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NOTE ON THE TRUNCATED GENERALIZATIONS OF GAUSS'S SQUARE EXPONENT THEOREM

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ABSTRACT. In this note, we investigate Liu's work on the truncated Gaussian square exponent theorem and obtain more truncations. We also discuss some possible multiple summation extensions of Liu's results.

1. Introduction. One major topic of q-series deals with various q-identities, most of which can be treated as the q-analog of combinatorial identities. Some renowned examples include Euler's pentagonal number theorem [**1,** Corollary 1.7]

$$\prod_{n\geq 1}(1-q^n)=\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2} \tag{1.1}$$

and Gauss's square exponent theorem [**1,** Corollary 2.10]

$$\prod_{n\geq 1}\frac{1-q^n}{1+q^n}=\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}. \tag{1.2}$$

Interestingly, some q-identities involving infinite sums and/or products also have the corresponding truncated version. Before presenting such truncations, we introduce some standard q-series notation:

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

$$(a;q)_\infty := \prod_{k\geq 0}(1-aq^k).$$

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We also adopt the q-binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} & \text{if } 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases}$$

In [**4**], Berkovich and Garvan combinatorially proved the following finite q-identity

$$\sum_{k=-L}^{L} (-1)^k q^{k(3k+1)/2} \begin{bmatrix} 2L-k \\ L+k \end{bmatrix} = 1. \tag{1.3}$$

If we let $L \to \infty$, then (1.3) becomes (1.1). In fact, (1.3) is a direct consequence of

$$\sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r q^{\binom{r}{2}} \begin{bmatrix} n-r \\ r \end{bmatrix} = \begin{cases} (-1)^{\lfloor n/3 \rfloor} q^{n(n-1)/6} & \text{if } n \not\equiv 2 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{1.4}$$

which first appears in [**6**]. To see this, we only need to replace n by $3L$, r by $L+k$ and q by $1/q$ in (1.4). On the other hand, Warnaar [**16**] observed that, if one replaces n by $3L+1$, r by $L+k$ and q by $1/q$ in (1.4), another truncated generalization of Euler's pentagonal number theorem can be derived

$$\sum_{k=-L}^{L} (-1)^k q^{k(3k-1)/2} \begin{bmatrix} 2L-k+1 \\ L+k \end{bmatrix} = 1. \tag{1.5}$$

Recently, Liu [**12**] obtained more truncated versions of (1.1) with a surprisingly elementary proof.

The truncations of Gauss's square exponent theorem (1.2), however, mainly come from a different direction. For example, in [**8**], Guo and Zeng showed that, for $L \ge 1$,

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{k=-L}^{L} (-1)^k q^{k^2} = 1 + (-1)^L \sum_{n=L+1}^{\infty} \frac{q^{(L+1)n} (-q;q)_L (-1;q)_{n-L}}{(q;q)_n} \begin{bmatrix} n-1 \\ L \end{bmatrix}. \tag{1.6}$$

The origin of this type of truncation comes from Andrews and Merca's work **[2]** on Euler's pentagonal number theorem. For other similar truncated theta series, the interested readers may refer to **[3, 5, 9, 11, 14, 17]**. Nonetheless, one should admit that (1.6) is complicated, especially compared with (1.3) and (1.5). Hence, we would expect truncated generalizations of Gauss's square exponent theorem as neat as (1.3) and (1.5). In **[13]**, Liu provided such truncations:

$$\sum_{k=-L}^{L}(-1)^k q^{k^2}(-q;q)_{L-k}\begin{bmatrix}3L-k+1\\ L+k\end{bmatrix}=1, \tag{1.7}$$

$$\sum_{k=-L}^{L}(-1)^k q^{k^2}(-q;q)_{L-k}\begin{bmatrix}3L-k\\ L+k\end{bmatrix}\frac{1-q^{2L}}{1-q^{3L-k}}=1, \tag{1.8}$$

$$\sum_{k=-L}^{L}(-1)^k q^{k^2}(-q;q)_{L-k}\begin{bmatrix}3L-k-1\\ L+k-1\end{bmatrix}=1. \tag{1.9}$$

