\}$ can be explicitly given in terms of the leading coefficient of p_n and p_{n-1}. Due to Nevai [**28,** Theorem 13], (see also Totik [**37**]), the condition that

$a_n \to a$ and $b_n \to b$ as $n \to \infty$, with $a \geq 0$ and $b \in \mathbb{R}$, is equivalent to

$$\lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)} = \frac{z - b + \sqrt{(z-b)^2 - 4a^2}}{2}, \tag{1.8}$$

with the convergence being locally uniformly valid for $z \notin \operatorname{supp} \mu$. When (1.8) holds for a sequence $\{p_n\}$ of OPRL, we say that the sequence is in the Nevai class. We note that this class is sometimes denoted as $M(a, b)$.

The three term recurrence relation [**35,** Theorem 1.5.4] for a sequence $\{\varphi_n\}$ of OPUC states

$$\varphi_{n+1}(z) = \frac{z\varphi_n(z) - \overline{\alpha}_n \varphi_n^*(z)}{\sqrt{1 - |\alpha_n|^2}}, \quad n = 0, 1, \ldots, \tag{1.9}$$

where the sequence of recurrence coefficients is $\{\alpha_n\} \subset \mathbb{D}$, and $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\overline{z})}$. From the recurrence relation, it can be seen that, when $\{\alpha_n\} \subset (-1, 1)$, the sequence $\{\varphi_n\}$ will be real-valued on the real line. Furthermore, in this case, it is known that there exists a unique conjugate-symmetric probability measure μ whose associated orthogonal polynomials satisfy (1.9) [**35,** Theorem 1.7.11]. Hence, we can refer to sequence $\{\varphi_n\}$ of OPUC as defined by either the measure μ or the recurrence coefficients $\{\alpha_n\}$. The ratio asymptotics [**35,** Theorem 1.7.4] in this case are:

$$\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi_n^*(z)} = 0, \tag{1.10}$$

where the convergence holds locally uniformly for $z \in \mathbb{D}$. When (1.10) holds for a sequence $\{\varphi_n\}$ of OPUC, we say that the sequence is from the Nevai class.

Corollary 1.2. *Let $P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z)$, where $\{\eta_j\}_{j=0}^n$ are complex-valued* iid *standard Gaussian random variables, and $\{f_j\}_{j=0}^n$ are orthogonal polynomials.*

(i) *When $\{p_j\}$ are* OPRL *from the Nevai class, the intensity function ρ_n from* (1.5) *for the random orthogonal polynomial satisfies*

$$\lim_{n\to\infty} \rho_n(z) = \frac{1}{4\pi(\operatorname{Im}(z))^2} - \frac{|z-b+\sqrt{(z-b)^2-4a^2}|^2}{4\pi|(z-b)^2-4a^2|(\operatorname{Im}(z+\sqrt{(z-b)^2-4a^2}\ ))^2}, \tag{1.11}$$

locally uniformly for all $z \notin \operatorname{supp} \mu$.

(ii) *Let* $\{\varphi_j\}$ *be* OPUC *from the Nevai class such that the associated recurrence coefficients satisfy* $\{\alpha_j\} \subset (-1,1)$. *Then, the intensity function* ρ_n *in* (1.6) *for the random orthogonal polynomial possesses*

$$\lim_{n\to\infty} \rho_n(z) = \frac{1}{\pi(1-|z|^2)^2}, \tag{1.12}$$

locally uniformly for all $z \in \mathbb{C} \setminus \mathbb{T}$.

When $a = 1/2$ and $b = 0$ in the definition of the Nevai class for the OPRL (1.8), it is known that this class contains the Chebyshev polynomials. The result of (1.11) extends the limiting value given by Farahmand and Grigorash **[15,** Section 4] in which the spanning functions of their random trigonometric polynomial can be modified to be the Chebyshev polynomials. We note that the result of (1.12) extends the limiting value of the first correlation function given by Peres and Virág **[29]** (i.e., taking $n = 1$ of their Theorem 1) when the spanning functions were the monomials to that of a very general basis of OPUC. The result further extends their work in that this limiting value also holds for the exterior of the unit circle.

