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ON THE STABILITY OF TWO NEURON POPULATIONS INTERACTING WITH EACH OTHER

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ABSTRACT. In this study, we deal with stability for the linearized neural field model for two neuron populations. We determine the asymptotic stability region by using the D-subdivision method for different delay terms including time delays. Also, we find the number of unstable characteristic exponents for the unstable regions. We observe that the stability region for the model becomes smaller as the delay term τ increases.

1. Introduction. Neural field models are used to model the brain activity at the level of neural tissue. Mathematically, they show the activity of neural populations consisting of infinitely many neurons. Neural field models are represented by using nonlocal integral or integro differential equations. Due to the finite speed of propagation of an action potential and the time for release of the neurotransmitter, delay terms are added in these models.

Studies about the neural field models made by Wilson and Cowan [**13**] and Amari [**1**] have an important role in the literature. Various techniques are used to perform the stability analysis of these models and to study the existence and uniqueness of their solutions. Stability changes of the model are studied by using some numerical methods in [**2, 5**], a center manifold result is given, and the effect of the delay is considered on the qualitative changes for the model in [**11**], while the effect of an added delay term is studied in [**3, 4, 6, 9, 10, 12**].

In this study, we are interested in the stability of a neural field model for two neuron populations. We find the asymptotic stability region for this model. We also investigate the change on stability for different delay terms τ.

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2. Stability analysis. The neural field model for the p neural population on the space $\Omega \subset R^d$, which presents the dynamics of mean membrane potential, is given in **[11, 12]** as

(2.1)
$$\left(\frac{d}{dt}+l_i\right)V_i(t,r)=\sum_{j=1}^{p}\int_{\Omega}J_{ij}(r,\overline{r})S[\sigma_j(V_j(t-\tau_{ij}(r,\overline{r}),\overline{r})-h_j)]\,d\overline{r}$$
$$+I_i^{\text{ext}}(r,t),\qquad t\geq 0,\ 1\leq i\leq p,$$
$$V_i(t,r)=\phi_i(t,r),\quad t\in[-\tau_{\max},0].$$

Here, we study the model for two neuron populations ($p = 2$), which is defined on a finite piece of cortex $\Omega \subset R$. We consider the case when nearby neurons inhibit each other (recurrent inhibition) while more distant neurons excite each other (lateral excitation), i.e., $J_{11}(x,y) = J_{22}(x,y) = 0$. For this model,

$$x,y\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right],$$

and the boundary conditions are periodic. The functions $V_1(x,t)$ and $V_2(x,t)$ describe the synaptic inputs for a large group of neurons at position x and time t, and the time derivatives of these functions are given by

$$\frac{d}{dt}V_1(x,t)\quad\text{and}\quad\frac{d}{dt}V_2(x,t).$$

The function S is a sigmoid function, which is a differentiable and monotonic activation function, playing an important role in the neural field models. The synaptic connectivity function $J_{ij}(x,y)$ is an even function which is π periodic. Here, $J_{12}(x,y)$ describes how neurons in the second neural population at position y affect the neurons in the first population at position x, and $J_{21}(x,y)$ describes how neurons in the first neural population at position y affect the neurons in the second population at position x. The stability of the solutions of this model can be determined by linearizing (2.1) about $(0,0)$ and using the D-subdivision method. Here, we define the synaptic inputs for a large group of neurons at position x and time t by functions $U_1(x,t)$ and $U_2(x,t)$ for the linearized system. Hence, the system is the following:

(2.2)

$$\frac{d}{dt}U_1(x,t) + l_1U_1(x,t) = \sigma_2 s_1 \int_{-\pi/2}^{\pi/2} J_{12}(x,y)U_2(y,t-\tau_{12}(x-y))\,dy,$$

$$\frac{d}{dt}U_2(x,t) + l_2U_2(x,t) = \sigma_1 s_1 \int_{-\pi/2}^{\pi/2} J_{21}(x,y)U_1(y,t-\tau_{21}(x-y))\,dy.$$