All three identities are, respectively, direct consequences of identities analogous to (1.4):

$$\sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix} = \begin{cases}0 & \text{if } n=2m-1,\\ (-1)^m q^{m(3m+1)} & \text{if } n=2m,\end{cases} \tag{1.10}$$

$$\sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r\\ r\end{bmatrix}\frac{1-q^n}{1-q^{2n-r}} = \begin{cases}0 & \text{if } n=2m-1,\\ (-1)^m q^{m(3m-1)} & \text{if } n=2m,\end{cases} \tag{1.11}$$

$$\sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r\\ r\end{bmatrix} = \begin{cases}(-1)^{m-1} q^{3m^2-3m+1} & \text{if } n=2m-1,\\ (-1)^m q^{m(3m-1)} & \text{if } n=2m.\end{cases} \tag{1.12}$$

For example, (1.7) is deduced by replacing n by $2L$, r by $L+k$ and q by $1/q$ in (1.10). We remark that (1.10) appears in an early paper of Jouhet **[10,** equation (2.7)].

We have two purposes in this note. The first purpose is to further investigate Liu's results. We then discuss some possible multiple summation extensions of (1.10) and (1.12), the idea of which originates from [**7**].

2. Further investigation of Liu's results. We begin with the following identity deduced from (1.10).

Theorem 2.1. *For $n \geq 1$,*

$$\sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\frac{1-q^{2n+1}}{1-q^{2n-r+1}} = \begin{cases}(-1)^m q^{m(3m-1)} & \text{if } n=2m-1,\\ (-1)^m q^{m(3m+1)} & \text{if } n=2m.\end{cases} \tag{2.1}$$

Proof. Following Liu's notation, we write the left-hand side of (1.10) as U_n, namely,

$$U_n = \sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}.$$

Then,

$$\begin{aligned}
&\sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\frac{1-q^{2n+1}}{1-q^{2n-r+1}}\\
&\quad= \sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\left(1+q^{2n-r+1}\frac{1-q^r}{1-q^{2n-r+1}}\right)\\
&\quad= U_n - q^{2n}\sum_{r=1}^{n}(-1)^{r-1} q^{\binom{r-1}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r\\ r-1\end{bmatrix}\\
&\quad= U_n - q^{2n}\sum_{r=0}^{n-1}(-1)^{r} q^{\binom{r}{2}}(-q;q)_{n-r-1}\begin{bmatrix}2n-r-1\\ r\end{bmatrix}\\
&\quad= U_n - q^{2n}U_{n-1}.
\end{aligned}$$

The desired result follows from (1.10). $\square$

If we replace n by $2L-1$ and r by $L+k$ in (2.1), then

$$\sum_{k=-L}^{L-1}(-1)^{L+k}q^{\binom{L+k}{2}}(-q;q)_{L-k-1}\begin{bmatrix}3L-k-1\\ L+k\end{bmatrix}\frac{1-q^{4L-1}}{1-q^{3L-k-1}} = (-1)^L q^{L(3L-1)}.$$

We then replace q by $1/q$ and note that

$$\begin{bmatrix}n\\ m\end{bmatrix}_{q^{-1}} = q^{m(m-n)}\begin{bmatrix}n\\ m\end{bmatrix}_q$$

and

$$(-q^{-1};q^{-1})_n = q^{-\binom{n+1}{2}}(-q;q)_n.$$

Hence,

$$(-1)^L q^{-L(3L-1)} = \sum_{k=-L}^{L-1}(-1)^{L+k}q^{-\binom{L+k}{2}}q^{-\binom{L-k}{2}}(-q;q)_{L-k} \times q^{(L+k)(-2L+2k+1)}\begin{bmatrix}3L-k+1\\ L+k\end{bmatrix}\frac{1-q^{-(4L-1)}}{1-q^{-(3L-k-1)}}.$$

This leads to a new truncation of Gauss's square exponent theorem, as follows.

Theorem 2.2. *For $L\geq 1$,*

$$\sum_{k=-L}^{L-1}(-1)^k q^{k^2}(-q;q)_{L-k-1}\begin{bmatrix}3L-k-1\\ L+k\end{bmatrix}\frac{1-q^{4L-1}}{1-q^{3L-k-1}} = 1. \tag{2.2}$$

We next observe that

$$\begin{aligned}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\frac{1-q^{2n+1}}{1-q^{2n-r+1}} &= \begin{bmatrix}2n-r\\ r-1\end{bmatrix}\frac{1-q^{2n+1}}{1-q^r}\\ &= \begin{bmatrix}2n-r\\ r-1\end{bmatrix}\left(1+\frac{q^r(1-q^{2n-r+1})}{1-q^r}\right)\\ &= \begin{bmatrix}2n-r\\ r-1\end{bmatrix}+q^r\begin{bmatrix}2n-r+1\\ r\end{bmatrix}.\end{aligned} \tag{2.3}$$