From (1.6) of Theorem 1.1 and (1.12) of Corollary 1.2 we see that the intensity function and its limiting value for the random orthogonal polynomial spanned by OPUC is singular on the unit circle. Assuming a little more on the measure μ associated to the OPUC, we can quantify how the zeros approach the unit circle.

Theorem 1.3. *Let* $P_n(z) = \sum_{j=0}^n \eta_j \varphi_j(z)$, *where* $\{\eta_j\}$ *are complex-valued* iid *standard Gaussian random variables, and* $\{\varphi_j\}$ *are* OPUC *such that their associated recurrence coefficients satisfy* $\{\alpha_j\} \subset (-1,1)$ *with* $\alpha_j \to 0$ *as* $j \to \infty$. *Let* S *be a compact subset of* $\mathbb{T} \setminus \{\pm 1\}$. *Assume, in addition, that the measure* μ *associated to the sequence* $\{\varphi_j\}$ *is absolutely continuous with respect to the arclength measure on an open set containing* S, *and its Radon-Nikodym derivative is positive and continuous at each point of* S. *Given* $-\infty < \tau_1 < \tau_2 < \infty$, *it follows that*

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[N(\Omega(S,\tau_1,\tau_2))] = \frac{|S|}{2\pi}\left(\frac{H'(\tau_2)}{H(\tau_2)} - \frac{H'(\tau_1)}{H(\tau_1)}\right), \tag{1.13}$$

where

$$\Omega(S,\tau_1,\tau_2) := \left\{ rz : z\in S,\, r \in \left(1+\frac{\tau_1}{2n}, 1+\frac{\tau_2}{2n}\right)\right\}$$

and

$$H(\tau) := \frac{e^{\tau}-1}{\tau}.$$

We note that H'/H is increasing on the real line with

$$\lim_{\tau\to-\infty} \frac{H'(\tau)}{H(\tau)} = 0 \quad \text{and} \quad \frac{H'(\tau)}{H(\tau)} = 1 - \frac{H'(-\tau)}{H(-\tau)}.$$

Thus, in the setting of Theorem 1.3, the zeros of a random orthogonal polynomial spanned by OPUC approaching S are expected to be contained in an annular band around S of width $n^{-1+\epsilon}$ for any $\epsilon > 0$.

We note that, when the coefficients of the random orthogonal polynomial spanned by OPUC satisfying the conditions of Theorem 1.3 are real-valued iid standard Gaussian random variables, the analog of the above result was recently proven by Yattselev and the author (cf., [**41,** Theorem 1.7]). Remarkably, both of the cases of random orthogonal polynomials with real-valued or complex-valued coefficients yield the same asymptotics in (1.13).

2. Proofs.

2.1. The intensity function for random orthogonal polynomials. In this section, we use the Christoffel-Darboux formula for OPRL and OPUC to simplify the kernels $K_n(z,z)$, $K_n^{(0,1)}(z,z)$ and $K_n^{(1,1)}(z,z)$, which make up the intensity function ρ_n from (1.4). For the convenience of the reader, we state the Christoffel-Darboux formula for OPRL [**36,** Theorem 3.2.2] for $z,w\in\mathbb{C}$ where $z\neq w$, and $\{p_j\}_{j\geq 0}$ OPRL, with k_j being the leading coefficient of p_j, we have

$$\sum_{j=0}^{n} p_j(z)p_j(w) = \frac{k_n}{k_{n+1}}\cdot\frac{p_{n+1}(z)p_n(w)-p_n(z)p_{n+1}(w)}{z-w}, \quad z\neq w. \tag{2.1}$$