For simplicity, we take $K_1 = \sigma_1 s_1$, $K_2 = \sigma_2 s_1$. The delay terms show the propagation delay for the first and second populations. We assume $\tau(x-y) = \tau$, a constant delay. In order to find the characteristic equation, we take

$$U_1(x,t) = u_1(t)e^{ikx} = c_1e^{\lambda t}e^{ikx},$$

and

$$U_2(x,t) = u_2(t)e^{ikx} = c_2e^{\lambda t}e^{ikx};$$

hence, we obtain

$$\lambda e^{ikx}u_1(t) + l_1e^{ikx}u_1(t) - K_2e^{-\lambda\tau}u_2(t) \int_{-\pi/2}^{\pi/2} J_{12}(x,y)e^{iky}dy = 0,$$

$$\lambda e^{ikx}u_2(t) + l_2e^{ikx}u_2(t) - K_1e^{-\lambda\tau}u_1(t) \int_{-\pi/2}^{\pi/2} J_{21}(x,y)e^{iky}dy = 0.$$

For $x = 0$, the system of equations becomes

(2.3)
$$\begin{aligned} \lambda u_1(t) + l_1u_1(t) - K_2u_2(t)e^{-\lambda\tau}F_1 = 0, \\ \lambda u_2(t) + l_2u_2(t) - K_1u_1(t)e^{-\lambda\tau}F_2 = 0, \end{aligned}$$

where

$$F_1 = \int_{-\pi/2}^{\pi/2} J_{12}(x,y)e^{iky}dy \quad \text{and} \quad F_2 = \int_{-\pi/2}^{\pi/2} J_{21}(x,y)e^{iky}dy.$$

Hence, the characteristic values λ satisfy the equation

$$\lambda^2 + \lambda l_2 + \lambda l_1 + l_1 l_2 - K_1 K_2 e^{-2\lambda\tau} F_1 F_2 = 0. \tag{2.4}$$

Writing $\lambda = \mu + i\nu$ in (2.4), then, for the real and imaginary parts, we have

$$\text{Re} : \mu^2 - \nu^2 + \mu l_2 + \mu l_1 + l_1 l_2 - K_1 K_2 e^{-2\mu\tau} \cos(2\tau\nu) F_1 F_2 = 0, \tag{2.5}$$

$$\text{Im} : 2\mu\nu + \nu l_2 + \nu l_1 + K_1 K_2 e^{-2\mu\tau} \sin(2\tau\nu) F_1 F_2 = 0. \tag{2.6}$$

To obtain the D-curves, we take $\mu = 0$, and then we have

$$P(\nu, l_1, K) = -\nu^2 + l_1 l_2 - K_1 K_2 \cos(2\tau\nu) F_1 F_2 = 0, \tag{2.7}$$

$$R(\nu, l_1, K) = \nu l_2 + \nu l_1 + K_1 K_2 \sin(2\tau\nu) F_1 F_2 = 0. \tag{2.8}$$

Since we choose the parameter space (l_1, K_1), we have the following expressions for $F_1 F_2 K_2 \neq 0$

$$\begin{aligned} l_1 &= \frac{\nu^2 \sin(2\tau\nu) - \nu l_2 \cos(2\tau\nu)}{l_2 \sin(2\tau\nu) + \nu \cos(2\tau\nu)}, \\ K_1 &= \frac{-\nu l_2^2 - \nu^3}{l_2 K_2 \sin(2\tau\nu) F_1 F_2 + \nu K_2 \cos(2\tau\nu) F_1 F_2}, \end{aligned} \tag{2.9}$$

as the boundaries of D regions. In addition, we have the singular line

$$l_2 l_1 - K_2 F_1 F_2 K_1 = 0 \tag{2.10}$$

for $\nu = 0$ as a boundary of D regions.

In order to determine the asymptotic stability region of this model, we use the properties of D-curves. It is obvious from (2.9) that $l_2 \sin(2\tau\nu) + \nu \cos(2\tau\nu) \neq 0$. Hence, we need to determine the roots of $\tan(2\tau\nu) = -\nu/l_2$. Let α denote the least positive root of this equation. The regions have been specified where the D-curves are sketched in the following form:

$$I_0 = (0, \alpha), \qquad I_n = \left(\alpha + \frac{(n-1)\pi}{2\tau}, \alpha + \frac{n\pi}{2\tau}\right), \quad n = 1, 2, \ldots .$$

Lemma 2.1. *The* D*-curves given in* (2.9) *do not intersect each other.*

Proof. Let $2\tau\nu_a \in I_a$, $2\tau\nu_b \in I_b$, $a \neq b$. If we consider $l_1(2\tau\nu_a) = l_1(2\tau\nu_b)$ and $K_1(2\tau\nu_a) = K_1(2\tau\nu_b)$ for the D-curves, we get $\nu_a = \nu_b$. Hence, we conclude that the D-curves do not intersect each other. □