On one hand, we have

Theorem 2.3. *For $n \geq 1$,*

(2.4)
$$\sum_{r=0}^{n}(-1)^r q^{\binom{r-1}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\frac{1-q^{2n+1}}{1-q^{2n-r+1}} = \begin{cases}(-1)^m q^{(m-1)(3m-2)} & \text{if } n=2m-1,\\ (-1)^m q^{m(3m+1)+1} & \text{if } n=2m.\end{cases}$$

Proof. It follows from (2.3) that

$$\begin{aligned}
&\sum_{r=0}^{n}(-1)^r q^{\binom{r-1}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\frac{1-q^{2n+1}}{1-q^{2n-r+1}}\\
&\quad=\sum_{r=0}^{n}(-1)^r q^{\binom{r-1}{2}}(-q;q)_{n-r}\left(\begin{bmatrix}2n-r\\ r-1\end{bmatrix}+q^r\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\right)\\
&\quad=-\sum_{r=1}^{n}(-1)^{r-1} q^{\binom{r-1}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r\\ r-1\end{bmatrix}+qU_n\\
&\quad=-\sum_{r=0}^{n-1}(-1)^{r} q^{\binom{r}{2}}(-q;q)_{n-r-1}\begin{bmatrix}2n-r-1\\ r\end{bmatrix}+qU_n\\
&\quad=-U_{n-1}+qU_n.
\end{aligned}$$

The desired result follows from (1.10). □

If we replace n by $2L$, r by $L+k$ and q by $1/q$ in (2.4), we obtain another new truncation of Gauss's square exponent theorem.

Theorem 2.4. *For $L \geq 1$,*

$$\sum_{k=-L}^{L}(-1)^k q^{k^2}(-q;q)_{L-k}\begin{bmatrix}3L-k+1\\ L+k\end{bmatrix}\frac{1-q^{4L+1}}{1-q^{3L-k+1}}=1. \tag{2.5}$$

On the other hand, we obtain, from (2.3),

Theorem 2.5. *For $n \geq 1$,*

(2.6)
$$\sum_{r=0}^{n}(-1)^r q^{\binom{r+1}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix} = \sum_{m=-\lfloor n/2\rfloor}^{\lfloor (n+1)/2\rfloor}(-1)^m q^{m(3m-1)}.$$

Proof. For convenience, we write

$$\widetilde{U}_n = \sum_{r=0}^{n}(-1)^r q^{\binom{r+1}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}.$$

It follows from (2.3) that

$$\begin{aligned}
&\sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\frac{1-q^{2n+1}}{1-q^{2n-r+1}}\\
&\quad= \sum_{r=0}^{n}(-1)^r q^{\binom{r}{2}}(-q;q)_{n-r}\left(\begin{bmatrix}2n-r\\ r-1\end{bmatrix} + q^r\begin{bmatrix}2n-r+1\\ r\end{bmatrix}\right)\\
&\quad= -\sum_{r=1}^{n}(-1)^{r-1} q^{\binom{r}{2}}(-q;q)_{n-r}\begin{bmatrix}2n-r\\ r-1\end{bmatrix} + \widetilde{U}_n\\
&\quad= -\sum_{r=0}^{n-1}(-1)^r q^{\binom{r+1}{2}}(-q;q)_{n-r-1}\begin{bmatrix}2n-r-1\\ r\end{bmatrix} + \widetilde{U}_n\\
&\quad= -\widetilde{U}_{n-1} + \widetilde{U}_n.
\end{aligned}$$

From this telescoping identity, along with (2.1) and the fact that $\widetilde{U}_0 = 1$, we arrive at the desired result. □

Remark 2.6. Letting $n \to \infty$ in (2.6) reduces it to

$$(-q;q)_\infty \sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r+1}{2}}}{(q;q)_r} = \sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m-1)}.$$

We further deduce from Euler's pentagonal number theorem (1.1) that

$$\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r+1}{2}}}{(q;q)_r} = \frac{(q^2;q^2)_\infty}{(-q;q)_\infty} = (q;q)_\infty.$$