Furthermore, on the diagonal $z=w$, the kernel takes the form

$$\sum_{j=0}^{n} (p_j(z))^2 = \frac{k_n}{k_{n+1}}\cdot (p'_{n+1}(z)p_n(z) - p'_n(z)p_{n+1}(z)). \tag{2.2}$$

For a collection of OPUC $\{\varphi_j\}_{j\geq 0}$, the Christoffel-Darboux formula for OPUC [**35,** Theorem 2.2.7] states that, for $z, w \in \mathbb{C}$ with $\overline{w}z \neq 1$, we have

$$\sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j(w)} = \frac{\overline{\varphi^*_{n+1}(w)}\varphi^*_{n+1}(z) - \overline{\varphi_{n+1}(w)}\varphi_{n+1}(z)}{1-\overline{w}z}, \tag{2.3}$$

where $\varphi^*_n(z) = z^n\overline{\varphi_n(1/\overline{z})}$.

Before obtaining our representations of the kernels, let us note that, since the polynomials $\{p_j\}$ are orthogonal on the real line, and since we are assuming that the recurrence coefficients $\{\alpha_j\}$ associated to $\{\varphi_j\}$ satisfy $\{\alpha_j\} \subset (-1,1)$, both classes of orthogonal polynomials have real coefficients. Thus, when using conjugation, we have that $\overline{p_j(z)} = p_j(\overline{z})$ and $\overline{\varphi_j(z)} = \varphi_j(\overline{z})$ for all $j = 0, 1, \ldots$, and all $z \in \mathbb{C}$.

Proof of Theorem 1.1 (1.5). For $z \neq w$, taking derivatives of (2.1) yields

$$\begin{aligned}\sum_{j=0}^{n} p_j(z)p_j'(w) &= \frac{k_n}{k_{n+1}}\left(\frac{p_{n+1}(z)p_n'(w)-p_n(z)p_{n+1}'(w)}{z-w} + \frac{p_{n+1}(z)p_n(w)-p_n(z)p_{n+1}(w)}{(z-w)^2}\right)\\ &= \frac{k_n}{k_{n+1}}\cdot\frac{p_{n+1}(z)p_n'(w)-p_n(z)p_{n+1}'(w)}{z-w} + \frac{\sum_{j=0}^{n} p_j(z)p_j(w)}{z-w},\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\sum_{j=0}^{n} p_j'(z)p_j'(w) = \frac{k_n}{k_{n+1}}\Bigg(&\frac{p_{n+1}'(z)p_n'(w)-p_n'(z)p_{n+1}'(w)}{z-w} - \frac{p_{n+1}(z)p_n'(w)-p_n(z)p_{n+1}'(w)}{(z-w)^2}\\ &+ \frac{p_{n+1}'(z)p_n(w)-p_n'(z)p_{n+1}(w)}{(z-w)^2}\\ &- \frac{2(p_{n+1}(z)p_n(w)-p_n(z)p_{n+1}(w))}{(z-w)^2}\Bigg)\end{aligned} \tag{2.5}$$

$$= \frac{k_n}{k_{n+1}} \cdot \frac{p'_{n+1}(z)p'_n(w) - p'_n(z)p'_{n+1}(w)}{z-w} - \frac{\sum_{j=0}^n p_j(z)p'_j(w)}{z-w} + \frac{\sum_{j=0}^n p'_j(z)p_j(w)}{z-w}.$$

Setting $w = \overline{z}$ in (2.1), (2.4) and (2.5), since the coefficients of $\{p_j\}$ are real, it follows that

$$K_n(z,z) = \sum_{j=0}^n p_j(z)\overline{p_j(z)} = \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}(z)p_n(\overline{z}) - p_n(z)p_{n+1}(\overline{z})}{2i\mathrm{Im}(z)}, \tag{2.6}$$