Lemma 2.2. *The* D*-curves given in* (2.9) *intersect the line* $l_1 = 0$ *only once. The* K_1 *coordinates of the* D*-curves above the* l_1 *axis are increasing as* n *increases, and the* K_1 *coordinates of the* D*-curves below the* l_1 *axis are decreasing as* n *increases.*

Proof. For each $2\tau\nu_n \in I_n$, considering that $l_1 = 0$, we obtain the K_{1n} coordinates as

$$K_{1n} = \frac{-\nu_n^2}{K_2 F_1 F_2 \cos(2\tau\nu_n)}.$$

Hence, the K_{1n} coordinates for the D-curves are uniquely determined. The condition $l_1 = 0$ yields

$$\tan(2\tau\nu) = \frac{l_2}{\nu} > 0$$

since $l_2 > 0$ in the model. When determining the sign of K_{1n} coordinates, the conditions $\sin(2\tau\nu_n) > 0$, $\cos(2\tau\nu_n) > 0$ and $\sin(2\tau\nu_n) < 0$, $\cos(2\tau\nu_n) < 0$ must be satisfied. For the first curve C_0,

$$2\tau\nu_0 \in \left(0, \frac{\pi}{4}\right),$$

and the sign for the K_1 coordinate is negative. In a similar manner, for the second curve C_1,

$$2\tau\nu_1 \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right),$$

and the sign for the K_1 coordinate is positive. Hence, we conclude that the K_1 coordinates of the D-curves above the l_1 axis are increasing as n increases, and the K_1 coordinates of the D-curves below the l_1 axis are decreasing as n increases. □

Lemma 2.3. *The following limits are satisfied for the* D*-curves*:

$$\lim_{2\tau\nu \to \alpha^-} l_1(2\tau\nu) = +\infty, \qquad \lim_{2\tau\nu \to \alpha^-} K_1(2\tau\nu) = -\infty,$$

$$\lim_{2\tau\nu \to \alpha + (2z\pi^-/2\tau)} l_1(2\tau\nu) = +\infty, \qquad \lim_{2\tau\nu \to \alpha + (2z\pi^+/2\tau)} l_1(2\tau\nu) = -\infty,$$

$$\lim_{2\tau\nu\to\alpha+(2z\pi^-/2\tau)} K_1(2\tau\nu)=-\infty, \qquad \lim_{2\tau\nu\to\alpha+(2z\pi^+/2\tau)} K_1(2\tau\nu)=+\infty,$$
$$\lim_{2\tau\nu\to\alpha+((2z+1)\pi^-/2\tau)} l_1(2\tau\nu)=+\infty, \qquad \lim_{2\tau\nu\to\alpha+((2z+1)\pi^+/2\tau)} l_1(2\tau\nu)=-\infty,$$
$$\lim_{2\tau\nu\to\alpha+((2z+1)\pi^+/2\tau)} K_1(2\tau\nu)=-\infty, \qquad \lim_{2\tau\nu\to\alpha+((2z+1)\pi^-/2\tau)} K_1(2\tau\nu)=+\infty.$$

Proof. The proof follows from (2.9). □

Lemma 2.4. *The only intersection point for the curve C_0 and the singular line C_* is the limit point.*

Proof.

$$\lim_{\upsilon\to 0} l_1(2\tau\upsilon) = \frac{-l_2}{2\tau l_2+1}$$
$$\lim_{\upsilon\to 0} K_1(2\tau\upsilon) = \frac{-l_2^2}{K_2F_1F_2(2\tau l_2+1)}.$$

Hence, the intersection point is

$$\left(\frac{-l_2}{2\tau l_2+1}, \frac{-l_2^2}{K_2F_1F_2(2\tau l_2+1)}\right). \qquad \square$$

Now, we sketch the graph of D-curves for the parameters $K_2 = F_1 = F_2 = l_2 = 1$, and we investigate the affect of the delay term τ on the stability of system (2.2).

To find the number of characteristic exponents with positive real parts, we use Stépán's formula **[7, 8]**. We use the functions $P(\nu, l_1, K_1)$ and $R(\nu, l_1, K_1)$ on the parameter space (l_1, K_1) and choose a point $B(l_0, K_0)$ in any subregion determined by the D-curves.

Let the positive real roots of $P(\nu, l_0, K_0)$ be $\omega = \rho_j$, $j = 1, \dots, s$, such that $\rho_1 \geq \cdots \geq \rho_s$, and let the nonnegative real roots of $R(\nu, l_0, K_0)$ be $\omega = \sigma_i$, $i = 1, \dots, s$ such that $\sigma_1 \geq \cdots \geq \sigma_s = 0$.