This identity, which is a special case of the q-binomial theorem (cf., [**1,** Theorem 2.1]), is another pioneering work of q-identities due to Euler;

see [**1,** Corollary 2.2]. The interested reader may refer to [**12,** Theorem 1.1] for the following, different truncation of this identity:

$$\sum_{r=0}^{\lfloor n/2\rfloor}(-1)^r q^{\binom{r+1}{2}}\begin{bmatrix}n-r\\ r\end{bmatrix}=\sum_{m=-\lfloor (n+1)/3\rfloor}^{\lfloor n/3\rfloor}(-1)^m q^{m(3m+1)/2}.$$

3. Multiple summations. In [**7**], Guo and Zeng obtained the multiple summation extensions of (1.3) and (1.5):

(3.1)
$$\sum_{j_1,\ldots,j_m=-L}^{2L}\prod_{k=1}^{m}(-1)^{j_k}q^{j_kj_{k+1}+\binom{j_k+1}{2}}\begin{bmatrix}2L-j_k\\ L+j_{k+1}\end{bmatrix}=\begin{cases}1 & \text{if } m\not\equiv 0 \pmod 3,\\ 3L+1 & \text{if } m\equiv 0 \pmod 3,\end{cases}$$

(3.2)
$$\sum_{j_1,\ldots,j_m=-L}^{2L+1}\prod_{k=1}^{m}(-1)^{j_k}q^{j_kj_{k+1}+\binom{j_k}{2}}\begin{bmatrix}2L-j_k+1\\ L+j_{k+1}\end{bmatrix}=\begin{cases}(-1)^{\lfloor m^2/3\rfloor} & \text{if } m\not\equiv 0 \pmod 3,\\ (-1)^{m/3}(3L+2) & \text{if } m\equiv 0 \pmod 3.\end{cases}$$

Here, we assume that $j_{m+1}=j_1$. The two multiple summations come from a multiple extension of (1.4). Motivated by their work, we study some possible multiple extensions of (1.10) and (1.12).

Parallel to Liu's notation in [**13**], for the positive integer m, we put

$$U_m(n)=\sum_{r_1,\ldots,r_m=0}^{2n+1}\prod_{k=1}^{m}(-1)^{r_k}q^{\binom{r_k}{2}}(-q;q)_{n-r_k}\begin{bmatrix}2n-r_k+1\\ r_{k+1}\end{bmatrix},$$

$$W_m(n)=\sum_{r_1,\ldots,r_m=0}^{2n}\prod_{k=1}^{m}(-1)^{r_k}q^{\binom{r_k}{2}}(-q;q)_{n-r_k}\begin{bmatrix}2n-r_k\\ r_{k+1}\end{bmatrix},$$

where, again, we assume that $r_{m+1}=r_1$. Hence, $U_1(n)$ and $W_1(n)$ reduce to Liu's U_n and W_n, respectively.

We shall show the following.

Theorem 3.1. *For $n \geq 1$,*

$$U_2(n) = 0, \tag{3.3}$$

$$U_3(n) = \begin{cases} 0 & \text{if } n = 2k-1, \\ \dfrac{(-1)^{k-1}}{2} q^{9k^2+3k} & \text{if } n = 2k, \end{cases} \tag{3.4}$$

$$W_2(n) = (-1)^n q^{n(3n-1)/2}, \tag{3.5}$$

$$W_3(n) = \begin{cases} \dfrac{(-1)^k}{2}(q^{9k^2-9k+3} - 3q^{9k^2-11k+3}) & \text{if } n = 2k-1, \\ \dfrac{(-1)^{k-1}}{2}(q^{9k^2-3k} - 3q^{9k^2-k}) & \text{if } n = 2k. \end{cases} \tag{3.6}$$

Instead of using the traditional q-series approach, we turn to a computer-assisted proof of Theorem 3.1. We recall that Riese implemented a powerful *Mathematica* package, `qMultiSum`, whose main function is generating recurrence relations for multiple summation q-identities. We refer to **[15]** or the url:

`http://www.risc.jku.at/research/combinat/software/ergosum/RISC/qMultiSum.html`

for an introduction to this package.

In our cases, the package gives us the following.