$$\begin{aligned} K_n^{(0,1)}(z,z) &= \sum_{j=0}^n p_j(z)\overline{p'_j(z)} \\ &= \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}(z)p'_n(\overline{z}) - p_n(z)p'_{n+1}(\overline{z})}{2i\mathrm{Im}(z)} + \frac{K_n(z,z)}{2i\mathrm{Im}(z)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} K_n^{(1,1)}(z,z) &= \sum_{j=0}^n p'_j(z)\overline{p'_j(z)} \\ &= \frac{k_n}{k_{n+1}} \cdot \frac{\mathrm{Im}(p'_{n+1}(z)p'_n(\overline{z}))}{\mathrm{Im}(z)} - \frac{K_n^{(0,1)}(z,z)}{2i\mathrm{Im}(z)} + \frac{\overline{K_n^{(0,1)}(z,z)}}{2i\mathrm{Im}(z)}. \end{aligned} \tag{2.8}$$

For our representation of $K_n(z,\overline{z})$, we simply use (2.2), and again that the coefficients of $\{p_j\}$ are real, to achieve

$$\begin{aligned} K_n(z,\overline{z}) &= \sum_{j=0}^n p_j(z)\overline{p_j(\overline{z})} = \sum_{j=0}^n p_j(z)p_j(z) \\ &= \frac{k_n}{k_{n+1}}(p'_{n+1}(z)p_n(z) - p'_n(z)p_{n+1}(z)). \end{aligned} \tag{2.9}$$

Using our derived expressions (2.6), (2.7), (2.8) and (2.9), the numerator of the intensity function ρ_n from (1.4) simplifies as

$$K_n^{(1,1)}(z,z)K_n(z,z) - |K_n^{(0,1)}(z,z)|^2 = \frac{(K_n(z,z))^2 - |K_n(z,\overline{z})|^2}{4(\mathrm{Im}(z))^2}.$$

Therefore, using the expression for the numerator above and recalling relations (2.6) and (2.9), we see that the intensity function given by (1.4) is

$$
\begin{aligned}
\rho_n(z) &= \frac{K_n^{(1,1)}(z,z)K_n(z,z) - |K_n^{(0,1)}(z,z)|^2}{\pi(K_n(z,z))^2} \\
&= \frac{1}{4\pi(\mathrm{Im}(z))^2}\left(1 - \frac{|K_n(z,\overline{z})|^2}{(K_n(z,z))^2}\right) \\
&= \frac{1}{4\pi(\mathrm{Im}(z))^2}\left(1 - \frac{(2i\mathrm{Im}(z))^2|p'_{n+1}(z)p_n(z) - p'_n(z)p_{n+1}(z)|^2}{(p_{n+1}(z)p_n(\overline{z}) - p_n(z)p_{n+1}(\overline{z}))^2}\right) \\
&= \frac{1}{4\pi(\mathrm{Im}(z))^2}\left(1 - \frac{(2i\mathrm{Im}(z))^2|(p_{n+1}(z)/p_n(z))'|^2}{((p_{n+1}(z)/p_n(z)) - (p_{n+1}(\overline{z})/p_n(\overline{z})))^2}\right) \\
&= \frac{1 - h_n(z)^2}{4\pi(\mathrm{Im}(z))^2},
\end{aligned}
$$

where

$$
h_n(z) = \frac{\mathrm{Im}(z)|a'_n(z)|}{\mathrm{Im}(a_n(z))}, \qquad a_n(z) = \frac{p_{n+1}(z)}{p_n(z)},
$$

which gives the result of (1.5) in Theorem 1.1. □

Proof of Theorem 1.1 (1.6). Applying the Christoffel-Darboux formula for OPUC from (2.3), and making derivations analogously as was done for the kernels for OPRL, our representations of $K_n(z,z)$, $K_n^{(0,1)}(z,z)$ and $K_n^{(1,1)}(z,z)$ are as follows:

$$
K_n(z,z) = \sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j(z)} = \frac{|\varphi^*_{n+1}(z)|^2 - |\varphi_{n+1}(z)|^2}{1-|z|^2}, \tag{2.10}
$$