If the dimension of (2.2) is even ($d = 2m$, $m \in Z^+$), then the number of characteristic roots with positive real parts in this subregion is given by

$$k = m + (-1)^m \sum_{j=1}^{s} (-1)^{j+1} \operatorname{sgn}(R(\rho_j, l_0, K_0)). \tag{2.11}$$

If the dimension of (2.2) is odd ($d = 2m+1$, $m \in Z^+$), then the number of characteristic roots with positive real parts in this subregion is given by

(2.12)
$$k = m + \frac{1}{2} + (-1)^m \left[\frac{1}{2}(-1)^s \operatorname{sgn}(P(0, l_0, K_0) + \sum_{j=1}^{s-1} (-1)^j \operatorname{sgn}(P(\sigma_i, l_0, K_0)) \right].$$

In the graphs (Figures 1–3), the regions where $k = 0$ denote the asymptotic stability regions.

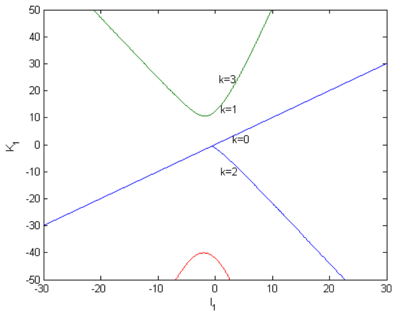

FIGURE 1. The stability region for $\tau = 0.5$.

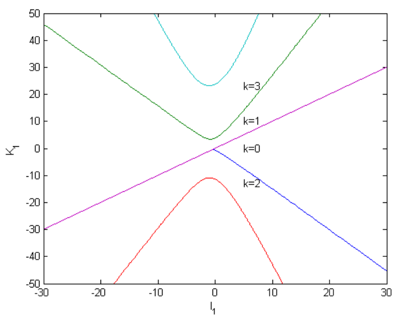

FIGURE 2. The stability region for $\tau = 1$.

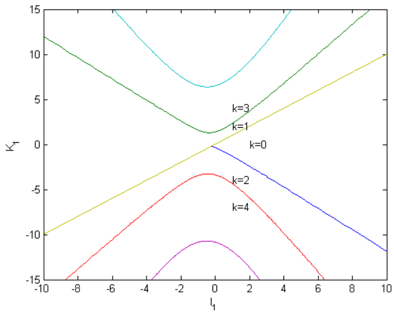

FIGURE 3. The stability region for $\tau = 2$.

As the delay term τ increases, the stability region becomes smaller. As a result, the delay term plays an important role in the stability of the system.

Now, we give a theorem using the above four lemmas and the graphics to define the asymptotic stability region.

Theorem 2.5. *The asymptotic stability region of system* (2.2) *is determined by*

$$l_1 > \frac{-l_2}{2\tau l_2 + 1}$$

and

$$\frac{-\nu l_2^2 - \nu^3}{l_2 K_2 \sin(2\tau\nu) F_1 F_2 + \nu K_2 \cos(2\tau\nu) F_1 F_2} < K_1 < \frac{l_2 l_1}{K_2 F_1 F_2},$$

where $\nu \in I_0$.

Proof. The asymptotic stability region is determined by the limit point

$$\left(\frac{-l_2}{2\tau l_2 + 1}, \frac{-l_2^2}{K_2 F_1 F_2 (2\tau l_2 + 1)}\right),$$

the singular line C_* and the D-curve C_0. For the coordinates of the asymptotic stability region, we write

$$l_1 > \frac{-l_2}{2\tau l_2 + 1}$$

and

$$\frac{-\nu l_2^2 - \nu^3}{l_2 K_2 \sin(2\tau\nu) F_1 F_2 + \nu K_2 \cos(2\tau\nu) F_1 F_2} < K_1 < \frac{l_2 l_1}{K_2 F_1 F_2},$$

where $\nu \in I_0$. □

3. Conclusions. In this study, we considered the neural field model for two neural population. We assumed that the neurons of these populations interact with each other. First, we linearized the system, and then we used the D-subdivision method to sketch the region where we investigated the stability of this system. After determining the D-regions, we used the Stépán's formula to find the asymptotic stability region. We also determined the number of unstable characteristic exponents for the unstable regions. As shown in the graphs, we conclude that the change in the delay term affects the stability of the system. The asymptotic stability region becomes smaller while the delay term τ increases.

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