Lemma 3.2. *For $n \geq 1$,*

(3.7)

$$\begin{aligned}
0 &= -q^{6n+8}U_2(n) + U_2(n+2), \\
0 &= q^{9n+12}U_3(n) + U_3(n+2), \\
0 &= -q^{9n+11}(1+q^{n+3})W_2(n) - q^{6n+10}(1+q^{n+3})W_2(n+1) \\
&\quad + q^{3n+7}(1+q^{n+1})W_2(n+2) + (1+q^{n+1})W_2(n+3), \\
0 &= -q^{15n+24}(1+q^{n+2})(-1+q^{n+3})(1+q^{n+3})(1+q^{n+4}) \\
&\qquad \times \Big(-1-2q^{n+2}+q^{n+3}\Big)W_3(n) \\
&\quad - q^{11n+23}(1+q^{n+3})(1+q^{n+4}) \\
&\qquad \times \Big(-1+q-2q^{n+2}-2q^{n+3}+2q^{n+4}-6q^{2n+4}+6q^{2n+5}
\end{aligned}$$

$$
\begin{aligned}
&\qquad - q^{2n+6} - q^{2n+7} + 2q^{3n+7} + 3q^{3n+8}\\
&\qquad - 2q^{3n+9} - q^{3n+10} + 5q^{4n+9} - 2q^{4n+10}\\
&\qquad - 3q^{4n+11} + 2q^{4n+12} - 2q^{5n+12} + 2q^{5n+13}\Big) W_3(n+1)\\
&+ q^{6n+18}(1+q^{n+1})(1+q^{n+4})(-1+q^{2n+5})\\
&\quad \times \Big(1 + 2q^{n+1} - q^{n+3} + 7q^{2n+3} - 6q^{2n+4} - 2q^{2n+5} + q^{2n+6}\\
&\qquad + 2q^{3n+6} - q^{3n+8} + q^{4n+10}\Big) W_3(n+2)\\
&+ q^{2n+8}(1+q^{n+1})(1+q^{n+2})\\
&\quad \times \Big(- 2 + 2q + 5q^{n+2} - 2q^{n+3} - 3q^{n+4} + 2q^{n+5} + 2q^{2n+5}\\
&\qquad + 3q^{2n+6} - 2q^{2n+7} - q^{2n+8} - 6q^{3n+7} + 6q^{3n+8}\\
&\qquad - q^{3n+9} - q^{3n+10} - 2q^{4n+10} - 2q^{4n+11} + 2q^{4n+12}\\
&\qquad - q^{5n+13} + q^{5n+14}\Big) W_3(n+3)\\
&+ (1+q^{n+1})(-1+q^{n+2})(1+q^{n+2})(1+q^{n+3})\\
&\quad \times \Big(2 - q + q^{n+3}\Big) W_3(n+4).
\end{aligned}
$$

Proof. We prove (3.7) by calling (with the initialization `<<RISC‘qMultiSum‘`)

```
stru = qFindStructureSet[qBinomial[2n-r1+1,r2,q]
     qBinomial[2n-r2+1,r1,q] (-1)^(r1+r2) q^(r1
  (r1-1)/2+r2(r2-1)/2) qPochhammer[-q,q,n-r1]
  qPochhammer[-q,q,n-r2], {n}, {r1,r2}, {1},
  {1,1}, {1,1}, qProtocol->True]
rec = qFindRecurrence [qBinomial[2n-r1+1,r2,q]
    qBinomial [2n-r2+1,r1,q] (-1)^(r1+r2) q^(r1
  (r1-1)/2+r2(r2-1)/2) qPochhammer [-q,q,n-r1]
  qPochhammer [-q,q,n-r2], {n}, {r1,r2}, {1},
  {1,1}, {1,1}, qProtocol->True, StructSet->
  stru[[1]]]
sumrec = qSumRecurrence[rec]
```

For the remaining three recurrence relations, apart from the corresponding summand, we may set other parameters as follows:

```
{n}, {r1,...,rm}, {1}, {1,...,1}, {1,...,1}
```

We remark that it costs over five hours to obtain the recurrence relation for $W_3(n)$. $\square$

Proof of Theorem 3.1. Theorem 3.1 is a direct consequence of Lemma 3.2 and several initial values. $\square$

Of course, it will be exciting to see traditional q-series proofs of identities in Theorem 3.1. We also notice from Theorem 3.1 that the multiple extensions of Gauss's square exponent theorem are not as neat as Guo and Zeng's multiple extensions of Euler's pentagonal number theorem (cf., [**7,** Corollary 2.3 and Theorem 2.4]). However, it would be appealing to see if there exist closed forms of $U_m(n)$ and $W_m(n)$ for arbitrary m.

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