(2.11)

$$
\begin{aligned}
K_n^{(0,1)}(z,z) &= \sum_{j=0}^{n} \varphi_j(z)\overline{\varphi'_j(z)} \\
&= \frac{\overline{\varphi^{*\,\prime}_{n+1}(z)}\varphi^*_{n+1}(z) - \overline{\varphi'_{n+1}(z)}\varphi_{n+1}(z)}{1-|z|^2} + \frac{zK_n(z,z)}{1-|z|^2},
\end{aligned}
$$

and

$$K_n^{(1,1)}(z,z) = \sum_{j=0}^{n} |\varphi_j'(z)|^2 = \frac{|\varphi_{n+1}^{*\,\prime}(z)|^2 - |\varphi_{n+1}'(z)|^2}{1-|z|^2} + \frac{\overline{z}K_n^{(0,1)}(z,z) + z\overline{K_n^{(0,1)}(z,z)} + K_n(z,z)}{1-|z|^2}. \tag{2.12}$$

Using (2.10), (2.11) and (2.12), the numerator of the intensity function ρ_n of (1.4) reduces to

$$\begin{aligned} &K_n^{(1,1)}(z,z)K_n(z,z) - |K_n^{(0,1)}(z,z)|^2 \\ &\qquad = \frac{(K_n(z,z))^2}{(1-|z|^2)^2} - \frac{|\varphi_{n+1}^*(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)|^2}{(1-|z|^2)^2}. \end{aligned}$$

From the above numerator and (2.10), the intensity function at (1.4) becomes

$$\begin{aligned} \rho_n(z) &= \frac{K_n^{(1,1)}(z,z)K_n(z,z) - |K_n^{(0,1)}(z,z)|^2}{\pi(K_n(z,z))^2} \\ &= \frac{1}{\pi(1-|z|^2)^2}\left(1 - \frac{|\varphi_{n+1}^*(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)|^2}{(K_n(z,z))^2}\right) \\ &= \frac{1}{\pi(1-|z|^2)^2} \\ &\quad \times \left(1 - \frac{(1-|z|^2)^2|\varphi_{n+1}^*(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)|^2}{(|\varphi_{n+1}(z)|^2 - |\varphi_{n+1}^*(z)|^2)^2}\right) \\ &= \frac{1}{\pi(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2|(\varphi_{n+1}(z)/\varphi_{n+1}^*(z))'|^2}{(|\varphi_{n+1}(z)/\varphi_{n+1}^*(z)|^2 - 1)^2}\right) \\ &= \frac{1-|k_n(z)|^2}{\pi(1-|z|^2)^2}, \end{aligned} \tag{2.13}$$

where

$$k_n(z) = \frac{(1-|z|^2)b_n'(z)}{1-|b_n(z)|^2}, \qquad b_n(z) = \frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)},$$

and hence, completes the proof of Theorem 1.1 (1.6). □

2.2. The limiting value of the intensity function for random orthogonal polynomials associated to the Nevai class.

Proof of Corollary 1.2 (1.11). Since the convergence of (1.8) is uniform on compact subsets away from the support of μ, for $z \notin \operatorname{supp} \mu$, we can differentiate to yield

$$\begin{aligned}\lim_{n\to\infty} a_n'(z) &= \lim_{n\to\infty}\left(\frac{p_{n+1}(z)}{p_n(z)}\right)' \\ &= \frac{d}{dz}\left(\frac{z-b+\sqrt{(z-b)^2-4a^2}}{2}\right) = \frac{z-b+\sqrt{(z-b)^2-4a^2}}{2\sqrt{(z-b)^2-4a^2}}.\end{aligned} \tag{2.14}$$

Also, from (1.8), we see that

$$\begin{aligned}\lim_{n\to\infty} \operatorname{Im}(a_n(z)) &= \lim_{n\to\infty}\frac{(p_{n+1}(z)/p_n(z)) - (p_{n+1}(\overline{z})/p_n(\overline{z}))}{2i} \\ &= \frac{z+\sqrt{(z-b)^2-4a^2}-(\overline{z}+\sqrt{(\overline{z}-b)^2-4a^2})}{4i}.\end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) gives

$$\begin{aligned}\lim_{n\to\infty} h_n(z)^2 &= \lim_{n\to\infty}\frac{(\operatorname{Im}(z))^2|a_n'(z)|^2}{(\operatorname{Im}(a_n(z)))^2} \\ &= \frac{(\operatorname{Im}(z))^2|z-b+\sqrt{(z-b)^2-4a^2}|^2}{|(z-b)^2-4a^2|(\operatorname{Im}(z+\sqrt{(z-b)^2-4a^2}\ ))^2}.\end{aligned}$$

Therefore, using the representation of the intensity function in Theorem 1.1 (1.5), from the above limit, we see that

$$\begin{aligned}\lim_{n\to\infty} \rho_n(z) &= \lim_{n\to\infty}\frac{1-h_n^2(z)}{4\pi(\operatorname{Im}(z))^2} \\ &= \frac{1}{4\pi(\operatorname{Im}(z))^2} - \frac{|z-b+\sqrt{(z-b)^2-4a^2}|^2}{4\pi|(z-b)^2-4a^2|(\operatorname{Im}(z+\sqrt{(z-b)^2-4a^2}))^2},\end{aligned}$$

locally uniformly for $z \notin \operatorname{supp} \mu$, and thus, the proof is complete. □

Proof of Corollary 1.2 (1.12). Under the assumption that $\{\varphi_j\}$ are OPUC in the Nevai class, (1.10) gives

$$\lim_{n\to\infty} b_n(z) = \lim_{n\to\infty}\frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)} = 0, \tag{2.16}$$

uniformly on compact subsets of $\mathbb{D}$. Since the convergence is locally uniform in $\mathbb{D}$, within $\mathbb{D}$, we can differentiate to achieve

$$\lim_{n\to\infty} b_n'(z) = \lim_{n\to\infty} \frac{d}{dz}\left(\frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)}\right) = 0. \tag{2.17}$$

Thus, combining (2.16) and (2.17), we see that

$$\lim_{n\to\infty} k_n(z) = \lim_{n\to\infty} \frac{(1-|z|^2)b_n'(z)}{1-|b_n(z)|^2} = 0. \tag{2.18}$$

This gives that the intensity function in Theorem 1.1 represented by (1.6) satisfies

$$\lim_{n\to\infty} \rho_n(z) = \lim_{n\to\infty} \frac{1-|k_n(z)|^2}{\pi(1-|z|^2)^2} = \frac{1}{\pi(1-|z|^2)^2}$$

locally uniformly on $\mathbb{D}$.

To see that the same limit holds in the exterior of the disk, observe that, for $z^{-1} \in \mathbb{D} \setminus \{0\}$,

$$0 = \lim_{n\to\infty} b_n\left(\frac{1}{z}\right) = \lim_{n\to\infty} \frac{\varphi_{n+1}(z^{-1})}{\varphi_{n+1}^*(z^{-1})} = \lim_{n\to\infty} \frac{\varphi_{n+1}^*(z)}{\varphi_{n+1}(z)}, \tag{2.19}$$

where, on the second equality, we have appealed to the hypothesis that the recurrence coefficients for the OPUC are such that $\{\alpha_j\} \subset (-1,1)$ so that the coefficients of $\{\varphi_j\}$ are real.

Note that, from (2.13), we can factor in a different manner to achieve

$$\begin{aligned}\rho_n(z) &= \frac{1}{\pi(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2|\varphi_{n+1}^*(z)\varphi_{n+1}'(z) - {\varphi_{n+1}^*}'(z)\varphi_{n+1}(z)|^2}{(|\varphi_{n+1}(z)|^2 - |\varphi_{n+1}^*(z)|^2)^2}\right)\\ &= \frac{1}{\pi(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2|(\varphi_{n+1}^*(z)/\varphi_{n+1}(z))'|^2}{(|\varphi_{n+1}^*(z)/\varphi_{n+1}(z)|^2 - 1)^2}\right) = \frac{1-|l_n(z)|^2}{\pi(1-|z|^2)^2},\end{aligned}$$

where

$$l_n(z) = \frac{(1-|z|^2)c_n'(z)}{1-|c_n(z)|^2}, \qquad c_n(z) = \frac{\varphi_{n+1}^*(z)}{\varphi_{n+1}(z)}.$$

Using (2.19) and continuing analogously as was done for the case of the unit circle, it follows that $l_n(z) \to 0$ locally uniformly for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{1-|l_n(z)|^2}{(1-|z|^2)^2} = \frac{1}{(1-|z|^2)^2}$$

uniformly on compact subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$, and hence, gives our desired result. $\square$

2.3. Zeros of random orthogonal polynomials spanned by OPUC in shrinking neighborhoods of the unit circle. To prove Theorem 1.3, we will rely on a universality result by Levin and Lubinsky [**24**]. One of the hypotheses of their result requires that the measure μ associated to the OPUC $\{\varphi_j\}$ is regular in the sense of Ullman-Stahl-Totik, that is,

$$\lim_{n\to\infty} \frac{\log|\kappa_n|}{n} = 0, \tag{2.20}$$

where κ_n is the leading coefficient of $\varphi_n(z)$. We note that, using [**35,** equation 1.5.22], it follows that

$$\kappa_n = \prod_{j=0}^{n} (1-\alpha_j^2)^{-1/2}. \tag{2.21}$$

In our hypothesis of Theorem 1.3, since we are assuming that the recurrence coefficients associated to $\{\varphi_j\}$ satisfy $\alpha_j \to 0$ as $j \to \infty$, appealing to (2.21) we see that

$$\lim_{n\to\infty} \frac{\log|\kappa_n|}{n} = \lim_{n\to\infty} \frac{-(1/2)\sum_{j=0}^{n} \log|1-\alpha_j^2|}{n} = 0, \tag{2.22}$$

so that the measure μ is regular in the sense of Ullman, Stahl and Totik. For the convenience of the reader, we will present the result by Levin and Lubinsky:

Theorem 2.1 ([**24,** Theorem 6.3])**.** *Let μ be a finite positive Borel measure on $[-\pi,\pi)$ that is Ullman-Stahl-Totik regular. Let $J \subset (-\pi,\pi)$ be compact and such that μ is absolutely continuous in an open interval containing J. Assume, moreover, that $w = \mu'$ is positive and continuous at each point of J. Then, uniformly for a, b in compact subsets of the plane and $z = e^{i\theta}$, $\theta \in J$, we have*

$$\lim_{n\to\infty} \frac{K_n(z(1+(i2\pi a/n)), z(1+(i2\pi\overline{b}/n)))}{K_n(z,z)} = e^{i\pi(a-b)} \frac{\sin\pi(a-b)}{\pi(a-b)}.$$

Changing the variables in the above by $a = u/(2\pi i)$ and $\overline{b} = \overline{v}/(2\pi i)$, the conclusion of the above result can be restated as

$$\lim_{n\to\infty} \frac{K_n(z(1+(u/n)), z(1+(\overline{v}/n)))}{K_n(z,z)} = \frac{e^{u+v}-1}{u+v} := H(u+v).$$

Proof of Theorem 1.3. It follows from the definition of the intensity function that

$$\begin{aligned}\frac{1}{n}\mathbb{E}[N_n(\Omega(S,\tau_1,\tau_2))] &= \frac{1}{n}\iint_{\Omega(S,\tau_1,\tau_2)} \rho_n(z)\, dA \\ &= \frac{1}{n}\int_S \int_{1+(\tau_1/2n)}^{1+(\tau_2/2n)} \rho_n(zr) r\, dr|dz|,\end{aligned}$$

where, after a change of variables $r = 1 + \tau/(2n)$, the previous integral becomes

$$\frac{1}{2n^2}\int_S \int_{\tau_1}^{\tau_2} \rho_n\left(z\left(1+\frac{\tau}{2n}\right)\right)\left(1+\frac{\tau}{2n}\right) d\tau|dz|.$$

Since

$$\frac{1}{2}\int_S |dz| = \frac{|S|}{2},$$

and, as $n \to \infty$, we have $1 + \tau/(2n) \to 1$ uniformly for τ on compact subsets of the real line. To complete the proof, it suffices to show

$$\lim_{n\to\infty} \frac{1}{n^2}\rho_n\left(z\left(1+\frac{\tau}{2n}\right)\right) = \frac{1}{\pi}\left(\frac{H'(\tau)}{H(\tau)}\right)' \tag{2.23}$$

uniformly for $z \in S$ and τ on compact subsets of the real line.

Under the conditions of the hypothesis, given (2.22), we can use Theorem 2.1 to achieve

$$\lim_{n\to\infty} K_n(z_{n,u}, z_{n,\overline{v}})K_n^{-1}(z,z) = H(u+v) \tag{2.24}$$

uniformly for $z \in S$ and u, v on compact subsets of $\mathbb{C}$, where $z_{n,a} := z(1 + a/n)$. Since the above convergence is uniform for $z \in S$ and u, v on compact subsets of $\mathbb{C}$, we can differentiate to yield

$$\begin{aligned}\lim_{n\to\infty} \frac{(\partial^{i+j}/\partial u^i \partial v^j)K_n(z_{n,u}, z_{n,\overline{v}})}{K_n(z,z)} &= \lim_{n\to\infty} \frac{z^{i-j}}{n^{i+j}} \frac{K_n^{(i,j)}(z_{n,u}, z_{n,\overline{v}})}{K_n(z,z)} \\ &= H^{(i+j)}(u+v), \qquad (2.25)\end{aligned}$$

where we retain the convergence uniformly for $z \in S$ and u, v on compact subsets of $\mathbb{C}$. Therefore, using the representation (1.4) of ρ_n and the two limits (2.24) and (2.25) yields

$$
\begin{aligned}
&\frac{1}{n^2}\rho_n\left(z\left(1+\frac{\tau}{2n}\right)\right)\\
&\quad= \frac{1}{\pi}\,\frac{K_n(z_{n,\tau/2}, z_{n,\overline{\tau}/2})K_n^{(1,1)}(z_{n,\tau/2}, z_{n,\overline{\tau}/2}) - |K_n^{(0,1)}(z_{n,\tau/2}, z_{n,\overline{\tau}/2})|^2}{n^2K_n(z_{n,\tau/2}, z_{n,\overline{\tau}/2})^2}\\
&\quad= \frac{1}{\pi}\,\frac{[K_n(z_{n,\tau/2}, z_{n,\overline{\tau}/2})K_n^{(1,1)}(z_{n,\tau/2}, z_{n,\overline{\tau}/2})]/[n^2K_n(z,z)^2]}{K_n(z_{n,\tau/2}, z_{n,\overline{\tau}/2})^2/K_n(z,z)^2}\\
&\qquad - \frac{|K_n^{(0,1)}(z_{n,\tau/2}, z_{n,\overline{\tau}/2})|^2/[n^2K_n(z,z)^2]}{K_n(z_{n,\tau/2}, z_{n,\overline{\tau}/2})^2/K_n(z,z)^2}\\
&\quad\longrightarrow \frac{1}{\pi}\,\frac{H(\tau)H''(\tau) - H'(\tau)^2}{H(\tau)^2} = \frac{1}{\pi}\left(\frac{H'(\tau)}{H(\tau)}\right)', \quad n \to \infty,
\end{aligned}
$$

and thus, completes the proof. $\square$

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